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A local Steiner–type formula for general closed sets and applications

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Abstract. We introduce support (curvature) measures of an arbitrary closed set A in $\mathbb{R}^d$ and establish a local Steiner–type formula for the localized parallel volume of A. We derive some of the basic properties of these support measures and explore how they are related to the curvature measures available in the literature. Then we use the support measures in analysing contact distributions of stationary random closed sets, with a particular emphasis on the Boolean model with general compact particles.

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1 Introduction

Let A denote a non-empty and closed subset of $\mathbb{R}^d$, $d \geq 2$, and let $A_{\oplus r}$ be the *parallel set* of A at distance $r \geq 0$, i.e. the set of all points $x \in \mathbb{R}^d$ the distance of which from A is at most r. If A is convex and compact, then the famous *Steiner formula* expresses the volume $V_d(A_{\oplus r})$ of $A_{\oplus r}$ as a polynomial in r,

$$V_d(A_{\oplus r}) = \sum_{j=0}^{d} r^{d-j} \kappa_{d-j} V_j(A). \tag{1.1}$$

Here, κ_j is the (j-dimensional) volume of the Euclidean unit ball in $\mathbb{R}^j$ and the coefficients $V_0(A), \ldots, V_d(A)$ are the *intrinsic volumes* of the convex body A (see e.g. [27, (4.2.27)]). Clearly, $V_d(A)$ is the volume of A, $V_{d-1}(A)$ is half the surface area and $V_0(A) = 1$. Formula (1.1) has been extended and refined in several ways.

The parallel set $A_{\oplus r}$ is equal to the *Minkowski sum* of A and a Euclidean ball of radius r. Expanding the volume of the Minkowski sum of several compact convex sets, one arrives at the theory of *mixed volumes* and *mixed area measures* (see [1], [6]). The *area measures* of just one compact convex set A can then be defined as the mixed area measures of A and a Euclidean ball. Alternatively, Fenchel and Jessen found a simpler approach to the area measures of A, which is based on the notion of a *local parallel set* and a corresponding local version of the Steiner formula (1.1). Federer [4] has introduced *curvature measures* for sets A with *positive reach* via a different localization of the parallel volume and again by means of a local Steiner formula. An important predecessor of Federer's work is Weyl's [34] *tube formula* for the volume of a tube around a submanifold of $\mathbb{R}^d$ (cf. [12]). Schneider [25] has defined *support measures* (or generalized curvature measures) of a convex set A by combining both the curvature measures and the area measures of A into one measure. Support measures can be extended to (locally finite) unions of convex sets (see [26]) and sets with positive reach ([24]), respectively.

Curvatures and curvature measures are fundamental geometric concepts and it is remarkable that they can be defined for very general sets. In fact, it is our main aim in this paper to use a local Steiner–type formula for introducing support measures of an *arbitrary* closed set $A \subset \mathbb{R}^d$. To reach this goal we will first refine (and clarify) the main results in [30]. In this important paper, Stachó shows that the (generalized) *normal bundle* $N(A)$ of A is *countably* $(d-1)$*-rectifiable* (see [5]) and expresses the local parallel volume of A in terms of signed measures on $N(A) \times (0, \infty)$. We will refine and extend this result by proving the existence of uniquely determined signed measures on $N(A)$, the *support measures* of the closed set A, which arise as coefficient measures of a local Steiner formula. As a consequence of our measure geometric approach to such a Steiner formula, we can give an explicit description of the support measures of a general closed set $A \subset \mathbb{R}^d$ in terms of *generalized principal curvatures* defined on $N(A)$ and the $(d-1)$-dimensional Hausdorff measure on $N(A)$. If A is a general closed set (a fractal, for instance), then these support measures need not have a locally finite total variation. Instead we will derive a crucial integrability property of the principal curvatures and the *reach function* of A which in turn leads to the appropriate property of local finiteness that the support measures need to satisfy. Having proved the existence of the support measures, we will then proceed with discussing some of their basic properties and with relating them to several other notions of curvature measures available in the literature.

Our second aim in this paper is the application of the general support measures in stochastic geometry, where curvature measures have proved to be very useful. On the one hand, curvature measures are used to define basic geometric mean values associated with random closed sets. On the other hand, they can be exploited to analyse some deeper geometric properties of random closed sets. Here we consider the *contact distributions* of a stationary random closed set Z in $\mathbb{R}^d$, i.e. the joint distributions of the distance between a point $x \in \mathbb{R}^d$ and Z and the associated normalized *contact vector* (see [31], [19]). Applying our general support measures, we are able to considerably generalize some of the recent results in [22] and [17] which have been proved under the assumption that Z is a countable union of (random) convex sets. We finally discuss the important special case of a stationary Boolean model Z with general compact particles. In this case, we will not only derive the form of the direction dependent contact distributions, but we will also obtain more detailed results on the relationship between *intensities* related to the support measures of Z and the corresponding mean values associated with a typical grain of the Boolean model. These formulae are significant extensions of some results in [23] and [17].

2 Steiner formula and support measures

2.1 Preliminaries

We are working in the d-dimensional space $\mathbb{R}^d$ with Euclidean norm $|\cdot|$. For a set $A \subset \mathbb{R}^d$, we denote by $\operatorname{int} A$ the interior, by $\operatorname{cl} A$ the closure, and by ∂A the boundary of A. The i-dimensional Hausdorff measure on $\mathbb{R}^d$ is denoted by $\mathcal{H}^i$. For $z \in \mathbb{R}^d$ and $r \geq 0$, $B^d(z, r) := \{y \in \mathbb{R}^d : |y - z| \leq r\}$ is the ball with centre z and radius r. The unit ball $B^d := B^d(0, 1)$ has volume κ_d and its boundary S^{d-1} (the unit sphere) has surface content $\omega_d = d\kappa_d$. The distance $d(A, z)$ between a set $A \subset \mathbb{R}^d$ and a point $z \in \mathbb{R}^d$ is defined as $\inf\{|y - z| : y \in A\}$, where $\inf \emptyset := \infty$. We write $p(A, z) := y$ whenever y is a uniquely determined point in A with $d(A, z) = |y - z|$. This is the *metric projection* of z on to A. If $0 < d(A, z) < \infty$ and $p(A, z)$ exists, then $p(A, z)$ lies on the boundary ∂A of A and we define $u(A, z) := (z - p(A, z))/d(A, z)$. Finally, for real numbers a, b we put $a \wedge b := \min\{a, b\}$.

In the following, we fix a non-empty closed set $A \subset \mathbb{R}^d$. The closed complement $\operatorname{cl}(\mathbb{R}^d \setminus A)$ of A is abbreviated by A^*. The *exoskeleton* $\operatorname{exo}(A)$ of A consists of all points of $\mathbb{R}^d \setminus A$ which do not admit a metric projection on to A. This is a measurable set (see Lemma 6.1) and it is well known that

$$\mathcal{H}^d(\operatorname{exo}(A)) = 0; \tag{2.1}$$

cf. [30], or [8], [16, Corollary 2.3] for more general results. It is convenient to extend the definition of $p(A, z)$ and $u(A, z)$ in a suitable and measurable way to all $z \in \mathbb{R}^d$. The *normal bundle* of A is defined by

$$N(A) := \{(p(A, z), u(A, z)) : z \notin A \cup \operatorname{exo}(A)\}.$$

It is a measurable subset of $\partial A \times S^{d-1}$ (see Lemma 6.2). Simple examples show that $\mathcal{H}^{d-1}(N(A) \cap (B \times S^{d-1}))$ can be infinite for compact sets $B \subset \mathbb{R}^d$. However, it will follow from Lemma 2.3 that $N(A)$ has σ-finite $(d-1)$-dimensional Hausdorff measure. In fact, $N(A)$ is countably $(d-1)$-rectifiable (in the sense of [5]). The *reach function* $\delta(A, \cdot) : \mathbb{R}^d \times S^{d-1} \to [0, \infty]$ of A is defined by

$$\delta(A, x, u) := \inf\{t \geq 0 : x + tu \in \operatorname{exo}(A)\}, \quad (x, u) \in N(A),$$

and $\delta(A, x, u) := 0$ for $(x, u) \notin N(A)$. Note that $\delta(A, \cdot) > 0$ on $N(A)$; moreover $\delta(A, \cdot) \equiv \infty$ on $N(A)$ if A is convex. By Lemma 6.2, $\delta(A, \cdot)$ is a measurable function. The number

$$\operatorname{reach}(A) := \inf\{\delta(A, x, u) : (x, u) \in N(A)\}$$

is called the *reach* of A. If $\mathrm{reach}(A) > 0$, then A is said to have *positive reach*. This definition is consistent with the one given in [4].

Let $(\mathbf{X}, \mathcal{X})$ denote a measurable space and consider a measurable function $h : \mathbf{X} \to [0, \infty)$. We call a subset of $\mathbf{X}$ h-bounded if it is contained in the sublevel set $\{x \in \mathbf{X} : h(x) \le c\}$, for some $c \in \mathbb{R}$. A $[-\infty, \infty]$-valued function μ which is defined on the system of h-bounded sets in $\mathcal{X}$ is called a *signed h-measure* if its restriction to each sublevel set of h is a signed measure of finite variation. In this case we obtain from the Hahn–decomposition a unique representation $\mu = \mu^+ - \mu^-$ with mutually singular σ-finite measures μ^+ and μ^- which are finite on each sublevel set. Although μ^+ and μ^- are defined on all measurable sets, it is in general not possible to extend μ to all measurable sets by $\mu = \mu^+ - \mu^-$. The measure $|\mu| := \mu^+ + \mu^-$ is the *total variation measure* of μ. For any measurable function $f : \mathbf{X} \to [-\infty, \infty]$, we define the integral $\int f d\mu$ as $\int f d\mu^+ - \int f d\mu^-$ whenever this difference is well defined, i.e. whenever the integrals $\int f d\mu^+$ and $\int f d\mu^-$ are both defined and the above difference is not of the form $-\infty + \infty$ or $\infty - \infty$.

For a closed set $A \subset \mathbb{R}^d$ and $(x, u) \in \mathbb{R}^d \times S^{d-1}$, let

$$h_A(x, u) := \mathbf{1}\{(x, u) \in N(A)\} \max\{|x|, \delta(A, x, u)^{-1}\}.$$

A *reach measure* μ of A is then a signed h_A-measure, where we require, in addition, that μ vanishes outside $N(A)$, hence $|\mu|(\{(x, u) \in \mathbb{R}^d \times S^{d-1} : (x, u) \notin N(A)\}) = 0$.

2.2 A general Steiner formula

Now we can state our general Steiner–type formula for arbitrary closed sets in Theorem 2.1. Apart from its generality concerning the class of sets considered, a crucial feature of this theorem is a new integrability condition, which is stated in terms of the total variation measures of the support measures and the reach function. As a consequence of our measure geometric approach, we can describe the support measures of a closed set A as integrals over the generalized normal bundle of A (see Corollary 2.5). We also take the opportunity to simplify and clarify some of the arguments in [30].

Theorem 2.1. *For any non-empty closed set $A \subset \mathbb{R}^d$, there exist uniquely determined reach measures $\mu_0(A; \cdot), \dots, \mu_{d-1}(A; \cdot)$ of A satisfying*

$$\int_{N(A)} \mathbf{1}\{x \in B\}(\delta(A, x, u) \wedge r)^{d-j} |\mu_j|(A; d(x, u)) < \infty, \tag{2.2}$$

$j = 0, \dots, d-1$, for all compact sets $B \subset \mathbb{R}^d$ and all $r > 0$, such that, for any measurable bounded function $f : \mathbb{R}^d \to \mathbb{R}$ with compact support,

$$\int_{\mathbb{R}^d \setminus A} f(x) \mathcal{H}^d(dx) = \sum_{i=0}^{d-1} \omega_{d-i} \int_0^\infty \int_{N(A)} t^{d-1-i} \mathbf{1}\{t < \delta(A, x, u)\} \times f(x + tu) \mu_i(A; d(x, u)) dt. \tag{2.3}$$

A more explicit description of the reach measures and of the normal bundle of a closed set is developed in the proof which we will give below.

The proof of Theorem 2.1 will be preceded by two lemmas. Lemma 2.2 provides a more direct approach to an auxiliary result in [30], which is also needed here. Lemma 2.3 describes the structure of the normal bundle of a general closed set and is based on the first lemma. We slightly modify and clarify the corresponding argument in [30]. Moreover, we will have to refer again to the proof of Lemma 2.3 in the proof of Proposition 4.1.

Following [4], we call a vector $u \in \mathbb{R}^d$ a tangent vector of a closed set A at $a \in A$ if $u = 0$ or if $u \neq 0$ and, for every $\varepsilon > 0$, there is some $b \in A$ such that

$$0 < |b - a| < \varepsilon \qquad \text{and} \qquad \left| \frac{b-a}{|b-a|} - \frac{u}{|u|} \right| < \varepsilon.$$

We write $\mathrm{Tan}(A, a)$ for the closed (but not necessarily convex) cone of all such tangent vectors. Moreover, for any $z \in \mathbb{R}^d$ we set

$$\Pi(A, z) := \{a \in A : d(A, z) = |a - z|\}.$$

The estimate of Lemma 2.2 below is sharp. This can be seen by choosing $A := \{a_1, a_2\}$ with $|a_1 - a_2| = 2r$.

Lemma 2.2. *If* $t > 0$ *and* $A \subset \mathbb{R}^d$ *is a non-empty compact set with circumradius* $r(A) < t$*, then* $\mathrm{reach}(A_{\oplus t})^* \geq \sqrt{t^2 - r(A)^2}$.

Proof. Put $r := r(A)$, hence $A \subset B^d(z, r)$ for some $z \in \mathbb{R}^d$. By Theorem 4.18 in [4], it is sufficient to show that

$$d(\mathrm{Tan}((A_{\oplus t})^*, x), y - x) \leq \frac{|y - x|^2}{2\sqrt{t^2 - r^2}}$$

whenever $x, y \in (A_{\oplus t})^*$. In the following, we fix $x, y \in (A_{\oplus t})^*$ and may clearly assume that $x \in \partial(A_{\oplus t})^*$. Further, by translation invariance we may assume that $x = 0$. We claim that

$$\mathrm{dual}(\Pi(A, 0)) \subset \mathrm{Tan}((A_{\oplus t})^*, 0), \tag{2.4}$$

where $\mathrm{dual}(S)$, the dual convex cone of a set $S \subset \mathbb{R}^d$, is the set of all $v \in \mathbb{R}^d$ such that $\langle v, s\rangle \leq 0$ for all $s \in S$.

Let us postpone the verification of (2.4) to the end of the proof. Then we can proceed as follows. Define $z_0 := \sqrt{t^2 - r^2}|z|^{-1}z$ and note that $\Pi(A, 0) \subset A \cap \partial B^d(0, t) \cap B^d(z_0, r)$. Hence, if $u \in \Pi(A, 0)$ and thus $u \in \partial B^d(0, t) \cap B^d(z_0, r)$, then

$$\langle u, z_0\rangle = \langle u, z_0/|z_0|\rangle\, |z_0| \geq |z_0|^2 = t^2 - r^2. \tag{2.5}$$

Furthermore, if $u \in \Pi(A, 0)$ and hence $u \in \partial B^d(0, t) \cap A$, then

$$\langle u, y\rangle \leq |y|^2/2. \tag{2.6}$$

To check this, we can assume that $y \neq 0$. Suppose that $\langle u, y/|y|\rangle > |y|/2$. But then $u \in \partial B^d(0,t)$ implies that $u \in A \cap \operatorname{int} B^d(y,t)$, which contradicts $A \cap \operatorname{int} B^d(y,t) = \emptyset$.

We define

$$\alpha := \frac{|y|^2}{2(t^2-r^2)} \qquad \text{and} \qquad v := y - \alpha z_0.$$

Then, for $u \in \Pi(A,0)$, the estimates (2.5) and (2.6) yield that

$$\langle u, v\rangle = \langle u, y\rangle - \alpha\langle u, z_0\rangle \leq \frac{|y|^2}{2} - \frac{|y|^2}{2(t^2-r^2)}(t^2-r^2) = 0.$$

Therefore, (2.4) shows that $v \in \operatorname{Tan}((A_{\oplus t})^*, 0)$. Hence,

$$d(\operatorname{Tan}((A_{\oplus t})^*, 0), y) \leq |y - v| = \alpha|z_0| = \frac{|y|^2}{2\sqrt{t^2-r^2}},$$

as required.

It remains to verify the inclusion (2.4). Assume that $v \in \operatorname{int}\operatorname{dual}(\Pi(A,0))$. Then $\langle v, a\rangle < 0$ for all $a \in \Pi(A,0)$, i.e. $\langle v, -a\rangle > 0$ whenever $a \in \Pi(A,0)$. With any $c \in \mathbb{R}^d$ we associate an arbitrary point $\xi(c) \in \Pi(A,c)$. Moreover, we define $f := d(A,\cdot)$ and, for $\lambda > 0$, $\psi_\lambda := |\xi(\lambda v) - \cdot|$. The function ψ_λ is convex and differentiable at 0. Then,

$$\begin{aligned}(f(\lambda v) - f(0))/\lambda &\geq (|\lambda v - \xi(\lambda v)| - |\xi(\lambda v)|)/\lambda = (\psi_\lambda(\lambda v) - \psi_\lambda(0))/\lambda \\ &\geq \langle v, \nabla\psi_\lambda(0)\rangle = \langle v, -\xi(\lambda v)/|\xi(\lambda v)|\rangle.\end{aligned}$$

Since $\{\xi(\lambda v) : \lambda \in (0,1]\}$ is bounded, there is a sequence $(\lambda_i)_{i\in\mathbb{N}}$ with $\lambda_i > 0$ such that $\lambda_i \to 0$ and $\xi(\lambda_i v) \to a_0 \in A$ as $i \to \infty$. From

$$d(A,0) = \lim_{i\to\infty} d(A, \lambda_i v) = \lim_{i\to\infty} |\xi(\lambda_i v) - \lambda_i v| = |a_0|,$$

we deduce that $a_0 \in \Pi(A,0)$, and therefore

$$\limsup_{i\to\infty}\,[(f(\lambda_i v) - f(0))/\lambda_i] \geq \langle v, -a_0/|a_0|\rangle > 0.$$

This shows that there are infinitely many $i \in \mathbb{N}$, such that $f(\lambda_i v) > f(0) = t$, i.e. $\lambda_i v \in (A_{\oplus t})^*$, hence $v \in \operatorname{Tan}((A_{\oplus t})^*, 0)$.

The convex cone $\operatorname{dual}(\Pi(A,0))$ has non-empty interior, since $\Pi(A,0) \subset \partial B^d(0,t) \cap B^d(z_0,r)$ and $r < t$. Furthermore, $\operatorname{Tan}((A_{\oplus t})^*, 0)$ is a closed set. Hence an approximation argument concludes the proof of (2.4). □

Lemma 2.3. *For a non-empty closed set $A \subset \mathbb{R}^d$, there exists a sequence A_n, $n \in \mathbb{N}$, of closed subsets of $\mathbb{R}^d$ with positive reach and compact boundary such that*

$$N(A) \subset \bigcup_{n=1}^{\infty} N(A_n) \tag{2.7}$$

and, for $(x,u) \in N(A)$,

$$\delta(A,x,u) \leq \sup\{\operatorname{reach}(A_n) : (x,u) \in N(A_n), n \in \mathbb{N}\}. \tag{2.8}$$

Proof. Let T be a countable dense subset of $(0,\infty)$. For $i \in \mathbb{N}$ and $t \in T$, let $(K(t,i,j))_{j\in\mathbb{N}}$ be a sequence of closed balls of radius $t/(2i)$ covering $\mathbb{R}^d$. We put

$$G(t,i,j) := \partial A_{\oplus t} \cap K(t,i,j)$$

and

$$A(t,i,j) := (G(t,i,j)_{\oplus t})^*.$$

If $G(t,i,j) \neq \emptyset$, we get from Lemma 2.2 that

$$\operatorname{reach}(A(t,i,j)) \geq \sqrt{t^2 - \left(\frac{t}{2i}\right)^2} = t\sqrt{1 - \frac{1}{4i^2}}. \tag{2.9}$$

If $G(t,i,j) = \emptyset$, this is trivially satisfied. Next, we show that

$$N(A) \subset \bigcup_{t\in T} \bigcup_{i,j\in\mathbb{N}} N(A(t,i,j)).$$

For this purpose, let $(x,u) \in N(A)$ and $\delta(A,x,u) > t$, for some $t \in T$. Since $\partial A_{\oplus t} = \cup_{j\geq 1} G(t,i,j)$ (for all $i \in \mathbb{N}$), we have $x + tu \in G(t,i,j')$, for some $j' = j(t,i) \in \mathbb{N}$. But then $d(A(t,i,j'), x+tu) \geq t$. Furthermore, $G(t,i,j') \subset \partial A_{\oplus t}$, and therefore $d(\partial A_{\oplus t}, x) \geq t$ yields $d(G(t,i,j'), x) \geq t$, i.e. $x \in A(t,i,j')$. This shows that $d(A(t,i,j'), x+tu) \leq t$. Thus, $d(A(t,i,j'), x+tu) = t$ and $(x,u) \in N(A(t,i,j'))$.

The above proof and (2.9) together imply that

$$\sup\{\operatorname{reach}(A(t,i,j)) : (x,u) \in N(A(t,i,j)) \text{ for some } i,j \in \mathbb{N}\} \geq t,$$

if $\delta(A,x,u) > t$; hence

$$\sup\{\operatorname{reach}(A(t,i,j)) : (x,u) \in N(A(t,i,j)), t \in T, i,j \in \mathbb{N}\} \geq \delta(A,x,u).$$

Finally, since $\partial A(t,i,j) = \partial(G(t,i,j)_{\oplus t}) \subset K(t,i,j)_{\oplus t}$ is compact, the countable family of sets $A(t,i,j)$, $t \in T$, $i,j \in \mathbb{N}$, satisfies all requirements. □

It is easy to see that one can also find a sequence of *compact* sets such that (2.7) and (2.8) are satisfied. For this, let $R_n > 0$ be such that $\partial A_n \subset B^d(0,R_n)$. Then $A'_n := A_n \cap B^d(0,R_n)$ is compact, $N(A_n) \subset N(A'_n)$ and $\operatorname{reach}(A'_n) = \operatorname{reach}(A_n)$.

The inclusion (2.7) in particular shows that $N(A)$ is countably $(d-1)$-rectifiable. An alternative (but less elementary) derivation of this special consequence follows from Theorem 2.31 in [14]. Conversely, Theorem 2.31 in [14] can be deduced from our proof of (2.7)

Proof of Theorem 2.1. Let $A \subset \mathbb{R}^d$ be non-empty and closed and let A_n, $n \in \mathbb{N}$, be chosen according to Lemma 2.3. Then, Lemma 2.3 implies that $N(A)$ is a countably $(d-1)$-rectifiable set in the sense of [5]. Subsequently, we use the notion of an *approximate tangent space* as defined in [28] (or [11]). Note that this concept is different from the one encountered in the proof of Lemma 2.2, although we employ the same notation. The rectifiability property of $N(A)$ implies that, for $\mathcal{H}^{d-1}$-a.e.

$(x, u) \in N(A)$, the approximate tangent space $\mathrm{Tan}(N(A), x, u)$ of $N(A)$ at (x, u) exists and is a $(d-1)$-dimensional vector space; moreover,

$$\mathrm{Tan}(N(A), x, u) = \mathrm{Tan}(N(A_n), x, u) \tag{2.10}$$

for $\mathcal{H}^{d-1}$-a.e. $(x, u) \in N(A) \cap N(A_n)$ and for each $n \in \mathbb{N}$. Due to these facts, it is possible to extend the definition of the (generalized) *principal curvatures*

$$k_1(A, x, u), \ldots, k_{d-1}(A, x, u) \in (-\infty, \infty]$$

in [35] (given for sets with positive reach) to our general setting. For $\mathcal{H}^{d-1}$-a.e. $(x, u) \in N(A)$, these are the uniquely determined numbers which have the property that the vectors

$$\left(\frac{1}{\sqrt{1 + k_i(A, x, u)^2}} u_i, \frac{k_i(A, x, u)}{\sqrt{1 + k_i(A, x, u)^2}} u_i \right), \qquad i = 1, \ldots, d-1,$$

span $\mathrm{Tan}(N(A), x, u)$. This can be easily deduced from (2.10). Here the unit vectors $u_i = u_i(x, u) \in S^{d-1}$, $i = 1, \ldots, d-1$, are the (generalized) principal directions of curvature (a sequence of orthonormal vectors lying in the orthogonal complement of u). Furthermore, here and in the sequel, expressions $a(k)$ with $k = \infty$ are defined as the corresponding limits $\lim_{k \to \infty} a(k) \in (-\infty, \infty]$, which will always be well-defined. For instance, $1/\sqrt{1 + k^2} = 0$ and $k/\sqrt{1 + k^2} = 1$ for $k = \infty$.

An alternative description of the generalized curvatures can be given as follows. For $\mathcal{H}^{d-1}$-a.e. $(x, u) \in N(A) \cap N(A_n)$ and $\varepsilon \in (0, \mathrm{reach}(A_n))$, $u(A_n, \cdot)$ is differentiable at $x + \varepsilon u$ and the ratios $k_i(A, x, u)/(1 + \varepsilon k_i(A, x, u))$, $i = 1, \ldots, d-1$, are the eigenvalues of the differential of $u(A_n, \cdot)$ at $x + \varepsilon u$ restricted to the orthogonal complement of u (and the u_i from above are the corresponding eigenvectors). Moreover, if $t < \delta(A, x, u)$, then we can find $n \in \mathbb{N}$ such that $(x, u) \in N(A_n)$ and $\mathrm{reach}(A_n) > t$. Therefore

$$1 + t k_i(A, x, u) \geq 0, \quad i = 1, \ldots, d-1, \tag{2.11}$$

holds for $\mathcal{H}^{d-1}$-a.e. $(x, u) \in N(A)$ with $\delta(A, x, u) > t$.

We set $M(A) := \{(x, u, t) \in N(A) \times (0, \infty) : \delta(A, x, u) > t\}$. For $\mathcal{H}^d$-a.e. $(x, u, t) \in M(A)$, the Jacobian of the map $T : N(A) \times (0, \infty) \to \mathbb{R}^d$, $(y, v, s) \mapsto y + sv$, is given by

$$JT(x, u, t) = \prod_{i=1}^{d-1} \frac{1 + t k_i(A, x, u)}{\sqrt{1 + k_i(A, x, u)^2}}.$$

In addition, T is injective on $M(A)$ and $\mathcal{H}^d(\mathbb{R}^d \setminus (A \cup T(M(A)))) = 0$. Injectivity easily follows from the definition of the normal bundle and the reach function, and the second assertion is implied by (2.1). Hence, using twice the coarea formula of Federer [5] in a slightly more general version (see [28] or [11]), we obtain similarly as in [35] that

$$\int_{\mathbb{R}^d\setminus A} f(z)\mathcal{H}^d(dz) = \int_{N(A)}\int_0^\infty \mathbf{1}\{t < \delta(A,x,u)\} f(x+tu) \times \prod_{i=1}^{d-1} \frac{1+tk_i(A,x,u)}{\sqrt{1+k_i(A,x,u)^2}} dt\mathcal{H}^{d-1}(d(x,u)), \tag{2.12}$$

for all measurable bounded functions $f : \mathbb{R}^d \to \mathbb{R}$ with compact support.

For $\mathcal{H}^{d-1}$-a.e. $(x,u) \in N(A)$, we define

$$H_j(A,x,u) := \prod_{i=1}^{d-1}(1+k_i(A,x,u)^2)^{-1/2} \sum_{|I|=j} \prod_{l\in I} k_l(A,x,u), \tag{2.13}$$

for $j \in \{0,\dots,d-1\}$, where the summation extends over all subsets $I \subset \{1,\dots,d-1\}$ of cardinality j. Note that for $j=0$, the product over the empty set is defined as 1, i.e.

$$H_0(A,x,u) := \prod_{i=1}^{d-1}(1+k_i(A,x,u)^2)^{-1/2}.$$

For the remainder of the proof, we often suppress the argument A in the curvatures k_i, the reach function δ and the functions H_j. By definition, $H_j(x,u) \in (-\infty,\infty)$, and $H_j(x,u) = 0$ if at least $j+1$ of the principal curvatures are infinite. We can now rewrite (2.12) as

$$\int_{\mathbb{R}^d\setminus A} f(z)\mathcal{H}^d(dz) = \int_{N(A)}\int_0^\infty \mathbf{1}\{t < \delta(x,u)\} f(x+tu) \times \left(\sum_{j=0}^{d-1} t^j H_j(x,u)\right) dt\mathcal{H}^{d-1}(d(x,u)). \tag{2.14}$$

Our next aim is to prove that

$$\int_{N(A)}\int_0^\infty \mathbf{1}\{x \in B\}\mathbf{1}\{t < \delta(x,u)\wedge r\} t^j \times |H_j(x,u)| dt\mathcal{H}^{d-1}(d(x,u)) < \infty, \tag{2.15}$$

for an arbitrary $r > 0$, for all compact sets $B \subset \mathbb{R}^d$ and all $j \in \{0,\dots,d-1\}$. The first main estimate used for proving the above integrability property follows from (2.14) applied to the function

$$f(z) := \mathbf{1}\{p(A,z) \in B, 0 < d(A,z) \le r\}.$$

Performing the inner integration in (2.14), we obtain that

$$\begin{aligned}\infty &> \int_{\mathbb{R}^d\setminus A} \mathbf{1}\{p(A,z) \in B, 0 < d(A,z) \le r\}\mathcal{H}^d(dz) \\ &= \int_{N(A)} \mathbf{1}\{x \in B\}\Bigg(\sum_{j=0}^{d-1}(j+1)^{-1}(\delta(x,u)\wedge r)^{j+1} \\ &\quad \times H_j(x,u)\Bigg)\mathcal{H}^{d-1}(d(x,u)).\end{aligned} \tag{2.16}$$

Our second tool in proving (2.15) is the following simple but crucial consequence of (2.11),

$$(\delta(x,u)\wedge r)k_i(x,u) \geq -1, \tag{2.17}$$

which is satisfied for $\mathcal{H}^{d-1}$-a.e. $(x,u)\in N(A)$.

We now use the decomposition $k_i(x,u) = k_i^+(x,u) + k_i^-(x,u)$, where $k_i^+(x,u) := \max(k_i(x,u),0)$ and $k_i^-(x,u) := \min(k_i(x,u),0)$, and define functions $H_0^+,\dots,H_{d-1}^+$ and $H_0^-,\dots,H_{d-1}^-$ on $N(A)$ as in (2.13) with $k_i(x,u)$ replaced by $k_i^+(x,u)$ resp. $k_i^-(x,u)$, $i=0,\dots,d-1$. We then get the decomposition

$$H_j = \sum_{l=0}^{j} H_l^- H_{j-l}^+, \quad j=0,\dots,d-1. \tag{2.18}$$

Since $H_l^- H_s^+ = 0$, for $l+s > d-1$, we deduce from this

$$\sum_{j=0}^{d-1}(j+1)^{-1}(\delta\wedge r)^{j+1}H_j = \sum_{s=0}^{d-1}\sum_{l=0}^{d-1}(s+l+1)^{-1}(\delta\wedge r)^l H_l^- H_s^+(\delta\wedge r)^{s+1}.$$

In order to proceed, we need the following lemma. It refers to the j-th elementary symmetric function h_j of m real variables $y_1,\dots,y_m$,

$$h_j(y_1,\dots,y_m) := \sum_{|I|=j}\prod_{i\in I} y_i,$$

$j\in\{0,\dots,m\}$, where the summation again extends over all subsets $I\subset\{1,\dots,m\}$ of cardinality j. Note that $h_0\equiv 1$.

Lemma 2.4. *For any $m\in\mathbb{N}$ and $k\in\mathbb{N}$, the function*

$$g_k := \sum_{i=0}^{m} h_i/(k+i)$$

is bounded from below on $[-1,0]^m$ by a positive constant depending only on m and k.

Proof. Since the function g_k is linear in each variable, it attains its minimum in a vertex $(y_1,\dots,y_m)$ of the cube $[-1,0]^m$. By symmetry, we can assume that $y_1=\dots=y_j=-1$ and $y_{j+1}=\dots=y_m=0$ for some $j\in\{0,\dots,m\}$. Then

$$g_k(y_1,\dots,y_m) = \sum_{i=0}^{j}(-1)^i\binom{j}{i}/(k+i) = \int_0^1 (1-t)^j t^{k-1}dt > 0,$$

and the lemma is proved. $\square$

For each pair (x, u), we apply Lemma 2.4 with $k = s + 1$ and $m := \operatorname{card}\{i : k_i(x, u) < 0\}$. Then

$$\sum_{j=0}^{d-1} (j+1)^{-1}(\delta \wedge r)^{j+1} H_j \geq \sum_{s=0}^{d-1} c_1(s, d) H_0^- H_s^+ (\delta \wedge r)^{s+1}, \tag{2.19}$$

for some positive constants $c_1(s, d)$. Formally, we first get this inequality for $H_j(x, u)$, $H_0^-(x, u)$, $H_s^+(x, u)$ with constants $c_1(s, d)$ depending on m. Since there are only finitely many values of m this implies the corresponding inequality for the functions H_j, H_0^-, H_s^+ with universal constants $c_1(s, d)$. The case $m = 0$ is not covered by Lemma 2.4, but follows directly since then $H_j = H_j^+$ and $H_0^- = 1$ and since we may choose $c_1(s, d) \leq 1$. Moreover, for $l \in \{0, \dots, d-1\}$ and $s \in \{0, \dots, l\}$, we deduce from (2.17) that

$$H_0^- \geq c_2(s, l, d)(\delta \wedge r)^{l-s} |H_{l-s}^-|, \tag{2.20}$$

where $c_2(s, l, d)$ is a positive constant. Therefore, using (2.19) and (2.20), as well as (2.18) again,

$$\begin{aligned}\sum_{j=0}^{d-1} (j+1)^{-1}(\delta \wedge r)^{j+1} H_j &\geq \sum_{s=0}^{l} c_1(s, d) c_2(s, l, d)(\delta \wedge r)^{l+1} |H_{l-s}^-| H_s^+ \\ &\geq c_3(l, d)(\delta \wedge r)^{l+1} |H_l|,\end{aligned} \tag{2.21}$$

where $c_3(l, d)$ is another positive constant. The desired integrability is now implied by (2.16) and (2.21).

It follows from (2.15) and Fubini's theorem that, for all $r > 0$,

$$\int_0^r \int_{N(A)} \mathbf{1}\{t \leq \delta(x, u), x \in B\} t^j |H_j(x, u)| \mathcal{H}^{d-1}(d(x, u)) dt < \infty \tag{2.22}$$

and hence

$$\int_{N(A)} \mathbf{1}\{\varepsilon \leq \delta(x, u), x \in B\} |H_j(x, u)| \mathcal{H}^{d-1}(d(x, u)) < \infty, \tag{2.23}$$

for all $\varepsilon > 0$ and all compact sets $B \subset \mathbb{R}^d$, where $j = 0, \dots, d-1$. Therefore

$$\mu_j(A; \cdot) := \frac{1}{\omega_{d-j}} \int_{N(A)} \mathbf{1}\{(x, u) \in \cdot\} H_{d-1-j}(x, u) \mathcal{H}^{d-1}(d(x, u)) \tag{2.24}$$

defines for each $j \in \{0, \dots, d-1\}$ a reach measure of A. In particular, the integrals

$$\int_0^\infty \int_{N(A)} t^{d-1-j} \mathbf{1}\{t < \delta(A, x, u)\} f(x + tu) \mu_j(A; d(x, u)) dt \tag{2.25}$$

are well-defined and finite for all measurable bounded functions $f : \mathbb{R}^d \to \mathbb{R}$ with compact support. To check this, let $\operatorname{supp}(f)$ denote the support of f. If $x + tu \in \operatorname{supp}(f)$, $0 \leq t < \delta(A, x, u)$ and $(x, u) \in N(A)$, then $x = p(A, x + tu)$ and hence $t = d(A, x + tu) \leq |x + tu - a|$ for an arbitrary but fixed $a \in A$. Since $\operatorname{supp}(f)$ is compact, $t \le \max\{|z-a| : z \in \operatorname{supp}(f)\} =: R < \infty$ and $x \in \operatorname{supp}(f) + RB^d$. The assertion then follows from (2.22). The local Steiner formula (2.3) is a consequence of (2.14).

To see that the above reach measures are uniquely determined by (2.3), we take $\varepsilon > 0$ and a compact set $B \subset \mathbb{R}^d$. Then we obtain for all $s \in (0, \varepsilon)$ and all measurable and bounded functions $f : \mathbb{R}^d \to \mathbb{R}$ that

$$\begin{aligned}\int_{\mathbb{R}^d\setminus A} & \mathbf{1}\{\varepsilon \le \delta(A, p(A,z), u(A,z)), p(A,z) \in B, d(A,z) \le s\} \\ & \times f(p(A,z), u(A,z))\mathcal{H}^d(dz) \\ = & \sum_{i=0}^{d-1} \kappa_{d-i} s^{d-i} \int_{N(A)} \mathbf{1}\{\varepsilon \le \delta(A,x,u), x \in B\} f(x,u) \mu_i(A; d(x,u)).\end{aligned} \tag{2.26}$$

Related issues of measurability are covered in Section 6. This determines the measures $\mu_i(A;\cdot)$ on $\{(x,u) \in N(A) : \varepsilon \le \delta(A,x,u), x \in B\}$, for any $\varepsilon > 0$, as asserted. □

2.3 Support measures

The signed measures $\mu_0(A;\cdot), \dots, \mu_{d-1}(A;\cdot)$ which have been introduced in Subsection 2.2, are called the *support measures* of the closed set $A \subset \mathbb{R}^d$. An integral representation for these support measures has been derived in the proof of Theorem 2.1 and will be stated explicitly in the next corollary.

Corollary 2.5. *For any non-empty closed set $A \subset \mathbb{R}^d$, the support measures of A are given by*

$$\mu_i(A;\cdot) = \frac{1}{\omega_{d-i}} \int_{N(A)} \mathbf{1}\{(x,u) \in \cdot\} H_{d-1-i}(A,x,u) \mathcal{H}^{d-1}(d(x,u)),$$

for $i = 0, \dots, d-1$, where $H_{d-1-i}(A,x,u)$ is defined by (2.13).

The integrability property (2.2) guarantees that all the integrals on the right-hand side of (2.3) are finite. By Fubini's theorem (applied to the measures $\mu_i^+(A;\cdot)$ and $\mu_i^-(A;\cdot)$), we then also have

$$\begin{aligned}\int_{\mathbb{R}^d\setminus A} f(z)\mathcal{H}^d(dz) = & \sum_{i=0}^{d-1} \omega_{d-i} \int_{N(A)} \int_0^{\delta(A,x,u)} t^{d-1-i} \\ & \times f(x+tu)dt\, \mu_i(A; d(x,u)).\end{aligned} \tag{2.27}$$

By definition, $\mu_j(A; D)$ is defined for all Borel subsets $D \subset \mathbb{R}^d \times S^{d-1}$ for which there is a $\varepsilon > 0$ and a compact subset B of $\mathbb{R}^d$ with $D \cap N(A) \subset \{(x,u) \in N(A) : \varepsilon \le \delta(A,x,u), x \in B\}$. However, we can extend the definition of $\mu_j(A;\cdot)$ by setting, for a Borel set $D \subset \mathbb{R}^d \times S^{d-1}$,

$$\mu_j(A; D) := \int_{N(A)} \mathbf{1}\{(x,u) \in D\} \mu_j(A; d(x,u))$$

whenever the integral exists. In general, the class of all such *admissible* Borel sets D need not be the whole Borel σ-field, but in many important cases it is. In the following, we always consider $\mu_j(A;\cdot)$ as a set function defined on admissible sets and, for simplicity, we speak of these set functions as measures.

Finally, we point out that the above construction of support measures includes the case $A=\mathbb{R}^d$. Here, $N(\mathbb{R}^d)=\emptyset$ and $\mu_j(\mathbb{R}^d;\cdot)=0$ for $j=0,\dots,d-1$. For later use, we also extend the support measures (and the other relevant notions) to the case $A=\emptyset$ by $N(\emptyset):=\emptyset$ and $\mu_j(\emptyset;\cdot):=0$, $j=0,\dots,d-1$.

3 Some special cases

Our aim in this section is to show how the support measures of general closed sets specialize for particular classes of sets and to discuss the connection with the literature.

If $A\subset\mathbb{R}^d$ is a *convex body*, i.e. an element of the set $\mathcal{K}^d$ of all non-empty, compact and convex subsets of $\mathbb{R}^d$, then $\delta(A,\cdot)\equiv\infty$ and the support measures of A are the (generalized) curvature measures $C_0(A;\cdot),\dots,C_{d-1}(A;\cdot)$ of A (see [26], [27]). These are finite measures on $\mathbb{R}^d\times S^{d-1}$ concentrated on $N(A)$. Their projections, the measures $C_0(A;\cdot\times S^{d-1}),\dots,C_{d-1}(A;\cdot\times S^{d-1})$ on $\mathbb{R}^d$, are the classical curvature measures of A.

The curvature measures enjoy an *additivity* property stating that

$$C_i(A_1\cup A_2;\cdot)+C_i(A_1\cap A_2;\cdot)=C_i(A_1;\cdot)+C_i(A_2;\cdot)$$

whenever $A_1, A_2, A_1\cup A_2\in\mathcal{K}^d$. Since they also depend continuously (with respect to the weak topology on the space of measures) on the convex bodies, they can be additively extended to the class $\mathcal{R}^d$ of all finite unions of convex bodies. In general, $C_i(A;\cdot)$ is a signed measure for $A\in\mathcal{R}^d$, in the special case $i=d-1$ this measure is non-negative. The curvature measures can be further extended to signed Radon measures $C_i(A;\cdot)$ on $\mathbb{R}^d\times S^{d-1}$ for any set $A\subset\mathbb{R}^d$ which can be represented as a *locally finite* union $\cup_n A_n$ of convex bodies A_n, $n\in\mathbb{N}$. Here the assumption of local finiteness means that each compact subset of $\mathbb{R}^d$ is intersected by only a finite number of the sets A_n. The class of all such sets A is denoted by $\mathcal{S}^d$ and called the *extended convex ring*. Theorem 3.3 in [17] shows that $\mu_i(A;\cdot)$ is a non-negative measure and

$$\mu_i(A;\cdot)=C_i(A;N(A)\cap\cdot) \tag{3.1}$$

whenever $A\in\mathcal{S}^d$ and $i\in\{0,\dots,d-1\}$; moreover, it is also shown in [17] that $\mu_{d-1}(A;\cdot)=C_{d-1}(A;\cdot)$. For $A\in\mathcal{S}^d$, Theorem 2.1 has been proved in [22] and [17].

If A is a set of positive reach, then $\mu_0(A;\cdot\times S^{d-1}),\dots,\mu_{d-1}(A;\cdot\times S^{d-1})$ are signed measures, the curvature measures introduced in [4]. The natural extension of these measures to $\mathbb{R}^d\times S^{d-1}$ as the underlying space and the explicit representation (2.24) have been established in [35].

Our next example are locally finite unions of sets with positive reach. More precisely, we introduce $\mathcal{U}^d$ as the system of all sets A which can be represented as a locally finite union $\cup A_n$ of sets A_n, $n \in \mathbb{N}$, of positive reach such that each intersection of any finite number of the sets A_n is also of positive reach. Note that $\mathcal{S}^d \subset \mathcal{U}^d$. However, $\mathcal{U}^d$ is a much broader class than $\mathcal{S}^d$, containing for instance the fibre and surface systems of [31]. The curvature measures are additive on the system of all sets of positive reach, and it has been proved in [24] that they can be additively extended to $\mathcal{U}^d$. For $A \in \mathcal{U}^d$, the measures $C_0(A;\cdot), \dots, C_{d-1}(A;\cdot)$ are signed Radon measures on $\mathbb{R}^d \times S^{d-1}$ concentrated on $\partial A \times S^{d-1}$. To compare these measures with the support measures, we provide some further details. As in [24], we define the *Schneider index* $i(A,\cdot) : \mathbb{R}^d \times S^{d-1} \to \mathbb{Z}$ by

$$i(A,x,u) := \mathbf{1}\{x \in A\} \left(1 - \lim_{\varepsilon\to 0+} \lim_{\delta \to 0+} \chi(A \cap B^d(x + (\varepsilon+\delta)u, \varepsilon))\right),$$

$(x,u) \in \mathbb{R}^d \times S^{d-1}$, where χ denotes the Euler characteristic and the existence of the double limit is proved in [24]. In this paper, the authors use a different notion of normal bundle for sets $A \in \mathcal{U}^d$, namely

$$N^*(A) := \{(x,u) \in \mathbb{R}^d \times S^{d-1} : i(A,x,u) \neq 0\}.$$

Let $S(\mathbb{N})$ denote the system of all non-empty finite subsets of $\mathbb{N}$. Then, by the additivity of the index function with respect to the set A, we obtain that

$$N^*(A) \subset \bigcup_{v \in S(\mathbb{N})} N\left(\bigcap_{i\in v} A_i\right) \tag{3.2}$$

if $A = \cup A_n$ is a locally finite representation of A as required in the definition of the class $\mathcal{U}^d$. From (3.2) one can derive the structure of the approximate tangent space $\mathrm{Tan}(N^*(A),x,u)$, for $\mathcal{H}^{d-1}$-a.e. $(x,u) \in N^*(A)$, and define generalized curvature functions $H_j^*(A,x,u)$ similarly as in (2.13) (see again [24]). Then the curvature measures for sets $A \in \mathcal{U}^d$ and $j \in \{0,\dots,d-1\}$ are defined by

$$\omega_{d-j} C_j(A;\cdot) := \int \mathbf{1}\{(x,u) \in \cdot\} i(A,x,u) H^*_{d-1-j}(A,x,u) \mathcal{H}^{d-1}(d(x,u)).$$

It is not difficult to infer from the definition of $N(A)$ that $i(A,x,u) = 1$ for all $(x,u) \in N(A)$, hence $N(A) \subset N^*(A)$; moreover, $H_j(A,x,u) = H_j^*(A,x,u)$ for $\mathcal{H}^{d-1}$-a.e. $(x,u) \in N(A)$ and $j = 0,\dots,d-1$. Therefore we obtain that $\mu_i(A;\cdot) = C_i(A; N(A) \cap \cdot)$ whenever $A \in \mathcal{U}^d$ and $i \in \{0,\dots,d-1\}$, which provides an extension of (3.1). The local Steiner formula (2.3) seems to be new in this setting. In contrast to the case $A \in \mathcal{S}^d$, the support measures $\mu_i(A;\cdot)$ may take negative values for $A \in \mathcal{U}^d \setminus \mathcal{S}^d$ and $i \le d-2$. In Section 4, the case $i = d-1$ is discussed for general closed sets.

In our next example, we consider the boundary of a convex body $K \in \mathcal{K}^d$ with interior points. Then,

$$\begin{aligned}\mu_j(\partial K;\cdot) &= \mu_j(K;\cdot)\\ &\quad + (-1)^{d-1-j} \int_{N(K)} \mathbf{1}\{(x,-u) \in \cdot \cap N(\partial K)\} \mu_j(K; d(x,u)).\end{aligned} \tag{3.3}$$

The proof of formula (3.3) follows from Corollary 2.5, since

$$k_i(\partial K, x, -u) = -k_i(K, x, u)$$

for $\mathcal{H}^{d-1}$-a.e. $(x, u) \in N(K)$ satisfying $(x, -u) \in N(\partial K)$. Similarly, we have

$$\mu_j(K^*; \cdot) = (-1)^{d-1-j} \int_{N(K)} \mathbf{1}\{(x, -u) \in \cdot \cap N(K^*)\} \mu_j(K; d(x, u)), \quad (3.4)$$

for the closed complement $K^* = \mathrm{cl}\,(\mathbb{R}^d \setminus K)$ of K. For a lower dimensional convex body K, we have $K^* = \mathbb{R}^d$ and relation (3.4) becomes trivial.

Equation (3.4) can be generalized to arbitrary closed sets A. We discuss the corresponding result in Section 5, in connection with extensions of the Steiner formula to the interior of a set A.

4 Basic properties of support measures

4.1 Support measures of order $d - 1$

In this section, we derive and discuss some of the basic properties of the support measures of closed sets $A \subset \mathbb{R}^d$. Our first aim is to provide an explicit formula for the support measure $\mu_{d-1}(A; \cdot)$ of order $d - 1$, which corresponds to the surface measure in the case of a smooth set A. In particular, it will turn out that $\mu_{d-1}(A; \cdot)$ is always a non-negative σ-finite measure on $N(A)$. The set

$$\partial^+ A := \{x \in \partial A : (x, u) \in N(A) \text{ for some } u \in S^{d-1}\}$$

is called the *positive boundary* of A. It follows from Lemma 6.3, that this is a measurable set. Moreover, as a consequence of Lemma 2.3, it is also countably $(d-1)$-rectifiable. In general, we may have $\mathcal{H}^{d-1}(\partial A \setminus \partial^+ A) > 0$. A simple example in $\mathbb{R}^2$ is a set A consisting of the x-axis L and a countable union of (disjoint) lines parallel to L which accumulate at L from both sides. In this case, $L \subset \partial A$ but $L \cap \partial^+ A = \emptyset$. Even for $A \in \mathcal{R}^d$, we may have $\partial A \setminus \partial^+ A \neq \emptyset$, but at least $\mathcal{H}^{d-1}(\partial A \setminus \partial^+ A) = 0$ whenever $A \in \mathcal{S}^d$. This can be proved analogously to Theorem 2.2 in [33]. Finally, we remark that $\partial^+ A = \partial A$ if A is a set with positive reach; moreover, for $A \in \mathcal{U}^d$, it can be shown that $\mathcal{H}^{d-1}(\partial A \setminus \partial^+ A) = 0$. The relevance of the positive boundary in connection with the support measure of order $d - 1$ will become clear in a moment.

For any $x \in \partial^+ A$, we define

$$N(A, x) := \{u \in S^{d-1} : (x, u) \in N(A)\}.$$

Then we call $n(A, x) := \{\lambda u : \lambda \geq 0, u \in N(A, x)\}$ the *normal cone* of A at x. It is easy to check that the normal cone of A is convex. Let

$$\partial^{++} A := \{x \in \partial^+ A : \dim n(A, x) = 1\},$$

where $\dim B$ denotes the dimension of the affine hull of a set $B \subset \mathbb{R}^d$. Clearly, $\partial^{++}A$ is the disjoint union of $\partial^1 A$ and $\partial^2 A$, where

$$\partial^i A := \{x \in \partial^{++}A : \operatorname{card} N(A,x) = i\}, \qquad i = 1,2.$$

Again by Lemma 6.3 all these sets are measurable. (Moreover, they are countably $(d-1)$-rectifiable.) For $x \in \partial^1 A$, we let $\nu(A,x)$ (the *outer normal of A at x*) denote the unique element of $N(A,x)$, and for $x \in \partial^2 A$, we choose $\nu(A,x)$ from $N(A,x)$ according to some measurable rule. The mapping $\nu(A,\cdot)$ is measurable on $\partial^{++}A$ (see Lemma 6.3).

Proposition 4.1. *The support measure of order $d-1$ of a closed set $A \subset \mathbb{R}^d$ is a non-negative σ-finite measure on $\mathbb{R}^d \times S^{d-1}$ satisfying*

$$\mu_{d-1}(A;\cdot) = \frac{1}{2}\int_{\partial^{++}A} \mathbf{1}\{(x,\nu(A,x)) \in \cdot\}\mathcal{H}^{d-1}(dx) + \frac{1}{2}\int_{\partial^2 A} \mathbf{1}\{(x,-\nu(A,x)) \in \cdot\}\mathcal{H}^{d-1}(dx). \tag{4.1}$$

Proof. From (2.24) we deduce for any measurable and bounded function $f : N(A) \to \mathbb{R}$ with compact support

$$\int_{N(A)} f(x,u)\mu_{d-1}(A;d(x,u)) = \frac{1}{2}\int_{N(A)} f(x,u)H_0(x,u)\mathcal{H}^{d-1}(d(x,u)).$$

It follows as in [35] that the function $H_0(x,u)$ is just the approximate Jacobian of the mapping $(x,u) \mapsto x$ from $N(A)$ to ∂A.

Excluding a set of $\mathcal{H}^{d-1}$-measure zero, we find that $H_0(x,u) = 0$ whenever $(x,u) \in N(A)$ and $x \notin \partial^{++}A$, i.e. $\dim N(A,x) \geq 2$ (see e.g. [15] for a similar assertion in a simpler situation). To justify this, we argue as follows. Let $K(t,i,j)$ denote a ball as defined in the proof of Lemma 2.3. Denoting by $\hat{K}(t,i,j)$ the ball with the same centre as $K(t,i,j)$ and with radius $t/(4i)$, we may require that already $\cup_{j=1}^{\infty}\hat{K}(t,i,j) = \mathbb{R}^d$. Let $(x,u) \in N(A)$ be such that (2.10) is satisfied and such that (x,u) is a point of differentiability of any of the strong Lipschitz submanifolds (cf. [32]) $N(A_n)$ which contain (x,u). Assume that $x \notin \partial^{++}A$, hence $\dim n(A,x) \geq 2$. Since $n(A,x)$ is a convex cone, there is a unit vector $v \in n(A,x)$ such that u and v are linearly independent. Clearly, there is some $t_0 \in T$ such that $\delta(A,x,w) > t_0$ for all $w \in [u,v]$, where $[u,v]$ denotes the spherical arc connecting u and v. Let $j \in \mathbb{N}$ be such that $x + t_0 u \in \hat{K}(t_0,1,j)$. Then we can find $w \in [u,v]\setminus\{u\}$ such that $x+t_0 w \in K(t_0,1,j)$, thus $x+t_0 u, x+t_0 w \in K(t_0,1,j)$. Therefore $x+t_0 u, x+t_0 w \in G(t_0,1,j)$, and we obtain as in the proof of Lemma 2.3 that $(x,u),(x,w) \in N(A(t_0,1,j)) = N(A_n)$, for some $n \in \mathbb{N}$. But then

$$\left(x, \frac{(1-s)u+sw}{|(1-s)u+sw|}\right) \in N(A_n)$$

for all $s \in [0,1]$, hence $(0,\bar{u}) \in \operatorname{Tan}(N(A_n),x,u)$ with some $\bar{u} \in S^{d-1} \cap u^{\perp}$. From (2.10) and the representation of $\operatorname{Tan}(N(A),x,u)$ in the proof of Theorem 2.1, it now follows that $k_i(A,x,u) = \infty$ for some $i \in \{1,\ldots,d-1\}$.

Therefore the coarea formula yields that

$$\int_{N(A)} f(x,u) H_0(x,u) \mathcal{H}^{d-1}(d(x,u)) = \int_{\partial^{++}A} \sum_{u \in N(A,x)} f(x,u) \mathcal{H}^{d-1}(dx),$$

which implies (4.1). This formula also shows that $\mu_{d-1}(A;\cdot)$ is a non-negative measure on $\mathbb{R}^d \times S^{d-1}$. □

From (2.2) for $j = d-1$ and (4.1) we obtain the following corollary.

Corollary 4.2. *Let $A \subset \mathbb{R}^d$ be a closed set. Then*

$$\int_{\partial^{++}A} \mathbf{1}\{x \in B\}(\delta(A,x,\nu(A,x)) \wedge r)\mathcal{H}^{d-1}(dx) + \int_{\partial^2 A} \mathbf{1}\{x \in B\}(\delta(A,x,-\nu(A,x)) \wedge r)\mathcal{H}^{d-1}(dx) < \infty, \tag{4.2}$$

for all compact sets $B \subset \mathbb{R}^d$ and all $r > 0$.

4.2 A Steiner formula for support measures

To motivate our next theorem, we first look at a special case. Let $K \subset \mathbb{R}^d$ be a non-empty compact convex set. For $s > 0$, we consider the map $T_s : \mathbb{R}^d \times S^{d-1} \to \mathbb{R}^d \times S^{d-1}$ which is defined by $T_s(x,u) := (x+su,u)$. Then, for $k \in \{0,\dots,d-1\}$, the support measures of K satisfy the Steiner formula

$$\mu_k(K_{\oplus s}; T_s(\cdot)) = \sum_{i=0}^{k} s^{k-i} \binom{d-i}{d-k} \frac{\kappa_{d-i}}{\kappa_{d-k}} \mu_i(K;\cdot);$$

see [27]. This result will be extended to general closed sets in the following Theorem and in Corollary 4.4.

Theorem 4.3. *Let $A \subset \mathbb{R}^d$ be a closed set, $k \in \{0,\dots,d-1\}$ and $0 < s < \varepsilon$. Then*

$$\int_{N(A_{\oplus s})} \mathbf{1}\{\varepsilon - s \le \delta(A_{\oplus s},y,u)\}\mathbf{1}\{(y-su,u) \in \cdot\}\mu_k(A_{\oplus s}; d(y,u)) = \sum_{i=0}^{k} s^{k-i} \binom{d-i}{d-k} \frac{\kappa_{d-i}}{\kappa_{d-k}} \int_{N(A)} \mathbf{1}\{\varepsilon \le \delta(A,x,u)\}\mathbf{1}\{(x,u) \in \cdot\}\mu_i(A; d(x,u)).$$

Proof. Let $\emptyset \ne A \ne \mathbb{R}^d$, and choose $t > 0$ so that $s+t < \varepsilon$. For the proof, we will use relation (2.26). Let $B \subset \mathbb{R}^d$ be compact and let $f : N(A) \to \mathbb{R}$ be a measurable and bounded function. Then we get

$$\begin{aligned}&\int \mathbf{1}\{\varepsilon \le \delta(A, p(A,z), u(A,z)),\ p(A,z) \in B\} \\ &\qquad \times \mathbf{1}\{0 < d(A,z) \le s+t\} f(p(A,z), u(A,z)) \mathcal{H}^d(dz) \\ &\quad = \sum_{i=0}^{d-1} \kappa_{d-i} (s+t)^{d-i} \int_{N(A)} \mathbf{1}\{\varepsilon \le \delta(A,x,u), x \in B\} f(x,u) \mu_i(A; d(x,u)) \\ &\quad = \sum_{k=0}^{d} t^{d-k} \sum_{i=0}^{k \wedge (d-1)} \binom{d-i}{d-k} \kappa_{d-i} s^{k-i} \int_{N(A)} \mathbf{1}\{\varepsilon \le \delta(A,x,u), x \in B\} \\ &\qquad \times f(x,u) \mu_i(A; d(x,u)).\end{aligned} \tag{4.3}$$

We write the integral in (4.3) as the sum of two integrals by means of

$$\mathbf{1}\{0 < d(A,z) \le s+t\} = \mathbf{1}\{0 < d(A,z) \le s\} + \mathbf{1}\{0 < d(A_{\oplus s}, z) \le t\}$$

and apply again (2.26) to obtain

$$\begin{aligned}&\sum_{i=0}^{d-1} \kappa_{d-i} s^{d-i} \int_{N(A)} \mathbf{1}\{\varepsilon \le \delta(A,x,u), x \in B\} f(x,u) \mu_i(A; d(x,u)) \\ &\quad + \int \mathbf{1}\{\varepsilon \le \delta(A, p(A,z), u(A,z)),\ p(A,z) \in B, 0 < d(A_{\oplus s}, z) \le t\} \\ &\quad \times f(p(A,z), u(A,z)) \mathcal{H}^d(dz).\end{aligned}$$

To rewrite the second term, recall that $s + t < \varepsilon$. Then, for $\mathcal{H}^d$-a.e. $z \in \mathbb{R}^d$ with $0 < d(A_{\oplus s}, z) \le t$,

$$\varepsilon \le \delta(A, p(A,z), u(A,z))$$

if and only if

$$\varepsilon - s \le \delta(A_{\oplus s}, p(A_{\oplus s}, z), u(A_{\oplus s}, z));$$

moreover, if these conditions are satisfied, then

$$p(A,z) = p(A_{\oplus s}, z) - s u(A_{\oplus s}, z) \quad \text{and} \quad u(A,z) = u(A_{\oplus s}, z).$$

Therefore, the last integral is equal to

$$\begin{aligned}&\int \mathbf{1}\{\varepsilon - s \le \delta(A_{\oplus s}, p(A_{\oplus s}, z), u(A_{\oplus s}, z)),\ p(A_{\oplus s}, z) - s u(A_{\oplus s}, z) \in B\} \\ &\qquad \times \mathbf{1}\{0 < d(A_{\oplus s}, z) \le t\} f(p(A_{\oplus s}, z) - s u(A_{\oplus s}, z), u(A_{\oplus s}, z)) \mathcal{H}^d(dz) \\ &\quad = \sum_{k=0}^{d-1} \kappa_{d-k} t^{d-k} \int_{N(A_{\oplus s})} \mathbf{1}\{\varepsilon - s \le \delta(A_{\oplus s}, y, u),\ y - su \in B\} \\ &\qquad \times f(y - su, u) \mu_k(A_{\oplus s}; d(y,u)).\end{aligned}$$

For the last step, we applied (2.26) to $A_{\oplus s}$ instead of A and used that $t < \varepsilon - s$. A comparison of coefficients now yields the assertion. □

Passing to the limit $\varepsilon \downarrow s$ in the formula of Theorem 4.3, we obtain the following important consequence.

Corollary 4.4. *Let $A \subset \mathbb{R}^d$ be a closed set and $k \in \{0, \ldots, d-1\}$. Then, for any $s > 0$, the total variation measure of $\mu_k(A_{\oplus s}; \cdot)$ is locally finite and*

$$\mu_k(A_{\oplus s}; \cdot) = \sum_{i=0}^{k} s^{k-i} \binom{d-i}{d-k} \frac{\kappa_{d-i}}{\kappa_{d-k}} \int_{N(A)} \mathbf{1}\{s < \delta(A, x, u)\} \times \mathbf{1}\{T_s(x, u) \in \cdot\} \mu_i(A; d(x, u)).$$

Proof. Applying Lebesgue's increasing convergence theorem in passing to the limit $\varepsilon \downarrow s$ in the equation of Theorem 4.3, we first obtain that the positive and the negative part of $\mu_k(A_{\oplus s}; \cdot)$ are both finite over compact sets. Hence the total variation measure of $\mu_k(A_{\oplus s}; \cdot)$ is finite on compact sets. Then one can apply the bounded convergence theorem to obtain the asserted equation. □

4.3 Some consequences

Corollary 4.4 in particular implies that if $A \subset \mathbb{R}^d$ is compact and $s > 0$, then $\mu_k(A_{\oplus s}; \cdot)$ has a finite total variation measure for $k = 0, \ldots, d-1$. Hence, $\mu_{d-1}(A_{\oplus s}; \cdot)$ is a finite measure. Using a special case of Proposition 4.1 and since $\partial^+ A_{\oplus s} = \partial^{++} A_{\oplus s}$ for $s > 0$, we obtain that

$$\mathcal{H}^{d-1}(\partial^+ A_{\oplus s}) < \infty.$$

We now derive some further consequences of Corollary 4.4. Let again $A \subset \mathbb{R}^d$ be compact. Then we set $V_A(r) := \mathcal{H}^d(A_{\oplus r})$.

Equation (2.3) implies in particular that for $r \geq 0$ the parallel volume of A can be written as

$$V_A(r) = V_A(0) + \sum_{i=0}^{d-1} \omega_{d-i} \int_0^r \int_{N(A)} t^{d-1-i} \times \mathbf{1}\{t < \delta(A, x, u)\} \mu_i(A; d(x, u)) dt, \tag{4.4}$$

which is the general counterpart to (1.1). Equation (4.4) implies that the right and left derivatives $V_A^{(+)}(r)$ and $V_A^{(-)}(r)$ of V_A exist for all $r \in (0, \infty)$. More specifically, since

$$t \mapsto \int_{N(A)} \mathbf{1}\{t < \delta(A, x, u)\} \mu_i(A; d(x, u))$$

is well defined and right continuous, we have

$$V_A^{(+)}(r) = \sum_{i=0}^{d-1} \omega_{d-i} r^{d-1-i} \int_{N(A)} \mathbf{1}\{r < \delta(A, x, u)\} \mu_i(A; d(x, u)) \tag{4.5}$$

for $r > 0$. On the other hand, the function

$$t \mapsto \int_{N(A)} \mathbf{1}\{t < \delta(A, x, u)\}\mu_i(A; d(x, u))$$

has the left limit

$$\int_{N(A)} \mathbf{1}\{t \le \delta(A, x, u)\}\mu_i(A; d(x, u))$$

at $t > 0$, hence

$$V_A^{(-)}(r) = \sum_{i=0}^{d-1} \omega_{d-i} r^{d-1-i} \int_{N(A)} \mathbf{1}\{r \le \delta(A, x, u)\}\mu_i(A; d(x, u)) \tag{4.6}$$

for $r > 0$. We summarize these considerations in the following corollary, which complements and refines previous work by Kneser [21] and Stachó [29].

Corollary 4.5. *Let $A \subset \mathbb{R}^d$ be compact. Then the derivative of V_A at $r > 0$ exists if*

$$\int_{N(A)} \mathbf{1}\{\delta(A, x, u) = r\}\mu_i(A; d(x, u)) = 0, \qquad i = 0, \dots, d-1. \tag{4.7}$$

In particular, the derivative exists for all $r \in (0, \infty)$ with the possible exception of an at most countable set.

It is clear that if the measures $\mu_i(A; \cdot)$ are all non-negative (cf. Section 3), then condition (4.7) is necessary and sufficient for the differentiability of the function V_A at $r > 0$. Moreover, for a non-empty compact convex set $A \subset \mathbb{R}^d$, condition (4.7) is satisfied for all $r > 0$; for a set A with $\text{reach}(A) > \varepsilon$, this condition is fulfilled at least for $0 < r \le \varepsilon$.

Let $r > 0$ and $A \subset \mathbb{R}^d$ be compact. Then, combining Equation (4.5) with Corollary 4.4, we find that

$$V_A^{(+)}(r) = 2\mu_{d-1}(A_{\oplus r}; \mathbb{R}^d \times S^{d-1}),$$

and hence

$$V_A^{(+)}(r) = \mathcal{H}^{d-1}(\partial^+ A_{\oplus r}),$$

where Proposition 4.1 and $\partial^+ A_{\oplus r} = \partial^{++} A_{\oplus r}$ were used. By the main result in [29], the $(d-1)$-dimensional Minkowski content (cf. [3], [5]) $\mathcal{M}^{d-1}(\partial A_{\oplus r})$ of $\partial A_{\oplus r}$ exists for all $r > 0$ and

$$\mathcal{M}^{d-1}(\partial A_{\oplus r}) = \frac{1}{2}\left(V_A^{(+)}(r) + V_A^{(-)}(r)\right).$$

Thus, as a consequence of Corollary 4.5 we obtain the next result.

Corollary 4.6. *Let $A \subset \mathbb{R}^d$ be compact. Then, for all $r \in (0, \infty)$,*

$$V_A^{(+)}(r) = \mathcal{H}^{d-1}(\partial^+ A_{\oplus r});$$

moreover,

$$\mathcal{M}^{d-1}(\partial A_{\oplus r}) = \mathcal{H}^{d-1}(\partial^+ A_{\oplus r})$$

for $r \in (0, \infty)$ with the possible exception of an at most countable set.

For $d \geq 4$, $\partial A_{\oplus r}$ need not be a rectifiable set (although it is a Hausdorff rectifiable set); see [7] for counterexamples and [10], [7], [9] for related work. Therefore, we cannot use Theorem 3.2.39 in [5] to conclude that $\mathcal{H}^{d-1}(\partial A_{\oplus r} \setminus \partial^+ A_{\oplus r}) = 0$. However, the desired conclusion can now be obtained in a more direct way.

Corollary 4.7. *Let $A \subset \mathbb{R}^d$ be compact. Then $\mathcal{H}^{d-1}(\partial A_{\oplus r} \setminus \partial^+ A_{\oplus r}) = 0$ for $\mathcal{H}^1$-almost all $r \in (0, \infty)$.*

Proof. We already know that $V_A^{(+)}(r) = \mathcal{H}^{d-1}(\partial^+ A_{\oplus r})$ for all $r > 0$. By Lemma 3.2.34 in [5] we also have

$$V_A(r) = V_A(0) + \int_0^r \mathcal{H}^{d-1}(\partial A_{\oplus t})dt,$$

hence $V_A^{(+)}(r) = \mathcal{H}^{d-1}(\partial A_{\oplus r})$ for $\mathcal{H}^1$-almost all $r \in (0, \infty)$. □

In general, the parallel volume is not differentiable at the point $r = 0$. If it were, then the boundary of each compact set with vanishing volume would admit a finite $(d-1)$-dimensional *Minkowski content* (see [5], [3]). Fractal sets of dimension strictly greater than $d - 1$ do not have this property. We wish to illustrate this point with just one very simple example.

Example 4.8. We consider the *Sierpiński gasket* $A \subset \mathbb{R}^2$ (see [3]). It is constructed by repeated removal of open equilateral triangles from an initial equilateral triangle A_1 with side length 1, say. In the first step one open triangle having side length 2^{-1} and its vertices on the boundary of A_1 is removed. The result is a set A_2 consisting of 3 triangles with side length 2^{-1}. Removing from each of the three triangles an open triangle with side length 2^{-2} yields the set A_3 consisting of 3^2 triangles each with side length 2^{-2}. The n-th set A_n is made up of 3^{n-1} equilateral triangles each with side length $2^{-(n-1)}$. The Sierpiński gasket A is then the intersection of all the sets A_n constructed in this way. Obviously this is a non-empty and compact set with vanishing area. It is easy to see that the positive boundary $\partial^+ A$ is the union of all the boundaries ∂A_n and that $\mathcal{H}^1(\partial^+ A) = \infty$. The set $\partial^{++} A$ is obtained from $\partial^+ A$ by removing the three vertices of A_1 and coincides with $\partial^1 A$. If $x \in \partial^{++} A \cap \partial A_n$ and x belongs exactly to one of the triangles of side length $2^{-(n-1)}$ forming A_n, then $\nu(A, x)$ is the outer normal of this particular triangle; otherwise, $\nu(A, x)$ is the unique common outer normal vector of the two triangles to which x belongs. If $x \in \partial^+ A \setminus \partial^{++} A$ (i.e. a vertex of A_1), then $k_1(A, x, u) = \infty$ for almost all

$u \in N(A, x)$. If $x \in \partial^{++}A$, then $k_1(A, x, \nu(A, x)) = 0$. Hence we obtain from (2.24) that

$$\mu_0(A; \cdot) = \frac{1}{2\pi} \sum_{x \in \partial^+ A \setminus \partial^{++} A} \int_{N(A,x)} \mathbf{1}\{(x, u) \in \cdot\} \mathcal{H}^1(du),$$

$$\mu_1(A; \cdot) = \frac{1}{2} \int_{\partial^{++} A} \mathbf{1}\{(x, \nu(A, x)) \in \cdot\} \mathcal{H}^1(dx).$$

Since it is also quite easy to determine the reach function, it is in fact possible to compute the right-hand side of (2.16) explicitly. Some calculus yields that

$$\mathcal{H}^2(A_{\oplus r}) \geq \pi r^2 + c r^{2-\log 3/\log 2}, \quad r > 0,$$

where c is an absolute constant. Hence

$$\liminf_{r \to 0+} r^{a-2} \mathcal{H}^2(A_{\oplus r}) = \infty,$$

whenever $a < \log 3/\log 2$, in accordance with the fact that the box-counting (or Minkowski) dimension of A is given by $\log 3/\log 2$ (see Definition 3.1 and Proposition 3.2 in [3]). In particular, we have $r^{-1}\mathcal{H}^2(A_{\oplus r}) \to \infty$ as $r \to 0+$.

4.4 Further properties

Equation (2.24) implies that the support measures are *locally defined*. This means that for any two non-empty closed sets $A_1, A_2 \subset \mathbb{R}^d$ satisfying $A_1 \cap U = A_2 \cap U$, for some open set $U \subset \mathbb{R}^d$,

$$\mu_i(A_1; D) = \mu_i(A_2; D), \quad i = 0, \dots, d-1,$$

for all Borel sets $D \subset U \times S^{d-1}$ for which one side (and hence both sides) of this equation are well-defined.

Another useful property of the support measures which follows immediately from Theorem 2.1 (or from (2.24)) is that they are *translation covariant*, i.e.

$$\mu_i(A + z; (B + z) \times C) = \mu_i(A; B \times C), \quad i = 0, \dots, d-1,$$

for all $z \in \mathbb{R}^d$, all non-empty closed sets $A \subset \mathbb{R}^d$ and all measurable sets $B \subset \mathbb{R}^d$, $C \subset S^{d-1}$, such that the right-hand side is well-defined.

As in the classical case, the support measures satisfy an important scaling property.

Proposition 4.9. *For any non-empty closed set $A \subset \mathbb{R}^d$ and any $c > 0$,*

$$\mu_j(cA; \cdot) = c^j \int \mathbf{1}\{(cx, u) \in \cdot\} \mu_j(A; d(x, u)), \quad j = 0, \dots, d-1.$$

Proof. Since $d(cA, z) = cd(A, c^{-1}z)$ for all $z \in \mathbb{R}^d$, we have that $z \in \text{exo}(cA)$ if and only if $c^{-1}z \in \text{exo}(A)$. It is now easy to check that $p(cA, z) = cp(A, c^{-1}z)$ and $u(cA, z) = u(A, c^{-1}z)$ for all $z \notin cA \cup \text{exo}(cA)$. Using this together with the scaling properties of Lebesgue measure, we obtain from the Steiner formula (2.3) that, for any bounded and measurable function $f : [0, \infty) \times \mathbb{R}^d \times S^{d-1}$ with compact support,

$$\int_{\mathbb{R}^d \setminus cA} f(d(cA, z), p(cA, z), u(cA, z))\mathcal{H}^d(dz) = \sum_{i=0}^{d-1} \omega_{d-i} c^i$$
$$\times \int_0^\infty \int_{N(A)} s^{d-1-i} \mathbf{1}\{s < c\delta(A, x, u)\} f(s, cx, u) \mu_i(A; d(x, u)) ds.$$

Noting that $c\delta(A, x, u) = \delta(cA, cx, u)$ for all $(x, u) \in N(A)$, we can conclude the assertion from the uniqueness part of Theorem 2.1. □

We conclude this section by providing a more explicit description of the support measures of order zero.

Proposition 4.10. *The support measure of order* 0 *of a closed set* $A \subset \mathbb{R}^d$ *is given by*

$$\omega_d \mu_0(A; \cdot) = \int_{S^{d-1}} \sum_{x:(x,u)\in N(A)} \mathbf{1}\{(x, u) \in \cdot\}(-1)^{j(A,x,u)} \mathcal{H}^{d-1}(du),$$

where $j(A, x, u) := \text{card}\{i \in \{1, \dots, d-1\} : k_i(A, x, u) < 0\}$. *Moreover,*

$$\int_{S^{d-1}} \sum_{x:(x,u)\in N(A)} \mathbf{1}\{x \in B\}(\delta(A, x, u) \wedge r)^d \mathcal{H}^{d-1}(du) < \infty$$

for all compact sets $B \subset \mathbb{R}^d$ *and* $r > 0$. *In particular, for* $\mathcal{H}^{d-1}$*-a.e.* $u \in S^{d-1}$ *there are at most countably many* $x \in \mathbb{R}^d$ *such that* $(x, u) \in N(A)$.

Proof. We are using Corollary 2.5 and apply the coarea formula to the mapping $(x, u) \mapsto u$ from $N(A)$ to S^{d-1}. Since the approximate Jacobian of this mapping equals $|H_{d-1}(A, x, u)|$ (see e.g. [15]) we obtain the first assertion. The second is then a consequence of (2.2). □

5 Interior reach

For a non-empty closed set $A \subset \mathbb{R}^d$, it is easy to see that

$$\text{exo}(\partial A) = \text{exo}(A) \cup \text{exo}(A^*),$$

and

$$N(\partial A) = N(A) \cup N(A^*), \quad N(A) \cap N(A^*) = \emptyset.$$

The elements $(x, u) \in N(A^*)$ consist of boundary points $x \in \partial A$ and normal vectors u reaching into the interior of A. Thus the reach function, the normal bundle and the support measures of ∂A can be used to extend the corresponding notions of A by taking boundary points with 'interior normals' into account. In this section, we discuss such extensions. Our final goal is to develop the total integral

$$\int f(z)\mathcal{H}^d(dz) \tag{5.1}$$

into a Steiner formula with respect to the given set A.

For this purpose, we introduce the *extended normal bundle* $N_e(A)$ of A as

$$N_e(A) := N(A) \cup T(N(A^*)),$$

where $T : \mathbb{R}^d \times S^{d-1} \to \mathbb{R}^d \times S^{d-1}$ is the reflection $(x, u) \mapsto (x, -u)$. Thus $N_e(A)$ consists of the *exterior normal bundle* $N(A)$ and the reflection of the *interior normal bundle* $N(A^*)$. The reach function of A is now called the *exterior reach function*, and denoted by $\delta^+(A, \cdot)$. We define a corresponding *interior reach function* $\delta^-(A, \cdot) : \mathbb{R}^d \times S^{d-1} \to [-\infty, 0]$ by

$$\delta^-(A, x, u) := -\delta(A^*, x, -u), \quad (x, u) \in \mathbb{R}^d \times S^{d-1}.$$

Note that $\delta^+(A, \cdot) = \delta(\partial A, \cdot)$ on $N(A)$ and $\delta^-(A, \cdot) = -\delta(\partial A, T(\cdot))$ on $T(N(A^*))$. In addition, we have $N_e(A) = T(N_e(A^*))$ if $A = \operatorname{cl}\operatorname{int} A$.

As a next step, we now use the support measures of ∂A to extend the support measures of A to the extended normal bundle $N_e(A)$. The starting point is the relation

$$\mu_j(\partial A; \cdot)\llcorner N(A) = \mu_j(A; \cdot), \quad \mu_j(\partial A; \cdot)\llcorner N(A^*) = \mu_j(A^*; \cdot),$$

$j = 0, \dots, d-1$, where $\mu\llcorner M$ denotes the restriction of a set function μ to the Borel set M. The idea is to combine $\mu_j(A; \cdot)$ with $(-1)^{d-1-j}T(\mu_j(A^*; \cdot))$, which is possible if both measures coincide on the intersection of their support. This is shown by the following result which extends equation (3.4) to general closed sets A.

Proposition 5.1. *For a non-empty closed set $A \subset \mathbb{R}^d$ and a measurable set $B \subset N(A) \cap T(N(A^*))$,*

$$\mu_j(A; B) = (-1)^{d-1-j}\mu_j(A^*; T(B)), \quad j = 0, \dots, d-1, \tag{5.2}$$

provided that one side, and hence both sides, are well-defined.

Proof. Using the explicit description of the tangent space of the normal bundles of A and A^*, we find that, for $\mathcal{H}^{d-1}$-a.e. $(x, u) \in N(A) \cap T(N(A^*))$,

$$k_i(A^*, T(x, u)) = -k_i(A, x, u), \qquad i = 1, \dots, d-1.$$

Since $JT(x, u) = 1$, we thus get that

$$\begin{aligned}\int_B & H_{d-1-j}(A, x, u)\mathcal{H}^{d-1}(d(x, u)) \\ &= (-1)^{d-1-j}\int_{T(B)} H_{d-1-j}(A^*, x, u)\mathcal{H}^{d-1}(d(x, u)),\end{aligned} \tag{5.3}$$

for all Borel sets $B \subset N(A) \cap T(N(A^*))$ for which one of the two sides of the required equation is well-defined. This is equivalent to the assertion. □

As a consequence of Proposition 5.1, we define the *extended support measure* $\nu_j(A;\cdot)$ on $N_e(A)$ by

$$\begin{aligned}\nu_j(A;\cdot) := \mu_j(A;\cdot) + (-1)^{d-1-j}T(\mu_j(A^*;\cdot))\\ -\mu_j(A;\cdot\cap N(A)\cap T(N(A^*)),\end{aligned}$$

for $j = 0,\dots,d-1$. It is now possible to combine the Steiner formula (2.3) (or its equivalent version (2.27)) for A with the one for A^*, which gives the following result.

Theorem 5.2. *For a non-empty closed proper subset $A \subset \mathbb{R}^d$ and a measurable bounded function $f : \mathbb{R}^d \to \mathbb{R}$ with compact support,*

$$\begin{aligned}&\int_{\mathbb{R}^d\setminus\partial A} f(x)\mathcal{H}^d(dx)\\ &\quad=\sum_{i=0}^{d-1}\omega_{d-i}\int_{N_e(A)}\int_{\delta^-(A,x,u)}^{\delta^+(A,x,u)} t^{d-1-i}f(x+tu)dt\,\nu_i(A;d(x,u)). \quad (5.4)\end{aligned}$$

In particular, if $\mathcal{H}^d(\partial A)=0$, then the integral in (5.1) is equal to the right-hand side of (5.4). However, the assumption $\mathcal{H}^d(\partial A)=0$ is not fulfilled automatically, it may fail, for example, for Cantor-type sets A.

6 Measurability and integrability properties

The applications of support measures in stochastic geometry, which will be considered in Sections 7 and 8, require some additional measurability properties related to the space $\mathcal{F}^d$ of all non-empty and closed subsets of $\mathbb{R}^d$. We endow $\mathcal{F}^d$ with the usual Fell-Matheron “hit-or-miss” topology (see [23]). Then, $\mathcal{F}^d$ is a locally compact, second-countable Hausdorff space. Measurability on this space does always refer to the Borel σ-field generated by the Fell-Matheron topology. A mapping h on $\mathcal{F}^d\times\mathbb{R}^d$ with values in some arbitrary set is called *covariant* if $h(A,z)=h(A-z,0)$ for all $(A,z)\in\mathcal{F}^d\times\mathbb{R}^d$. A subset of $\mathcal{F}^d\times\mathbb{R}^d$ is called covariant if its indicator function is covariant.

Lemma 6.1. *The set $\{(A,z)\in\mathcal{F}^d\times\mathbb{R}^d : z\in \mathrm{exo}(A)\}$ is measurable and covariant. Moreover, the mappings $(A,z)\mapsto p(A,z)$ and $(A,z)\mapsto u(A,z)$ from $\mathcal{F}^d\times\mathbb{R}^d$ to $\mathbb{R}^d$ are measurable.*

Lemma 6.2. *The map $\delta : \mathcal{F}^d\times\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty]$, $(A,x,u)\mapsto\delta(A,x,u)$, is measurable, and covariant with respect to the first two arguments. In particular, the map $(A,x,u)\mapsto \mathbf{1}\{(x,u)\in N(A)\}$ is measurable, and covariant with respect to the first two arguments.*

In the following Lemma we extend the definition of $\nu(A,x)$ by giving $\nu(A,x)$ some fixed value in S^{d-1} whenever $x\in\mathbb{R}^d\setminus\partial^{++}A$.

Lemma 6.3. *The mapping*

$$(A, x) \mapsto (\mathbf{1}\{x \in \partial^+ A\}, \mathbf{1}\{x \in \partial^{++} A\}, \mathbf{1}\{x \in \partial^1 A\}, \nu(A, x))$$

from $\mathcal{F}^d \times \mathbb{R}^d$ *to* $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times S^{d-1}$ *is measurable and covariant.*

The proofs of the preceding three lemmas follow from the results and arguments provided at the end of Section 3 in [17].

Lemma 6.4. *For any compact set* $B \subset \mathbb{R}^d$, $\varepsilon > 0$, *for any measurable set* $D \subset B \times S^{d-1}$, *and* $j \in \{0, \dots, d-1\}$ *the mapping*

$$A \mapsto \mu_j(A; \{(x, u) \in N(A) : (x, u) \in D, \delta(A, x, u) > \varepsilon\})$$

from $\mathcal{F}^d$ *to* $\mathbb{R}$ *is measurable. Furthermore, the map* $A \mapsto |\mu_j|(A; \cdot)$ *from* $\mathcal{F}^d$ *to* $[0, \infty]$ *is measurable.*

Proof. The first assertion is implied by Lemmas 6.1 and 6.2, Fubini's theorem and by relation (2.26).

For the second assertion, it is sufficient to consider the case where $\mu_j(A; \cdot)$ has finite total variation. Let C_0 denote a countable and dense (with respect to the maximum norm) set of continuous functions $f : \mathbb{R}^d \times S^{d-1} \to [0, 1]$ with compact support. Then

$$|\mu_j|(A; C) = \sup \left\{ \int_C f(x, u) \mu_j(A; d(x, u)) : f \in C_0 \right\},$$

which yields the required measurability. □

The following improvement of the integrability property (2.2) will be useful in the next section.

Theorem 6.5. *Let* $\mathbb{V}$ *be a* σ*-finite measure on* $\mathcal{F}^d$ *and* $B \subset \mathbb{R}^d$ *a measurable set. Then*

$$\int_{\mathcal{F}^d} \int_{\mathbb{R}^d \setminus A} \mathbf{1}\{0 < d(A, z) \le r, p(A, z) \in B\} \mathcal{H}^d(dz) \mathbb{V}(dA) < \infty \tag{6.1}$$

for some $r > 0$ *if and only if*

$$\int_{\mathcal{F}^d} \int_{\mathbb{R}^d \times S^{d-1}} \mathbf{1}\{x \in B\} (\delta(A, x, u) \wedge r)^{d-j} |\mu_j|(A; d(x, u)) \mathbb{V}(dA) < \infty \tag{6.2}$$

for some $r > 0$ *and all* $j = 0, \dots, d-1$. *In this case, both* (6.1) *and* (6.2) *are satisfied by any* $r > 0$.

Proof. The constants $c_3(l, d)$ appearing in (2.21) do not depend on A. Therefore, if (6.1) holds for some fixed $r = r_0 > 0$, then (6.2) holds with the same $r = r_0$. But then

$$\int_{\mathcal{F}^d} \int_{\mathbb{R}^d \times S^{d-1}} \mathbf{1}\{\varepsilon \le \delta(A, x, u), x \in B\} |\mu_j|(A; d(x, u)) \mathbb{V}(dA) < \infty$$

for $j = 0, \dots, d-1$, first for $0 < \varepsilon < r_0$ and then for all $\varepsilon > 0$. Together with (6.2), for $r = r_0$, this implies that (6.2) is true for all $r > 0$. Conversely, if (6.2) holds for just one $r_0 > 0$ and for $j = 0, \dots, d-1$, then it holds for all $r > 0$ and (6.1) follows from the local Steiner formula. □

The inclusion

$$\{z \in \mathbb{R}^d : 0 < d(A, z) \le r,\ p(A, z) \in B\} \subset A_{\oplus r} \cap B_{\oplus r}$$

implies that (6.1) is trivially satisfied if B is bounded and $\mathbb{V}$ is a finite measure.

7 Contact distributions of random closed sets

In this section, we consider a random closed set Z defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Formally, Z is a random element of the measurable space $\mathcal{F}^d$. The requirement $Z \neq \emptyset$ is no restriction of generality. If Z is a random element of $\mathcal{F}^d \cup \{\emptyset\}$, then we can apply the results of this section to the conditional probability measure $\mathbb{P}(\cdot | Z \neq \emptyset)$. Our basic assumption on Z is *stationarity*, i.e. the distributional invariance of Z under all translations.

Due to stationarity, the *volume fraction*

$$p := \mathbb{P}(0 \in Z)$$

of Z can be expressed as $p = \mathbb{E}[\mathcal{H}^d(Z \cap B)]$ for any Borel set B of volume 1. We study the distribution of the distance $d(Z, z)$ of $z \in \mathbb{R}^d$ from Z. If $p = 1$, then $\mathbb{P}(d(Z, z) = 0) = 1$ for all $z \in \mathbb{R}^d$ and hence $Z = \mathbb{R}^d$ is satisfied $\mathbb{P}$-almost surely. To exclude this trivial case we assume here that $p < 1$. The *spherical contact distribution function* of Z (see e.g. [31]) is defined by

$$H(t) := \mathbb{P}(d(Z, z) \le t \,|\, z \notin Z), \quad t \ge 0.$$

Again by stationarity, H is independent of z. More generally, we define (see [22], [17], [19])

$$H(t, C) := \mathbb{P}(d(Z, z) \le t, u(Z, z) \in C \,|\, z \notin Z), \tag{7.1}$$

for any measurable $C \subset S^{d-1}$.

It follows from Theorem 6.5 that

$$\mathbb{E}\left[\int \mathbf{1}\{x \in B\}(\delta(Z, x, u) \wedge r)^{d-j} |\mu_j|(Z; d(x, u))\right] < \infty \tag{7.2}$$

for $j = 0, \dots, d-1$, for all compact sets $B \subset \mathbb{R}^d$ and all $r > 0$. This is the expected value version of (2.2). Define

$$\beta_j(\cdot) := \int \mathbf{1}\{(x, u, \delta(Z, x, u)) \in \cdot\}\mu_j(Z; d(x, u)), \quad j = 0, \dots, d-1.$$

Then $\beta_j(\mathbb{R}^d \times S^{d-1} \times \{0\}) = 0$ so that β_j can be interpreted as a random signed measure on $\mathbb{R}^d \times S^{d-1} \times (0, \infty]$. Since Z is assumed to be stationary, it can be easily seen from translation covariance and from the equation $\delta(Z + y, x, u) = \delta(Z, x - y, u)$, $y \in \mathbb{R}^d$, that β_j is stationary, i.e. its distribution is invariant under shifts in the first variable. Therefore, if $B \subset \mathbb{R}^d$ is a Borel set with $0 < \mathcal{H}^d(B) < \infty$, then

$$\Gamma_j(\cdot) := \frac{1}{\mathcal{H}^d(B)} \mathbb{E}\left[\int \mathbf{1}\{x \in B\}\mathbf{1}\{(u, \delta(Z, x, u)) \in \cdot\}\mu_j(Z; d(x, u))\right] \tag{7.3}$$

is a signed measure which does not depend on the choice of B. In general we may have $|\Gamma_j(S^{d-1} \times (s, \infty])| \to \infty$ as $s \to 0$, but equation (7.2) implies that

$$\int_0^r s^{d-1-j} |\Gamma_j|(S^{d-1} \times (s, \infty])ds < \infty \tag{7.4}$$

for $r \geq 0$ and $j = 0, \dots, d-1$. The following result can now be proved similarly to Theorem 5.1 and Corollary 5.2 in [17]. In that paper the random closed set Z was assumed to be $\mathcal{S}^d$-valued. A first version of a result of this type has been established in [22].

Theorem 7.1. *Let Z be a stationary random closed set. Then*

$$(1-p)H(t, C) = \sum_{i=0}^{d-1} \omega_{d-i} \int_0^t s^{d-1-i} \Gamma_i(C \times (s, \infty])ds$$

for any $t \geq 0$ and any measurable set $C \subset S^{d-1}$.

Absolute continuity of the contact distribution $H(t)$ has been proved (independently of [22]) in [2] (see also [13]) using Federer's coarea theorem. The new and very pleasing fact here is that the density of $(1-p)H(\cdot, C)$ is of the same explicit form as in [22] and [17] where the case of a random set taking values in the extended convex ring is studied. Note in particular that we do not need to impose any integrability condition on Z.

Since $\Gamma_i(C \times (s, \infty])$ is a right continuous function of $s \in (0, \infty)$, it follows that $(1-p)H(\cdot, C)$ admits a right derivative on $(0, \infty)$. Moreover, since the left limit of $\Gamma_i(C \times (\cdot, \infty])$ exists, the contact distribution also has a left derivative. It is even differentiable with the possible exception of at most countably many points. Example 8.2 below shows that $(1-p)H(\cdot, C)$ need not be differentiable at the point 0. Such a differentiability property can be deduced under the additional assumption

$$\mathbb{E}[|\mu_i|(Z; B \times S^{d-1})] < \infty, \quad i = 0, \dots, d-1, \tag{7.5}$$

for some Borel set B with positive volume. Under this assumption (which certainly excludes fractal behaviour of Z), the curvature measures $\mu_i(Z; \cdot)$ can be considered as random signed measures on $\mathbb{R}^d \times S^{d-1}$. The associated total variation measures $|\mu_i|(Z; \cdot)$ are (locally finite) random measures.

Corollary 7.2. *Let Z be a stationary random closed set satisfying* (7.5) *for some Borel set B with positive volume. Then*

$$\lim_{t \to 0+} t^{-1}(1-p)H(t, C) = 2\lambda_{d-1}(C)$$

for any measurable set $C \subset S^{d-1}$, where

$$\lambda_{d-1}(C) := \mathbb{E}[\mu_{d-1}(Z; [0, 1]^d \times C)] < \infty. \tag{7.6}$$

Proof. Assumption (7.5) ensures that the Γ_j are finite signed measures. Therefore the assertion is an immediate consequence of Theorem 7.1. □

Under (7.5) we have in particular that

$$\Lambda_{d-1} := \mathbb{E}[\mu_{d-1}(Z;\cdot)] \tag{7.7}$$

is a locally finite measure on $\mathbb{R}^d \times S^{d-1}$. According to Proposition 4.1 we may interpret $\Lambda_{d-1}(\cdot \times S^{d-1})$ as the *surface intensity measure* of Z. From stationarity we obtain that

$$\Lambda_{d-1} = \mathcal{H}^d \otimes \lambda_{d-1}. \tag{7.8}$$

Again by Proposition 4.1 we can interpret the number $\lambda_{d-1}(S^{d-1})$ as the *surface intensity* of Z and (assuming $\lambda_{d-1}(S^{d-1}) > 0$) the probability measure

$$R := \lambda_{d-1}/\lambda_{d-1}(S^{d-1})$$

as the *rose of directions* of Z.

In some applications one might have $\lambda_{d-1}(S^{d-1}) = 0$. An example are fibre processes in $\mathbb{R}^3$ (see [31]). If the surface intensity is 0, the first (right) derivative of $(1-p)H(\cdot, C)$ vanishes at the point 0. Since the first derivative is itself differentiable with the exception of only countably many points, we then can consider the second (right) derivative at 0. This yields the following result.

Corollary 7.3. *Let Z be a non-empty stationary random closed set satisfying* (7.5) *and $\lambda_{d-1}(S^{d-1}) = 0$. Then, for any measurable set $C \subset S^{d-1}$, $(1-p)H(\cdot, C)$ has a second derivative at the point* 0 *which is given by*

$$2\pi\mathbb{E}[\mu_{d-2}(Z;[0,1]^d \times C)].$$

8 Contact distributions of Boolean models

We finally discuss the important special case of a *Boolean model* with compact particles. Hence we assume now that

$$Z = \bigcup_{n\in\mathbb{N}} (Z_n + \xi_n),$$

where the $\xi_n, n \in \mathbb{N}$, build a stationary Poisson process Φ in $\mathbb{R}^d$ with positive and finite intensity γ and where the *grains* $Z_1, Z_2, \dots$ form a sequence of independent, identically distributed random elements of $\mathcal{C}^d$ (the space of non-empty compact subsets of $\mathbb{R}^d$) which is independent of Φ (see [23] and [31] for more details). Denoting the common distribution of the Z_n by $\mathbb{Q}$ we make the standard assumption

$$\int \mathcal{H}^d(A \oplus B)\mathbb{Q}(dA) < \infty \tag{8.1}$$

for all compact sets $B \subset \mathbb{R}^d$, where $A \oplus B := \{x + y : x \in A, y \in B\}$. Since any bounded set can be covered by finitely many balls of a fixed radius, condition (8.1) is equivalent to

$$\int \mathcal{H}^d(A_{\oplus r})\mathbb{Q}(dA) < \infty \tag{8.2}$$

for just one $r > 0$. Assumption (8.1) guarantees that each compact set is intersected by only a finite number of the (shifted) grains $Z_n + \xi_n$, $n \in \mathbb{N}$.

A good starting point for the analysis of the spherical contact distribution function is the formula for the *capacity functional* of Z stating that

$$\mathbb{P}(Z \cap B \neq \emptyset) = 1 - \exp\left[-\gamma \int \mathcal{H}^d(A \oplus (-B))\mathbb{Q}(dA)\right], \tag{8.3}$$

for all Borel sets $B \subset \mathbb{R}^d$, where $-B := \{-z : z \in B\}$. By (8.1), this number is strictly less than 1, if B is bounded. In particular, we obtain for the volume fraction that

$$p = 1 - \exp\left[-\gamma \int \mathcal{H}^d(A)\mathbb{Q}(dA)\right] < 1.$$

Taking $B = B^d$ in (8.3) and using the preceding formula for p, we obtain

$$1 - H(t) = \exp\left[-\gamma \int \mathcal{H}^d(A_{\oplus t} \setminus A)\mathbb{Q}(dA)\right]. \tag{8.4}$$

By Theorem 6.5, assumption (8.1) implies that

$$\iint (\delta(A, x, u) \wedge r)^{d-j} |\mu_j|(A; d(x, u))\mathbb{Q}(dA) < \infty \tag{8.5}$$

for $r > 0$ and $j = 0, \ldots, d-1$. Therefore we can apply the local Steiner formula (2.3) to deduce from (8.4) that

$$H(t) = 1 - \exp\left[-\int_0^t \lambda(s)ds\right], \quad t \geq 0,$$

where

$$\lambda(s) := \sum_{i=0}^{d-1} \omega_{d-i} s^{d-1-i} \gamma \iint \mathbf{1}\{s < \delta(A, x, u)\}\mu_i(A; d(x, u))\mathbb{Q}(dA).$$

Using different methods we can generalize this result to contact distribution functions $H(\cdot, C)$.

Theorem 8.1. *Let Z be the stationary Boolean model defined above and let $C \subset S^{d-1}$ be measurable. Then $H(\cdot, C)$ is absolutely continuous with density*

$$t \mapsto (1 - H(t))\gamma \sum_{i=0}^{d-1} \omega_{d-i} t^{d-1-i} \iint \mathbf{1}\{t < \delta(A, x, u)\}\mathbf{1}\{u \in C\} \times \mu_i(A; d(x, u))\mathbb{Q}(dA).$$

Proof. It follows exactly as in the proof of Theorem 3.1 in [18] that

$$H(t, C) = \gamma \iint (1 - H(d(A, z)))\mathbf{1}\{0 < d(A, z) \leq t, u(A, z) \in C\} \times \mathcal{H}^d(dz)\mathbb{Q}(dA) \tag{8.6}$$

for all $t \geq 0$ and all measurable $C \subset S^{d-1}$. The only additional argument, which is required, concerns Lemma 3.1 of [18]. In order to get the corresponding result in the present context, we need to show that the boundary of $A_{\oplus s}$ has volume 0, for all $A \in \mathcal{C}^d$ and $s > 0$. This can be seen as follows. Let $x \in \partial A_{\oplus s}$ be fixed for the moment. Then there is some $a \in \operatorname{cl} A$ such that $|x - a| = s$ and $\partial A_{\oplus s} \cap \operatorname{int} B^d(a, s) = \emptyset$. Therefore, we find that

$$\limsup_{r \to 0+} \frac{\mathcal{H}^d(\partial A_{\oplus s} \cap B^d(x, r))}{\mathcal{H}^d(B^d(x, r))} < 1,$$

for any $x \in \partial A_{\oplus s}$. The assertion now follows from Theorem 2.9.11 in [5].

Since $z \mapsto 1 - H(d(A, z))$ is a bounded function and A is compact, we can exploit the local Steiner formula to express the inner integral of (8.6) in terms of the support measures. By (8.5) we can then use Fubini's theorem to conclude the desired result. □

Example 8.2. Assume that $\mathbb{Q}$ is the distribution of a random multiple ξA_0 of some fixed compact set A_0, where ξ is a positive random variable with $\mathbb{E}[\xi^d] < \infty$. Using the scaling properties in Proposition 4.9 we obtain from Theorem 8.1 that $H(t, C)$ has the density

$$(1 - H(t))\gamma \sum_{i=0}^{d-1} \omega_{d-i} t^{d-1-i} \times \mathbb{E}\left[\xi^i \int \mathbf{1}\{t < \xi\delta(A_0, x, u)\}\mathbf{1}\{u \in C\}\mu_i(A_0; d(x, u))\right].$$

If A_0 is a fractal, we might have that the above density tends to ∞ as $t \to 0$. Indeed, it follows directly from (8.4) that

$$H(t) = 1 - \exp\left(-\gamma\mathbb{E}[\xi^d \mathcal{H}^d(A_0 + \xi^{-1} t B^d)]\right).$$

Assume now that A_0 is a fractal with box-counting dimension $a \in (d - 1, d)$. For instance A_0 could be the Sierpiński gasket introduced and discussed in Example 4.8. Further, we assume that $\xi \geq t_0 > 0$ holds $\mathbb{P}$-a.s. Then we choose $\varepsilon \in (0, a - d + 1)$. By definition, if $t > 0$ is sufficiently small, we get that

$$d - a + \varepsilon > \frac{\log \mathcal{H}^d(A_0 + \xi^{-1} t B^d)}{\log(\xi^{-1} t)},$$

hence

$$\xi^d \mathcal{H}^d(A_0 + \xi^{-1} t B^d) > t^{d-a+\varepsilon} t_0^{a-\varepsilon}.$$

This shows that

$$H(t) \geq (\gamma/2) t_0^{a-\varepsilon} t^{d-a+\varepsilon}$$

if $t > 0$ is sufficiently small. In particular, $t^{-1}H(t) \to \infty$ as $t \to 0$.

Finally, we turn to the relationships between the measures Γ_j introduced in Section 7 for a general stationary closed set and corresponding mean values with respect to $\mathbb{Q}$.

Theorem 8.3. *Let Z be a stationary Boolean model as above. Then*

$$\Gamma_j(C \times (s,\infty]) = (1-p)(1-H(s))\gamma \iint \mathbf{1}\{s < \delta(A,x,u)\} \times \mathbf{1}\{u \in C\} \mu_j(A; d(x,u)) \mathbb{Q}(dA), \quad (8.7)$$

for all measurable sets $C \subset S^{d-1}$, $s > 0$, and $j \in \{0, \dots, d-1\}$.

The proof of this theorem requires the following lemma. Recall that $S(\mathbb{N})$ denotes the system of all non-empty finite subsets of $\mathbb{N}$.

Lemma 8.4. *Let $A \subset \mathbb{R}^d$ be the union set of the locally finite family of compact sets $A_i \subset \mathbb{R}^d$, $i \in \mathbb{N}$. Let $j \in \{0, \dots, d-1\}$, $s > 0$, and let $B \subset \mathbb{R}^d$ be measurable and bounded. Then, for all measurable and bounded functions $f : \mathbb{R}^d \times S^{d-1} \to \mathbb{R}$,*

$$\begin{aligned}\int & f(x,u)\mathbf{1}\{s < \delta(A,x,u), x \in B\}\mu_j(A; d(x,u)) \\ &= \sum_{v \in S(\mathbb{N})} \int \mathbf{1}\{B^d(x+su,s) \cap A^{(v)} = \emptyset\} \prod_{i \in v} \mathbf{1}\{s < \delta(A_i,x,u), x \in B\} \\ &\quad \times f(x,u)\mu_j(A_v; d(x,u)),\end{aligned}$$

where

$$A_v := \bigcap_{i \in v} A_i \qquad \text{and} \qquad A^{(v)} := \bigcup_{i \notin v} A_i.$$

An analogous relationship is satisfied for the total variation measures $|\mu_j|(A;\cdot)$ and $|\mu_j|(A_v;\cdot)$, $v \in S(\mathbb{N})$.

Proof. It is an easy consequence of the definition that $N(A)$ is the disjoint union of the sets D_v, $v \in S(\mathbb{N})$, where

$$D_v := ((\mathbb{R}^d \setminus A^{(v)}) \times S^{d-1}) \cap \bigcap_{i \in v} N(A_i).$$

Moreover, if $(x,u) \in D_v$ and $s > 0$, then $\delta(A,x,u) > s$ if and only if $B^d(x+su,s) \cap A^{(v)} = \emptyset$ and $\delta(A_i,x,u) > s$ for $i \in v$. Since clearly $\bigcap_{i \in v} N(A_i) \subset N(A_v)$, we have (in obvious notation) $H_{d-1-j}(A,x,u) = H_{d-1-j}(A_v,x,u)$, for $\mathcal{H}^{d-1}$-a.e. $(x,u) \in D_v$. Hence the result follows from (2.24). □

Proof of Theorem 8.3. Let $B \subset \mathbb{R}^d$ denote a bounded Borel set of volume 1. Further, let $s > 0$ and an arbitrary Borel set $C \subset S^{d-1}$ be given. Then, by definition (7.3) and Lemma 8.4,

$$\begin{aligned}
&\Gamma_j(C \times (s, \infty]) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \mathbb{E}\Big[\sum\nolimits^* \mathbf{1}\{B^d(x+su, s) \cap Z(A_1, x_1, \ldots, A_n, x_n) = \emptyset\} \int_{B \times C} \\
&\quad \times \prod_{i=1}^{n} \mathbf{1}\{s < \delta(A_i + x_i, x, u)\} \mu_j((A_1 + x_1) \cap \ldots \cap (A_n + x_n); d(x, u)) \Big],
\end{aligned}$$

where the sum $\sum^*$ extends over all n-tuples $((A_1, x_1), \ldots, (A_n, x_n))$ of mutually different elements of $\{(Z_m, \xi_m) : m \in \mathbb{N}\}$ and where, for any such tuple, $Z(A_1, x_1, \ldots, A_n, x_n)$ is the union of all $Z_m + \xi_m$ such that (Z_m, ξ_m) does not pertain to $\{(A_1, x_1), \ldots, (A_n, x_n)\}$. To justify the interchange of summation and expectation in deducing the previous equation, one first derives the corresponding equation for the total variation measure $|\Gamma_j|$, which is finite for the sets considered.

By a fundamental property of the Poisson process (see e.g. [20]) we conclude that

$$\begin{aligned}
\Gamma_j(C \times (s, \infty]) &= (1-p)(1-H(s)) \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \int \cdots \int \int \cdots \int \int_{B \times C} \\
&\quad \times \prod_{i=1}^{n} \mathbf{1}\{s < \delta(A_i + x_i, x, u)\} \mu_j((A_1 + x_1) \cap \ldots \cap (A_n + x_n); d(x, u)) \\
&\quad \times dx_1 \ldots dx_n \mathbb{Q}(dA_1) \ldots \mathbb{Q}(dA_n),
\end{aligned} \tag{8.8}$$

where $dx_1, \ldots, dx_n$ denote integration with respect to Lebesgue measure. Using this result for $C = S^{d-1}$ and comparing Theorem 8.1 with Theorem 7.1, we obtain that

$$\begin{aligned}
0 = \sum_{j=0}^{d-1} \sum_{n=2}^{\infty} & \omega_{d-j} s^{d-1-j} \frac{\gamma^n}{n!} \int \cdots \int \int \cdots \int \int_{B \times S^{d-1}} \\
& \times \prod_{i=1}^{n} \mathbf{1}\{s < \delta(A_i + x_i, x, u)\} \\
& \times \mu_j((A_1 + x_1) \cap \ldots \cap (A_n + x_n); d(x, u)) \\
& \times dx_1 \ldots dx_n \mathbb{Q}(dA_1) \ldots \mathbb{Q}(dA_n).
\end{aligned} \tag{8.9}$$

We are now fixing j and n in the above formula, as well as the sets $B_i := A_i + x_i$, $i = 1, \ldots, n$. It is easy to check that

$$\delta'(B_1, \ldots, B_n, x, u) := \min\{\delta(B_i, x, u) : i = 1, \ldots, n\} \le \delta(B_1 \cap \ldots \cap B_n, x, u)$$

for all $(x, u) \in N(B_1) \cap \ldots \cap N(B_n)$. Therefore we can apply (2.21) to obtain from (8.9) that

$$0 = \sum_{n=2}^{\infty} \frac{\gamma^n}{n!} \int_0^r s^{d-1-k} \int \cdots \int \int \cdots \int \int_{B \times S^{d-1}}$$
$$\times \mathbf{1}\{s < \delta'(A_1 + x_1, \ldots, A_n + x_n, x, u)\}$$
$$\times |\mu_k|((A_1 + x_1) \cap \ldots \cap (A_n + x_n); d(x, u))$$
$$\times dx_1 \ldots dx_n \mathbb{Q}(dA_1) \ldots \mathbb{Q}(dA_n) ds$$

for all $k \in \{0, \ldots, d-1\}$ and all $r > 0$. Hence

$$0 = \int \cdots \int \int \cdots \int \int_{B \times S^{d-1}}$$
$$\times \mathbf{1}\{s < \delta'(A_1 + x_1, \ldots, A_n + x_n, x, u)\}$$
$$\times |\mu_k|((A_1 + x_1) \cap \ldots \cap (A_n + x_n); d(x, u))$$
$$\times dx_1 \ldots dx_n \mathbb{Q}(dA_1) \ldots \mathbb{Q}(dA_n)$$

for all $n \geq 2$, $k \in \{0, \ldots, d-1\}$ and $s > 0$. Inserting this relation into (8.8), we obtain the desired result. □

It is tempting to take the limit $s \to 0$ in (8.7). This requires the integrability condition

$$\int |\mu_j|(A; \mathbb{R}^d \times S^{d-1}) \mathbb{Q}(dA) < \infty, \quad j = 0, \ldots, d-1. \tag{8.10}$$

As we will see, assumption (8.10) implies that (7.5) is satisfied (the converse is also true). Hence, under this condition

$$\Lambda_i := \mathbb{E}[\mu_i(Z; \cdot)], \quad i = 0, \ldots, d-1,$$

are signed Radon measures on $\mathbb{R}^d \times S^{d-1}$. Since Z is stationary,

$$\Lambda_i = \mathcal{H}^d \otimes \lambda_i, \quad i = 0, \ldots, d-1, \tag{8.11}$$

where $\lambda_i(C) = \Lambda_i([0, 1]^d \times C)$ for any Borel set $C \subset S^{d-1}$. Note that λ_{d-1} has already been introduced by (7.8).

Together with $\int \mathcal{H}^d(A)\mathbb{Q}(dA) < \infty$, condition (8.10) is a stronger assumption than (8.1). Fractal grains are excluded this way. Our final theorem generalizes results in [23] and [17].

Theorem 8.5. *Let Z be the stationary Boolean model defined above and assume that* (8.10) *holds. Then* (7.5) *is satisfied and*

$$\lambda_j(C) = (1 - p)\gamma \int \mu_j(A; \mathbb{R}^d \times C) \mathbb{Q}(dA) \tag{8.12}$$

for $j = 0, \ldots, d-1$ and all measurable sets $C \subset S^{d-1}$.

Proof. Fix a Borel set $C \subset S^{d-1}$ and $j \in \{0, \dots, d-1\}$, and let $B \subset \mathbb{R}^d$ be a bounded and measurable set of volume 1. It follows as in the proof of Theorem 8.3 that

$$\mathbb{E}\left[\int \mathbf{1}\{x \in B\}\mathbf{1}\{s < \delta(Z, x, u)\}|\mu_j|(Z; d(x, u))\right]$$
$$= (1-p)(1-H(s))\gamma \iint \mathbf{1}\{s < \delta(A, x, u)\}|\mu_j|(A; d(x, u))\mathbb{Q}(dA), \quad (8.13)$$

for all $s > 0$. Letting $s \to 0$ we obtain (7.5) by dominated convergence. The assumption of stationarity implies that

$$\lambda_j(C) = \mathbb{E}[\mu_j(Z; B \times C)] = \Gamma_j(C \times (0, \infty]). \quad (8.14)$$

Letting $s \to 0$ in (8.7), the assertion (8.12) now follows again by dominated convergence. □

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