Large Poisson-Voronoi Cells and Crofton Cells

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Abstract. It is proved that the shape of the typical cell of a stationary Poisson-Voronoi tessellation in Euclidean space, under the condition that the volume of the typical cell is large, must be close to spherical shape, with high probability. The same holds if the volume is replaced by the surface area or other suitable functionals. Similar results are established for the zero cell of a stationary and isotropic Poisson hyperplane tessellation.

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1 Introduction

In this paper, we continue a line of research that began with a problem of D.G. Kendall (see the foreword to the first edition of [17]). He conjectured that the shape of the zero cell (or Crofton cell) of a stationary and isotropic Poisson line process in the plane, given that the area of the cell tends to infinity, must become circular. Contributions to Kendall's question are due to Miles [11] and Goldman [3], and the conjecture was proved by Kovalenko [8], [10]. In [7], Kovalenko's result was extended to higher dimensions and to stationary, but not necessarily isotropic Poisson hyperplane processes. It was also strengthened, by estimating the probability of large deviations from spherical shape, given that the volume of the zero cell lies in a prescribed interval. In the present paper, we prove an analogous result for the typical cell of a stationary Poisson-Voronoi tessellation (mosaic) of d-dimensional space. Thus we extend and strengthen a result of Kovalenko [9] in the planar case. We further extend this result by considering, in addition to the volume functional, also the kth intrinsic volume, $k = 1, \ldots, d-1$. This includes cells of large surface area or of large mean width. The result from [7] on Crofton cells of stationary Poisson hyperplane processes with large volume is also extended to the kth intrinsic volume, but only for $k \geq 2$ and under the additional assumption of isotropy. For both types of random polytopes, Poisson-Voronoi cells and isotropic Crofton cells, we can also replace (somewhat easier) the condition of large volume by the condition of

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large inradius. This is suggested by considerations of Miles [11] on Crofton cells in the plane and by the work of Calka [1] on planar Poisson-Voronoi tessellations. Finally, we mention here that cells of large volume in Poisson-Delaunay tessellations were treated in [6]; such cells tend to be regular simplices.

Let A be a locally finite point set in Euclidean space $\mathbb{R}^d$ (with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$), where $d \geq 2$. For $\boldsymbol{x} \in A$, the *Voronoi cell* of $\boldsymbol{x}$ with respect to A is defined by

$$C(\boldsymbol{x}, A) := \{\boldsymbol{y} \in \mathbb{R}^d : \|\boldsymbol{y} - \boldsymbol{x}\| \leq \|\boldsymbol{y} - \boldsymbol{a}\| \text{ for all } \boldsymbol{a} \in A\}.$$

Let $\tilde{X}$ be a stationary Poisson point process of intensity $\lambda > 0$ in $\mathbb{R}^d$. (In treating simple point processes, we conveniently identify a simple counting measure with its support.) Then

$$X := \{C(\boldsymbol{x}, \tilde{X}) : \boldsymbol{x} \in \tilde{X}\}$$

is the Poisson-Voronoi tessellation derived from $\tilde{X}$. Let Z denote the typical cell of X (we recall its definition in Section 2).

For a convex body (non-empty, compact, convex set) $K \subset \mathbb{R}^d$, we denote the volume by $v_d(K)$. The (conveniently renormalized) *intrinsic volumes* $v_0(K), \dots, v_{d-1}(K)$ can be defined by means of the Steiner formula

$$v_d(K + \epsilon B^d) = \sum_{k=0}^{d} \epsilon^{d-k} \binom{d}{k} v_k(K), \qquad \epsilon \geq 0.$$

Here $B^d := \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| \leq 1\}$ is the unit ball. Equivalently, $v_i(K)$ is the mixed volume $V(K, \dots, K, B^d, \dots, B^d)$, where K appears i times and B^d appears $d-i$ times. The functional $W_j = v_{d-j}$ is known as the jth *quermassintegral*. In particular, dv_{d-1} is the surface area, and $(2/\kappa_d)v_1$ is the mean width; here $\kappa_d := v_d(B^d)$. More information is found in [15].

Let $K \subset \mathbb{R}^d$ be a compact set with $\boldsymbol{o} \in K$ and containing more than one point. In order to measure the deviation of K from a ball with centre $\boldsymbol{o}$, we define

$$\vartheta(K) := \frac{R_o - \rho_o}{R_o + \rho_o},$$

where R_o is the radius of the smallest ball with centre $\boldsymbol{o}$ containing K and ρ_o is the radius (possibly zero) of the largest ball with centre $\boldsymbol{o}$ contained in K.

By P we denote the underlying probability, and $\mathrm{P}(\cdot \mid \cdot)$ is a conditional probability.

Theorem 1. *Let X denote the Poisson-Voronoi tessellation derived from a stationary Poisson point process with intensity $\lambda > 0$ in $\mathbb{R}^d$; let Z be its typical cell. Let $k \in \{1, \dots, d\}$. There is a positive constant c_0 depending only on the dimension d such that the following is true. If $\epsilon \in (0, 1)$ and $I = [a, b)$ is any interval (possibly $b = \infty$) with $a^{d/k}\lambda \geq \sigma_0$, for some constant $\sigma_0 > 0$, then*

$$\mathrm{P}(\vartheta(Z) \geq \epsilon \mid v_k(Z) \in I) \leq c \exp\left\{-c_0 \epsilon^{(d+3)/2} a^{d/k} \lambda\right\},$$

where c is a constant depending only on d, ϵ and σ_0.

In particular,

$$\lim_{a \to \infty} \mathrm{P}(\vartheta(Z) \geq \epsilon \mid v_k(Z) \geq a) = 0 \qquad \text{for every } \epsilon > 0, \tag{1}$$

but Theorem 1 provides much stronger information. The relation (1) can equivalently be formulated as follows (see Section 2 for further explanation).

Corollary. *The conditional law for the shape of Z, given a lower bound for $v_k(Z)$, converges weakly, as that lower bound tends to infinity, to the law concentrated at the shape of a ball.*

Theorem 1 will be proved in Section 6, after preliminary explanations in Section 2 and preparations in Sections 3 to 5.

In [7], a similar result was obtained for the volume of the zero cell (also called Crofton cell) of the tessellation generated by a stationary Poisson hyperplane process. We will indicate in Section 7 how, under the additional assumption of isotropy, this result can be extended to the kth intrinsic volume, $k = 2, \ldots, d$. As in [7], we measure the deviation of the shape of a convex body $K \subset \mathbb{R}^d$ with interior points from spherical shape by r_{B^d}, which we abbreviate by r_d, thus

$$r_d(K) := \min\{s/r - 1 : rB^d + \boldsymbol{z} \subset K \subset sB^d + \boldsymbol{z},\ \boldsymbol{z} \in \mathbb{R}^d,\ r, s > 0\}.$$

Theorem 2. *Let Z_o be the zero cell of the tessellation induced by a stationary isotropic Poisson hyperplane process in $\mathbb{R}^d$ with intensity $\lambda > 0$. Let $k \in \{2, \ldots, d\}$. There is a positive constant c_0 depending only on the dimension d such that the following is true. If $\epsilon \in (0, 1)$ and $I = [a, b)$ is any interval (possibly $b = \infty$) with $a^{1/k}\lambda \geq \sigma_0$, for some constant $\sigma_0 > 0$, then*

$$\mathrm{P}(r_d(Z_o) \geq \epsilon \mid v_k(Z_o) \in I) \leq c \exp\left\{-c_0 \epsilon^{(d+3)/2} a^{1/k} \lambda\right\},$$

where c is a constant depending only on d, ϵ and σ_0.

The case of the volume is included here for $k = d$. We remark that in this case the inequality of Theorem 2 is sharper (in its dependence on ϵ) than Theorem 1 of [7], specialized to the isotropic case. The reason for this improvement lies in the fact that in the isotropic case sharper stability estimates from convex geometry are available.

Unfortunately, our method of proof does not permit us to treat the case $k = 1$, which in the plane is the case of the perimeter, already studied by Miles [11] in his heuristic approach.

In addition to $\rho_o(K)$ defined above, we denote by $\rho(K)$ the radius of the largest ball contained in the convex body K.

Theorem 3. *Let Z and Z_o be defined as in Theorems 1 and 2, respectively. There is a positive constant c_0 depending only on the dimension d such that the following is true. If $\epsilon \in (0, 1)$ and $I = [a, b)$ is any interval (possibly $b = \infty$) with $a^d\lambda \geq \sigma_0$ in the case of Z, respectively $a\lambda \geq \sigma_0$ in the case of Z_o, with some constant $\sigma_0 > 0$, then*

$$\mathrm{P}(\vartheta(Z) \geq \epsilon \mid \rho_o(Z) \in I) \leq c \exp\left\{-c_0 \epsilon^{(d+1)/2} a^d \lambda\right\}$$

and

$$\mathrm{P}(r_d(Z_o) \geq \epsilon \mid \rho(Z_o) \in I) \leq c \exp\left\{-c_0 \epsilon^{(d+1)/2} a \lambda\right\},$$

where c is a constant depending only on d, ϵ and σ_0.

The proof will be sketched in Section 8.

A few words about the choices of the shape parameters $\vartheta(K)$ and $r_d(K)$ seem in order. For the formulation of our estimates, we want a simple, similarity invariant measure for the deviation of the shape of a convex body from spherical shape. Such a measure appears implicitly in relation (2) of Miles [11], and explicitly in the paper of Kovalenko [10]. The number $r_d(K)$ used above is the extension of the latter to higher dimensions. For an interior point $\boldsymbol{z}$ of the compact convex set K, let $R_{\boldsymbol{z}}$ be the radius of the smallest ball with centre $\boldsymbol{z}$ containing K, and $\rho_{\boldsymbol{z}}$ the radius of the largest ball with centre $\boldsymbol{z}$ contained in K. Then $r_d(K)$ is the minimum, over all $\boldsymbol{z}$ in the interior of K, of the quotient $(R_{\boldsymbol{z}} - \rho_{\boldsymbol{z}})/\rho_{\boldsymbol{z}}$. In the case of the typical cell of a Poisson-Voronoi tessellation, for which the nucleus $\boldsymbol{o}$ is a distinguished point, we get a sharper result if we use instead the deviation measure $(R_{\boldsymbol{o}} - \rho_{\boldsymbol{o}})/\rho_{\boldsymbol{o}}$, or, which for bodies close to a ball with centre $\boldsymbol{o}$ is essentially equivalent, $(R_{\boldsymbol{o}} - \rho_{\boldsymbol{o}})/(R_{\boldsymbol{o}} + \rho_{\boldsymbol{o}}) = \vartheta(K)$. Here we have chosen the latter, since for this the crucial stability estimate (13) below takes a simple form. Finally, we mention that Goldman [3], in the planar case, considers the first eigenvalue of the Laplacian for the Dirichlet problem and finds the same asymptotic behaviour for large Crofton cells and their incircles, but this does apparently not lead to explicit geometric estimates for the deviation from circular shape.

2 The typical cell of a Poisson-Voronoi tessellation

In general, the notion of the typical cell of a stationary random tessellation requires the choice of a centroid function (e.g., see Møller [12, Section 3.2]), but for Voronoi cells there is a canonical choice, the nucleus. Let X be the Poisson-Voronoi tessellation generated by a stationary Poisson point process $\tilde{X}$. Let $\mathcal{K}_o^d$ denote the space of convex bodies K in $\mathbb{R}^d$ with $\boldsymbol{o} \in K$, equipped with the Hausdorff metric and corresponding Borel structure. The distribution $\mathbb{Q}$ of the typical cell of X can be defined by

$$\mathbb{Q}(\mathcal{A}) = \frac{1}{\lambda} \mathrm{E} \sum_{\boldsymbol{x} \in \tilde{X}} \mathbf{1}_{\mathcal{A}}(C(\boldsymbol{x}, \tilde{X}) - \boldsymbol{x}) \mathbf{1}_B(\boldsymbol{x})$$

(E denotes mathematical expectation) for Borel sets $\mathcal{A} \subset \mathcal{K}_o^d$, where $B \subset \mathbb{R}^d$ is an arbitrary Borel set with $\lambda_d(B) = 1$; here λ_d denotes Lebesgue measure in $\mathbb{R}^d$. The distribution $\mathbb{Q}$ is commonly interpreted as the conditional distribution of $C(\boldsymbol{o}, \tilde{X})$ given that $\boldsymbol{o} \in \tilde{X}$. An intuitive interpretation follows from the fact that stationary Poisson-Voronoi tessellations are mixing ([16, Satz 6.4.1]) and hence ergodic. This implies that

$$\mathbb{Q}(\mathcal{A}) = \lim_{r \to \infty} \frac{\mathrm{card}\,\{\boldsymbol{x} \in \tilde{X} \cap rB^d : C(\boldsymbol{x}, \tilde{X}) - \boldsymbol{x} \in \mathcal{A}\}}{\mathrm{card}\,(\tilde{X} \cap rB^d)}$$

holds with probability one.

Since $\tilde{X}$ is a stationary Poisson process, it follows from Slivnyak's theorem that the typical cell of the Poisson-Voronoi tessellation X is equal in distribution to the random polytope

$$Z = C(\boldsymbol{o}, \tilde{X} \cup \{\boldsymbol{o}\})$$

(see [12, Remark 4.1.1]; see also [13]). Hence, we can consider Z as the typical cell of X, and for this we obtain a convenient representation. For $\boldsymbol{x} \in \mathbb{R}^d$, we define

$$\begin{aligned} H(\boldsymbol{x}) &:= \{\boldsymbol{y} \in \mathbb{R}^d : \langle \boldsymbol{y}, \boldsymbol{x} \rangle = \|\boldsymbol{x}\|^2/2\}, \\ H^-(\boldsymbol{x}) &:= \{\boldsymbol{y} \in \mathbb{R}^d : \langle \boldsymbol{y}, \boldsymbol{x} \rangle \le \|\boldsymbol{x}\|^2/2\}, \end{aligned}$$

so that $H(\boldsymbol{x})$ is the mid hyperplane of $\boldsymbol{o}$ and $\boldsymbol{x}$. Then

$$Z = \bigcap_{\boldsymbol{x} \in \tilde{X}} H^-(\boldsymbol{x}),$$

thus Z is the zero cell of the tessellation induced by the Poisson hyperplane process

$$Y := \{H(\boldsymbol{x}) : \boldsymbol{x} \in \tilde{X}\}.$$

The intensity measure $\mathrm{E}\, Y(\cdot)$ of this process can be represented as follows (recall that Y denotes a random simple counting measure on the space of hyperplanes as well as its support). For a Borel set $\mathcal{A}$ in the space of hyperplanes, we have

$$\begin{aligned} \mathrm{E}\, Y(\mathcal{A}) &= \mathrm{E}\, \mathrm{card}\, (\mathcal{A} \cap Y) = \mathrm{E}\, \mathrm{card}\, \{\boldsymbol{x} \in \tilde{X} : H(\boldsymbol{x}) \in \mathcal{A}\} \\ &= \lambda \cdot \lambda_d(\{\boldsymbol{x} \in \mathbb{R}^d : H(\boldsymbol{x}) \in \mathcal{A}\}). \end{aligned}$$

Writing

$$H(\boldsymbol{u}, t) := \{\boldsymbol{y} \in \mathbb{R}^d : \langle \boldsymbol{y}, \boldsymbol{u} \rangle = t\}, \qquad H^-(\boldsymbol{u}, t) := \{\boldsymbol{y} \in \mathbb{R}^d : \langle \boldsymbol{y}, \boldsymbol{u} \rangle \le t\}$$

for $\boldsymbol{u} \in S^{d-1} := \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| = 1\}$ and $t \in \mathbb{R}$, and introducing polar coordinates, we get

$$\mathrm{E}\, Y(\cdot) = 2^d \lambda \int_{S^{d-1}} \int_0^\infty \mathbf{1}\{H(\boldsymbol{u}, t) \in \cdot\} t^{d-1} \mathrm{d}t\, \sigma(\mathrm{d}\boldsymbol{u}), \tag{2}$$

where σ denotes spherical Lebesgue measure on the unit sphere S^{d-1}.

In particular, for $K \in \mathcal{K}_o^d$ let $\mathcal{H}_K$ be the set of all hyperplanes $H \subset \mathbb{R}^d$ with $H \cap K \neq \emptyset$. Then (2) gives

$$\mathrm{E}\, Y(\mathcal{H}_K) = 2^d \lambda U(K) \tag{3}$$

with

$$U(K) := \frac{1}{d} \int_{S^{d-1}} h(K, \boldsymbol{u})^d \, \sigma(\mathrm{d}\boldsymbol{u}), \tag{4}$$

where $h(K, \cdot)$ is the support function of K. Writing

$$\Phi(K) := \{\boldsymbol{y} \in \mathbb{R}^d : H(2\boldsymbol{y}) \cap K \neq \emptyset\},$$

we have $U(K) = \lambda_d(\Phi(K))$. The star body $\Phi(K)$ is the union of all closed balls having a diameter segment $[\boldsymbol{o}, \boldsymbol{x}]$ with $\boldsymbol{x} \in K$.

The relation (3), together with the Poisson property of the hyperplane process Y, implies that

$$\mathrm{P}(Y(\mathcal{H}_K) = n) = \frac{[2^d U(K) \lambda]^n}{n!} \exp\{-2^d U(K) \lambda\} \tag{5}$$

for $K \in \mathcal{K}_o^d$ and $n \in \mathbb{N}_0$.

It is now clear that we are in a similar situation as in [7]. There, the zero cell of a stationary (not necessarily isotropic) Poisson hyperplane process, with intensity measure given by [7, (2)], was studied. This process is now replaced by the isotropic, non-stationary Poisson hyperplane process Y, with intensity measure given by (2). The functional $U(K)$, defined by (4), will play the role of the mixed volume $V_1(B, K)$ in [7] (up to dimensional factors). In addition to the volume functional considered in [7], we now treat general intrinsic volumes. All these differences require a number of changes and new arguments, but also some parallel reasoning is possible. In the latter cases, we will be brief and just list how the arguments of [7] have to be modified.

In analogy to Kendall's original conjecture, of which a more general version was treated in [7], we have formulated the Corollary after Theorem 1. We make this more precise. As a space of 'shapes', suitable for our purpose, we may take the space $\mathcal{S}$ (with the Hausdorff metric) of convex bodies containing $\boldsymbol{o}$ and with circumradius one. The dilated version of Z contained in $\mathcal{S}$ is denoted by Z^*. The conditional law for the shape of Z, given the lower bound a for $v_k(Z)$, can be defined as the probability measure on $\mathcal{S}$ given by $\mu_a(\cdot) := \mathrm{P}(Z^* \in \cdot \mid v_k(Z) \geq a)$. We are asserting that $\lim_{a\to\infty} \mu_a = \delta_{B^d}$ weakly, where δ_{B^d} is the Dirac measure on $\mathcal{S}$ concentrated at the ball B^d (as we are prescribing the centre of the ball, this is a slightly stronger assertion than formulated in the Corollary). Thus, we have to prove that

$$\limsup_{a\to\infty} \mu_a(\mathcal{C}) \leq \delta_{B^d}(\mathcal{C}) \tag{6}$$

for every closed set $\mathcal{C} \subset \mathcal{S}$. This is trivial if $B^d \in \mathcal{C}$. Let $B^d \notin \mathcal{C}$. Since the deviation measure ϑ is continuous and positive on $\mathcal{C}$, there exists $\epsilon > 0$ such that $\mathcal{C} \subset \{K \in \mathcal{S} : \vartheta(K) \geq \epsilon\}$. This implies

$$\mu_a(\mathcal{C}) \leq \mu_a(\{K \in \mathcal{S} : \vartheta(K) \geq \epsilon\}) = \mathrm{P}(\vartheta(Z) \geq \epsilon \mid v_k(Z) \geq a) \to 0$$

for $a \to \infty$, by (1). This proves the Corollary.

Conversely, the assertion of the Corollary implies (1): given $\epsilon > 0$, let $\mathcal{C} := \{K \in \mathcal{S} : \vartheta(K) \geq \epsilon\}$, then $\mathcal{C}$ is closed and $B^d \notin \mathcal{C}$, hence (6) yields (1).

As the details of the proofs for Theorems 1 – 3 are a bit technical, we want to sketch here the main lines of the reasoning, taking Theorem 1 as an example. Let $k \in \{1, \dots, d\}$. In the first instance, we are interested in bounding the probability $\mathrm{P}(\vartheta(Z) \geq \epsilon \mid v_k(Z) \geq a)$ from above, for given $\epsilon > 0$ and large a. This conditional probability is a quotient. A lower bound for the denominator $\mathrm{P}(v_k(Z) \geq a)$ follows immediately from (5): the ball $B_a := (a/\kappa_d)^{1/k} B^d$ satisfies $v_k(B_a) = a$, hence

$$\mathrm{P}(v_k(Z) \geq a) \geq \mathrm{P}(Y(\mathcal{H}_{B_a}) = 0) = \exp\left\{-2^d U(B_a)\lambda\right\} = \exp\left\{-2^d \kappa_d^{1-d/k} a^{d/k} \lambda\right\}. \tag{7}$$

To obtain an upper bound for the numerator $\mathrm{P}(\vartheta(Z) \geq \epsilon,\ v_k(Z) \geq a)$, we use the geometric inequality

$$U(K) \geq \kappa_d^{1-d/k} v_k(K)^{d/k}, \tag{8}$$

in a strengthened form. Equality in (8) holds if and only if K is a ball with centre $\boldsymbol{o}$. Let $K \in \mathcal{K}_o^d$ be a convex body satisfying $\vartheta(K) \geq \epsilon > 0$. It can be proved (Lemma 1) that

$$U(K) \geq (1 + f(\epsilon)) \kappa_d^{1-d/k} v_k(K)^{d/k}$$

with $f(\epsilon) > 0$. If now K satisfies

$$\vartheta(K) \geq \epsilon \quad \text{and} \quad v_k(K) \geq a, \tag{9}$$

this gives

$$\mathrm{P}(Y(\mathcal{H}_K) = 0) = \exp\left\{-2^d U(K)\lambda\right\} \leq \exp\left\{-(1 + f(\epsilon)) 2^d \kappa_d^{1-d/k} a^{d/k} \lambda\right\}.$$

Since a convex body contained in the interior of the cell Z does not meet any hyperplane of Y, one might now hope that this estimate remains essentially true if the fixed convex body K

is replaced by the cell Z, in the cases where the latter satisfies the inequalities (9). Although this replacement is not legitimate, a similar and slightly weaker inequality of the form

$$\mathrm{P}(\vartheta(Z) \geq \epsilon,\ v_k(Z) \geq a) \leq c \exp\left\{-(1+g(\epsilon))2^d \kappa_d^{1-d/k} a^{d/k} \lambda\right\}, \tag{10}$$

with a constant $c > 0$ and $g(\epsilon) > 0$, might be true. If this holds, then dividing (10) by (7) immediately gives

$$\mathrm{P}(\vartheta(Z) \geq \epsilon \mid v_k(Z) \geq a) \leq c \exp\left\{-g(\epsilon) 2^d \kappa_d^{1-d/k} a^{d/k} \lambda\right\},$$

which is of the required type.

An estimate of type (10) can indeed be proved, but at first not for $v_k(Z)$ in intervals $[a, \infty)$, but in intervals $a(1,2)$. This is done in Lemmas 4 and 6. The distinction of two cases is necessary since it turns out that cells which are in a sense 'too elongated' need an extra treatment. The convex bodies are classified by an integer parameter m such that increasing m means increasing elongation. The auxiliary geometric Lemma 3 provides inner and outer inclusion estimates for bodies of given elongation. For large m, the estimate of Lemma 4 can be used; here the condition $\vartheta(Z) \geq \epsilon$ does not play a role. Its proof uses only elementary geometric arguments. For small m, the condition $\vartheta(Z) \geq \epsilon$ is essential. Lemma 6 contains the relevant estimate. It is here that the geometric stability result of Lemma 1 is needed. Moreover, the approximation result for polytopes expressed in Lemma 5 is required for the reduction to a situation involving only a fixed number of hyperplanes. The further Lemmas 7, 8, 9 are needed to pass from intervals $a(1,2)$ to intervals $a(1, 1+h)$, with fixed h; the extension is achieved by a transformation. The obtained estimates are then combined into Lemma 10, which gives an upper estimate for the probability $\mathrm{P}(\vartheta(Z) \geq \epsilon,\ v_k(Z) \in a(1, 1+h))$. Since this upper bound contains h as a factor, it is necessary to estimate the denominator $\mathrm{P}(v_k(Z) \in a(1, 1+h))$ from below by a suitable bound which is also linear in h. This is achieved in Lemma 2. Its proof is essentially constructive, exhibiting sufficiently many realizations of Y for which the event $v_k(Z) \in a(1, 1+h)$ occurs. In both, Lemmas 2 and 10, the number h must be sufficiently small. The final proof of Theorem 1 extends the estimates from the special intervals $a(1, 1+h)$, with small h, to general intervals $[a, b)$, by a covering argument, as in the proof of Theorem 1 in [7].

3 A stability estimate

For $K \in \mathcal{K}_o^d$, we trivially have $K \subset \Phi(K)$, hence $U(K) \geq v_d(K)$. Here equality holds if and only if K is a ball with centre $\boldsymbol{o}$ (this follows from the considerations below, but can also be shown directly). A similar inequality can be obtained for the other intrinsic volumes. In the following, we write $h(K, \cdot) =: h_K$ for the support function of K. Integrations with respect to σ extend over S^{d-1}. By Hölder's inequality,

$$U(K) \geq \frac{1}{d}(d\kappa_d)^{1-d} \left(\int h_K \, \mathrm{d}\sigma \right)^d,$$

and since

$$\int h_K \, \mathrm{d}\sigma = d v_1(K),$$

we get

$$U(K)^{1/d} \geq \kappa_d^{(1-d)/d} v_1(K).$$

A well-known inequality (e.g., [15, p. 334]) says that

$$v_1(K)^k \geq \kappa_d^{k-1} v_k(K) \tag{11}$$

for $k = 1, \ldots, d$. Hence,

$$U(K) \geq \kappa_d^{1-d/k} v_k(K)^{d/k}. \tag{12}$$

Equality for a number $k \in \{1, \ldots, d\}$ holds if and only if K is a ball with centre $\boldsymbol{o}$. We will need an improved version of (12), in the form of a stability estimate. The following proof combines techniques developed for obtaining stability results related to Hölder's inequality and to isoperimetric inequalities such as (11).

Lemma 1. *For $K \in \mathcal{K}_o^d$ and $k \in \{1, \ldots, d\}$,*

$$U(K) \geq (1 + \gamma \vartheta(K)^{(d+3)/2}) \kappa_d^{1-d/k} v_k(K)^{d/k}, \tag{13}$$

where γ is a positive constant depending only on the dimension d.

Proof. We assume that K contains more than one point; otherwise the assertion is trivial. In order to improve Hölder's inequality, we use Lemma 4.2 of Gardner and Vassallo [2]. There we put $m = 2$, $f_0 = f_2 = 1$, $f_1 = h_K$, $w_1 = 1/d$, $w_2 = (d-1)/d$ (hence $w = 1/d$), and obtain

$$\begin{aligned} & 1 - \frac{\int h_K \, \mathrm{d}\sigma}{\left(\int h_K^d \, \mathrm{d}\sigma\right)^{1/d} \left(\int 1 \, \mathrm{d}\sigma\right)^{(d-1)/d}} \\ & \geq \frac{1}{d} \int \left[\frac{h_K^{d/2}}{\left(\int h_K^d \, \mathrm{d}\sigma\right)^{1/2}} - \frac{1}{\left(\int 1 \, \mathrm{d}\sigma\right)^{1/2}} \right]^2 \mathrm{d}\sigma =: \beta(K). \end{aligned}$$

Using $(1-\beta)(1+\beta) \leq 1$, we deduce that

$$U(K)^{1/d} \geq (1 + \beta(K)) \kappa_d^{(1-d)/d} v_1(K). \tag{14}$$

Next, we establish an estimate of the form $\beta(K) \geq c \vartheta(K)^\alpha$ with $\alpha > 0$. For this, we argue similarly as in the proof of Lemma 1 in [5] (which is reproduced in [4], see inequality (2.3.3)).

From now on in this paper, $c_1, c_2, \ldots$ denote constants depending only on the dimension, except in those cases where other dependences are explicitly indicated.

Let $K \in \mathcal{K}_o^d$ be given. Since β and ϑ are invariant under dilatations, we can normalize K and assume that

$$\int h_K^d \, \mathrm{d}\sigma = \int 1 \, \mathrm{d}\sigma = d\kappa_d, \tag{15}$$

which permits us to obtain the following relations. From (12) we get

$$\kappa_d = U(K) \geq \kappa_d^{1-d} v_1(K)^d \geq c_1 D(K)^d,$$

where D denotes the diameter, hence $D(K) \leq c_2$ and therefore

$$h_K \leq c_2. \tag{16}$$

Moreover,

$$\beta(K) = c_3 \int \left(h_K^{d/2} - 1\right)^2 \mathrm{d}\sigma.$$

Suppose that $\rho_o B^d \subset K \subset R_o B^d$, where ρ_o is maximal and R_o is minimal. It follows from (15) that $\rho_o \leq 1 \leq R_o$.

Case 1: $R_o - 1 \geq 1 - \rho_o$.

We put $R_o = 1 + h$, then

$$\vartheta(K) = \frac{R_o - \rho_o}{R_o + \rho_o} \leq 2h. \tag{17}$$

There exists a vector $\boldsymbol{u}_0 \in S^{d-1}$ such that $h_K(\boldsymbol{u}_0) = R_o = 1+h$, and the point $\boldsymbol{p} = (1+h)\boldsymbol{u}_0$ belongs to K. For $\boldsymbol{u} \in S^{d-1}$, let $E(\boldsymbol{u})$ be the hyperplane through $\boldsymbol{p}$ and orthogonal to $\boldsymbol{u}$. Let ω denote the angle between $\boldsymbol{u}$ and $\boldsymbol{u}_0$, and let ω_0 be the angle between $\boldsymbol{u}_0$ and any $\boldsymbol{v}$ such that $E(\boldsymbol{v})$ is tangent to B^d. Then $\cos\omega_0 = 1/(1+h)$ and (with $\sigma_{d-1} := (d-1)\kappa_{d-1}$)

$$\int \left(h_K^{d/2} - 1\right)^2 \mathrm{d}\sigma \geq \sigma_{d-1} \int_0^{\omega_0} \left\{[(1+h)\cos\omega]^{d/2} - 1\right\}^2 (\sin\omega)^{d-2}\, \mathrm{d}\omega.$$

We set $(1+h)^{d/2} - 1 =: \alpha$ and substitute $[(1+h)\cos\omega]^{d/2} - 1 = \alpha x$, to obtain

$$\int \left(h_K^{d/2} - 1\right)^2 \mathrm{d}\sigma$$

$$\geq \frac{2\sigma_{d-1}\alpha^3}{d(1+h)^{d-2}} \int_0^1 x^2[(1+h)^2 - (\alpha x+1)^{4/d}]^{(d-3)/2}(\alpha x+1)^{(2-d)/d}\, \mathrm{d}x.$$

By (16), we can estimate

$$(1+h)^{2-d} \geq c_4$$

and, for $0 \leq x \leq 1$,

$$(\alpha x+1)^{(2-d)/d} \geq (\alpha+1)^{(2-d)/d} = (1+h)^{(2-d)/2} \geq c_4^{1/2}.$$

This gives

$$\beta(K) \geq c_5\alpha^3 \int_0^1 x^2[(1+h)^2 - (\alpha x+1)^{4/d}]^{(d-3)/2}\, \mathrm{d}x.$$

The function

$$f(x) := (1+h)^2 - (\alpha x+1)^{4/d}, \qquad 0 \leq x \leq 1,$$

satisfies $f(0) = h(h+2)$, $f(1) = 0$, and $f'(1) = -(4/d)\alpha(1+h)^{(4-d)/2}$. It is convex for $d \geq 4$ and concave for $d = 2, 3$. For $d \geq 4$ we deduce that

$$f(x) \geq \frac{4}{d}\alpha(1+h)^{(4-d)/2}(1-x) \geq c_6\alpha(1-x),$$

and for $d = 2, 3$ we get

$$f(x) \geq h(h+2)(1-x) \geq h(1-x).$$

Together with

$$\alpha = (1+h)^{d/2} - 1 \geq \frac{d}{2}h$$

this yields

$$\beta(K) \geq c_7 h^{(d+3)/2}.$$

From (17) we now get

$$\beta(K) \geq c_8\vartheta(K)^{(d+3)/2}.$$

Case 2: $R_o - 1 < 1 - \rho_o$.

We put $\rho_o = 1 - h$, then $R_o < 1 + h$, hence

$$K \subset (1+h)B^d \tag{18}$$

and

$$\vartheta(K) \leq 2h. \tag{19}$$

There is a vector $\boldsymbol{u}_0 \in S^{d-1}$ such that $h_K(\boldsymbol{u}_0) = \rho_o = 1 - h$, and the hyperplane G through $\rho_o \boldsymbol{u}_0$ orthogonal to $\boldsymbol{u}_0$ supports K. Let $\boldsymbol{p} \in (\partial(1+h)B^d) \cap G$. Let $\boldsymbol{u}_1 \in S^{d-1}$ be the vector, positively spanned by $\boldsymbol{u}_0$ and $\boldsymbol{p}$, that is orthogonal to a support plane of B^d through $\boldsymbol{p}$. Let ω_0 be the angle between $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$. For $\boldsymbol{u} \in S^{d-1}$, let $\omega_{\boldsymbol{u}}$ be the angle between $\boldsymbol{u}_0$ and $\boldsymbol{u}$. If $\omega_{\boldsymbol{u}} \leq \omega_0$, then $h_K(\boldsymbol{u}) \leq 1$, hence $1 - h_K(\boldsymbol{u})^{d/2} \geq 1 - h_K(\boldsymbol{u}) \geq 0$ and thus

$$\beta(K) \geq c_3 \int (1 - h_K(\boldsymbol{u}))^2 \, \mathbf{1}\{\omega_{\boldsymbol{u}} \leq \omega_0\} \, \sigma(\mathrm{d}\boldsymbol{u}).$$

Now exactly the argument of *Case 2* in [5, pp. 71–72] leads to

$$\beta(K) \geq c_9 h^{(d+3)/2},$$

which together with (19) gives

$$\beta(K) \geq c_{10} \vartheta(K)^{(d+3)/2}.$$

In each case, we conclude from (14) and (11) the inequality (13). □

4 Probabilities involving small intervals

In this section we prove an inequality which replaces the easily obtained estimate (7) in the case where $v_k(Z)$ is contained in a bounded interval.

For $A \subset \mathbb{R}^d$ we define

$$Z(A) := \bigcap_{\boldsymbol{x} \in A} H^-(\boldsymbol{x})$$

and set $Z(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) := Z(\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_n\})$. For the random polytope $Z(\tilde{X})$, the typical cell of X, we retain the notation Z.

Let $k \in \{1, \dots, d\}$ be fixed.

A main feature of the following estimate is the linear dependence of the lower bound on the length h of the interval $(1, 1+h)$. A similar result, for the zero cell of a stationary Poisson hyperplane process and the function v_d, is given in [7] as Lemma 3.2. It is an important advantage of the present argument that it merely uses the monotoneity, continuity and homogeneity properties of the function v_k.

Lemma 2. *For each $\beta > 0$, there are constants $h_0 > 0$, $N \in \mathbb{N}$ and $c_{11} > 0$, depending only on β and d, such that for $a > 0$ and $0 < h < h_0$,*

$$\mathrm{P}(v_k(Z) \in a(1, 1+h)) \geq c_{11} h \left(a^{d/k} \lambda\right)^N \exp\left\{-(1+\beta)^d 2^d \kappa_d^{1-d/k} a^{d/k} \lambda\right\}.$$

Proof. Let $\beta > 0$ and $a > 0$ be given. For $n \geq 2$ we define

$$\begin{aligned} \mathcal{Q}_n := \quad & \Big\{(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}, \boldsymbol{u}) \in \Big((1+\beta/2)B^d\Big)^{n-1} \times S^{d-1} : \\ & Z(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}, \boldsymbol{u}) \subset 2^{-1}(1+\beta/2)B^d,\ v_k(Z(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}, \boldsymbol{u})) \geq 2^{-k}\kappa_d\Big\}. \end{aligned}$$

A continuity argument shows that we can choose $N = N(\beta, d) \in \mathbb{N}$ (sufficiently large) so that

$$\int_{S^{d-1}} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \mathbf{1}\{(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-1}, \boldsymbol{u}) \in \mathcal{Q}_N\}\, \mathrm{d}\boldsymbol{x}_1 \dots \mathrm{d}\boldsymbol{x}_{N-1} \sigma(\mathrm{d}\boldsymbol{u}) =: c_{12}(d, \beta) > 0.$$

If $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-1}, \boldsymbol{u}) \in \mathcal{Q}_N$, then

$$v_k(Z(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-1}, \boldsymbol{u})) \leq (1+\beta/2)^k 2^{-k} \kappa_d. \tag{20}$$

We define $h_0 = h_0(\beta) > 0$ by

$$(1+h_0)^{1/k}(1+\beta/2) = 1+\beta \tag{21}$$

and suppose that $0 < h < h_0$. As before, we put $B_a := (a/\kappa_d)^{1/k} B^d$.

We start with the trivial estimate (again we use that a realization of $\tilde{X}$ denotes a simple counting measure and also its support)

$$\begin{aligned} & \mathrm{P}(v_k(Z) \in a(1, 1+h)) \\ & \geq \mathrm{P}\Big(\tilde{X}(2(1+\beta)B_a) = N,\ Z(\tilde{X} \cap 2(1+\beta)B_a) \subset (1+\beta)B_a, \\ & \quad v_k(Z(\tilde{X} \cap 2(1+\beta)B_a)) \in a(1, 1+h)\Big) \\ & = \mathrm{P}\Big(\tilde{X}(2(1+\beta)B_a) = N\Big)\mathrm{P}\Big(Z(\tilde{X} \cap 2(1+\beta)B_a) \subset (1+\beta)B_a, \\ & \quad v_k(Z(\tilde{X} \cap 2(1+\beta)B_a)) \in a(1, 1+h) \;\Big|\; \tilde{X}(2(1+\beta)B_a) = N\Big). \end{aligned}$$

Since $\tilde{X}$ is a stationary Poisson process with intensity λ, we obtain (using Satz 3.2.3(b) of [16])

$$\begin{aligned} & \mathrm{P}(v_k(Z) \in a(1, 1+h)) \\ & \geq \frac{\lambda^N}{N!} \exp\{-v_d(2(1+\beta)B_a)\lambda\} \\ & \quad \times \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \mathbf{1}\{\forall i:\ \boldsymbol{x}_i \in 2(1+\beta)B_a\}\, \mathbf{1}\{Z(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N) \subset (1+\beta)B_a\} \\ & \quad \times \mathbf{1}\{v_k(Z(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N)) \in a(1, 1+h)\}\, \mathrm{d}\boldsymbol{x}_1 \dots \mathrm{d}\boldsymbol{x}_N. \end{aligned}$$

Assume that $\boldsymbol{x}_1, \dots, \boldsymbol{x}_N \in \mathbb{R}^d$ satisfy the conditions

(i) $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N) \in \|\boldsymbol{x}_N\| \mathcal{Q}_N$;

(ii) $v_k(Z(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N)) \in a(1, 1+h)$.

Then, using (ii), (i) and the definition of $\mathcal{Q}_N$, we get

$$\frac{a(1+h)}{\|\boldsymbol{x}_N\|^k} \geq \frac{v_k(Z(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N))}{\|\boldsymbol{x}_N\|^k} \geq 2^{-k}\kappa_d,$$

hence

$$\|\boldsymbol{x}_N\| \leq 2(a/\kappa_d)^{1/k}(1+h)^{1/k}. \tag{22}$$

By (21) and (22),

$$\|\boldsymbol{x}_N\| \leq 2(1+\beta)(a/\kappa_d)^{1/k}.$$

Further, using (i), the definition of $\mathcal{Q}_N$, (22) and (21), we find that, for $i = 1, \ldots, N-1$,

$$\|\boldsymbol{x}_i\| \leq \|\boldsymbol{x}_N\|(1+\beta/2) \leq 2(a/\kappa_d)^{1/k}(1+h)^{1/k}(1+\beta/2) \leq 2(1+\beta)(a/\kappa_d)^{1/k},$$

thus $\boldsymbol{x}_i \in 2(1+\beta)B_a$. Finally, (i), the definition of $\mathcal{Q}_N$, (22) and (21) imply that

$$\begin{aligned} Z(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \quad &\subset \quad 2^{-1}\|\boldsymbol{x}_N\|(1+\beta/2)B^d \subset (a/\kappa_d)^{1/k}(1+h)^{1/k}(1+\beta/2)B^d \\ &\subset \quad (1+\beta)B_a. \end{aligned}$$

Hence, introducing polar coordinates, we obtain

$$\begin{aligned} &\mathrm{P}(v_k(Z) \in a(1, 1+h)) \\ &\geq \frac{\lambda^N}{N!} \exp\{-v_d(2(1+\beta)B_a)\lambda\} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \|\boldsymbol{x}_N\|\mathcal{Q}_N\} \\ &\quad \times \mathbf{1}\{v_k(Z(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)) \in a(1, 1+h)\} \mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{x}_N \\ &= \frac{\lambda^N}{N!} \exp\{-v_d(2(1+\beta)B_a)\lambda\} \\ &\quad \times \int_{S^{d-1}} \int_0^\infty \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}, r\boldsymbol{u}) \in r\mathcal{Q}_N\} \\ &\quad \times \mathbf{1}\{v_k(Z(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}, r\boldsymbol{u})) \in a(1, 1+h)\} r^{d-1} \mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{x}_{N-1} \mathrm{d}r\, \sigma(\mathrm{d}\boldsymbol{u}). \end{aligned}$$

Substituting $\boldsymbol{x}_i = r\boldsymbol{y}_i$ for $i = 1, \ldots, N-1$, we get

$$\begin{aligned} &\mathrm{P}(v_k(Z) \in a(1, 1+h)) \\ &\geq \frac{\lambda^N}{N!} \exp\{-v_d(2(1+\beta)B_a)\lambda\} \\ &\quad \times \int_{S^{d-1}} \int_0^\infty \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\{(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{N-1}, \boldsymbol{u}) \in \mathcal{Q}_N\} \\ &\quad \times \mathbf{1}\{r^k v_k(Z(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{N-1}, \boldsymbol{u})) \in a(1, 1+h)\} r^{Nd-1} \mathrm{d}\boldsymbol{y}_1 \ldots \mathrm{d}\boldsymbol{y}_{N-1} \mathrm{d}r\, \sigma(\mathrm{d}\boldsymbol{u}) \\ &= \frac{\lambda^N}{N!} \exp\{-v_d(2(1+\beta)B_a)\lambda\} \\ &\quad \times \int_{S^{d-1}} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\{(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{N-1}, \boldsymbol{u}) \in \mathcal{Q}_N\} \\ &\quad \times \frac{a^{Nd/k}}{Nd} v_k(Z(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{N-1}, \boldsymbol{u}))^{-Nd/k} \left((1+h)^{Nd/k} - 1\right) \mathrm{d}\boldsymbol{y}_1 \ldots \mathrm{d}\boldsymbol{y}_{N-1} \sigma(\mathrm{d}\boldsymbol{u}) \\ &\geq \frac{\lambda^N}{N!} \exp\{-v_d(2(1+\beta)B_a)\lambda\} \frac{a^{Nd/k}}{Nd} \left(2^k(1+\beta/2)^{-k}\kappa_d^{-1}\right)^{Nd/k} \frac{Nd}{k} h\, c_{12}(d, \beta) \\ &\geq h \left(a^{d/k}\lambda\right)^N \exp\{-v_d(2(1+\beta)B_a)\lambda\}\, c_{13}(d, \beta), \end{aligned}$$

where also (20) was used. Since $v_d(2(1+\beta)B_a) = (1+\beta)^d 2^d \kappa_d^{1-d/k} a^{d/k}$, this proves the assertion. □

5 Probabilities involving elongated cells

We will later need estimates showing that typical cells which, compared to their value of v_k, are 'too long', occur only with small probability. This requires the following preparations, in the course of which convex bodies are classified according to their degree of elongation.

We denote by $\mathcal{P}_o^d \subset \mathcal{K}_o^d$ the subset of convex polytopes and by $G(d,k)$ the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^d$. For $K \in \mathcal{K}_o^d$ and $L \in G(d,k)$, the set $K|L$ is the image of K under orthogonal projection to L. For $k \in \{1,\dots,d\}$, we define

$$\eta_k(K) := \min\{D(K)/\Delta(K,L) : L \in G(d,k)\},$$

where $D(K)$ denotes the diameter of K and $\Delta(K,L)$ is the width of $K|L$ evaluated in L. The simplest cases are $k = d$, where $\eta_d(K)$ is the ratio of the diameter and the width of K, and $k = 1$, where $\eta_1(K) = 1$.

Let $a > 0$ be given. For $m \in \mathbb{N}$, we set

$$\mathcal{K}_a^{d,k}(m) := \left\{ K \in \mathcal{K}_o^d : v_k(K) \in a(1,2),\, \eta_k(K) \in [m^k, (m+1)^k) \right\}.$$

The information that $K \in \mathcal{K}_a^{d,k}(m)$ can be used to derive estimates for the size of K from above and below.

Lemma 3. *Let $m \in \mathbb{N}$ and $k \in \{1,\dots,d\}$. Then*

(a) $K \in \mathcal{K}_a^{d,k}(m)$ *implies that* $K \subset c_{14} m^k a^{1/k} B^d =: C$*;*

(b) *there exists a measurable map* $\mathcal{K}_a^{d,k}(m) \cap \mathcal{P}_o^d \ni P \mapsto \boldsymbol{v}(P)$ *such that* $\boldsymbol{v}(P)$ *is a vertex of* P *with* $\|\boldsymbol{v}(P)\| \geq c_{15} m a^{1/k}$.

Proof. We use repeatedly that $v_k(K|L)$ is a constant multiple of the k-dimensional volume of $K|L$.

(a) A special case of equation (5.3.23) in [15] and the monotoneity of mixed volumes imply that

$$v_k(K|L) \leq c_{16} v_k(K)$$

holds for all $K \in \mathcal{K}_o^d$ and $L \in G(d,k)$. Let $K \in \mathcal{K}_a^{d,k}(m)$, and choose $L_0 \in G(d,k)$ such that $\eta_k(K) = D(K)/\Delta(K,L_0)$. Then

$$2a \geq v_k(K) \geq c_{16}^{-1} v_k(K|L_0) \geq c_{17} \Delta(K,L_0)^{k-1} D(K|L_0),$$

where the estimate (16) from [7] was used. Thus

$$2a \geq c_{17} \Delta(K,L_0)^k \geq c_{17}(m+1)^{-k^2} D(K)^k$$

and therefore

$$D(K) \leq c_{18} m^k a^{1/k}.$$

Since $\boldsymbol{o} \in K$, this implies (a).

(b) For any $L \in G(d,k)$, we enclose $K|L$ in a rectangular parallelepiped in L with one edge length equal to $\Delta(K,L)$ and the other edge lengths at most $D(K|L)$. Then

$$v_k(K|L) \leq c_{19} \Delta(K,L) D(K|L)^{k-1} \leq c_{19} m^{-k} D(K) D(K)^{k-1}$$

and hence, by an integral-geometric projection formula ([15], (5.3.27)),

$$a \leq v_k(K) \leq c_{20} m^{-k} D(K)^k.$$

Therefore, K has a point at distance at least $2^{-1} c_{20}^{-1/k} m a^{1/k}$ from the origin. If K is a polytope, such a point can be chosen as a vertex. That a measurable selection is possible, follows as in [7, Lemma 4.3(c)]. □

Remark. We have $\eta_1(K) = 1$; moreover, $\mathcal{K}_a^{d,1}(m) \neq \emptyset$ only for $m = 1$. Therefore, some of the subsequent arguments simplify considerably, or can be omitted, in the case $k = 1$.

Let $a > 0$, $\epsilon > 0$ be given. For $m \in \mathbb{N}$, we define

$$\mathcal{K}_{a,\epsilon}^{d,k}(m) := \{K \in \mathcal{K}_a^{d,k}(m) : \vartheta(K) \geq \epsilon\} \tag{23}$$

and

$$q_{a,\epsilon}^k(m) := \mathrm{P}(Z \in \mathcal{K}_{a,\epsilon}^{d,k}(m)). \tag{24}$$

Similarly as in [7], we prove two estimates concerning the decay of $q_{a,\epsilon}^k(m)$ as $a^{d/k}\lambda \to \infty$. The dependence on ϵ will not play a role until Lemma 6.

For $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in S^{d-1}$ and $t_1, \ldots, t_n \in (0, \infty)$ we introduce the abbreviation

$$\bigcap_{i=1}^{n} H^-(\boldsymbol{u}_i, t_i) =: P(\boldsymbol{u}_{(n)}, t_{(n)}).$$

Our next lemma corresponds to Lemma 5.1 in [7]; its proof is a modified and slightly simplified version of the proof of the latter.

Lemma 4. *For $m \in \mathbb{N}$ and $a^{d/k}\lambda \geq \sigma_0$, where $\sigma_0 > 0$ is a constant,*

$$q_{a,\epsilon}^k(m) \leq c_{21}(d, \sigma_0) \exp\{-c_{22} m^d a^{d/k} \lambda\}. \tag{25}$$

Proof. Let C be the ball defined in Lemma 3(a). Then

$$q_{a,\epsilon}^k(m) = \sum_{N=d+1}^{\infty} \mathrm{P}(Y(\mathcal{H}_C) = N)\mathrm{P}(Z \in \mathcal{K}_{a,\epsilon}^{d,k}(m) \mid Y(\mathcal{H}_C) = N). \tag{26}$$

Here,

$$\begin{aligned} p_N &:= \mathrm{P}(Z \in \mathcal{K}_{a,\epsilon}^{d,k}(m) \mid Y(\mathcal{H}_C) = N) \\ &= \frac{1}{U(C)^N} \int_{S^{d-1}} \int_0^{\infty} \cdots \int_{S^{d-1}} \int_0^{\infty} \mathbf{1}\left\{P(\boldsymbol{u}_{(N)}, t_{(N)}) \in \mathcal{K}_{a,\epsilon}^{d,k}(m)\right\} \\ &\quad \times \mathbf{1}\{\forall i :\ H(\boldsymbol{u}_i, t_i) \cap C \neq \emptyset\} (t_1 \cdots t_N)^{d-1} \, \mathrm{d}t_1 \, \sigma(\mathrm{d}\boldsymbol{u}_1) \ldots \mathrm{d}t_N \, \sigma(\mathrm{d}\boldsymbol{u}_N). \end{aligned} \tag{27}$$

Suppose that $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N, t_1, \ldots, t_N$ are such that the indicator functions occurring in the multiple integral are all equal to one; then $P := P(\boldsymbol{u}_{(N)}, t_{(N)}) \in \mathcal{K}_{a,\epsilon}^{d,k}(m)$ has a vertex $\boldsymbol{v}(P)$

according to Lemma 3(b). This vertex is the intersection of d facets of P. Hence, there exists an index set $J \subset \{1, \dots, N\}$ with d elements such that

$$\{\boldsymbol{v}(P)\} = \bigcap_{i \in J} H(\boldsymbol{u}_i, t_i).$$

The segment $S := [\boldsymbol{o}, \boldsymbol{v}(P)]$ satisfies

$$\operatorname{relint} S \cap H(\boldsymbol{u}_j, t_j) = \emptyset \qquad \text{for } j \in \{1, \dots, N\} \setminus J,$$

where relint denotes the relative interior. Since $S \subset C$, we have

$$\begin{aligned} &\int_{S^{d-1}} \int_0^\infty \mathbf{1}\{H(\boldsymbol{u}, t) \cap C \neq \emptyset,\ H(\boldsymbol{u}, t) \cap S = \emptyset\} t^{d-1} \,\mathrm{d}t \,\sigma(\mathrm{d}\boldsymbol{u}) \\ &= U(C) - U(S) = U(C) - 2^{-d} \kappa_d |S|^d, \end{aligned} \tag{28}$$

where $|S|$ denotes the length of S. Similarly as in the proof of [7, Lemma 5.1] we obtain

$$\begin{aligned} p_N \;&\leq\; \binom{N}{d} \frac{1}{U(C)^N} \int_{S^{d-1}} \int_0^\infty \cdots \int_{S^{d-1}} \int_0^\infty \mathbf{1}\{H(\boldsymbol{u}_i, t_i) \cap C \neq \emptyset,\ i = 1, \dots, d\} \\ &\quad \times [U(C) - c_{23} m^d a^{d/k}]^{N-d} (t_1 \cdots t_d)^{d-1} \,\mathrm{d}t_1 \,\sigma(\mathrm{d}\boldsymbol{u}_1) \dots \mathrm{d}t_d \,\sigma(\mathrm{d}\boldsymbol{u}_d) \\ &= \binom{N}{d} \left(1 - \frac{c_{23} m^d a^{d/k}}{U(C)}\right)^{N-d}. \end{aligned} \tag{29}$$

This leads to the estimate

$$\begin{aligned} q_{a,\epsilon}^k(m) \;&\leq\; \sum_{N=d+1}^{\infty} \frac{[2^d U(C) \lambda]^N}{N!} \exp\{-2^d U(C) \lambda\} \binom{N}{d} \left(1 - \frac{c_{23} m^d a^{d/k}}{U(C)}\right)^{N-d} \\ &= \frac{1}{d!} [2^d U(C) \lambda]^d \exp\{-2^d U(C) \lambda\} \\ &\quad \times \sum_{N=d+1}^{\infty} \frac{1}{(N-d)!} \left[2^d U(C) \lambda - c_{24} m^d a^{d/k} \lambda\right]^{N-d} \\ &\leq \frac{1}{d!} [2^d U(C) \lambda]^d \exp\left\{-c_{24} m^d a^{d/k} \lambda\right\} \\ &\leq c_{25} m^{kd^2} \left(a^{d/k} \lambda\right)^d \exp\left\{-c_{24} m^d a^{d/k} \lambda\right\} \\ &\leq c_{27}(d, \sigma_0) \exp\left\{-c_{26} m^d a^{d/k} \lambda\right\}, \end{aligned}$$

which completes the proof. □

The following result allows us to approximate a given convex polytope P by a polytope $L \subset P$ with a restricted number of vertices such that $U(L)$ is not much smaller than $U(P)$.

For a polytope P, let $\operatorname{ext} P$ be the set of vertices and $f_0(P)$ the number of vertices of P.

Lemma 5. *Let $\alpha > 0$ be given. There is a number $\nu \in \mathbb{N}$ depending only on d and α such that the following is true. For $P \in \mathcal{P}_o^d$ there exists a polytope $L = L(P) \in \mathcal{P}_o^d$ satisfying* ext$L \subset$ ext$P$, $f_0(L) \leq \nu$, *and* $U(L) \geq (1-\alpha)U(P)$. *Moreover, there exists a measurable selection* $P \mapsto L(P)$.

Proof. The following can be extracted from the proof of Lemma 4.2 in [7]. There exist numbers $k_0 = k_0(d)$ and $b_0 = b_0(d)$ such that the following is true. Let $P \in \mathcal{P}_o^d$ be a polytope and let $P \subset RB^d$, where R is minimal. Let $k \geq k_0$. There is a measurable map $P \mapsto L(P)$ such that $L = L(P)$ is the convex hull of at most $(k+1)d$ vertices of P, $\boldsymbol{o} \in L$, and $P \subset L + \kappa RB^d$ with $\kappa = b_0 k^{-2/(d-1)}$.

There is a unit vector $\boldsymbol{u}$ such that $R[\boldsymbol{o}, \boldsymbol{u}] \subset P$ and hence $U(P) \geq U(R[\boldsymbol{o}, \boldsymbol{u}]) = 2^{-d}\kappa_d R^d$. By a suitable choice of k, depending only on d and α, we can achieve that $\kappa \leq 1$ and $4^d\kappa \leq \alpha$. Since $h_L \leq R$, we get

$$U(P) \leq \frac{1}{d}\int_{S^{d-1}} [h_L + \kappa R]^d \mathrm{d}\sigma \leq U(L) + \kappa \cdot 2^d \kappa_d R^d \leq U(L) + 4^d \kappa U(P),$$

thus $U(L) \geq (1-\alpha)U(P)$, and $\nu = (k+1)d$ is the required number. □

For the probabilities $q_{a,\epsilon}^k$, we now state another upper bound, which is based on the stability estimate in Lemma 1 and on the preceding approximation result.

Lemma 6. *For* $m \in \mathbb{N}$, $\epsilon \in (0,1)$ *and* $a, \lambda > 0$,

$$q_{a,\epsilon}^k(m) \leq c_{28}(d,\epsilon) m^{kd^2\nu} \exp\left\{ -\left(1 + c_{29}\epsilon^{(d+3)/2}\right) 2^d \kappa_d^{1-d/k} a^{d/k} \lambda \right\},$$

where ν *depends only on* d *and* ϵ.

Proof. We define C as in Lemma 3(a) and use (26) and (27) again. Assume that $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N, t_1, \ldots, t_N$ are such that the indicator functions in (27) are all equal to one. Then, by Lemma 1,

$$U\left(P(\boldsymbol{u}_{(N)}, t_{(N)})\right) \geq (1 + \gamma\epsilon^{(d+3)/2})\kappa_d^{1-d/k} a^{d/k}. \tag{30}$$

Let $\alpha := \gamma\epsilon^{(d+3)/2}/(2 + \gamma\epsilon^{(d+3)/2})$; then $(1-\alpha)(1+\gamma\epsilon^{(d+3)/2}) = 1 + \alpha$. Put $c_{30} := \gamma/(2+\gamma)$; then $\alpha > c_{30}\epsilon^{(d+3)/2}$. By Lemma 5, there are $\nu = \nu(d,\epsilon)$ vertices of $P(\boldsymbol{u}_{(N)}, t_{(N)})$ such that the convex hull $L = L\left(P(\boldsymbol{u}_{(N)}, t_{(N)}\right)$ of these vertices satisfies

$$U(L) \geq (1-\alpha) U\left(P(\boldsymbol{u}_{(N)}, t_{(N)})\right). \tag{31}$$

The inequalities (30) and (31) imply that

$$U(L) \geq (1+\alpha)\,\kappa_d^{1-d/k} a^{d/k}. \tag{32}$$

Now the same argument as in the proof of Lemma 5.2 in [7], with the obvious modifications, yields

$$\begin{aligned} &\mathrm{P}(Z \in \mathcal{K}_{a,\epsilon}^{d,k}(m) \mid Y(\mathcal{H}_C) = N)[U(C)]^N \\ &\leq \sum_{j=d+1}^{d\nu} \binom{N}{j}\binom{j}{d}^{\nu} \left[U(C) - (1+\alpha)\kappa_d^{1-d/k} a^{d/k}\right]^{N-j} [U(C)]^j. \end{aligned}$$

Here j denotes the number of hyperplanes generating the vertices of L, and $\binom{j}{d}$ bounds the number of points of intersection of these hyperplanes; thus $\binom{j}{d}^{\nu}$ estimates the possibilities to choose the vertices of L. The probability that the other $N-j$ hyperplanes intersecting C do not meet L is given by $[U(C)-U(L)]^{N-j}U(C)^{-N+j}$, which is estimated using (32).

Inserting the inequality in (26), we can continue as in [7], finally using $U(C) = (c_{14}m^k a^{1/k})^d \kappa_d$ and $\alpha > c_{30}\epsilon^{(d+3)/2}$. This completes the proof. □

6 Proof of Theorem 1

From now on, the proofs follow essentially the lines of those given in [7]. We will, therefore, state only the necessary lemmas in their modified forms and refer to the corresponding proofs in [7].

Let $a > 0$ and $\epsilon \in (0,1)$ be given. For $h \in (0,1]$ and $m \in \mathbb{N}$ we define

$$\mathcal{K}^{d,k}_{a,\epsilon,h}(m) := \left\{K \in \mathcal{K}^d_o : v_k(K) \in a(1,1+h),\ \eta_k(K) \in [m^k,(m+1)^k),\ \vartheta(K) \geq \epsilon\right\}; \tag{33}$$

thus

$$\mathrm{P}(v_k(Z) \in a(1,1+h),\ \vartheta(Z) \geq \epsilon) = \sum_{m=1}^{\infty} q^{k,h}_{a,\epsilon}(m)$$

with

$$q^{k,h}_{a,\epsilon}(m) := \mathrm{P}\left(Z \in \mathcal{K}^{d,k}_{a,\epsilon,h}(m)\right). \tag{34}$$

Moreover, we put

$$q^{k,h}_{a,\epsilon}(m,n) := \mathrm{P}\left(Z \in \mathcal{K}^{d,k}_{a,\epsilon,h}(m),\ f_{d-1}(Z) = n\right) \tag{35}$$

for $n \in \mathbb{N}$; here $f_{d-1}(P)$ denotes the number of facets of a polytope P. Then we have

$$\mathrm{P}(v_k(Z) \in a(1,1+h),\ \vartheta(Z) \geq \epsilon) = \sum_{m=1}^{\infty}\sum_{n=d+1}^{\infty} q^{k,h}_{a,\epsilon}(m,n).$$

Finally, we define

$$\begin{aligned} R^{k,h}_{a,\epsilon}(m,n) \ &:= \ \Big\{(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n,t_1,\dots,t_n) \in (S^{d-1})^n \times (0,\infty)^n : \\ &\qquad P(\boldsymbol{u}_{(n)},t_{(n)}) \in \mathcal{K}^{d,k}_{a,\epsilon,h}(m),\ \ f_{d-1}\left(P(\boldsymbol{u}_{(n)},t_{(n)})\right) = n, \\ &\qquad H(\boldsymbol{u}_i,t_i) \cap C \neq \emptyset \quad \text{for } i = 1,\dots,n\Big\}, \end{aligned}$$

where the ball C is again defined as in Lemma 3(a), for the given a, ϵ, m.

Lemma 7. *For $m,n \in \mathbb{N}$, $n \geq d+1$ and $h \in (0,1]$,*

$$\begin{aligned} q^{k,h}_{a,\epsilon}(m,n) \ &= \ \frac{(2^d\lambda)^n}{n!} \underbrace{\int\cdots\int}_{R^{k,h}_{a,\epsilon}(m,n)} \exp\left\{-2^d U\left(P(\boldsymbol{u}_{(n)},t_{(n)})\right)\lambda\right\} \\ &\qquad \times (t_1\cdots t_n)^{d-1}\,\mathrm{d}t_1\dots\mathrm{d}t_n\,\sigma(\mathrm{d}\boldsymbol{u}_1)\dots\sigma(\mathrm{d}\boldsymbol{u}_n). \end{aligned}$$

The proof is the same as that for Lemma 6.1 in [7], with the obvious necessary changes.

Lemma 8. *Let $w > 0$, $h \in (0, 1/2)$ and $r \geq d-1$. Then*

$$\begin{aligned} &\int_1^{\sqrt[k]{1+h}} x^r \exp\{-wx^d\}\,\mathrm{d}x \\ \leq\ & c_{20} hw[1 + (\exp\{w/2\} - 1)^{-1}] \int_1^{\sqrt[k]{2}} x^r \exp\{-wx^d\}\,\mathrm{d}x. \end{aligned}$$

After the substitution $x^d = y$, one can imitate the proof of Lemma 6.2 in [7] to obtain the result.

The next (technical) lemma states that each bound for $q_{a,\epsilon}^{k,1}$ yields a bound for $q_{a,\epsilon}^{k,h}$ which is linear in h. This should be compared to the bound in Lemma 2 which is also linear in h.

Lemma 9. *For $m \in \mathbb{N}$, $h \in (0, 1/2)$ and $a^{d/k}\lambda \geq \sigma_0 > 0$,*

$$q_{a,\epsilon}^{k,h}(m) \leq c_{31}(d, \sigma_0) h\, a^{d/k} \lambda\, m^{dk} q_{a,\epsilon}^{k,1}(m).$$

Again, the proof is obtained by adapting the corresponding one from [7], namely that of Lemma 6.3. After applying Lemmas 7 and 8, we arrive at the inequality

$$\begin{aligned} q_{a,\epsilon}^{k,h}(m,n) \ \leq\ & \frac{(2^d\lambda)^n}{n!} \int_{U(m,n)} t(\zeta)^{nd} c_{32} h U(K(\zeta, t(\zeta)))\lambda \\ & \times \left(1 + \left(\exp\left\{2^{d-1} U(K(\zeta, t(\zeta)))\lambda\right\} - 1\right)^{-1}\right) \\ & \times \int_1^{\sqrt[k]{2}} s^{nd-1} \exp\left\{-2^d U(K(\zeta, t(\zeta)))\lambda s^d\right\} \mathrm{d}s \\ & \times (t_1 \cdots t_{n-1})^{d-1} \mathrm{d}t_1 \ldots \mathrm{d}t_{n-1} \sigma(\mathrm{d}\boldsymbol{u}_1) \ldots \sigma(\mathrm{d}\boldsymbol{u}_n), \end{aligned}$$

where $U(m,n)$, $t(\zeta)$ and $K(\cdot,\cdot)$ are defined as in [7], with the obvious changes. Now we have to observe that $v_k(K(\zeta, t(\zeta))) = a$, hence $K(\zeta, t(\zeta)) \in \mathcal{K}_{a,\epsilon}^{d,k}(m)$, which implies that

$$c_{33} m^d a^{d/k} \leq U(K(\zeta, t(\zeta))) \leq c_{34} m^{kd} a^{d/k},$$

by Lemma 3. The estimation can now be completed as in [7]. □

The following lemma establishes an upper estimate for an unconditional probability.

Lemma 10. *Let $\epsilon \in (0,1)$, $h \in (0, 1/2)$ and $a^{d/k}\lambda \geq \sigma_0 > 0$. Then*

$$\begin{aligned} &\mathrm{P}(v_k(Z) \in a(1, 1+h),\ \vartheta(Z) \geq \epsilon) \\ &\leq c_{35}(d, \epsilon, \sigma_0) h \exp\left\{-\left(1 + (c_{29}/2)\epsilon^{(d+3)/2}\right) 2^d \kappa_d^{1-d/k} a^{d/k} \lambda\right\}. \end{aligned}$$

Here c_{29} is the constant appearing in Lemma 6. The proof of Lemma 10 follows the one of Proposition 7.1 in [7] and uses Lemmas 9, 4 and 6, in this order.

The choice $(1+\beta)^d = 1+(c_{29}/4)\epsilon^{(d+3)/2}$ in Lemma 2 immediately proves Theorem 1 with $b = a(1+h)$ in the case $h \le \min(h_0, 1/2)$. As to arbitrary $b \ge a$, we observe that Lemmas 2 and 10 have the same structure as Lemma 3.2 and Proposition 7.1, respectively, in [7]; they differ only by the values of some parameters. It is, therefore, clear that Theorem 1 now follows precisely in the same way as Theorem 1 of [7] was proved. □

7 Proof of Theorem 2

In this section, Y denotes a stationary isotropic Poisson hyperplane process in $\mathbb{R}^d$ with intensity $\lambda > 0$. For a convex body $K \subset \mathbb{R}^d$ and for $n \in \mathbb{N}_0$, we have

$$\mathrm{P}(Y(\mathcal{H}_K) = n) = \frac{[2\kappa_d^{-1} v_1(K)\lambda]^n}{n!} \exp\{-2\kappa_d^{-1} v_1(K)\lambda\}, \tag{36}$$

by [7, (4)], where B is now the ball with surface area 1, thus $B = (d\kappa_d)^{-1/(d-1)} B^d$. Let Z_o be the zero cell of the tessellation induced by Y.

Lemma 11. *Let $k \in \{1, \dots, d\}$. For each $\beta > 0$, there are constants $h_0 > 0$, $N \in \mathbb{N}$ and $c_{36} > 0$, depending only on β and d, such that for $a > 0$ and $0 < h < h_0$,*

$$\mathrm{P}(v_k(Z_o) \in a(1, 1+h)) \ge c_{36} h \left(a^{1/k}\lambda\right)^N \exp\left\{-(1+\beta) 2\kappa_d^{-1/k} a^{1/k} \lambda\right\}.$$

Proof. In order to be able to essentially copy the proof of Lemma 2, we define a measure ψ on $\mathbb{R}^d$ by

$$\psi(A) := \frac{1}{d\kappa_d} \int_{S^{d-1}} \int_0^\infty \mathbf{1}_A(t\boldsymbol{u}) \,\mathrm{d}t \,\sigma(\mathrm{d}\boldsymbol{u})$$

for Borel sets $A \subset \mathbb{R}^d$. Let $\tilde{X}$ be the Poisson process in $\mathbb{R}^d$ with intensity measure $\lambda\psi$, and let Y' be the hyperplane process defined by $Y' := \{H(\boldsymbol{x}) : \boldsymbol{x} \in \tilde{X}\}$. Then Y' is a Poisson process in the space $\mathcal{H}$ of hyperplanes, and for a Borel set $\mathcal{A} \subset \mathcal{H}$ we have

$$\begin{aligned} \mathrm{E}\, Y'(\mathcal{A}) &= \lambda\psi(\{\boldsymbol{x} \in \mathbb{R}^d : H(\boldsymbol{x}) \in \mathcal{A}\}) \\ &= \frac{2\lambda}{d\kappa_d} \int_{S^{d-1}} \int_0^\infty \mathbf{1}_{\mathcal{A}}(H(\boldsymbol{u}, t)) \,\mathrm{d}t \,\sigma(\mathrm{d}\boldsymbol{u}). \end{aligned}$$

This shows that Y' has the same intensity measure as Y. Since Y and Y' are Poisson processes, they are stochastically equivalent. We can now repeat the proof of Lemma 2, where we replace Z by Z_o and $\mathrm{d}\boldsymbol{x}$ by $\psi(\mathrm{d}\boldsymbol{x})$. Further, we observe that $\psi(\mathrm{d}(r\boldsymbol{u})) = (d\kappa_d)^{-1} \mathrm{d}r \,\sigma(\mathrm{d}\boldsymbol{u})$ for $\|\boldsymbol{u}\| = 1$ and $\psi(\mathrm{d}(r\boldsymbol{y})) = r\psi(\mathrm{d}\boldsymbol{y})$ for $r > 0$. In the exponential, $v_d(2(1+\beta)B_a)$ (where $B_a = (a/\kappa_d)^{1/k} B^d$) has to be replaced by $\psi(2(1+\beta)B_a) = 2(1+\beta)(a/\kappa_d)^{1/k}$. With these changes, the proof of Lemma 2 yields the assertion of Lemma 11. □

We will need a stability version of the inequality (11).

Lemma 12. *There is a positive constant γ, depending only on the dimension d, such that for $\epsilon \in (0,1)$ and every convex body $K \subset \mathbb{R}^d$ with $r_d(K) \ge \epsilon$, the inequality*

$$v_1(K) \ge \left(1 + \gamma\epsilon^{(d+3)/2}\right) \kappa_d^{1-1/k} v_k(K)^{1/k} \tag{37}$$

holds for $k = 2, \dots, d$.

Proof. Without loss of generality, we assume that K has interior points, mean width 2 (the same as the unit ball), and Steiner point $\boldsymbol{o}$. We put $v_i := v_i(K)$ for $i = 0, \dots, d$ ($v_0 = \kappa_d$) and use the Aleksandrov-Fenchel inequalities $v_i^2 \geq v_{i-1}v_{i+1}$ for $i = 1, \dots, d-1$ (see [15]). Theorem 6.6.6 and Lemma 6.6.5 of [15] provide an improvement of the first of these inequalities, namely

$$v_1^2 - v_0 v_2 \geq c_{37}\delta(K, B^d)^{(d+3)/2},$$

where δ is the Hausdorff distance. This can be rewritten as

$$\frac{v_1}{v_0} \geq \frac{v_2}{v_1}\left(1 + \frac{c_{37}}{v_0 v_2}\delta(K, B^d)^{(d+3)/2}\right) \geq \frac{v_2}{v_1}(1+\alpha)$$

with $\alpha := c_{38}\delta(K, B^d)^{(d+3)/2}$, since $v_0 v_2 \leq v_1^2 = c_{39}$ (v_1 being a constant multiple of the mean width). From

$$\frac{v_1}{v_0} \geq (1+\alpha)\frac{v_2}{v_1} \geq \cdots \geq (1+\alpha)\frac{v_k}{v_{k-1}}$$

we get

$$\left(\frac{v_1}{v_0}\right)^{k-1} \geq (1+\alpha)^{k-1}\frac{v_k}{v_1}$$

and thus

$$\frac{v_1^k}{v_0^{k-1}v_k} \geq 1 + c_{38}\delta(K, B^d)^{(d+3)/2}.$$

If $\delta := \delta(K, B^d) \geq 1/2$, then (since $\epsilon < 1$)

$$\frac{v_1^k}{v_0^{k-1}v_k} \geq 1 + c_{40}\epsilon^{(d+3)/2}.$$

If $\delta \leq 1/2$, let $rB^d \subset K \subset sB^d$ where r is maximal and s is minimal. Then (by the definition of the Hausdorff metric) $s \leq 1 + \delta$ and $r \geq 1 - \delta \geq 1/2$, hence $\epsilon \leq r_d(K) \leq (s/r) - 1 \leq 4\delta$ and thus

$$\frac{v_1^k}{v_0^{k-1}v_k} \geq 1 + c_{41}\epsilon^{(d+3)/2}.$$

Both cases together give

$$v_1 \geq \left(1 + c_{42}\epsilon^{(d+3)/2}\right)^{1/k} v_0^{1-1/k}v_k^{1/k} \geq \left(1 + c_{43}\epsilon^{(d+3)/2}\right)\kappa_0^{1-1/k}v_k^{1/k}.$$

□

Theorem 2 can now be proved in essentially the same way as Theorem 1, and we list only the necessary changes in Sections 5 and 6, in addition to those already mentioned in the proof of Lemma 11. Definitions (23) and (24) are replaced by

$$\mathcal{K}_{a,\epsilon}^{d,k}(m) := \{K \in \mathcal{K}_a^{d,k}(m) : r_d(K) \geq \epsilon\}, \qquad q_{a,\epsilon}^k(m) := \mathrm{P}(Z_o \in \mathcal{K}_{a,\epsilon}^{d,k}(m)).$$

Lemma 13. *Let $k \in \{1, \dots, d\}$. For $m \in \mathbb{N}$ and $a^{1/k}\lambda \geq \sigma_0$, where $\sigma_0 > 0$ is a given constant,*

$$q_{a,\epsilon}^k(m) \leq c_{44}(d, \sigma_0)\exp\{-c_{45}ma^{1/k}\lambda\}. \tag{38}$$

Proof. In the proof of Lemma 4, the number $U(C)$ is replaced by $dv_1(C)$. In the integrations, $t^{d-1}\mathrm{d}t$ is replaced by $\mathrm{d}t$. Equation (28) now reads

$$\int_{S^{d-1}}\int_0^\infty \mathbf{1}\{H(\boldsymbol{u},t)\cap C\neq\emptyset,\ H(\boldsymbol{u},t)\cap S=\emptyset\}\,\mathrm{d}t\,\sigma(\mathrm{d}\boldsymbol{u})$$
$$= dv_1(C)-dv_1(S)=dv_1(C)-\kappa_{d-1}|S|,$$

hence (29) reads

$$p_N\le\binom{N}{d}\left(1-\frac{c_{46}ma^{1/k}}{v_1(C)}\right)^{N-d}.$$

Continuing as in the proof of Lemma 4, we arrive at (38). □

Lemma 14. *Let $k\in\{2,\dots,d\}$. For $m\in\mathbb{N}$, $\epsilon\in(0,1)$ and $a,\lambda>0$,*

$$q^k_{a,\epsilon}(m)\le c_{47}(d,\epsilon)m^{kd\nu}\exp\left\{-\left(1+c_{48}\epsilon^{(d+3)/2}\right)2\kappa_d^{-1/k}a^{1/k}\lambda\right\},$$

where ν depends only on d and ϵ.

Proof. We use (26) and (27), with the changes already made for the proof of Lemma 13. Assume that $\boldsymbol{u}_1,\dots,\boldsymbol{u}_N,t_1,\dots,t_N$ are such that the indicator functions in (27) are all equal to one. By Lemma 12,

$$v_1(P(\boldsymbol{u}_{(N)},t_{(N)}))\ge\left(1+\gamma\epsilon^{(d+3)/2}\right)\kappa_d^{1-1/k}a^{1/k}.$$

We define α as in the proof of Lemma 6 and put $c_{49}:=\gamma/(2+\gamma)$, so that $\alpha>c_{49}\epsilon^{(d+3)/2}$. By Lemma 4.2 of [7] (with $B=(d\kappa_d)^{-1/(d-1)}B^d$), there are $\nu=\nu(d,\epsilon)$ vertices of $P(\boldsymbol{u}_{(N)},t_{(N)})$ such that their convex hull $L=L(P(\boldsymbol{u}_{(N)},t_{(N)}))$ satisfies

$$v_1(L)\ge(1-\alpha)v_1(P(\boldsymbol{u}_{(N)},t_{(N)})).$$

This gives

$$v_1(L)\ge(1+\alpha)\kappa_d^{1-1/k}a^{1/k}.$$

As in the proof of Lemma 6 (and of Lemma 5.2 in [7]) we deduce that

$$\mathrm{P}(Z_o\in\mathcal{K}^{d,k}_{a,\epsilon}(m)\mid Y(\mathcal{H}_C)=N)[dv_1(C)]^N$$
$$\le\sum_{j=d+1}^{d\nu}\binom{N}{j}\binom{j}{d}^{\nu}\left[dv_1(C)-d(1+\alpha)\kappa_d^{1-1/k}a^{1/k}\right]^{N-j}[dv_1(C)]^j.$$

By Lemma 3,

$$v_1(C)=c_{50}m^ka^{1/k}.$$

The assertion is now obtained as in the proofs quoted above. □

The further proof again follows the lines of [7] and of the proof of Theorem 1. Definitions (33), (34) and (35) are replaced by

$$\mathcal{K}^{d,k}_{a,\epsilon,h}(m):=\left\{K\in\mathcal{K}^d_o: v_k(K)\in a(1,1+h),\ \eta_k(K)\in[m^k,(m+1)^k),\ r_d(K)\ge\epsilon\right\},$$

$$q_{a,\epsilon}^{k,h}(m) := \mathrm{P}\left(Z_o \in \mathcal{K}_{a,\epsilon,h}^{d,k}(m)\right),$$

$$q_{a,\epsilon}^{k,h}(m,n) := \mathrm{P}\left(Z_o \in \mathcal{K}_{a,\epsilon,h}^{d,k}(m),\ f_{d-1}(Z_o) = n\right).$$

Lemma 7 is replaced by

$$q_{a,\epsilon}^{k,h}(m,n) = \frac{(2\lambda)^n}{(d\kappa_d)^n n!} \underbrace{\int \cdots \int}_{R_{a,\epsilon}^{k,h}(m,n)} \exp\left\{-2\kappa_d^{-1} v_1\left(P(\boldsymbol{u}_{(n)}, t_{(n)})\right)\lambda\right\}$$
$$\times \, \mathrm{d}t_1 \dots \mathrm{d}t_n \, \sigma(\mathrm{d}\boldsymbol{u}_1) \dots \sigma(\mathrm{d}\boldsymbol{u}_n).$$

This is a special case of Lemma 6.1 in [7]. Instead of Lemma 8, we use Lemma 6.2 from [7], with d replaced by $k \geq 2$. Lemma 9 is replaced by the following assertion. For $m \in \mathbb{N}$, $h \in (0, 1/2)$ and $a^{1/k}\lambda \geq \sigma_0 > 0$,

$$q_{a,\epsilon}^{k,h}(m) \leq c_{51}(d, \sigma_0) h \, a^{1/k} \lambda \, m^k q_{a,\epsilon}^{k,1}(m).$$

The proof is obtained by adapting the proof of Lemma 6.3 in [7]. In the course of the proof, one has to use that Lemma 3 implies

$$c_{52} m a^{1/k} \leq v_1(K) \leq c_{53} m^k a^{1/k}$$

for $K \in \mathcal{K}_{a,\epsilon}^{d,k}(m)$. The counterpart of Lemma 10 now reads as follows. Let $\epsilon \in (0,1)$, $h \in (0, 1/2)$ and $a^{1/k}\lambda \geq \sigma_0 > 0$. Then

$$\mathrm{P}(v_k(Z_o) \in a(1, 1+h),\ r_d(Z_o) \geq \epsilon)$$
$$\leq c_{54}(d, \epsilon, \sigma_0) h \exp\left\{-\left(1 + (c_{48}/2)\epsilon^{(d+3)/2}\right) 2\kappa_d^{-1/k} a^{1/k} \lambda\right\}.$$

With these preliminaries, the proof of Theorem 2 can now be completed in the same way as that of Theorem 1.

8 Proof of Theorem 3

In large parts of the proofs of Theorems 1 and 2, only the following properties of the functional v_k are used: it is monotone under set inclusion, i.e., $v_k(K_1) \leq v_k(K_2)$ if $K_1 \subset K_2$, positively homogeneous of degree k, i.e., $v_k(rK) = r^k v_k(K)$ for $r \geq 0$, and continuous with respect to the Hausdorff metric. These properties are shared, with $k = 1$, by the inradius functionals ρ_o and ρ. Hence, the corresponding parts of the proofs apply also to Theorem 3. Additional properties of the function v_k were only needed for the stability estimates of Lemmas 1 and 12 and for Lemma 3. We replace these lemmas by the following ones.

Lemma 15. *There is a positive constant γ, depending only on the dimension d, such that for $\epsilon \in (0,1)$ and every convex body $K \in \mathcal{K}_o^d$ with $\vartheta(K) \geq \epsilon$ the inequality*

$$U(K) \geq (1 + \gamma\epsilon^{(d+1)/2}) \kappa_d \rho_o(K)^d$$

holds.

Proof. Without loss of generality, we assume that $\rho_o(K) = 1$. Let $K \subset R_o B^d$, where R_o is minimal. First we assume that $R_o \leq 3$. Put $R_o = 1 + h$, then $\epsilon \leq \vartheta(K) \leq h \leq 2$. Proceeding similarly as in Case 1 of the proof of Lemma 1, we find that

$$U(K) - U(B^d) \geq c_{55} h^{(d+1)/2} \geq c_{55} \epsilon^{(d+1)/2}.$$

Now suppose that $R_o > 3$. Then there is a spherical cap $A \subset S^{d-1}$ with $\sigma(A) = c_{56} > 0$ on which $h_K \geq 2$. It follows that

$$U(K) - U(B^d) \geq \frac{1}{d} \int_A \left(h_K^d - 1 \right) \mathrm{d}\sigma \geq c_{57} \epsilon^{(d+1)/2},$$

since $\epsilon < 1$. Hence, both cases yield

$$U(K) \geq U(B^d) + c_{58} \epsilon^{(d+1)/2} = \left(1 + c_{59} \epsilon^{(d+1)/2} \right) \kappa_d \rho_o(K)^d.$$

□

Lemma 16. *There is a positive constant γ, depending only on the dimension d, such that for $\epsilon \in (0,1)$ and every convex body $K \in \mathcal{K}_o^d$ with $r_d(K) \geq \epsilon$ the inequality*

$$v_1(K) \geq (1 + \gamma \epsilon^{(d+1)/2}) \kappa_d \rho(K)$$

holds.

The proof is similar to that of Lemma 15.

For a replacement of Lemma 3, let $a > 0$ be given. For $m \in \mathbb{N}$ we set

$$\mathcal{K}_a^d(m) := \{K \in \mathcal{K}_o : \rho(K) \in a(1,2),\ D(K)/\rho(K) \in [m, m+1)\}.$$

Lemma 17. *Let $m \in \mathbb{N}$. Then*

(a) $K \in \mathcal{K}_a^d(m)$ *implies that* $K \subset c_{60} m a B^d$*;*

(b) *there exists a measurable map* $\mathcal{K}_a^d(m) \cap \mathcal{P}_o^d \ni P \mapsto \boldsymbol{v}(P)$ *such that* $\boldsymbol{v}(P)$ *is a vertex of* P *with* $\|\boldsymbol{v}(P)\| \geq c_{61} m a$*.*

A similar result holds with ρ_o instead of $\rho(K)$. The proof is immediate.

With these changes, the proofs of Theorems 1 and 2 yield the proof of Theorem 3.

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