Gaussian polytopes: variances and limit theorems

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Abstract

The convex hull of n independent random points in $\mathbb{R}^d$ chosen according to the normal distribution is called a Gaussian polytope. Estimates for the variance of the number of i-faces and for the variance of the i-th intrinsic volume, of a Gaussian polytope in $\mathbb{R}^d$, $d \in \mathbb{N}$, are established by means of the Efron-Stein jackknife inequality and a new formula of Blaschke-Petkantschin type. These estimates imply laws of large numbers for the number of i-faces and for the i-th intrinsic volume of a Gaussian polytope as $n \to \infty$.

1 Introduction and statements of results

Let $X_1, \dots, X_n$ be a Gaussian sample in $\mathbb{R}^d$ ($d \in \mathbb{N}$), i.e., independent random points chosen according to the d-dimensional standard normal distribution with mean zero and covariance matrix $\frac{1}{2} I_d$. Denote by $P_n = [X_1, \dots, X_n]$ the convex hull of these random points, and call P_n a *Gaussian polytope*. We are interested in geometric functionals such as volume, intrinsic volumes, and the number of i-dimensional faces of Gaussian polytopes. Most of the previous investigations were concerned with *expectations* of such functionals. The starting point of this line of research is marked by a classical paper by Rényi and Sulanke [21] in which the asymptotic behaviour of the expected number of vertices, $\mathbb{E} f_0(P_n)$, of P_n is determined in the plane, and thus also the expected number of edges, $\mathbb{E} f_1(P_n)$, as n tends to infinity. This result was generalized by Raynaud [20] who investigated the asymptotic behaviour of the mean number of facets, $\mathbb{E} f_{d-1}(P_n)$, for arbitrary dimensions. Both results are only particular cases of the formula

$$\mathbb{E} f_i(P_n) = \frac{2^d}{\sqrt{d}} \binom{d}{i+1} \beta_{i,d-1} (\pi \ln n)^{\frac{d-1}{2}} \, (1 + o(1)), \tag{1.1}$$

where $i \in \{0, \dots, d-1\}$ and $d \in \mathbb{N}$, as $n \to \infty$. This follows, in arbitrary dimensions, from work of Affentranger and Schneider [2] and Baryshnikov and Vitale [3]. Here $f_i(P_n)$ denotes the number of i-faces of P_n, and $\beta_{i,d-1}$ is the internal angle of a regular $(d-1)$-simplex at one of its i-dimensional faces. Recently, a more direct proof of (1.1) and some additional relations, which cannot be derived from [2] and [3], were given in [12]. However, it turned out to be difficult to extend these results to higher moments of $f_i(P_n)$, and thus to prove limit theorems. An exception is the particular case $i = 0$, where Hueter [10], [11] states a Central Limit Theorem,

$$\frac{f_0(P_n) - \mathbb{E} f_0(P_n)}{\sqrt{\operatorname{Var} f_0(P_n)}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \tag{1.2}$$

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as n tends to infinity; here $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution and $\mathcal{N}(0,1)$ is the (one-dimensional) normal distribution. The asymptotic behaviour of the variance is asserted to be of the form

$$\operatorname{Var} f_0(P_n) = \bar{c}_d\,(\ln n)^{\frac{d-1}{2}}\,(1+o(1)),$$

as $n \to \infty$. Most probably it is difficult to establish such a precise limit relation for all $f_i(P_n)$, $i \in \{1,\dots,d-1\}$. Our first result provides an upper bound for the order of the variance of $f_i(P_n)$, for all $i \in \{0,\dots,d-1\}$, which is of the same order.

Theorem 1.1. *Let $f_i(P_n)$ be the number of i-dimensional faces of a d-dimensional Gaussian polytope P_n, $d \in \mathbb{N}$. Then there exists a positive constant c_d, depending only on the dimension, such that*

$$\operatorname{Var} f_i(P_n) \le c_d\,(\ln n)^{\frac{d-1}{2}} \tag{1.3}$$

for all $i \in \{0,\dots,d-1\}$.

Combining Chebyshev's inequality and (1.3), we obtain

$$\mathbb{P}\left(|f_i(P_n) - \mathbb{E}f_i(P_n)|(\ln n)^{-\frac{d-1}{2}} \ge \varepsilon\right) \le \varepsilon^{-2}(\ln n)^{-(d-1)}\operatorname{Var} f_i(P_n) \le \varepsilon^{-2}c_d\,(\ln n)^{-\frac{d-1}{2}},$$

and thus the random variable $f_i(P_n)$ satisfies a (weak) law of large numbers for all $d \in \mathbb{N}$ (the case $d=1$ is trivial). In fact, the law for $f_i(P_n)(\ln n)^{-(d-1)/2}$ converges in probability to the law concentrated at a constant.

Corollary 1.2. *For $d \in \mathbb{N}$ and $i \in \{0,\dots,d-1\}$, the number of i-dimensional faces, $f_i(P_n)$, of a Gaussian polytope P_n in $\mathbb{R}^d$ satisfies*

$$f_i(P_n)(\ln n)^{-\frac{d-1}{2}} \longrightarrow \frac{2^d}{\sqrt{d}}\binom{d}{i+1}\beta_{i,d-1}\,\pi^{\frac{d-1}{2}}$$

in probability as $n \to \infty$.

Massé [16] deduces a corresponding weak law of large numbers for $d=2$ and $i=0$ from Hueter's central limit theorem (1.2).

Our method of proof also works for the volume and, more generally, the intrinsic volumes. Denote by $V_i(P_n)$ the i-th intrinsic volume of the Gaussian polytope P_n; hence, for instance, $V_d(P_n)$ is the volume, $2V_{d-1}(P_n)$ is the surface area and $V_1(P_n)$ is a multiple of the mean width of P_n. The expected values of the i-th intrinsic volumes were investigated by Affentranger [1] who proved that

$$\mathbb{E}V_i(P_n) = \binom{d}{i}\frac{\kappa_d}{\kappa_{d-i}}\,(\ln n)^{\frac{i}{2}}\,(1+o(1)), \tag{1.4}$$

for $i \in \{1,\dots,d\}$, as n tends to infinity, where κ_j denotes the volume of the j-dimensional unit ball. The case $d=1$ which has been excluded in [1] can be checked directly. Relation (1.4) was expected to hold, since a result of Geffroy [6] implies that the Hausdorff distance between P_n and the d-dimensional ball of radius $(\ln n)^{1/2}$ and centred at the origin converges almost surely to zero. But it seems that (1.4) cannot be deduced directly from Geffroy's result.

In the planar case, Hueter also states Central Limit Theorems for $V_1(P_n)$ and $V_2(P_n)$,

$$\frac{V_1(P_n) - \mathbb{E}V_1(P_n)}{\sqrt{\operatorname{Var} V_1(P_n)}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \qquad \text{and} \qquad \frac{V_2(P_n) - \mathbb{E}V_2(P_n)}{\sqrt{\operatorname{Var} V_2(P_n)}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

The variances suggested by Hueter are of the form $\operatorname{Var} V_i(P_n) = \frac{1}{2}\,\pi^{3/2}\,(\ln n)^i\,\left(1 + o(1)\right)$. That her result cannot be correct can be seen from the following: if the stated asymptotic behaviour of the variances were true, this would immediately imply $\mathbb{P}(V_1(P_n) \le 0) \to \Phi(-\sqrt[4]{4\pi})$ and $\mathbb{P}(V_2(P_n) \le 0) \to \Phi(-\sqrt[4]{4\pi})$. But then the stated probabilities would be positive for large n, which obviously cannot hold. In the next theorem we give an upper bound for the variances, for all $i = 1, \ldots, d$ and $d \in \mathbb{N}$.

Theorem 1.3. *Let $V_i(P_n)$ be the i-th intrinsic volume of a Gaussian polytope P_n in $\mathbb{R}^d$, $d \in \mathbb{N}$. Then there exists a positive constant c_d, depending only on the dimension, such that*

$$\operatorname{Var} V_i(P_n) \le c_d (\ln n)^{\frac{i-3}{2}} \tag{1.5}$$

for all $i \in \{1, \ldots, d\}$.

Let $X_1, X_2, \ldots$ be a sequence of independent random points which are identically distributed according to the d-dimensional normal distribution, and let $P_n = [X_1, \ldots, X_n]$. For $d = 1$, the quantity $V_1(P_n)$ is the *sample range* of $X_1, \ldots, X_n$. Although its distribution and moments can be expressed as multiple integrals (see [19, Chapter 8], [13, Chapter 14] and [15]), explicit values are not available in general. The asymptotic behaviour of Var $V_1(P_n)$ for $d = 1$ is deduced in Chapter 14 (see equation (14.100)) of [13], which yields (with the present normalization) that $\operatorname{Var} V_1(P_n) = \frac{\pi^2}{12}\,(\ln n)^{-1}\,(1 + o(1))$ as $n \to \infty$. A different extension of the univariate sample range to higher dimensions is given by the largest interpoint distance of the given random points. Limiting distributions have been considered in [17] and in a more general framework in [8]. Still annother extension is discussed in [9].

From (1.4) and (1.5) we obtain an additive weak law of large numbers for $i \in \{1, 2\}$, i.e.

$$V_i(P_n) - \binom{d}{i}\frac{\kappa_d}{\kappa_{d-i}}(\ln n)^{\frac{i}{2}} \to 0$$

in probability as $n \to \infty$. In order to derive a (multiplicative) strong law of large numbers for $i \in \{1, \ldots, d\}$, we put $n_k = 2^k$. From the upper bound for the variance and Chebyshev's inequality we deduce that

$$\mathbb{P}\left(|V_i(P_{n_k}) - \mathbb{E}V_i(P_{n_k})|(\ln n_k)^{-\frac{i}{2}} \ge \varepsilon\right) \le \varepsilon^{-2} c_d\,(\ln n_k)^{-\frac{i+3}{2}}.$$

Since

$$\sum_{k\ge 1}(\ln n_k)^{-(i+3)/2} = (\ln 2)^{-(i+3)/2}\sum_{k\ge 1} k^{-(i+3)/2} < \infty,$$

(1.4) and the Borell-Cantelli Lemma imply that

$$V_i(P_{n_k})(\ln n_k)^{-\frac{i}{2}} \to \binom{d}{i}\frac{\kappa_d}{\kappa_{d-i}} \tag{1.6}$$

with probability one as k tends to infinity. Moreover, since $n \mapsto V_i(P_n)$ is increasing,

$$V_i(P_{n_{k-1}})(\ln n_k)^{-\frac{i}{2}} \le V_i(P_n)(\ln n)^{-\frac{i}{2}} \le V_i(P_{n_k})(\ln n_{k-1})^{-\frac{i}{2}}$$

for $n_{k-1} \le n \le n_k$, where by definition $(\ln n_{k+1})/(\ln n_k) \to 1$. Thus (1.6) implies a strong law of large numbers.

Corollary 1.4. *Let $V_i(P_n)$ be the i-th intrinsic volume of a Gaussian polytope P_n in $\mathbb{R}^d$, $d \in \mathbb{N}$. Then, for $i \in \{1, \ldots, d\}$,*

$$V_i(P_n)(\ln n)^{-\frac{i}{2}} \to \binom{d}{i}\frac{\kappa_d}{\kappa_{d-i}}$$

with probability one as $n \to \infty$.

This law of large numbers can also be deduced from a result of Geffroy in [6].

The estimates for the variances obtained in Theorems 1.1 and 1.3 are based on the solution of another problem which is of independent interest. Consider the random polytope P_n and choose another independent random point X according to the normal distribution. The question we are interested in is the following: if $X \notin P_n$, how many facets of P_n can be seen from X? We will determine the asymptotic behaviour of the expectation of the corresponding random variable, as $n \to \infty$, and we will provide upper and lower bounds for its second moment.

In the following, let $F_n(X)$ be the number of facets of P_n which can be seen from X, i.e., which are up to $(d-2)$-dimensional faces contained in the interior of the convex hull of P_n and X. Note that $F_n(X) = 0$ if X is contained in P_n.

Theorem 1.5. *Let $X, X_1, \dots, X_n$ be independent random points in $\mathbb{R}^d$, $d \in \mathbb{N}$, which are identically distributed according to the d-dimensional normal distribution. Let $P_d^{(d-1)}$ denote a Gaussian polytope in $\mathbb{R}^{d-1}$. Then*

$$\lim_{n\to\infty} \mathbb{E}F_n(X)\, n(\ln n)^{-\frac{d-1}{2}} = 2^{d-1}\kappa_d\,\Gamma(d+1)\mathbb{E}V_{d-1}(P_d^{(d-1)}). \tag{1.7}$$

Further, there is a positive constant c_d, depending only on the dimension, such that

$$c_d^{-1} n^{-1}(\ln n)^{\frac{d-1}{2}} \le \mathbb{E}F_n(X)^2 \le c_d\, n^{-1}(\ln n)^{\frac{d-1}{2}}\,. \tag{1.8}$$

For more information on random polytopes, we refer to the recent survey article by Schneider [24].

2 Projections of high-dimensional simplices

We want to give two interpretations of our results. The first one uses the fact that any orthogonal projection of a Gaussian sample again is a Gaussian sample. So we make our notation more precise by writing $P_n^{(d)}$ for a Gaussian polytope in $\mathbb{R}^d$ which is generated as the convex hull of n normally distributed random points in $\mathbb{R}^d$. Let $\Pi_i : \mathbb{R}^d \to \mathbb{R}^i$ be the projection to the first i components ($i < d$). For an arbitrary i-dimensional subspace of $\mathbb{R}^d$, which we identify with $\mathbb{R}^i$, we then obtain

$$\varphi(\Pi_i P_n^{(d)}) \stackrel{d}{=} \varphi(P_n^{(i)}), \tag{2.1}$$

where $\stackrel{d}{=}$ means equality in distribution and φ is any (measurable) functional on the convex polytopes.

Now let $P_{n+1}^{(n)}$ be a Gaussian simplex in $\mathbb{R}^n$. As a consequence of Corollary 1.4 and (2.1), we obtain a law of large numbers for projections of high-dimensional random simplices: for a fixed integer $i \ge 1$,

$$V_i(\Pi_i P_{n+1}^{(n)})\,(\ln n)^{-\frac{i}{2}} \to \kappa_i$$

in probability as $n \to \infty$. Moreover, for a fixed integer $i \ge 1$, (1.4) implies that

$$\mathbb{E}V_i(\Pi_i P_{n+1}^{(n)}) = \mathbb{E}V_i(P_{n+1}^{(i)}) = \kappa_i\,(\ln n)^{\frac{i}{2}}\,(1 + o(1)),$$

as $n \to \infty$. An estimate of the variance can be deduced from Theorem 1.3. Thus, for $i \ge 1$, we have

$$\operatorname{Var} V_i(\Pi_i P_{n+1}^{(n)}) \le c_i\,(\ln n)^{\frac{i-3}{2}}.$$

Finally, Kubota's theorem (see [25, (4.6)], [23, (5.3.27)]), Hölder's inequality, and Theorem 1.3 yield the following asymptotic result for the i-th intrinsic volume of a high-dimensional Gaussian simplex (cf. the proof of Theorem 1.3 in Section 7 for a similar argument).

Corollary 2.1. *Let $V_i(P_{n+1}^{(n)})$ be the i-th intrinsic volume of a Gaussian simplex in $\mathbb{R}^n$. Then, for any fixed integer $i \geq 1$,*

$$V_i(P_{n+1}^{(n)})\, c_{n,i}^{-1}\, (\ln n)^{-\frac{i}{2}} \to \kappa_i$$

in probability as $n \to \infty$, where $c_{n,i} = \binom{n}{i}\kappa_n/(\kappa_i\kappa_{n-i})$.

Another method of generating $n+1$ random points in $\mathbb{R}^d$ goes back to a suggestion of Goodman and Pollack. Let R denote a random rotation of $\mathbb{R}^n$, put $\Pi_d^* := \Pi_d \circ R$, (recall that Π_d denotes the projection onto $\mathbb{R}^d$) and let $T^{(n)}$ be a regular simplex in $\mathbb{R}^n$. Then $\Pi_d^*(T^{(n)})$ is a random polytope in $\mathbb{R}^d$ in the *Goodman-Pollack model*. It was proved by Baryshnikov and Vitale [3] that

$$\varphi(\Pi_d^* T^{(n)}) \stackrel{d}{=} \varphi(P_{n+1}^{(d)}), \tag{2.2}$$

for any affine invariant (measurable) functional φ on the convex polytopes. Thus, if f_i denotes the number of i-faces, (1.1) is equivalent to

$$\mathbb{E} f_i(\Pi_d^* T^{(n)}) = \frac{2^d}{\sqrt{d}}\binom{d}{i+1}\beta_{i,d-1}\,(\pi \ln n)^{\frac{d-1}{2}}\,(1+o(1))$$

as $n \to \infty$, which is what was actually proved by Affentranger and Schneider [2]. By (2.2), the bound for the variance in Theorem 1.1 and the law of large numbers in Corollary 1.2 now give bounds and a law of large numbers, respectively, for $\Pi_d^* T^{(n)}$, i.e.

$$\operatorname{Var} f_i(\Pi_d^* T^{(n)}) \leq c_d\, (\ln n)^{\frac{d-1}{2}}$$

and, for $d \in \mathbb{N}$,

$$f_i(\Pi_d^* T^{(n)})(\ln n)^{-\frac{d-1}{2}} \longrightarrow \frac{2^d}{\sqrt{d}}\binom{d}{i+1}\beta_{i,d-1}\,\pi^{\frac{d-1}{2}}$$

in probability, as n tends to infinity.

For further information on the ‘Goodman-Pollack model’ and related work of Vershik and Sporyshev [27], we refer to [2].

3 A new formula of Blaschke-Petkantschin type

We work in the d-dimensional Euclidean spaces $\mathbb{R}^d$ with scalar product $\langle\cdot\,,\cdot\rangle$ and induced norm $\|\cdot\|$. The d-dimensional Lebesgue measure in $\mathbb{R}^d$ will be denoted by λ_d. We write $\mathbb{S}^{d-1}$ for the Euclidean unit sphere and σ for spherical Lebesgue measure (the dimension will be clear from the context). Recall that for points $x_1, \dots, x_m \in \mathbb{R}^d$, the convex hull of these points is denoted by $[x_1, \dots, x_m]$. If $P \subset \mathbb{R}^d$ is a (convex) polytope, then we write $\mathcal{F}_k(P)$ for the set of its k-dimensional faces and $f_k(P)$ for the number of these k-faces, where $k \in \{0, \dots, d\}$. The k-dimensional Lebesgue measure in a k-dimensional flat $E \subset \mathbb{R}^d$ is denoted by λ_E. Subspaces are endowed with the induced scalar product and norm. Finally, we write $\Gamma(\cdot)$ for the Gamma function, especially $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.

An important tool in our investigations will be a new formula of Blaschke-Petkantschin type. The classical *affine Blaschke-Petkantschin formula* (see [22], II. 12. 3., [25], § 6.1) states that

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f(x_1, \dots, x_d)\, \prod_{j=1}^d d\lambda_d(x_j)$$

$$= \Gamma(d) \int_{\mathcal{H}_{d-1}^d} \int_H \cdots \int_H f(x_1, \dots, x_d)\, \lambda_H([x_1, \dots, x_d])\, \prod_{j=1}^d d\lambda_H(x_j)\, d\bar{\mu}(H)$$

for any nonnegative measurable function $f: (\mathbb{R}^d)^d \to \mathbb{R}$. Here $\bar{\mu}$ is the motion invariant Haar measure on the affine Grassmannian $\mathcal{H}^d_{d-1}$ of hyperplanes in $\mathbb{R}^d$ normalized such that the measure of all hyperplanes hitting the Euclidean unit ball is equal to $d\kappa_d$. Any hyperplane H with $0 \notin H$ can be parameterized (uniquely) by one of its unit normal vectors $u \in \mathbb{S}^{d-1}$ and its distance $t \geq 0$ to the origin such that $H = \{y \in \mathbb{R}^d : \langle y, u\rangle = t\}$. Then we have

$$\bar{\mu}(\cdot) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbf{1}\{tu + u^\perp \in \cdot\} dt\, d\sigma(u),$$

where $u^\perp$ denotes the $(d-1)$-dimensional subspace of $\mathbb{R}^d$ totally orthogonal to u.

The affine Blaschke-Petkantschin formula relates the d-dimensional volume elements $d\lambda_d(x_j)$ of points $x_1, \dots, x_d$ to the differential $d\bar{\mu}(H)$ of a hyperplane H and the $(d-1)$-dimensional volume elements $d\lambda_H(x_j)$ of points $x_j \in H$, $j = 1, \dots, d$. Intuitively speaking, instead of choosing d random points in $\mathbb{R}^d$, we first choose a random hyperplane and then, in a second step, we choose d random points in this hyperplane. More precisely, the corresponding transformation involves a Jacobian of the form $[x_1, \dots, x_d]$.

In this paper we need an analogous formula for two sets of points. The points $x_1, \dots, x_{2d-k}$ determine two hyperplanes H_1, H_2 which are the affine span of $x_1, \dots, x_d$ and $x_{d-k+1}, \dots, x_{2d-k}$, respectively. The following formula of Blaschke-Petkantschin type relates the d-dimensional volume elements $d\lambda_d(x_j)$, $j = 1, \dots, 2d-k$, to the differentials $d\bar{\mu}(H_1)$, $d\bar{\mu}(H_2)$ of the hyperplanes H_1, H_2, to the $(d-1)$-dimensional volume elements $d\lambda_{H_1}(x_j)$ and $d\lambda_{H_2}(x_l)$ of points x_j, $j = 1, \dots, d-k$ and x_l, $l = d+1, \dots, 2d-k$, which are contained in exactly one hyperplane, and to the $(d-2)$-dimensional volume elements $d\lambda_{H_1 \cap H_2}(x_j)$ of points x_j, $j = d-k+1, \dots, d$, which are contained in both hyperplanes. Again such a transformation involves a Jacobian which takes into account the angle between the hyperplanes. This angle is defined as the angle between the normal vectors of the hyperplanes. Since we only consider the sinus of this angle, the choice of the orientation of the normal vectors need not be specified.

Lemma 3.1. *Let $0 \leq k \leq d-1$, and let $g: (\mathbb{R}^d)^{2d-k} \to \mathbb{R}$ be a nonnegative measurable function. Then*

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} g(x_1, \dots, x_{2d-k}) \prod_{j=1}^{2d-k} d\lambda_d(x_j) \tag{3.1}$$

$$= \Gamma(d)^2 \int_{\mathcal{H}^d_{d-1}} \int_{\mathcal{H}^d_{d-1}} \int_{H_1} \cdots \int_{H_1} \int_{H_1 \cap H_2} \cdots \int_{H_1 \cap H_2} \int_{H_2} \cdots \int_{H_2} g(x_1, \dots, x_{2d-k})$$

$$\times \lambda_{H_1}([x_1, \dots, x_d]) \lambda_{H_2}([x_{d-k+1}, \dots, x_{2d-k}]) (\sin\varphi)^{-k}$$

$$\times \prod_{j=d+1}^{2d-k} d\lambda_{H_2}(x_j) \prod_{j=d-k+1}^{d} d\lambda_{H_1 \cap H_2}(x_j) \prod_{j=1}^{d-k} d\lambda_{H_1}(x_j)\, d\bar{\mu}(H_1)\, d\bar{\mu}(H_2),$$

where $\sin\varphi$ denotes the sinus of the angle between H_1 and H_2.

Proof. The Blaschke-Petkantschin formula applied to $x_1, \dots, x_d$ and Fubini's theorem show that

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} g(x_1, \dots, x_{2d-k}) \prod_{j=1}^{2d-k} d\lambda_d(x_j)$$
$$= \Gamma(d) \int_{\mathcal{H}^d_{d-1}} \int_{H_1} \cdots \int_{H_1} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} g(x_1, \dots, x_{2d-k}) \lambda_{H_1}([x_1, \dots, x_d])$$
$$\times \prod_{j=d+1}^{2d-k} d\lambda_d(x_j) \prod_{j=1}^{d} d\lambda_{H_1}(x_j)\, d\bar{\mu}(H_1).$$

We fix H_1 and set

$$\mathcal{I}(f) = \int_{H_1} \cdots \int_{H_1} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f(x_{d-k+1}, \dots, x_{2d-k}) \prod_{j=d+1}^{2d-k} d\lambda_d(x_j) \prod_{j=d-k+1}^{d} d\lambda_{H_1}(x_j)$$

for nonnegative measurable functions $f : (\mathbb{R}^d)^d \to \mathbb{R}$.

An essential ingredient of our proof is a special case of a generalized linear Blaschke-Petkantschin formula due to Vedel Jensen and Kiêu [14] (see also [26, Theorem 5.6, p. 135]). For all nonnegative measurable functions h and for a (fixed) hyperplane H, we thus have

$$\int_{H} \cdots \int_{H} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} h(y_1, \dots y_{d-1}) \prod_{j=k+1}^{d-1} d\lambda_d(y_j) \prod_{j=1}^{k} d\lambda_H(y_j)$$
$$= \Gamma(d) \int_{\mathcal{L}^d_{d-1}} \int_{H\cap L} \cdots \int_{H\cap L} \int_{L} \cdots \int_{L} h(y_1, \dots y_{d-1})$$
$$\times \lambda_L([0, y_1, \dots, y_{d-1}])\, (\sin\varphi)^{-k} \prod_{j=k+1}^{d-1} d\lambda_L(y_j) \prod_{j=1}^{k} d\lambda_{H\cap L}(y_j)\, d\bar{\nu}(L),$$

where φ denotes the angle between H and L, and $\bar{\nu}$ is the rotation invariant Haar measure on the Grassmannian $\mathcal{L}^d_{d-1}$ of $(d-1)$-dimensional linear subspaces in $\mathbb{R}^d$ with total measure $d\kappa_d/2$.

Using a standard argument (cf., e.g., Schneider and Weil [25], § 6.1) this immediately gives a generalized affine Blaschke-Petkantschin formula for $\mathcal{I}(f)$. Put $H = H_1 - x_{2d-k}$ and $x_{d-k+j} - x_{2d-k} = y_j$; then

$$\mathcal{I}(f) = \int_{\mathbb{R}^d} \int_{H} \cdots \int_{H} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f(y_1 + x_{2d-k}, \dots, y_{d-1} + x_{2d-k}, x_{2d-k})$$
$$\times \prod_{j=k+1}^{d-1} d\lambda_d(y_j) \prod_{j=1}^{k} d\lambda_H(y_j)\, d\lambda_d(x_{2d-k})$$
$$= \Gamma(d) \int_{\mathbb{R}^d} \int_{\mathcal{L}^d_{d-1}} \int_{H\cap L} \cdots \int_{H\cap L} \int_{L} \cdots \int_{L} f(y_1 + x_{2d-k}, \dots, y_{d-1} + x_{2d-k}, x_{2d-k})$$
$$\times \lambda_L([0, y_1, \dots, y_{d-1}])\, (\sin\varphi)^{-k}$$
$$\times \prod_{j=k+1}^{d-1} d\lambda_L(y_j) \prod_{j=1}^{k} d\lambda_{H\cap L}(y_j)\, d\bar{\nu}(L)\, d\lambda_d(x_{2d-k})$$

$$=\Gamma(d) \int\limits_{\mathcal{H}^d_{d-1}} \int\limits_{H_1\cap H_2} \cdots \int\limits_{H_1\cap H_2} \int\limits_{H_2} \cdots \int\limits_{H_2} f(x_{d-k+1}, \ldots, x_{2d-k})$$

$$\times \lambda_{H_2}([x_{d-k+1}, \ldots, x_{2d-k}]) (\sin\varphi)^{-k}$$

$$\times \prod_{j=d+1}^{2d-k} d\lambda_{H_2}(x_j) \prod_{j=d-k+1}^{d} d\lambda_{H_1\cap H_2}(x_j)\, d\bar{\mu}(H_2).$$

Setting $f(x_{d-k+1} \ldots, x_{2d-k}) = \Gamma(d) g(x_1, \ldots, x_{2d-k}) \lambda_{H_1}([x_1, \ldots, x_d])$ for fixed $x_1, \ldots x_{d-k}$, one can easily complete the proof of the lemma. □

4 Some auxiliary estimates

A random point X in $\mathbb{R}^d$ is said to be normally distributed with positive definite $d \times d$-covariance matrix Σ and mean 0, i.e., $X \sim N(0, \Sigma)$, if it is chosen according to the density

$$f_\Sigma(x) = \frac{1}{\sqrt{(2\pi)^d \det \Sigma}}\, e^{-\frac{1}{2}x^T\Sigma^{-1}x}, \qquad x \in \mathbb{R}^d.$$

For simplicity, we will exclusively consider the case $\Sigma = \frac{1}{2} I_d$. In this case we put $f_\Sigma(x) = \phi_d(x)$, or simply $f_\Sigma(x) = \phi(x)$ if $d = 1$. The one-dimensional normal distribution $N(0, \frac{1}{2})$ is given by

$$\Phi(z) := \int\limits_{-\infty}^{z} \phi(x)\, dx = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{z} e^{-x^2}\, dx, \qquad z \in \mathbb{R}.$$

The corresponding measures having these functions as densities with respect to the appropriate Lebesgue measure, will be denoted by $d\phi_d(x)$ instead of $\phi_d(x)dx$, etc.

We will repeatedly use the following well known asymptotic expansions concerning the density of the normal distribution:

Lemma 4.1. *Let $j \geq 0$ and $\gamma > 0$. Then, as $h_1 \to \infty$,*

$$\int\limits_{h_1}^{\infty} (h_2 - h_1)^j \phi(h_2)^\gamma\, dh_2 = \frac{\Gamma(j+1)}{(2\gamma)^{j+1}}\, h_1^{-(j+1)} \phi(h_1)^\gamma \left(1 + O\left(h_1^{-2}\right)\right).$$

Lemma 4.2. *For $n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$ and $\gamma > 0$,*

$$\int\limits_{1}^{\infty} \Phi(h_1)^{n-\alpha} h_1^\beta \phi(h_1)^\gamma\, dh_1 = \Gamma(\gamma) 2^{\gamma-1} n^{-\gamma} (\ln n)^{\frac{\beta+\gamma-1}{2}} (1 + o(1))$$

as $n \to \infty$.

The proof of Lemma 4.1 is immediate by the substitution $t = 2\gamma h_1(h_2 - h_1)$. The proof of Lemma 4.2 is a direct generalization of an argument given by Affentranger [1].

We provide two useful estimates which will be needed later.

Lemma 4.3. *Let $j, l \geq 0$ and $\gamma > 0$. Then there exists a constant $c > 0$ depending only on j, l, γ such that, for $h_1 \geq 1$,*

$$\int_{h_1}^{\infty}\int_{0}^{\pi}(h_2 - h_1)^j \phi(h_2)^\gamma \phi\left(\frac{h_1 \sin\varphi}{2}\right)(\sin\varphi)^l \, d\varphi\, dh_2 \leq c\, h_1^{-(j+l+2)}\phi(h_1)^\gamma.$$

Proof. Since $\sin$ is symmetric with respect to $\frac{\pi}{2}$, by Fubini's theorem and Lemma 4.1, we get

$$\begin{aligned}
&\int_{h_1}^{\infty}\int_{0}^{\pi}(h_2 - h_1)^j \phi(h_2)^\gamma \phi\left(\frac{h_1 \sin\varphi}{2}\right)(\sin\varphi)^l \, d\varphi\, dh_2 \\
&\leq 2\int_{h_1}^{\infty}(h_2 - h_1)^j \phi(h_2)^\gamma \, dh_2 \int_{0}^{\frac{\pi}{2}} \phi\left(\frac{h_1 \sin\varphi}{2}\right)(\sin\varphi)^l \, d\varphi \\
&\leq 2c_1 h_1^{-(j+1)}\phi(h_1)^\gamma \int_{0}^{\frac{\pi}{2}} \phi\left(\frac{h_1\varphi}{\pi}\right)\varphi^l \, d\varphi,
\end{aligned}$$

where $\frac{2}{\pi}\varphi \leq \sin\varphi \leq \varphi$ for $0 \leq \varphi \leq \frac{\pi}{2}$ was used in the last step. Here the constant c_1 depends only on j, γ. Substituting $h_1\varphi = t$, we obtain

$$\int_{0}^{\frac{\pi}{2}} \phi\left(\frac{h_1\varphi}{\pi}\right)\varphi^l \, d\varphi \leq h_1^{-(l+1)}\int_{0}^{\infty}\phi\left(\frac{t}{\pi}\right)t^l \, dt \leq c_2\, h_1^{-(l+1)},$$

where c_2 is a constant depending only on l. Thus the assertion follows. □

Lemma 4.4. *Let $j, l \geq 0$ and $\gamma > 0$. Then there exists a constant $c > 0$ depending only on j, l, γ such that, for $h_1 \geq 1$,*

$$\int_{h_1}^{\infty}\int_{0}^{\pi}(h_2 - h_1)^j \phi(h_2)^\gamma \phi\left(\frac{h_1 - h_2\cos\varphi}{2\sin\varphi}\right)(\sin\varphi)^{l-1} \, d\varphi\, dh_2 \leq c\, h_1^{-(j+l+1)}\phi(h_1)^\gamma.$$

Proof. We will use Fubini's theorem repeatedly and let $c_1, c_2, \dots$ denote constants depending only on j, l, γ. Then, first substituting (for fixed h_2)

$$u = \frac{h_1 - h_2\cos\varphi}{\sin\varphi}, \quad du = \frac{h_2 - h_1\cos\varphi}{\sin^2\varphi}\, d\varphi = \sqrt{u^2 + h_2^2 - h_1^2}\, \sin^{-1}\varphi\, d\varphi$$

with

$$\sin\varphi = \frac{1}{u^2 + h_2^2}\left(h_1 u + h_2\sqrt{u^2 + h_2^2 - h_1^2}\right),$$

and then $h_2 = h_1 + \frac{s^2}{2h_1}$, we get

$$\begin{aligned}
&\int_{h_1}^{\infty}\int_{0}^{\pi}(h_2 - h_1)^j \phi(h_2)^\gamma \phi\left(\frac{h_1 - h_2\cos\varphi}{2\sin\varphi}\right)(\sin\varphi)^{l-1} \, d\varphi\, dh_2 \\
&= \int_{h_1}^{\infty}\int_{-\infty}^{\infty}(h_2 - h_1)^j \phi(h_2)^\gamma \phi\left(\frac{u}{2}\right)\frac{\left(h_1 u + h_2\sqrt{u^2 + h_2^2 - h_1^2}\right)^l}{\sqrt{u^2 + h_2^2 - h_1^2}(u^2 + h_2^2)^l}\, du\, dh_2
\end{aligned}$$

$$
\begin{aligned}
&\le c_1\, h_1^{-j} \phi(h_1)^\gamma \int_{-\infty}^{\infty} \int_0^{\infty} \phi(s)^\gamma \phi\left(\frac{s^2}{2h_1}\right)^\gamma \phi\left(\frac{u}{2}\right) s^{2j} h_1^{-2l} \\
&\qquad\qquad \times \frac{\left(h_1 u + \left(h_1 + \frac{s^2}{2h_1}\right)\sqrt{u^2 + s^2 + \frac{s^4}{4h_1^2}}\right)^l}{\sqrt{u^2+s^2+\frac{s^4}{4h_1^2}}} \frac{s}{h_1}\, ds\, du \\
&\le c_2\, h_1^{-(j+l+1)} \phi(h_1)^\gamma \int_{-\infty}^{\infty} \int_0^{\infty} \phi(s)^\gamma \phi\left(\frac{u}{2}\right) s^{2j} \frac{s}{\sqrt{u^2+s^2+\frac{s^4}{4h_1^2}}} \\
&\qquad\qquad \times \left(|u| + \left(1 + \frac{s^2}{2h_1^2}\right)\sqrt{u^2+s^2+\frac{s^4}{4h_1^2}}\right)^l ds\, du \\
&\le c_2\, h_1^{-(j+l+1)} \phi(h_1)^\gamma \int_{-\infty}^{\infty} \int_0^{\infty} \phi(s)^\gamma \phi\left(\frac{u}{2}\right) \left[s^{2j}\left(|u| + (1+s^2)(|u|+s+s^2)\right)^l\right] ds\, du \\
&\le c\, h_1^{-(j+l+1)} \phi(h_1)^\gamma,
\end{aligned}
$$

since the last double integral is finite. □

5 Reduction of Theorem 1.1 to Theorem 1.5

The essential tool for estimating the variance of functionals of random polytopes is the Efron-Stein jackknife inequality [5], see also Efron [4] and Hall [7].

If $S = S(Y_1, \dots, Y_n)$ is any real symmetric function of the independent identically distributed random vectors Y_j, $1 \le j < \infty$, we set $S_i = S(Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_{n+1})$ and $S_{(.)} = \frac{1}{n+1}\sum_{i=1}^{n+1} S_i$. The Efron-Stein jackknife inequality then says

$$
\operatorname{Var} S \le \mathbb{E} \sum_{i=1}^{n+1} (S_i - S_{(.)})^2 = (n+1)\mathbb{E}\left(S_{n+1} - S_{(.)}\right)^2. \tag{5.1}
$$

Note that the right-hand side is not decreased if $S_{(.)}$ is replaced by any other function of $Y_1, \dots, Y_{n+1}$.

We apply this inequality to the random variable $f(P_n)$ where $f(\cdot)$ is a measurable function of convex polytopes. Then $S = f([X_1, \dots, X_n]) = f(P_n)$, and we replace $S_{(.)}$ by $f(P_{n+1})$ which is a function of the convex hull of P_n and a further random point X_{n+1}. The Efron-Stein jackknife inequality then yields that

$$
\operatorname{Var} f(P_n) \le (n+1)\mathbb{E}\left(f(P_n) - f(P_{n+1})\right)^2. \tag{5.2}
$$

In the case that $f(\cdot)$ is the number of i-faces of P_n, we obtain

$$
\operatorname{Var} f_i(P_n) \le (n+1)\mathbb{E}\left(f_i(P_{n+1}) - f_i(P_n)\right)^2.
$$

Let P_n be fixed and choose the additional random point X_{n+1}. If the point X_{n+1} is contained in P_n, the random variable $f_i(P_{n+1}) - f_i(P_n)$ equals 0. If $X_{n+1} \notin P_n$, the relative interior of some of the i-dimensional faces of P_n is contained in the interior of $[P_n, X_{n+1}]$, let $f_i^-(X_{n+1})$ be the number of theses faces, and some of the i-dimensional faces of $[P_n, X_{n+1}]$ are not contained in P_n, let $f_i^+(X_{n+1})$ be the number of those faces. Then we have

$$
|f_i([P_n, X_{n+1}]) - f_i(P_n)| = \left|f_i^+(X_{n+1}) - f_i^-(X_{n+1})\right| \le f_i^+(X_{n+1}) + f_i^-(X_{n+1}).
$$

Since P_n is simplicial with probability one, this number can easily be estimated in terms of the number $F_n(X_{n+1})$ of facets of P_n which can be seen from X_{n+1}. Here $F_n(X_{n+1}) = 0$ if X_{n+1} is contained in P_n, and if $X_{n+1} \notin P_n$ then $F_n(X_{n+1}) > 0$ is the number of facets of P_n which are – up to $(d-2)$-dimensional faces – contained in the interior of the convex hull of P_n and X_{n+1}. Now each i-dimensional "new" face of $[P_n, X_{n+1}]$ not contained in P_n is the convex hull of X_{n+1} and an $(i-1)$-dimensional face of P_n. Since this $(i-1)$-dimensional face is also a face of a facet of P_n which can be seen from X_{n+1}, and each facet is a simplex, we obtain

$$f_i^+(X_{n+1}) \leq \binom{d}{i} F_n(X_{n+1}) \,.$$

On the other hand each i-dimensional face of P_n which is – up to $(i-1)$-dimensional faces – contained in the interior of $[P_n, X_{n+1}]$ is also a face of a facet contained in the interior of $[P_n, X_{n+1}]$. Hence

$$f_i^-(X_{n+1}) \leq \binom{d}{i+1} F_n(X_{n+1})$$

and combining these estimates proves

$$\mathbb{E}(f_i(P_{n+1}) - f_i(P_n))^2 \leq \binom{d+1}{i+1}^2 \mathbb{E} F_n(X_{n+1})^2 \,. \tag{5.3}$$

Thus each estimate for the second moment of $F_n(X_{n+1})$ yields an estimate for $\operatorname{Var} f_i(P_n)$, and hence Theorem 1.1 follows from Theorem 1.5. □

6 Proof of Theorem 1.5

6.1 Asymptotic expansion of the expectation

We start with the proof of the first part of Theorem 1.5. The case $d = 1$ is included by proper interpretations of the subsequent arguments. Choose $n+1$ independent normally distributed random points $X_1, \dots X_n, X$ in $\mathbb{R}^d$. The convex hull of the first n points is a Gaussian polytope P_n, and with probability one P_n is simplicial. For $I \subset \{1, \dots, n\}$ with $|I| = d$, denote by F_I the convex hull of $\{X_i : i \in I\}$ which is a $(d-1)$-dimensional simplex. The affine hull of F_I is denoted by $H(F_I)$. With probability one, this affine hull is a hyperplane which dissects $\mathbb{R}^d$ into two (closed) halfspaces. The halfspace which contains the origin will be denoted by $H_0(F_I)$, the other by $H_+(F_I)$. The origin is contained in exactly one halfspace with probability one. In the following, we want to assume that P_n contains the origin. This happens with high probability, since by Wendel's theorem [28]

$$\mathbb{P}(0 \notin P_n) = O(n^d 2^{-n}),$$

an thus the condition that P_n contains the origin can be inserted by adding a suitable error term. Moreover, we can also assume that the points $X_1, \dots, X_n, X$ are in general relative position (i.e. any subset of at most $d+1$ of these random points is affinely independent).

We are interested in the number of facets of P_n which can be seen from the additional random point $X \notin P_n$. Denote the set of these facets by $\mathcal{F}_n(X)$, i.e.,

$$\begin{aligned} \mathcal{F}_n(X) &= \mathcal{F}(X_1, \dots, X_n;\, X) \\ &= \{F_I \,:\, P_n \subset H_0(F_I),\, X \in H_+(F_I),\, I \subset \{1, \dots, n\},\, |I| = d\} \,. \end{aligned}$$

Here we can define $\mathcal{F}_n(X)$ as the empty set, if the origin is not contained in the interior of P_n or if the random points are not in general relative position. Similar definitions can be given for deterministic points $x_1, \dots, x_n$ and x in general relative position such that the convex hull of $x_1, \dots, x_n$ contains the origin. When applying Wendel's theorem in considering $\mathbb{E}F_n(X)$, the error term is of the order $n^{2d}2^{-n} = O(c^{-n})$ with a suitable constant $c > 1$, since $F_n(X)$ is bounded by $\binom{n}{d}$. Using this we have

$$\mathbb{E}F_n(X) = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \sum \mathbf{1}\{F_I \in \mathcal{F}_n(x)\} \prod_{j=1}^{n} d\phi_d(x_j)\, d\phi_d(x) + O(c^{-n}), \tag{6.1}$$

where the summation extends over all subsets $I \subset \{1, \dots, n\}$ with $|I| = d$. Denote by F_1 the convex hull of $x_1, \dots, x_d$. Then

$$\mathbb{E}F_n(X) = \binom{n}{d} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\{F_1 \in \mathcal{F}_n(x)\} \prod_{j=1}^{n} d\phi_d(x_j)\, d\phi_d(x) + O(c^{-n}).$$

The probability content of the halfspace $H_+(F_1)$ is

$$\int_{H_+(F_1)} d\phi_d(x) = 1 - \Phi(h_1),$$

where h_1 is the distance of $H(F_1)$ to the origin. If $F_1 \in \mathcal{F}_n(X)$, then X is contained in the halfspace $H_+(F_1)$ with probability content $1-\Phi(h_1)$, and the random points X_j, $j \in \{d+1, \dots, n\}$, are contained in the halfspace $H_0(F_1)$ with probability content $\Phi(h_1)$. Hence we obtain

$$\mathbb{E}F_n(X) = \binom{n}{d} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \Phi(h_1)^{n-d} \left(1 - \Phi(h_1)\right) \prod_{j=1}^{d} d\phi_d(x_j) + O(c^{-n}).$$

Parameterizing the hyperplane $H(F_1) =: H_1$ by its distance $h_1 \geq 0$ from the origin and its unit normal vector u_1 in the form $H_1 = H(u_1, h_1)$, and using the affine Blaschke-Petkantschin formula, we find that

$$\begin{aligned}\mathbb{E}F_n(X) =& \Gamma(d)\binom{n}{d} \int_{\mathbb{S}^{d-1}} \int_0^{\infty} \Phi(h_1)^{n-d} \left(1 - \Phi(h_1)\right) \\ &\times \left\{ \int_{H_1} \cdots \int_{H_1} \lambda_{H_1}([x_1, \dots, x_d]) \prod_{j=1}^{d} \left(\phi_d(x_j)\, d\lambda_{H_1}(x_j)\right) \right\} dh_1\, d\sigma(u_1) + O(c^{-n}).\end{aligned}$$

The inner integral (in brackets) is the expected volume $\mathbb{E}V_{d-1}(P_d^{(d-1)})$ of a random $(d-1)$-dimensional Gaussian simplex in $\mathbb{R}^{d-1}$ times $\phi(h_1)^d$, which gives

$$\mathbb{E}F_n(X) = \Gamma(d)\binom{n}{d} \mathbb{E}V_{d-1}(P_d^{(d-1)}) \int_{\mathbb{S}^{d-1}} \int_0^{\infty} \Phi(h_1)^{n-d} \left(1 - \Phi(h_1)\right) \phi(h_1)^d\, dh_1\, d\sigma(u_1) + O(c^{-n}).$$

The expected volume of a random Gaussian simplex was computed explicitly by Miles [18]. It now follows from Lemma 4.1 and Lemma 4.2 that

$$\mathbb{E}F_n(X) = 2^{d-1}\kappa_d\, \Gamma(d+1) \mathbb{E}V_{d-1}(P_d^{(d-1)})\, n^{-1} (\ln n)^{\frac{d-1}{2}} (1 + o(1)). \tag{6.2}$$

This proves the first part of Theorem 1.5.

6.2 Estimate of the variance

The main part of the proof is devoted to estimating the second moment of $F_n(X)$. In the case $d = 1$, we have $F_n(X) = F_n(X)^2$, hence the assertion follows. Now let $d \geq 2$.

As in (6.1) we have

$$\mathbb{E}F_n(X)^2 = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \left(\sum_I \mathbf{1}\{F_I \in \mathcal{F}_n(x)\} \right)^2 \prod_{j=1}^n d\phi_d(x_j)\, d\phi_d(x) + O(c^{-n})$$

with some $c > 1$. The summation extends over all subsets $I \subset \{1, \ldots, n\}$ with $|I| = d$. We expand the integrand and get

$$\mathbb{E}F_n(X)^2 = \sum_I \sum_J \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\{F_I,\, F_J \in \mathcal{F}_n(x)\} \prod_{j=1}^n d\phi_d(x_j)\, d\phi_d(x) + O(c^{-n}), \tag{6.3}$$

where the summation extends over all subsets $I, J \subset \{1, \ldots, n\}$ with $|I| = |J| = d$. If we fix the number $k = |I \cap J| \in \{0, \ldots, d\}$, then the corresponding term in (6.3) depends only on k and not on the particular choice of I and J. For given $k \in \{0, \ldots, d\}$, we put $F_1 = [X_1, \ldots, X_d]$ and $F_2^{(k)} = [X_{d-k+1}, \ldots, X_{2d-k}]$. Note that for $k = d$ we have $F_1 = F_2^{(d)}$. Hence $\mathbb{E}F_n(X)^2$ can be rewritten as

$$\mathbb{E}F_n(X)^2 = \sum_{k=0}^d \binom{n}{d}\binom{d}{k}\binom{n-d}{d-k} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\{F_1,\, F_2^{(k)} \in \mathcal{F}_n(x)\} \prod_{j=1}^n d\phi_d(x_j)\, d\phi_d(x) + O(c^{-n}).$$

The summand corresponding to $k = d$ is just $\mathbb{E}F_n(X)$, and thus (6.2) yields

$$\begin{aligned}\mathbb{E}F_n(X)^2 \leq c_1 \sum_{k=0}^{d-1} n^{2d-k} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\{F_1,\, F_2^{(k)} \in \mathcal{F}_n(x)\} \prod_{j=1}^n d\phi_d(x_j)\, d\phi_d(x) \\ + O\left(n^{-1} (\ln n)^{\frac{d-1}{2}}\right);\end{aligned}$$

here and in the following $c_1, c_2, \ldots$ denote constants which are independent of n. The summand corresponding to $k = d$ also yields the asserted lower bound for $\mathbb{E}F_n(X)^2$.

Let h_1, h_2 be the distance to the origin and u_1, u_2 the unit normal vector of $H(F_1), H(F_2^{(k)})$, respectively, such that $H(F_1) = H(u_1, h_1)$ and $H(F_2^{(k)}) = H(u_2, h_2)$. Since the integrand is symmetric in F_1 and $F_2^{(k)}$, we restrict our integration to $h_1 \leq h_2$. Thus we get

$$\begin{aligned}\mathbb{E}F_n(X)^2 \leq c_2 \sum_{k=0}^{d-1} n^{2d-k} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\{h_1 \leq h_2\}\, \mathbf{1}\{F_1,\, F_2^{(k)} \in \mathcal{F}_n(x)\} \prod_{j=1}^n d\phi_d(x_j)\, d\phi_d(x) \\ + O\left(n^{-1} (\ln n)^{\frac{d-1}{2}}\right).\end{aligned}$$

If $F_1,\, F_2^{(k)} \in \mathcal{F}_n(x)$, then the points $x_{2d-k+1}, \ldots, x_n$ are contained in $H_0(F_1) \cap H_0(F_2^{(k)})$, and the corresponding measure of the set of these points is at most $\Phi(h_1)^{n-2d+k}$. Moreover, $F_1,\, F_2^{(k)} \in \mathcal{F}_n(x)$ implies that x is contained in $H_+(F_1) \cap H_+(F_2^{(k)})$. Denote the distance of $H_+(F_1) \cap H_+(F_2^{(k)})$ to the origin by h_{12}. Then the corresponding measure is at most $1 - \Phi(h_{12})$. This yields

$$\begin{aligned}\mathbb{E}F_n(X)^2 \leq c_2 \sum_{k=0}^{d-1} n^{2d-k} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\{h_1 \leq h_2\}\, \Phi(h_1)^{n-2d+k} \left(1 - \Phi(h_{12})\right) \prod_{j=1}^{2d-k} d\phi_d(x_j) \\ + O\left(n^{-1} (\ln n)^{\frac{d-1}{2}}\right).\end{aligned}$$

The range of integration can be reduced further to $\mathbf{1}\{1 \le h_1 \le h_2\}$, since the additional error term is of the order $n^{2d}\Phi(1)^{n-2d+k} = O(c^{-n})$ with a suitable $c > 1$. For $h_1 \ge 1$ and since $h_{12} \ge h_1$, Lemma 4.1 implies that $1 - \Phi(h_{12}) \le c_3 h_1^{-1}\phi(h_{12})$. Therefore we can conclude that

$$\mathbb{E}F_n(X)^2 \le c_4 \sum_{k=0}^{d-1} n^{2d-k} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\{1 \le h_1 \le h_2\}\, \Phi(h_1)^{n-2d+k}\, h_1^{-1}\phi(h_{12}) \prod_{j=1}^{2d-k} d\phi_d(x_j) + O\left(n^{-1}(\ln n)^{\frac{d-1}{2}}\right). \tag{6.4}$$

To develop an estimate for the integral

$$\mathcal{I}_k = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\{1 \le h_1 \le h_2\}\, \Phi(h_1)^{n-2d+k}\, h_1^{-1}\phi(h_{12}) \prod_{j=1}^{2d-k} d\phi_d(x_j)$$

in the cases where $0 \le k \le d-1$, we apply the Blaschke-Petkantschin formula (3.1) and obtain

$$\begin{aligned}\mathcal{I}_k =& \Gamma(d)^2 \int_{\mathbb{S}^{d-1}} \int_1^\infty \int_{\mathbb{S}^{d-1}} \int_{h_1}^\infty \Phi(h_1)^{n-2d+k}\, h_1^{-1}\phi(h_{12})\,(\sin\varphi)^{-k} \\ & \times \mathcal{J}_k(u_1,h_1,u_2,h_2)\, dh_2\, d\sigma(u_2)\, dh_1\, d\sigma(u_1)\end{aligned} \tag{6.5}$$

with

$$\mathcal{J}_k(u_1,h_1,u_2,h_2) = \left\{ \int_{H_1} \cdots \int_{H_1} \int_{H_1\cap H_2} \cdots \int_{H_1\cap H_2} \int_{H_2} \cdots \int_{H_2} \lambda_{H_1}(F_1)\lambda_{H_2}(F_2^{(k)}) \prod_{j=1}^{2d-k} \phi_d(x_j) \right.$$
$$\left. \times \prod_{j=d+1}^{2d-k} d\lambda_{H_2}(x_j) \prod_{j=d-k+1}^{d} d\lambda_{H_1\cap H_2}(x_j) \prod_{j=1}^{d-k} d\lambda_{H_1}(x_j) \right\}. \tag{6.6}$$

In the following, we distinguish the cases $k = 0$ and $k \in \{1, \dots, d-1\}$.

First, let $k \in \{1, \dots, d-1\}$. We denote by $h_{12}^{(d-2)}$ the distance of the origin to the $(d-2)$-dimensional intersection $H_1 \cap H_2$ and define an angle $\varphi^* \in (0, \pi/2)$ by $h_1 = h_2 \cos\varphi^*$. Then $h_{12} = h_2 \le h_{12}^{(d-2)}$ if $0 \le \varphi \le \varphi^*$, and $h_{12} = h_{12}^{(d-2)}$ if $\varphi \ge \varphi^*$. By v_1 and v_2 we denote unit vectors parallel to H_1 and H_2, respectively, orthogonal to $H_1 \cap H_2$. The $(d-1)$-volumes of the simplices F_1 and $F_2^{(k)}$ can be bounded from above by the $(d-2)$-volumes of their projections to $H_1 \cap H_2$, $\mathrm{proj}_{H_1\cap H_2} F_1$ and $\mathrm{proj}_{H_1\cap H_2} F_2^{(k)}$, and their heights in direction orthogonal to $H_1 \cap H_2$. Hence we get

$$\lambda_{H_1}(F_1) \le \lambda_{H_1\cap H_2}(\mathrm{proj}_{H_1\cap H_2} F_1) \left(\max_{j=1,\dots,d-k} |\langle x_j, v_1\rangle| + \sqrt{\left(h_{12}^{(d-2)}\right)^2 - h_1^2} \right),$$

where the height of F_1 was estimated by the distance of $H_1 \cap H_2$ to $h_1 u_1$, which is $\left(\left(h_{12}^{(d-2)}\right)^2 - h_1^2\right)^{1/2}$, and the maximal distance of the points x_j to the point $h_1 u_1$ in direction v_1. Analogously,

$$\lambda_{H_2}(F_2^{(k)}) \le \lambda_{H_1\cap H_2}(\mathrm{proj}_{H_1\cap H_2} F_2^{(k)}) \left(\max_{j=d+1,\dots,2d-k} |\langle x_j, v_2\rangle| + \sqrt{\left(h_{12}^{(d-2)}\right)^2 - h_2^2} \right).$$

Writing $x_j \in H_1$, $j = 1, \dots d-k$, as the orthogonal sum $x_j = h_1 u_1 + x_j^1 v_1 + y_j$, where y_j is contained in the $(d-2)$-dimensional linear subspace parallel to $H_1 \cap H_2$, which we identify with $\mathbb{R}^{d-2}$, we have $\langle x_j, v_1 \rangle = x_j^1$ and $\phi_d(x_j) = \phi(h_1)\phi(x_j^1)\phi_{d-2}(y)$. For the integration with respect x_j^1, we obtain

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \max_{j=1,\dots,d-k} |x_j^1| \prod_{j=1}^{d-k} (\phi_d(x_j)\, dx_j^1) \le c_5 \prod_{j=1}^{d-k} (\phi(h_1)\phi_{d-2}(y_j)),$$

and an analogous result holds for $x_j \in H_2$, $j = d+1, \dots, 2d-k$. Recall that $h_1 \le h_2$. This shows that

$$\begin{aligned} \mathcal{J}_k(u_1, h_1, u_2, h_2) \le &\left(c_5 + \sqrt{\left(h_{12}^{(d-2)}\right)^2 - h_1^2} \right)^2 \phi(h_1)^{d-k} \phi(h_2)^{d-k} \phi(h_{12}^{(d-2)})^k \\ & \times \left\{ \int_{\mathbb{R}^{d-2}} \cdots \int_{\mathbb{R}^{d-2}} \lambda_{d-2}([y_1, \dots, y_d]) \lambda_{d-2}([y_{d-k+1}, \dots y_{2d-k}]) \right. \\ & \left. \prod_{j=1}^{2d-k} (\phi_{d-2}(y_j)\, d\lambda_{d-2}(y_j)) \right\}. \end{aligned} \tag{6.7}$$

Using that $(a+b)^2 \le 2(a^2+b^2)$ and $(1+x/2)e^{-x/2} \le 1$, for $a, b \in \mathbb{R}$ and $x \ge 0$, we deduce that

$$\mathcal{J}_k(u_1, h_1, u_2, h_2) \le c_6\, \phi(h_1)^{d-k+\frac{1}{2}} \phi(h_2)^{d-k} \phi\left(h_{12}^{(d-2)}\right)^{k-\frac{1}{2}}, \tag{6.8}$$

and thus

$$\begin{aligned} \mathcal{I}_k \le c_7 \int_{\mathbb{S}^{d-1}} \int_1^\infty \int_{\mathbb{S}^{d-1}} \int_{h_1}^\infty & \Phi(h_1)^{n-2d+k} h_1^{-1} \phi(h_1)^{d-k+\frac{1}{2}} \phi(h_2)^{d-k} \\ & \times \phi(h_{12}) \phi\left(h_{12}^{(d-2)}\right)^{k-\frac{1}{2}} (\sin\varphi)^{-k}\, dh_2\, d\sigma(u_2)\, dh_1\, d\sigma(u_1). \end{aligned}$$

Since the integrand is rotation invariant, we may assume that $u_1 = e_d$, and hence arrive at

$$\begin{aligned} \mathcal{I}_k \le c_8 \int_1^\infty \int_{\mathbb{S}^{d-1}} \int_{h_1}^\infty & \Phi(h_1)^{n-2d+k} h_1^{-1} \phi(h_1)^{d-k+\frac{1}{2}} \phi(h_2)^{d-k} \\ & \times \phi(h_{12}) \phi\left(h_{12}^{(d-2)}\right)^{k-\frac{1}{2}} (\sin\varphi)^{-k}\, dh_2\, d\sigma(u_2)\, dh_1. \end{aligned} \tag{6.9}$$

Elementary calculations show that

$$\left(h_{12}^{(d-2)}\right)^2 = h_1^2 + \frac{(h_2 - h_1\cos\varphi)^2}{(\sin\varphi)^2} = h_2^2 + \frac{(h_1 - h_2\cos\varphi)^2}{(\sin\varphi)^2}, \qquad \varphi \in (0, \pi); \tag{6.10}$$

moreover,

$$\left(h_{12}^{(d-2)}\right)^2 \ge h_{12}^2 = h_2^2 \ge h_1^2 \left(1 + \frac{(\sin\varphi)^2}{4}\right), \qquad \varphi \in (0, \varphi^*], \tag{6.11}$$

and

$$h_{12}^2 = \left(h_{12}^{(d-2)}\right)^2 \ge h_1^2 \left(1 + \frac{(\sin\varphi)^2}{4}\right), \qquad \varphi \in [\varphi^*, \pi). \tag{6.12}$$

We parameterize $u_2 \in \mathbb{S}^{d-1} \setminus \{e_d, -e_d\}$ by the angle φ enclosed by u_2 and e_d and by its normalized projection v to $\mathbb{R}^{d-1}$, i.e. $v \in \mathbb{S}^{d-1} \cap e_d^\perp$ and $\varphi \in (0, \pi)$. Let σ_{d-2} denote the spherical Lebesgue measure on $\mathbb{S}^{d-2}$.

Then we can estimate the inner double integral in (6.9) by using that $k - \frac{1}{2} \geq \frac{1}{4}$ and Lemma 4.4

$$\begin{aligned}
&\int_{h_1}^{\infty} \int_{\mathbb{S}^{d-1}} \phi(h_2)^{d-k} \phi(h_{12}) \phi\left(h_{12}^{(d-2)}\right)^{k-\frac{1}{2}} (\sin\varphi)^{-k} \, d\sigma(u_2) \, dh_2 \\
&\leq c_9 \, \phi(h_1)^{k+\frac{1}{2}} \int_{h_1}^{\infty} \int_{S^{d-2}} \int_0^{\pi} \phi(h_2)^{d-k} \phi\left(\frac{h_1 - h_2 \cos\varphi}{2 \sin\varphi}\right) (\sin\varphi)^{d-k-2} \, d\varphi \, \sigma_{d-2}(v) \, dh_2 \\
&\leq c_{10} \, \phi(h_1)^{k+\frac{1}{2}} \int_{h_1}^{\infty} \int_0^{\pi} \phi(h_2)^{d-k} \phi\left(\frac{h_1 - h_2 \cos\varphi}{2 \sin\varphi}\right) (\sin\varphi)^{d-k-2} \, d\varphi \, dh_2 \\
&\leq c_{11} \, h_1^{-(d-k)} \phi(h_1)^{d+\frac{1}{2}}. \qquad (6.13)
\end{aligned}$$

Then, for $k \in \{1, \ldots, d-1\}$, we deduce from (6.9) and (6.13) that

$$\mathcal{I}_k \leq c_{12} \int_1^{\infty} \Phi(h_1)^{n-2d+k} h_1^{-d+k-1} \phi(h_1)^{2d+1-k} \, dh_1 \leq c_{13} \, n^{-(2d-k+1)} \, (\ln n)^{\frac{d-1}{2}}. \qquad (6.14)$$

Finally, we consider the case $k = 0$. Using the previous notation and (6.11) and (6.12), we get directly from (6.5) and (6.6)

$$\begin{aligned}
\mathcal{I}_0 &\leq c_{14} \int_{\mathbb{S}^{d-1}} \int_1^{\infty} \int_{\mathbb{S}^{d-1}} \int_{h_1}^{\infty} \Phi(h_1)^{n-2d} {h_1}^{-1} \phi(h_{12}) \phi(h_1)^d \phi(h_2)^d \, dh_2 \, d\sigma(u_2) \, dh_1 \, d\sigma(u_1) \\
&\leq c_{15} \int_1^{\infty} \Phi(h_1)^{n-2d} {h_1}^{-1} \phi(h_1)^{d+1} \int_{h_1}^{\infty} \int_0^{\pi} \phi(h_2)^d \phi\left(\frac{h_1 \sin\varphi}{2}\right) (\sin\varphi)^{d-2} \, d\varphi.
\end{aligned}$$

By Lemma 4.3, we obtain

$$\mathcal{I}_0 \leq \; c_{16} \int_1^{\infty} \Phi(h_1)^{n-2d} {h_1}^{-(d+1)} \phi(h_1)^{2d+1} \, dh_1 \leq \; c_{17} \, n^{-(2d+1)} \, (\ln n)^{\frac{d-1}{2}}. \qquad (6.15)$$

Combining the estimates (6.4), (6.14) and (6.15), we arrive at

$$\mathbb{E} F_n(X)^2 \leq c_{18} \sum_{k=0}^{d} n^{2d-k} \, n^{-2d+k-1} (\ln n)^{\frac{d-1}{2}} \leq c_{19} \, n^{-1} (\ln n)^{\frac{d-1}{2}}.$$

This completes the proof of Theorem 1.5. □

7 Proof of Theorem 1.3

The proof is based on arguments similar to those involved in the proofs of Theorems 1.1 and 1.5. Therefore we use the same notation as before. In particular, $c_1, c_2, \ldots$ denote constants which merely depend on the dimension.

Denote by ν_i the Haar probability measure on the set $\mathcal{L}_i^d$ of i-dimensional linear subspaces in $\mathbb{R}^d$. Kubota's theorem and the rotation invariance of the normal distribution immediately give

$$\mathbb{E}V_i(P_n) = \mathbb{E}\left(c_{d,i}\int_{\mathcal{L}_i^d} V_i(\mathrm{proj}_L P_n)\, d\nu_i(L)\right) = c_{d,i}\mathbb{E}V_i(\mathrm{proj}_{L_0} P_n)$$

with $c_{d,i} = \binom{d}{i}\kappa_d / (\kappa_i \kappa_{d-i})$, for an arbitrary i-dimensional linear subspace L_0 which we identify with $\mathbb{R}^i$. The projection to $\mathbb{R}^i$ of a Gaussian sample in $\mathbb{R}^d$ again is a Gaussian sample. Hence

$$\mathbb{E}V_i(P_n) = c_{d,i}\mathbb{E}V_i(P_n^{(i)}),$$

where $P_n^{(i)}$ is the convex hull of a Gaussian sample in $\mathbb{R}^i$. For the variance we obtain by Hölder's inequality

$$\begin{aligned}\mathrm{Var}\, V_i(P_n) &= \mathbb{E}(V_i(P_n) - \mathbb{E}V_i(P_n))^2 \\ &= \mathbb{E}\left(c_{d,i}\int_{\mathcal{L}_i^d}\left(V_i(\mathrm{proj}_L P_n) - \mathbb{E}V_i(P_n^{(i)})\right)\, d\nu_i(L)\right)^2 \\ &\le c_{d,i}^2\mathbb{E}\left(\int_{\mathcal{L}_i^d}\left(V_i(\mathrm{proj}_L P_n) - \mathbb{E}V_i(P_n^{(i)})\right)^2\, d\nu_i(L)\right).\end{aligned}$$

Thus Fubini's theorem and the rotation invariance imply

$$\mathrm{Var}\, V_i(P_n) \le c_{d,i}^2\mathbb{E}\left(V_i(P_n^{(i)}) - \mathbb{E}V_i(P_n^{(i)})\right)^2 = c_{d,i}^2 \mathrm{Var}\, V_i(P_n^{(i)}). \tag{7.1}$$

Therefore by the Efron-Stein jackknife inequality (5.2) it suffices to prove

$$\mathrm{Var}\, V_d(P_n) \le (n+1)\mathbb{E}\left(V_d(P_{n+1}) - V_d(P_n)\right)^2 \le c_1(\ln n)^{\frac{d-3}{2}} \tag{7.2}$$

(i.e., the case $i = d$), which then yields the general result.

We will assume that P_n contains the origin, which leads to an error term of the order $\sqrt{n^{5d}2^{-n}} = O(c^{-n})$ with a suitable constant $c > 1$. To see this, one can use Cauchy's inequality

$$\mathbb{E}\,\left((V_d(P_{n+1}) - V_d(P_n))^2\mathbf{1}\{0 \notin P_n\}\right) \le \sqrt{\mathbb{E}\,\left((V_d(P_{n+1}) - V_d(P_n))^4\right)\mathbb{P}(0 \notin P_n)}$$

and Hölder's inequality

$$\mathbb{E}\left(V_d(P_{n+1}) - V_d(P_n)\right)^4 \le \mathbb{E}\, V_d(P_{n+1})^4 \le \mathbb{E}\left(\sum_I V_d([0, F_I])\right)^4 \le \binom{n+1}{d}^4 \mathbb{E}V_d([0, F_1])^4;$$

here the summation extends over all sets $I \subset \{1, \dots, n+1\}$ with $|I| = d$. Hence we obtain

$$\begin{aligned}\mathbb{E}\left(V_d(P_{n+1}) - V_d(P_n)\right)^2 &= \mathbb{E}\left(\sum_I \mathbf{1}\{F_I \in \mathcal{F}_n(X)\}\, V_d([F_I, X])\right)^2 + O(c^{-n}) \\ &= 2\sum_{k=0}^d \binom{n}{d}\binom{d}{k}\binom{n-d}{d-k}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} \mathbf{1}\{1 \le h_1 \le h_2\} \\ &\qquad \times \mathbf{1}\{F_1, F_2^{(k)} \in \mathcal{F}_n(x)\} V_d([F_1, x]) V_d([F_2^{(k)}, x]) \\ &\qquad \times \prod_{j=1}^n d\phi_d(x_j)\, d\phi_d(x) + O(c^{-n})\end{aligned}$$

$$\leq c_2 \sum_{k=0}^{d} n^{2d-k} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\{1 \leq h_1 \leq h_2\} \, \mathbf{1}\{F_1, F_2^{(k)} \in \mathcal{F}_n(x)\}$$
$$\times V_d([F_1, x]) V_d([F_2^{(k)}, x]) \prod_{j=1}^{n} d\phi_d(x_j) \, d\phi_d(x) + O(c^{-n}).$$

For the integration with respect to x, observe that $V_d([F_1, x])$ equals $\frac{1}{d}\lambda_{H_1}(F_1)$ – which is independent of x – times the distance $\langle x, u_1\rangle - h_1$ of x to $H_1 = H(F_1)$, and a similar assertion holds for $F_2^{(k)}$.

Let us first consider the summand corresponding to $k = d$. In this case, we can estimate

$$\begin{aligned}
& n^d \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\{1 \leq h_1\} \, \mathbf{1}\{F_1 \in \mathcal{F}_n(x)\} V_d([F_1, x])^2 \prod_{j=1}^{n} d\phi_d(x_j) \, d\phi_d(x) \\
\leq \; & n^d \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \Phi(h_1)^{n-d} \mathbf{1}\{1 \leq h_1\} \\
& \quad \times \int_{\mathbb{R}^d} \mathbf{1}\{x \in H^+(F_1)\} \left(\langle x, u_1\rangle - h_1\right)^2 d\phi_d(x) \lambda_{H_1}(F_1)^2 \prod_{j=1}^{d} d\phi_d(x_j) \\
= \; & n^d \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \Phi(h_1)^{n-d} \mathbf{1}\{1 \leq h_1\} \int_{h_1}^{\infty} (h - h_1)^2 \phi(h) \, dh \, \lambda_{H_1}(F_1)^2 \prod_{j=1}^{d} d\phi_d(x_j) \\
\leq \; & c_3 \, n^d \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \Phi(h_1)^{n-d} \mathbf{1}\{1 \leq h_1\} h_1^{-3} \phi(h_1) \lambda_{H_1}(F_1)^2 \prod_{j=1}^{d} d\phi_d(x_j) \\
\leq \; & c_4 \, n^d \int_1^{\infty} \Phi(h_1)^{n-d} h_1^{-3} \phi(h_1)^{d+1} \, dh_1 \\
\leq \; & c_5 \, n^{-1} \left(\ln n\right)^{\frac{d-3}{2}} .
\end{aligned}$$

where the affine Blaschke-Petkantschin formula and Lemma 4.2 were applied in the fourth step.

If $d = 1$, then only the case $k = 1$ can occur, and the proof is finished at this point. Subsequently, we consider the case $d \geq 2$. In order to estimate the summands corresponding to $0 \leq k \leq d-1$, we need some preparations. Denote again by φ the angle between u_1 and u_2, by h_{12} the distance of $H_+(F_1) \cap H_+(F_2^{(k)})$ to the origin, and by u_{12} the unit vector pointing from the origin to the nearest point of $H_+(F_1) \cap H_+(F_2^{(k)})$. Then $H_+(F_1) \cap H_+(F_2^{(k)})$ is contained in $\{y \in \mathbb{R}^d : \langle y, u_{12}\rangle \geq h_{12}\}$. We use the parameterization $x = hu_{12} + z$ with $z \in u_{12}^{\perp}$, where $h \geq h_{12}$, and have

$$\langle x, u_i\rangle - h_i = h\langle u_{12}, u_i\rangle + \langle z, u_i\rangle - h_i \leq h - h_{12} + \|z\| \sin\varphi + h_{12}\langle u_{12}, u_i\rangle - h_i$$

for $\varphi \leq \frac{\pi}{2}$, since the angle between u_i and u_{12} is bounded by the angle between u_1 and u_2. If $\varphi^* \leq \varphi \leq \frac{\pi}{2}$, then $h_{12} = h_{12}^{(d-2)}$ and $h_{12}u_{12} \in H(F_1) \cap H(F_2^{(k)})$, hence $h_{12}\langle u_{12}, u_i\rangle = h_i$ for $i = 1, 2$. If $0 < \varphi < \varphi^*$, then $h_{12}u_{12} \in H(F_2^{(k)})$, $h_{12} = h_2$, $u_{12} = u_2$, and $h_{12}\langle u_{12}, u_1\rangle - h_1 \leq h_2 - h_1$. Hence, in any case we have

$$\int_{\mathbb{R}^d} \mathbf{1}\left\{x \in H_+(F_1) \cap H_+(F_2^{(k)})\right\} \left(\langle x, u_1\rangle - h_1\right)\left(\langle x, u_2\rangle - h_2\right) d\phi_d(x)$$

$$\le \int_{h_{12}}^{\infty} \int_{u_{12}^{\perp}} (h - h_{12} + \|z\| \sin\varphi + h_2 - h_1)(h - h_{12} + \|z\| \sin\varphi)\, d\phi_{d-1}(z)\, d\phi(h)$$
$$\le c_6 \left(h_{12}^{-3} + h_{12}^{-2}(\sin\varphi + (h_2 - h_1)) + h_{12}^{-1}(\sin^2\varphi + (h_2 - h_1)\sin\varphi) \right) \phi(h_{12})$$
$$=: S_1(h_1, h_2, \varphi)$$

for $\varphi \le \frac{\pi}{2}$, where we used Lemma 4.1. For $\varphi \ge \frac{\pi}{2}$, (6.10) implies that $h_{12}^{(d-2)} = h_{12} \ge \sqrt{2} h_1$. Thus, for $\varphi \ge \pi/2$ we obtain

$$\int_{\mathbb{R}^d} \mathbf{1}\left\{ x \in H_+(F_1) \cap H_+(F_2^{(k)}) \right\} (\langle x, u_1 \rangle - h_1)(\langle x, u_2 \rangle - h_2)\, d\phi_d(x)$$
$$\le \int_{h_{12}}^{\infty} (h - h_1)^2 \phi(h)\, dh \le 3\phi(h_{12}) + \frac{(h_{12} - h_1)^2}{h_{12}} \phi(h_{12})$$
$$\le c_7 \phi(\eta h_1) =: S_2(h_1, h_2, \varphi),$$

where $\eta := (1 + \sqrt{2})/2$. In addition, we define $S_1(h_1, h_2, \varphi) = 0$ for $\varphi > \pi/2$ and $S_2(h_1, h_2, \varphi) = 0$ for $\varphi < \pi/2$, and then we put $S(h_1, h_2, \varphi) := S_1(h_1, h_2, \varphi) + S_2(h_1, h_2, \varphi)$.

The probability that the random points $X_{2d-k+1}, \dots, X_n$ are contained in $H_0(F_1) \cap H_0(F_2^{(k)})$ can be bounded from above by $\Phi(h_1)^{n-2d+k}$. Hence we get

$$\mathbb{E}\left(V_d(P_{n+1}) - V_d(P_n)\right)^2 \le c_8 \sum_{k=0}^{d-1} n^{2d-k} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\{1 \le h_1 \le h_2\} \Phi(h_1)^{n-2d+k} S(h_1, h_2, \varphi)$$
$$\times \lambda_{H_1}(F_1) \lambda_{H_2}(F_2^{(k)}) \prod_{j=1}^{2d-k} d\phi_d(x_j) + O\left(n^{-1} (\ln n)^{\frac{d-3}{2}} \right). \qquad (7.3)$$

For $k \in \{0, \dots, d-1\}$, we apply the Blaschke-Petkantschin formula (3.1) to the integral

$$\mathcal{I}_k = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbf{1}\{1 \le h_1 \le h_2\} \Phi(h_1)^{n-2d+k} S(h_1, h_2, \varphi)\, \lambda_{H_1}(F_1) \lambda_{H_2}(F_2^{(k)}) \prod_{j=1}^{2d-k} d\phi_d(x_j). \qquad (7.4)$$

Thus we obtain

$$\mathcal{I}_k = \Gamma(d)^2 \int_{\mathbb{S}^{d-1}} \int_1^{\infty} \int_{\mathbb{S}^{d-1}} \int_{h_1}^{\infty} \Phi(h_1)^{n-2d+k} S(h_1, h_2, \varphi) (\sin\varphi)^{-k}$$
$$\times \mathcal{J}_k(u_1, h_1, u_2, h_2)\, dh_2\, d\sigma(u_2)\, dh_1\, d\sigma(u_1)$$

with

$$\mathcal{J}_k(u_1, h_1, u_2, h_2) = \left\{ \int_{H_1} \cdots \int_{H_1} \int_{H_1 \cap H_2} \cdots \int_{H_1 \cap H_2} \int_{H_2} \cdots \int_{H_2} \lambda_{H_1}(F_1)^2 \lambda_{H_2}(F_2^{(k)})^2 \prod_{j=1}^{2d-k} \phi_d(x_j) \right.$$
$$\left. \times \prod_{j=d+1}^{2d-k} d\lambda_{H_2}(x_j) \prod_{j=d-k+1}^{d} d\lambda_{H_1 \cap H_2}(x_j) \prod_{j=1}^{d-k} d\lambda_{H_1}(x_j) \right\}.$$

The two cases $k = 0$ and $k \in \{1, \dots, d-1\}$ will be treated separately. We decompose $\mathcal{I}_k$ in the form $\mathcal{I}_k = \mathcal{I}_k^1 + \mathcal{I}_k^2$ corresponding to the decomposition $S = S_1 + S_2$.

We start with the case $k = 0$ and obtain

$$\begin{aligned}
\mathcal{I}_0^1 = c_9 \int\limits_{\mathbb{S}^{d-1}} \int\limits_1^\infty \int\limits_{\mathbb{S}^{d-1}} \int\limits_{h_1}^\infty \Phi(h_1)^{n-2d}\, S_1(h_1,h_2,\varphi) \int\limits_{H_1} \cdots \int\limits_{H_1} \lambda_{H_1}(F_1)^2 \prod_{j=1}^d \phi_d(x_j) \prod_{j=1}^d d\lambda_{H_1}(x_j) \\
\times \int\limits_{H_2} \cdots \int\limits_{H_2} \lambda_{H_2}(F_2^{(0)})^2 \prod_{j=d+1}^{2d} \phi_d(x_j) \prod_{j=d+1}^{2d} d\lambda_{H_2}(x_j) dh_2\, d\sigma(u_2)\, dh_1\, d\sigma(u_1) \\
\le c_{10} \int\limits_{\mathbb{S}^{d-1}} \int\limits_1^\infty \int\limits_{\mathbb{S}^{d-1}} \int\limits_{h_1}^\infty \Phi(h_1)^{n-2d}\, S_1(h_1,h_2,\varphi)\phi(h_1)^d \phi(h_2)^d\, dh_2\, d\sigma(u_2)\, dh_1\, d\sigma(u_1) \\
\le c_{11} \int\limits_1^\infty \Phi(h_1)^{n-2d} \phi(h_1)^d \int\limits_0^{\frac{\pi}{2}} \int\limits_{h_1}^\infty S_1(h_1,h_2,\varphi)(\sin\varphi)^{d-2} \phi(h_2)^d\, dh_2\, d\varphi\, dh_1.
\end{aligned}$$

Now we have to consider the five different summands into which $S_1(h_1, h_2, \varphi)$ naturally decomposes and estimate the corresponding integrals. Using that $h_{12} \ge h_1$ and applying repeatedly Lemma 4.3, we thus obtain

$$\int\limits_{h_1}^\infty \int\limits_0^{\frac{\pi}{2}} S_1(h_1,h_2,\varphi)\,(\sin\varphi)^{d-2}\,\phi(h_2)^d\, dh_2\, d\varphi \le c_{12}\, h_1^{-(d+3)} \phi(h_1)^{d+1}.$$

Hence we arrive at

$$\mathcal{I}_0^1 \le c_{13} \int_1^\infty \Phi(h_1)^{n-2d} h_1^{-(d+3)} \phi(h_1)^{2d+1}\, dh_1 \le c_{14}\, n^{-(2d+1)}\, (\ln n)^{\frac{d-3}{2}}\,.$$

Moreover,

$$\begin{aligned}
\mathcal{I}_0^2 &\le c_{15} \int\limits_{\mathbb{S}^{d-1}} \int\limits_1^\infty \int\limits_{\mathbb{S}^{d-1}} \int\limits_{h_1}^\infty \Phi(h_1)^{n-2d}\, S_2(h_1,h_2,\varphi)\phi(h_1)^d \phi(h_2)^d\, dh_2\, d\sigma(u_2)\, dh_1\, d\sigma(u_1) \\
&\le c_{16} \int\limits_1^\infty \Phi(h_1)^{n-2d} \phi(h_1)^d \int\limits_{\frac{\pi}{2}}^{\pi} \int\limits_{h_1}^\infty \phi(\eta h_1)(\sin\varphi)^{d-2} \phi(h_2)^d\, dh_2\, d\varphi\, dh_1 \\
&\le c_{17} \int\limits_1^\infty \Phi(h_1)^{n-2d} h_1^{-1} \phi(h_1)^{2d+\eta^2}\, dh_1 \\
&\le c_{18}\, n^{-(2d+\eta^2)}\, (\ln n)^{\frac{2d+\eta^2-1-1}{2}} \\
&\le c_{19}\, n^{-(2d+1)}\, (\ln n)^{\frac{d-3}{2}}\,,
\end{aligned}$$

as $1 < \eta^2 < 2$. Thus we have

$$\mathcal{I}_0 \le c_{20}\, n^{-(2d+1)}\, (\ln n)^{\frac{d-3}{2}}\,. \tag{7.5}$$

Next we consider the cases $k \in \{1, \dots, d-1\}$. First, we obtain

$$\mathcal{J}_k(u_1,h_1,u_2,h_2) \le c_{21} \phi(h_1)^{d-k+\frac{1}{2}} \phi(h_2)^{d-k} \phi(h_{12}^{(d-2)})^{k-\frac{1}{2}}$$

by arguing in a similar way as for proving (6.8). Thus we have to bound from above

$$\int_1^\infty \int_{\mathbb{S}^{d-1}} \int_{h_1}^\infty \Phi(h_1)^{n-2d+k} S(h_1,h_2,\varphi)(\sin\varphi)^{-k} \phi(h_1)^{d-k+\frac{1}{2}} \phi(h_2)^{d-k} \phi(h_{12}^{(d-2)})^{k-\frac{1}{2}}\, dh_2\, d\sigma(u_2)\, dh_1,$$

where φ is the angle between e_1 and u_2. Parameterizing the unit sphere as before, we see that this integral can be estimated from above by

$$\int_1^\infty \int_{h_1}^\infty \int_0^\infty \Phi(h_1)^{n-2d+k} \phi(h_1)^{d-k+\frac{1}{2}} S(h_1,h_2,\varphi) \phi(h_2)^{d-k} \phi(h_{12}^{(d-2)})^{k-\frac{1}{2}} (\sin\varphi)^{d-k-2}\, d\varphi\, dh_2\, dh_1$$

up to a constant multiplier. Using (6.10), Lemma 4.4 and $h_{12} \geq h_1$, $h_2 \geq h_1$, $k - \frac{1}{2} \geq \frac{1}{4}$, this multiple integral can be estimated from above by

$$\begin{aligned} &\int_1^\infty \Phi(h_1)^{n-2d+k} \phi(h_1)^{d-k+\frac{1}{2}} \phi(h_1)^{k-\frac{1}{2}} \phi(h_1) \\ &\qquad \times \int_{h_1}^\infty \int_0^\infty \phi(h_2)^{d-k} S(h_1,h_2,\varphi) \phi(-h_{12}) \phi\left(\frac{h_1 - h_2\cos\varphi}{2\sin\varphi}\right) (\sin\varphi)^{d-k-2}\, d\varphi\, dh_2\, dh_1 \\ &\leq c_{22} \int_1^\infty \Phi(h_1)^{n-2d+k} \phi(h_1)^{d+1} h_1^{-(d-k+3)} \phi(h_1)^{d-k}\, dh_1 \\ &\leq c_{23}\, n^{-2d+k-1} (\ln n)^{\frac{d-3}{2}} . \end{aligned} \tag{7.6}$$

Thus, combining (7.3), (7.4), (7.5) and (7.6), we finally obtain

$$\mathbb{E}\left(V_d(P_{n+1} - V_d(P_n)\right)^2 \leq c_{24}\, n^{-1} (\ln n)^{\frac{d-3}{2}} ,$$

which completes the proof. □

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