Random Mosaics

Daniel Hug

Mathematisches Institut, Albert-Ludwigs-Universität
Eckerstr. 1, D-79104 Freiburg, Germany
daniel.hug@math.uni-freiburg.de

Mosaics arise frequently as models in materials science, biology, geology, telecommunications, and in data and point pattern analysis. Very often the data from which a mosaic is constructed is random, hence we arrive at a random mosaic. Such a random mosaic can be considered as a special particle process or as a random closed set having a special structure. Random mosaics provide a convenient and important model for illustrating some of the general concepts discussed in previous parts of these lecture notes. In particular, the investigation of random mosaics naturally requires a combination of probabilistic, geometric and combinatorial ideas.

In this brief survey of selected results and methods related to random mosaics, we will mainly concentrate on classical topics which are contained in the monographs [9], [12] and [14]. A survey with a different emphasis is provided in [10]. Some of the more recent developments related to modelling of communication networks, iterated constructions of random mosaics and related central limit theorems cannot be discussed here. But there is also interesting recent research which is concerned with obtaining quantitative information about various shape characteristics of random mosaics. In the final part of this appendix, we will report on work carried out in this direction.

1 General Results

In this section, we collect general information which is available for arbitrary mosaics. Later we will concentrate on results for special types of mosaics such as hyperplane, Voronoi and Delaunay mosaics.

1.1 Basic Notions

In a purely deterministic setting, a mosaic can be defined in a rather general way. For instance, we may request a mosaic (tessellation) in $\mathbb{R}^d$ to be a system of closed sets which cover $\mathbb{R}^d$ and have no common interior points. It turns out, however, that for many applications this definition is unnecessarily general. Therefore we use the following more restricted notion of mosaic which is sufficient for our present introductory purposes. Thus we define a **mosaic** in $\mathbb{R}^d$ as a locally finite system of compact convex sets (cells) with nonempty interiors which cover $\mathbb{R}^d$ and have mutually no common interior points. Let $\mathcal{M}$ denote the set of mosaics.

Lemma 1.1. *The cells of a mosaic* m *are convex polytopes.*

This follows from basic separation properties of convex sets. In fact, if $K \in \mathsf{m}$ and $K_1, \ldots, K_k \in \mathsf{m}\setminus\{K\}$ satisfy $K \cap K_i \neq \emptyset$ and $\partial K = \bigcup_{i=1}^{k}(K \cap K_i)$, then $K = \bigcap_{i=1}^{k} H_i^+$. Here, for $i = 1, \ldots, k$, H_i^+ is a halfspace which contains K and is bounded by a hyperplane which separates K and K_i.

As a consequence of the lemma, it is natural to consider the facial structure of the cells. A k-face of $\mathsf{m} \in \mathcal{M}$ is a k-face of a cell of m. These faces contain additional explicit information about the metric and combinatorial structure of the mosaic. For the mathematical analysis of mosaics and their faces, it is often convenient to consider more restricted classes of tessellations which enjoy additional properties. For instance, we will exclude the possibility that the intersection of two faces of different cells is not a face of both cells. Thus we use the following terminology. For $k = 0, \ldots, d$ and a polytope $P \subset \mathbb{R}^d$, let $\mathcal{S}_k(P)$ denote the set of k-faces of P, and let $\mathcal{S}(P)$ be the set of faces of P (of any dimension). Then a mosaic $\mathsf{m} \in \mathcal{M}$ is **face-to-face** if $P \cap P' \in (\mathcal{S}(P) \cap \mathcal{S}(P')) \cup \{\emptyset\}$, for all $P, P' \in \mathsf{m}$. We write $\mathcal{S}_k(\mathsf{m}) := \bigcup_{P \in \mathsf{m}} \mathcal{S}_k(P)$ for the set of all k-faces of m. Let $\mathcal{M}_s$ denote the set of face-to-face mosaics. A mosaic $\mathsf{m} \in \mathcal{M}_s$ is called **normal** if each k-face lies in exactly $d - k + 1$ cells.

All mosaics considered here will be face-to-face. A line mosaic is an example of a mosaic which is not normal. It easily follows from the definition that if m is a normal mosaic, then each j-face of m lies in exactly $\binom{d-j+1}{d-k+1} = \binom{d-j+1}{k-j}$ different k-faces of m.

1.2 Random Mosaics

A **random mosaic** X in $\mathbb{R}^d$ is a particle process $X : (\Omega, \mathsf{A}, \mathbb{P}) \to \mathsf{N}_e(\mathcal{F}')$ satisfying $X \in \mathcal{M}_s$ with probability one. Put

$$X^{(k)} := \mathcal{S}_k(X), \qquad \Theta^{(k)} := \mathbb{E}X^{(k)}, \qquad k = 0, \ldots, d.$$

The **intensity measure** $\Theta^{(k)}$ of the **process of k-faces** $X^{(k)}$ is assumed to be locally finite. If X is stationary, then $X^{(k)}$ is stationary, and hence we obtain the decomposition

$$\Theta^{(k)} = \gamma^{(k)} \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \mathbf{1}\{x + K \in \cdot\} \lambda_d(\mathrm{d}x) \mathbb{P}_0^{(k)}(\mathrm{d}K),$$

where $\mathcal{K}_0 := \{K \in \mathcal{K} : c(K) = 0\}$ with a centre function c, $\mathbb{P}_0^{(k)}$ is a probability measure, and the factor $\gamma^{(k)}$ is the *intensity* of $X^{(k)}$. Any random polytope Z_k with distribution $\mathbb{P}_0^{(k)}$ is called *typical k-face* of X. The *quermass densities* (the densities of the intrinsic volumes) of $X^{(k)}$ are

$$d_j^{(k)} := \overline{V}_j(X^{(k)}) := \gamma^{(k)} \int_{\mathcal{K}_0} V_j(K)\mathbb{P}_0^{(k)}(\mathrm{d}K).$$

For these densities alternative representations and various relationships are known. For instance, Theorem 14 in Weil's lecture notes in this volume implies

$$\overline{V}_j(X) = \lim_{r\to\infty} \frac{1}{V_d([-r,r]^d)} \mathbb{E} \sum_{K\in X} V_j(K \cap [-r,r]^d),$$

and hence $d_d^{(d)} = \overline{V}_d(X) = 1$, which shows that $\mathbb{E}V_d(Z) = 1/\gamma^{(d)}$. Moreover, the intensities of the processes of i-faces satisfy the Euler relation

$$\sum_{i=0}^{d} (-1)^i \gamma^{(i)} = 0,$$

which is the special case $j = 0$ of the following theorem.

Theorem 1.1. *Let X be a stationary random mosaic in $\mathbb{R}^d$. Then, for $j \in \{0, \dots, d-1\}$,*

$$\sum_{i=j}^{d} (-1)^i d_j^{(i)} = 0.$$

Proof. For $\mathsf{m} \in \mathcal{M}$, let $S_1, \dots, S_p$ be the cells of m hitting the unit ball B^d with centre at the origin. By additivity,

$$\begin{aligned} V_j(B^d) = V_j\left(B^d \cap \bigcup_{l=1}^{p} S_l\right) &= \sum_{r=1}^{p} (-1)^{r-1} \sum_{|I|=r} V_j\left(\bigcap_{l\in I} S_l \cap B^d\right) \\ &= \sum_{i=j}^{d} \sum_{F\in\mathcal{S}_i(\mathsf{m})} V_j(F \cap B^d) \sum_{r=1}^{p} (-1)^{r-1} \nu(F,r), \end{aligned} \tag{1}$$

where

$$\nu(F,r) = \mathrm{card}(\{I \subset \{1,\dots,p\} : |I| = r, \bigcap_{l\in I} S_l = F\}).$$

Below we will show that

$$\sum_{r=1}^{p} (-1)^{r-1} \nu(F,r) = (-1)^{d-\dim(F)}. \tag{2}$$

From (1) and (2) with ρB^d instead of B^d, we get

$$V_j(\rho B^d)/V_d(\rho B^d) = \sum_{i=j}^{d}(-1)^{d-i} \sum_{F\in\mathcal{S}_i(X)} V_j(F\cap\rho B^d)/V_d(\rho B^d). \qquad (3)$$

By Theorem 14 in Weil's lecture notes in this volume and since $j<d$,

$$0 = \lim_{\rho\to\infty}\frac{V_j(\rho B^d)}{V_d(\rho B^d)} = \lim_{\rho\to\infty}\frac{1}{V_d(\rho B^d)}\mathbb{E}\sum_{i=j}^{d}(-1)^{d-i}\sum_{F\in X^{(i)}} V_j(F\cap\rho B^d)$$
$$= \sum_{i=j}^{d}(-1)^{d-i}\overline{V}_j(X^{(i)}).$$

We verify (2) in the special case $F=\{x\}$, the general case can be deduced from the special one. Let $T_1,\dots,T_q$ be the cells containing x, choose a polytope $P\subset\mathbb{R}^d$ with $x\in\operatorname{int}(P)$ and $P\subset\operatorname{int}(T_1\cup\ldots\cup T_q)$. Then, for $|I|=r$,

$$\chi\left(\bigcap_{l\in I} T_l\cap\partial P\right) = \begin{cases} 0, & \bigcap_{l\in I} T_l = \{x\},\\ 1, & \text{otherwise,}\end{cases}$$

and hence

$$\begin{aligned}\sum_{r=1}^{p}(-1)^{r-1}\nu(\{x\},r) &= \sum_{r=1}^{q}(-1)^{r-1}\sum_{|I|=r}\left(1-\chi\left(\bigcap_{l\in I}T_l\cap\partial P\right)\right)\\ &= 1-\chi(\partial P)\\ &= (-1)^d,\end{aligned}$$

wich completes the proof. □

1.3 Face-Stars

In order to admit a finer analysis of the combinatorial relations (incidences) between faces of different dimensions, we use the notion of a **face-star**. For a j-face $T\in\mathcal{S}_j(\mathsf{m})$, let

$$\mathcal{S}_k(T,\mathsf{m}) := \begin{cases}\mathcal{S}_k(T-c(T)), & k\le j,\\ \{S-c(T): S\in\mathcal{S}_k(\mathsf{m}), T\subset S\}, & k>j.\end{cases}$$

Then we call $(T,\mathcal{S}_k(T,\mathsf{m}))$ a (j,k)-face-star. The special centering used here turns out to be appropriate in view of the symmetric form of the basic relation provided by Theorem 1.2. Let X be a stationary random mosaic X. From X we derive a stationary marked particle process by considering the (j,k)-face-stars of the j-faces of X. For its intensity measure we obtain the decomposition

$$\mathbb{E} \sum_{T \in \mathcal{S}_j(X)} h(T, \mathcal{S}_k(T, X)) = \gamma^{(j,k)} \int_{\mathbb{R}^d} \int_{\mathcal{K}_0 \times \mathcal{F}(\mathcal{K}')} h(x + T, \mathcal{S}) \mathbb{P}_0^{(j,k)}(\mathrm{d}(T, \mathcal{S})) \lambda_d(\mathrm{d}x),$$

where h is a nonnegative measurable function. The probability measure $\mathbb{P}_0^{(j,k)}$ is called the shape-mark distribution of X. A random (j,k)-face-star with this distribution is said to be a typical (j,k)-face star. Specialization leads to

$$\gamma^{(j,k)} = \gamma^{(j)} \qquad \text{and} \qquad \mathbb{P}_0^{(j,k)}(\cdot \times \mathcal{F}(\mathcal{K}')) = \mathbb{P}_0^{(j)}.$$

Moreover, the characteristics

$$v_i^{(j,k)} := \int_{\mathcal{K}_0 \times \mathcal{F}(\mathcal{K}')} \sum_{S \in \mathcal{S}} V_i(S) \mathbb{P}_0^{(j,k)}(\mathrm{d}(T, \mathcal{S})), \qquad d_i^{(j,k)} := \gamma^{(j)} v_i^{(j,k)}$$

are useful. For instance, the mean number of k-faces of a typical (j,k)-face-star is given by

$$n_{jk} := v_0^{(j,k)} = \int_{\mathcal{K}_0 \times \mathcal{F}(\mathcal{K}')} \mathrm{card}(\mathcal{S})\, \mathbb{P}_0^{(j,k)}(\mathrm{d}(T, \mathcal{S})).$$

If $k \leq j$, then

$$n_{jk} = \int_{\mathcal{K}_0} \mathrm{card}(\mathcal{S}_k(T))\, \mathbb{P}_0^{(j)}(\mathrm{d}T),$$

and hence

$$\sum_{k=0}^{j} (-1)^k n_{jk} = 1, \qquad j = 0, \ldots, d, \tag{4}$$

which is an Euler relation for random mosaics. A general symmetry result is provided by the following theorem.

Theorem 1.2. *Let X be a stationary random mosaic in $\mathbb{R}^d$, let $f : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ be a nonnegative, measurable, translation invariant function. Then, for $j, k \in \{0, \ldots, d\}$,*

$$\gamma^{(j)} \int_{\mathcal{K}_0 \times \mathcal{F}(\mathcal{K}')} \sum_{S \in \mathcal{S}} f(S, T) \mathbb{P}_0^{(j,k)}(\mathrm{d}(T, \mathcal{S})) = \gamma^{(k)} \int_{\mathcal{K}_0 \times \mathcal{F}(\mathcal{K}')} \sum_{T \in \mathcal{T}} f(S, T) \mathbb{P}_0^{(k,j)}(\mathrm{d}(S, \mathcal{T})).$$

For the proof one uses an extension of Theorem 14 in Weil's lecture notes in this volume for stationary marked particle processes and the symmetry relation

$$\{(S+c(T),T) : T \in \mathcal{S}_j(\mathsf{m}), S \in \mathcal{S}_k(T,\mathsf{m})\} \\ = \{(S,T+c(S)) : S \in \mathcal{S}_k(\mathsf{m}), T \in \mathcal{S}_j(S,\mathsf{m})\}.$$

The usefulness of Theorem 1.2 can be seen from the consequence

$$\gamma^{(j)} n_{jk} = \gamma^{(k)} n_{kj}, \tag{5}$$

which will be used later. Further, if X is normal and $j \leq k$, then we have $n_{jk} = \binom{d+1-j}{k-j}$, and hence

$$\sum_{j=0}^{k} (-1)^j \binom{d+1-j}{k-j} \gamma^{(j)} = \sum_{j=0}^{k} (-1)^j \gamma^{(j)} n_{jk} = \gamma^{(k)} \sum_{j=0}^{k} (-1)^j n_{kj} = \gamma^{(k)},$$

where (4) was used for the last equality. Thus we arrive at

$$\left(1-(-1)^k\right) \gamma^{(k)} = \sum_{j=0}^{k-1} (-1)^j \binom{d+1-j}{k-j} \gamma^{(j)},$$

which is another Euler-type relation for the intensities of the processes of faces of the given random mosaic.

1.4 Typical Cell and Zero Cell

The **typical cell** Z of a stationary random mosaic X can be considered as a spatial average, since its distribution can be written in the form

$$\mathbb{P}_0^{(d)} = \lim_{r\to\infty} \frac{\mathbb{E}\sum_{K\in X} \mathbf{1}\{K \subset rB^d\}\mathbf{1}\{K - c(K) \in \cdot\}}{\mathbb{E}\sum_{K\in X} \mathbf{1}\{K \subset rB^d\}}.$$

Instead of such an average shape we may also consider a specific cell such as the **zero cell** Z_0, i.e.

$$Z_0 = \bigcup_{K\in X} \mathbf{1}\{0 \in \operatorname{int}(K)\}\, K.$$

Is there any relation between the sizes of Z and Z_0? For a first answer, observe that by **Campbell's theorem**

$$\mathbb{E} f(Z_0) = \gamma^{(d)} \mathbb{E}\left[f(Z) V_d(Z)\right], \quad \gamma^{(d)} = \left(\mathbb{E} V_d(Z)\right)^{-1}$$

for any nonnegative, measurable, translation invariant function $f : \mathcal{K} \to \mathbb{R}$. The Cauchy-Schwarz inequality then implies

$$\mathbb{E} V_d(Z_0) = \mathbb{E} V_d(Z)^2 / \mathbb{E} V_d(Z) \geq \mathbb{E} V_d(Z).$$

A stronger result is given below.

Theorem 1.3. *Let X be a stationary random mosaic in $\mathbb{R}^d$. Let F_0, F denote the distribution functions of $V_d(Z_0)$, $V_d(Z)$. Then $F_0(x) \le F(x)$ for $x \ge 0$, and thus $\mathbb{E}V_d(Z_0)^k \ge \mathbb{E}V_d(Z)^k$, $k \in \mathbb{N}$.*

Proof. The first assertion follows by verifying

$$(F(x) - F_0(x))\,\mathbb{E}V_d(Z) = F(x)\int_x^\infty (1 - F(t))\mathrm{d}t + (1 - F(x))\int_0^x F(t)\mathrm{d}t \ge 0,$$

the second then is clear from

$$\mathbb{E}V_d(Z)^k = k\int_0^\infty x^{k-1}(1 - F(x))\mathrm{d}x,$$

which completes the proof. □

2 Voronoi and Delaunay Mosaics

The two most important mosaics are the Voronoi and the Delaunay mosaics. First, we introduce them separately, but then we also point out their combinatorial equivalence.

2.1 Voronoi mosaics

Let $A \subset \mathbb{R}^d$ be locally finite. We will write $\|\cdot\|$ for the Euclidean norm in $\mathbb{R}^d$. Then, for $x \in A$, the **Voronoi cell** of A with centre x is defined by

$$C(x, A) := \{z \in \mathbb{R}^d : \|z - x\| = d(z, A)\},$$

where $d(z, A) := \min\{\|z - a\| : a \in A\}$. It is clear from the definition that the Voronoi cells are always polyhedral sets.

Lemma 2.1. *If* $\operatorname{conv}(A) = \mathbb{R}^d$*, then* $\mathcal{V}(A) := \{C(x, A) : x \in A\}$ *is a mosaic which is face-to-face.*

If $\tilde{X}$ is a stationary point process in $\mathbb{R}^d$, then $X := \mathcal{V}(\tilde{X})$ is a stationary random mosaic, the **Voronoi mosaic** of $\tilde{X}$. If $\tilde{X}$ is a Poisson process, we call X the **Poisson-Voronoi mosaic** induced by $\tilde{X}$.

It will be useful to have a characterization of the flats spanned by the faces of a Voronoi mosaic. To describe this, let $A \subset \mathbb{R}^d$ be a locally finite set in general position. The latter means that any $k+1$ points of A do not lie in a $(k-1)$-flat ($k = 1, \dots, d+1$). Then, for $m \in \{1, \dots, d\}$ and $x_0, \dots, x_m \in A$, let $B^m(x_0, \dots, x_m)$ denote the m-ball having $x_0, \dots, x_m$ in its boundary. Let $z(x_0, \dots, x_m)$ be the centre of this ball, $F(x_0, \dots, x_m)$ the orthogonal flat through it, and write $S(x_0,\dots,x_m;A)$ for the set of all $y \in F(x_0,\dots,x_m)$ satisfying $\|y-x_0\| = d(y,A)$. Then

$$F(x_0,\dots,x_m) = \operatorname{aff}(S) \quad \text{for some } S \in \mathcal{S}_{d-m}(\mathcal{V}(A))$$

if and only if

$$S(x_0,\dots,x_m;A) \neq \emptyset.$$

If either condition is satisfied, then $S = S(x_0,\dots,x_m;A)$. Moreover, any face $S \in \mathcal{S}_{d-m}(\mathcal{V}(A))$ can be obtained in this way.

Theorem 2.1. *Let X be the Poisson-Voronoi mosaic induced by $\tilde{X}$ with intensity $\tilde{\gamma}$. Then X is a normal mosaic. For $0 \le k \le j \le d$,*

$$d_k^{(j,k)} = \binom{d-k+1}{j-k} d_k^{(k)} \quad \text{and} \quad d_k^{(k)} = c(d,k)\, \tilde{\gamma}^{\frac{d-k}{d}},$$

where $c(d,k)$ is an explicitly known constant.

Proof. By Theorem 1.2, we obtain

$$\begin{aligned} d_k^{(j,k)} &= \gamma^{(k)} \int\limits_{\mathcal{K}_0 \times \mathcal{F}(\mathcal{K}')} V_k(S) N_j(S,\mathcal{T})\, \mathbb{P}_0^{(k,j)}(\mathrm{d}(S,\mathcal{T})) \\ &= \binom{d-k+1}{j-k} d_k^{(k)}, \end{aligned}$$

where $N_j(S,\mathcal{T})$ is the number of j-faces in the (k,j)-face-star $(S,\mathcal{T})$. Thus it remains to calculate $d_k^{(k)}$. For this task, we will use the (extended) **Slivnyak-Mecke formula**

$$\begin{aligned} &\mathbb{E} \sum_{(x_1,\dots,x_m) \in \tilde{X}^m_{\neq}} f(\tilde{X}; x_1,\dots,x_m) \\ &= \tilde{\gamma}^m \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \mathbb{E} f(\tilde{X} \cup \{x_1,\dots,x_m\}; x_1,\dots,x_m) \lambda_d(\mathrm{d}x_1) \dots \lambda_d(\mathrm{d}x_m), \end{aligned}$$

which holds for any nonnegative, measurable function f (cf. Theorems 3.1 and 3.2 in Baddeley's lecture notes in this volume, [9, Proposition 4.1], [11, Theorem 3.3]). Hence we get

$$\begin{aligned} d_k^{(k)} &= \overline{V}_k(X^{(k)}) = \frac{1}{\kappa_d} \mathbb{E} \sum_{S \in X^{(k)}} V_k(S \cap B^d) \\ &= \frac{1}{\kappa_d (d-k+1)!} \mathbb{E} \sum_{(x_0,\dots,x_{d-k}) \in \tilde{X}^{d-k+1}_{\neq}} V_k(S(x_0,\dots,x_{d-k};\tilde{X}) \cap B^d) \\ &= \frac{\tilde{\gamma}^{d-k+1}}{\kappa_d (d-k+1)!} \int \mathbb{E} V_k(S(x_0,\dots,x_{d-k}; \tilde{X} \cup \{x_0,\dots,x_{d-k}\}) \cap B^d) \\ &\qquad \times \lambda_d^{\otimes(d-k+1)}(\mathrm{d}(x_0,\dots,x_{d-k})). \end{aligned}$$

For $x_0, \dots, x_{d-k} \in \mathbb{R}^d$ in general position, we obtain

$$\begin{aligned}
&\mathbb{E}V_k(S(x_0,\dots,x_{d-k};\tilde{X}\cup\{x_0,\dots,x_{d-k}\})\cap B^d)\\
&= \iint \mathbf{1}\{y\in F(x_0,\dots,x_{d-k})\cap B^d\}\mathbf{1}\{\|y-x_0\| = d(y,\tilde{X})\}\lambda_k(\mathrm{d}y)\mathrm{d}\mathbb{P}\\
&= \int_{F(x_0,\dots,x_{d-k})\cap B^d} \mathbb{P}(\tilde{X}\cap \operatorname{int}(B^d(y,\|y-x_0\|)) = \emptyset)\lambda_k(\mathrm{d}y)\\
&= \int_{F(x_0,\dots,x_{d-k})\cap B^d} \mathrm{e}^{-\tilde{\gamma}\kappa_d\|y-x_0\|^d}\lambda_k(\mathrm{d}y).
\end{aligned}$$

Thus we have expressed $d_k^{(k)}$ in purely geometric terms. Now we can use **integral geometric transformation formulae** due to Blaschke, Petkantschin and Miles (see [13]) to complete the proof. □

2.2 Delaunay mosaics

Let $A \subset \mathbb{R}^d$ be locally finite and assume that $\operatorname{conv}(A) = \mathbb{R}^d$. We put $\mathsf{m} := \mathcal{V}(A)$. For any vertex $e \in \mathcal{S}_0(\mathsf{m})$, the **Delaunay cell** of e is defined by

$$D(e,A) := \operatorname{conv}\{x \in A : e \in \mathcal{S}_0(C(x,A))\}.$$

In the following, we assume that A is in general position and that any $d+2$ points of A do not lie on a sphere (in this case, A is said to be in general quadratic position). Then the Delaunay cells are simplices which can be defined without reference to the Voronoi mosaic. The convex hull of $d+1$ points of A is a Delaunay simplex if and only if its circumscribed sphere does not contain any other point of A in its interior.

Lemma 2.2. *If $A \subset \mathbb{R}^d$ is locally finite and* $\operatorname{conv}(A) = \mathbb{R}^d$*, then*

$$\mathsf{d} := \mathcal{D}(A) := \{D(e,A) : e \in \mathcal{S}_0(\mathsf{m})\} \in \mathcal{M}_s.$$

If A is in general quadratic position, then the Delaunay cells are simplices and

$$\Sigma : \mathcal{S}(\mathsf{m}) \to \mathcal{S}(\mathsf{d}), \quad S(x_0,\dots,x_{d-m};A) \mapsto \operatorname{conv}\{x_0,\dots,x_{d-m}\},$$

is a combinatorial anti-isomorphism.

If $\tilde{X}$ is a stationary point process in $\mathbb{R}^d$, then $Y := \mathcal{D}(\tilde{X})$ is a stationary random mosaic, the **Delaunay mosaic** of $\tilde{X}$. In case $\tilde{X}$ is a Poisson process, we call Y the induced **Poisson-Delaunay mosaic**. Moreover, we write $Y^{(j)}$ and $\beta^{(j)}$ for the process of j-faces and its intensity, respectively. The corresponding notation for $X := \mathcal{V}(\tilde{X})$ is $X^{(j)}$ and $\gamma^{(j)}$. If $\tilde{X}$ is a Poisson process, then $\tilde{X}$ is in general quadratic position almost surely. This yields the first assertion of the next theorem.

Theorem 2.2. *A Poisson-Delaunay mosaic Y is simplicial and $\beta^{(j)} = \gamma^{(d-j)}$.*

Proof. The following brief argument is based on [14, Theorem 4.3.1] which contains useful information about the intensities of stationary particle processes and associated marked point processes with respect to general centre functions. Consider the stationary marked point process

$$\tilde{X}^{(d-j)} := \{(z(x_0,\dots,x_j), S(x_0,\dots,x_j;\tilde{X}) - z(x_0,\dots,x_j)) : \\ (x_0,\dots,x_j) \in \tilde{X}^{j+1}, S(x_0,\dots,x_j;\tilde{X}) \neq \emptyset\}.$$

The associated particle process is $X^{(d-j)}$ with intensity $\gamma^{(d-j)}$. Hence,

$$\tilde{Y}^{(j)} := \{z(x_0,\dots,x_j) : (x_0,\dots,x_j) \in \tilde{X}^{j+1}, S(x_0,\dots,x_j;\tilde{X}) \neq \emptyset\}$$

has the same intensity. But $\tilde{Y}^{(j)}$ consists of the circumcentres of the polytopes in $Y^{(j)}$. □

Let $\triangle^{(d)}$ be the set of d-simplices in $\mathbb{R}^d$. Let $z(S)$ denote the centre of the circumscribed sphere of $S \in \triangle^{(d)}$. The distribution of the **typical cell** of a Poisson-Delaunay mosaic Y is the shape distribution $\mathbb{Q}_0$ of Y on the space $\triangle_0^{(d)} := \{S \in \triangle^{(d)} : z(S) = 0\}$ with respect to the centre function z. Hence, for a measurable set $\mathcal{A} \subset \triangle_0^{(d)}$, we have

$$\begin{aligned}\mathbb{Q}_0(\mathcal{A}) &= \mathbb{E}\sum_{S\in Y} \mathbf{1}\{z(S) \in [0,1]^d\}\mathbf{1}\{S - z(S) \in \mathcal{A}\} \\ &= \lim_{r\to\infty} \frac{\operatorname{card}\{S \in Y : z(S) \in [0,r]^d, S - z(S) \in \mathcal{A}\}}{\operatorname{card}\{S \in Y : z(S) \in [0,r]^d\}}.\end{aligned}$$

The first equation is true by definition, the second holds almost surely and follows from the ergodic properties of Poisson-Delaunay mosaics (see [14, Section 6.4]).

Theorem 2.3. *Let Y be a Poisson-Delaunay mosaic associated with a stationary Poisson point process of intensity $\tilde{\gamma}$. Then, for a measurable set $\mathcal{A} \subset \triangle_0^{(d)}$,*

$$\mathbb{Q}_0(\mathcal{A}) = c(d)\tilde{\gamma}^d \int_0^\infty \int_{S^{d-1}} \dots \int_{S^{d-1}} \mathbf{1}\{r\operatorname{conv}\{u_0,\dots,u_d\} \in \mathcal{A}\} \\ \times\ \exp(-\tilde{\gamma}\kappa_d r^d) r^{d^2-1} \Delta_d(u_0,\dots,u_d)\sigma(\mathrm{d}u_0)\dots\sigma(\mathrm{d}u_d)\mathrm{d}r,$$

where $\Delta_d(u_0,\dots,u_d) := \lambda_d(\operatorname{conv}\{u_0,\dots,u_d\})$ and $c(d)$ is an explicitly known constant.

Proof. Since $\beta^{(d)} = \gamma^{(0)} = c(d)^{-1}\tilde{\gamma}$ by Theorems 2.2 and 2.1, we can again use the Slivnyak-Mecke formula to get

$$\mathbb{Q}_0(\mathcal{A}) = \frac{c(d)\tilde{\gamma}^{-1}}{(d+1)!}\mathbb{E} \sum_{(x_0,\ldots,x_d)\in\tilde{X}^{d+1}_{\neq}} \mathbf{1}\{\mathrm{conv}\{x_0,\ldots,x_d\} - z(x_0,\ldots,x_d) \in \mathcal{A}\}$$
$$\times\ \mathbf{1}\{z(x_0,\ldots,x_d) \in [0,1]^d\}\mathbf{1}\{\tilde{X} \cap \mathrm{int}(B^d(x_0,\ldots,x_d)) = \emptyset\}$$
$$= \frac{c(d)\tilde{\gamma}^{d}}{(d+1)!}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} \mathbf{1}\{\mathrm{conv}\{x_0,\ldots,x_d\} - z(x_0,\ldots,x_d) \in \mathcal{A}\}$$
$$\times\ \mathbf{1}\{z(x_0,\ldots,x_d) \in [0,1]^d\}\,\mathbb{P}(\tilde{X} \cap \mathrm{int}(B^d(x_0,\ldots,x_d)) = \emptyset)$$
$$\times\ \lambda_d(\mathrm{d}x_0)\ldots\lambda_d(\mathrm{d}x_d).$$

Finally, an application of an integral geometric transformation formula due to Miles (see [14, Satz 7.2.2]) completes the proof. □

3 Hyperplane Mosaics

Let $\hat{X}$ be a stationary **hyperplane process** in $\mathbb{R}^d$. Each realization of $\hat{X}$ decomposes $\mathbb{R}^d$ into polyhedral cells. If all cells are bounded, we obtain a **stationary hyperplane mosaic** X. We always assume that X (i.e. $\hat{X}$) is in **general position**. The j-th **intersection process** of $\hat{X}$, which is a process of $(d-j)$-flats, is denoted by $\hat{X}_j$. We collect some information about the quermass densities of the processes of faces of the given hyperplane mosaic.

Theorem 3.1. *Let X be a stationary hyperplane mosaic in $\mathbb{R}^d$. Assume that the intersection processes of $\hat{X}$ have finite intensity. Then, for $0 \le j \le k \le d$,*

$$d_j^{(k)} = \binom{d-j}{d-k} d_j^{(j)}, \qquad \gamma^{(k)} = \binom{d}{k}\gamma^{(0)}, \qquad n_{kj} = 2^{k-j}\binom{k}{j}.$$

Proof. Define $\nu_k := \mathrm{card}(\{F \in \mathcal{S}_k(X) : F \cap rB^d \neq \emptyset\})$, $\alpha_0 := 0$ and, for $1 \le j \le d$,

$$\alpha_j := \frac{1}{j!}\mathrm{card}(\{(H_1,\ldots,H_j) \in \hat{X}^j_{\neq} : H_1 \cap \ldots \cap H_j \cap rB^d \neq \emptyset\}).$$

Miles has shown that

$$\nu_k = \sum_{j=d-k}^{d} \binom{j}{d-k}\alpha_j.$$

Hence $\mathbb{E}\nu_k < \infty$ follows from the assumption of finite intersection densities. Let $0 \le j < k \le d-1$ and $r > 0$. For $E \in \hat{X}_{d-k}$, a stationary random mosaic is induced in E by $\hat{X} \cap E$ for which, by applying (3) in E, we get

$$\sum_{E\in\hat{X}_{d-k}} V_j(E\cap rB^d) = \sum_{i=j}^{k}(-1)^{k-i} \sum_{E\in\hat{X}_{d-k}} \sum_{F\in X^{(i)},F\subset E} V_j(F\cap rB^d)$$
$$= \sum_{i=j}^{k}(-1)^{k-i}\binom{d-i}{d-k} \sum_{F\in X^{(i)}} V_j(F\cap rB^d).$$

We estimate

$$\frac{1}{V_d(rB^d)}\mathbb{E}\sum_{E\in\hat{X}_{d-k}} V_j(E\cap rB^d) \le \frac{1}{V_d(rB^d)}\mathbb{E}\sum_{E\in\hat{X}_{d-k}} r^j V_j(B^k)\chi(E\cap rB^d)$$
$$\le \frac{r^j V_j(B^k)}{V_d(rB^d)}\kappa_{d-k} r^{d-k}\hat{\gamma}_{d-k} \to 0$$

as $r\to\infty$. Thus, for $0\le j<k\le d$, we arrive at

$$\sum_{i=j}^{k}(-1)^{k-i}\binom{d-i}{d-k} d_j^{(i)} = 0.$$

It remains to solve this triangular system of linear equations.

As to the third assertion, as a consequence of Theorem 1.2 we obtained relation (5), i.e. $\gamma^{(k)}n_{kj}=\gamma^{(j)}n_{jk}$. Here we have $n_{jk}=2^{k-j}\binom{d-j}{d-k}$. Combining this with the second assertion, the required result follows. □

Let $\hat{X}$ be a stationary hyperplane process in $\mathbb{R}^d$. Let $\langle\cdot,\cdot\rangle$ denote the scalar product in $\mathbb{R}^d$ and put $H(u,t):=\{x\in\mathbb{R}^d : \langle x,u\rangle = t\}$ whenever $u\in S^{d-1}$ and $t\in\mathbb{R}$. Then the translation invariant intensity measure Θ of $\hat{X}$ can be written in the form

$$\Theta := \mathbb{E}\hat{X} = \hat{\gamma}\int_{S^{d-1}}\int_{\mathbb{R}} \mathbf{1}\{H(u,t)\in\cdot\}\mathrm{d}t\,\tilde{\mathbb{P}}(\mathrm{d}u),$$

where $\hat{\gamma}$ is called the **intensity** and $\tilde{\mathbb{P}}$ the **direction distribution** (an even probability measure on S^{d-1}) of $\hat{X}$. We call $\hat{X}$ **non-degenerate** if $\tilde{\mathbb{P}}$ is not concentrated on a great subsphere. For $K\in\mathcal{C}^d$, let $\mathcal{H}_K := \{E\in\mathcal{E}^d_{d-1} : K\cap E\neq\emptyset\}$.

A **Poisson hyperplane process** $\hat{X}$ satisfies, for $K\in\mathcal{C}^d$ and $k\in\mathbb{N}_0$,

$$\mathbb{P}(\hat{X}(\mathcal{H}_K)=k) = \frac{\Theta(\mathcal{H}_K)^k}{k!}\mathrm{e}^{-\Theta(\mathcal{H}_K)}.$$

It is in general position and induces a **Poisson hyperplane mosaic** X.

The probabilistic information contained in the direction distribution of a stationary hyperplane process $\hat{X}$ can be expressed in geometric terms by associating with $\hat{X}$ a suitable convex body. More precisely, the **associated zonoid** $\Pi_{\hat{X}}$ of $\hat{X}$ is defined by its support function

$$h(\Pi_{\hat{X}}, \cdot) = \frac{\hat{\gamma}}{2} \int_{S^{d-1}} |\langle \cdot, v \rangle| \, \tilde{\mathbb{P}}(\mathrm{d}v).$$

Introducing geometric tools and methods for describing hyperplane processes turns out to be a fruitful strategy. We mention two of the simpler applications which are suggested by this method. A different geometric idea used for other purposes will be described in the next section.

Let X be a stationary Poisson hyperplane mosaic in $\mathbb{R}^d$, derived from a stationary Poisson hyperplane process $\hat{X}$ with intensity $\hat{\gamma}$.

- In the special case of a Poisson hyperplane mosaic, the formulas of Theorem 3.1 take a more specific form. In particular, they can be interpreted by means of geometric functionals. Thus known inequalities from geometry become available for the study of extremal problems. For $0 \le j \le k \le d$, we have

  $$d_j^{(k)} = \binom{d-j}{d-k} V_{d-j}(\Pi_{\hat{X}}), \quad \gamma^{(k)} = \binom{d}{k} V_d(\Pi_{\hat{X}}).$$

  If X is isotropic, then even

  $$d_j^{(k)} = \binom{d-j}{d-k}\binom{d}{j} \frac{\kappa_{d-1}^{d-j}}{d^{d-j}\kappa_d^{d-1-j}\kappa_j} \hat{\gamma}^{d-j}, \quad \gamma^{(k)} = \binom{d}{k} \frac{\kappa_{d-1}^{d}}{d^d \kappa_d^{d-1}} \hat{\gamma}^d.$$

- Let Z_0 be the zero cell of X. Then

  $$\mathbb{E}V_d(Z_0) = 2^{-d} d! V_d((\Pi_{\hat{X}})^\circ) \ge d! \kappa_d \left(\frac{2\kappa_{d-1}}{d\kappa_d} \hat{\gamma} \right)^{-d}$$

  with equality if and only if $\hat{X}$ is isotropic. Here $(\Pi_{\hat{X}})^\circ$ denotes the polar body of $\Pi_{\hat{X}}$. For a direct approach to this inequality one can proceed as follows. The **radial function** $\rho(Z_0, v)$ of Z_0 at $v \in S^{d-1}$, which is the distance from the origin to the boundary point of Z_0 in direction v, is exponentially distributed, that is

  $$\mathbb{P}(\rho(Z_0, v) \le r) = \mathbb{P}(\hat{X}(\mathcal{F}_{[0,rv]}) > 0) = 1 - \exp\left(-2rh(\Pi_{\hat{X}}, v)\right),$$

  where we use that

  $$\begin{aligned} \mathbb{E}\hat{X}(\mathcal{F}_{[0,rv]}) &= 2\hat{\gamma} \int_{S^{d-1}} \int_0^\infty \mathbf{1}\{H(u,t) \cap [0, rv] \ne \emptyset\} \mathrm{d}t \, \tilde{\mathbb{P}}(\mathrm{d}u) \\ &= \hat{\gamma} r \int_{S^{d-1}} |\langle u, v \rangle| \, \tilde{\mathbb{P}}(\mathrm{d}u) = 2rh(\Pi_{\hat{X}}, v). \end{aligned}$$

  Hence, by Jensen's inequality

$$
\begin{aligned}
\mathbb{E}V_d(Z_0) &= \frac{1}{d}\int_{S^{d-1}} \mathbb{E}\rho(Z_0,v)^d\sigma(\mathrm{d}v) = \frac{1}{d}\int_{S^{d-1}} \frac{d!}{2^d} h(\Pi_{\hat{X}},v)^{-d}\sigma(\mathrm{d}v),\\
&\geq \frac{d!\kappa_d(d\kappa_d)^d}{2^d}\left(\int_{S^{d-1}} h(\Pi_{\hat{X}},u)\sigma(\mathrm{d}u)\right)^{-d}\\
&= \frac{d!\kappa_d(d\kappa_d)^d}{2^d}\left(\frac{\hat{\gamma}}{2}\int_{S^{d-1}}\int_{S^{d-1}} |\langle u,v\rangle|\sigma(\mathrm{d}u)\tilde{\mathbb{P}}(\mathrm{d}v)\right)^{-d},
\end{aligned}
$$

which yields the required estimate.

4 Kendall's Conjecture

In this section, we will consider the shape of large cells in Poisson hyperplane, in Poisson-Voronoi and in Poisson-Delaunay mosaics. We start with the description of a classical problem which was first formulated in the 1940s in a more restricted framework. For the purpose of introduction, let $\hat{X}$ be a stationary isotropic Poisson line process in $\mathbb{R}^2$. Further, let Z_0 denote the zero cell (Crofton cell) of the associated Poisson-line mosaic. Then D. Kendall's classical conjecture (Kendall's problem) can be stated as follows.

The conditional law for the shape of Z_0, given a lower bound for $A(Z_0)$, converges weakly, as that lower bound tends to infinity, to the law concentrated at the circular shape.

Recently, various contributions to this subject have been made by R. Miles (heuristically) [8], I. Kovalenko [6], [7], A. Goldman [2], P. Calka [1], and in the papers [3], [4], [5]. For a more detailed description of the relevant literature we refer to [3], [5].

In order to motivate a crucial ingredient for the solution of Kendall's problem, we start with a related, but much simpler extremal problem. Let $\hat{X}$ be a stationary Poisson hyperplane process with intensity measure

$$
\Theta = 2\hat{\gamma}\int_{S^{d-1}}\int_0^\infty \mathbf{1}\{H(u,t)\in\cdot\}\mathrm{d}t\,\tilde{\mathbb{P}}(\mathrm{d}u).
$$

Now we wish to know which convex bodies $K\subset\mathbb{R}^d$ with $0\in K$ and fixed $V_d(K)>0$ maximize the inclusion probability $\mathbb{P}(K\subset Z_0)$.

By **Minkowski's existence theorem** there is a unique centred convex body $B\subset\mathbb{R}^d$ such that $\tilde{\mathbb{P}} = S_{d-1}(B,\cdot)$, where $S_{d-1}(B,\cdot)$ is the surface area measure of order $d-1$ of B. The convex body B, the **direction body** associated with the given hyperplane process, thus is another example of a convex auxiliary body which can be used to translate the given probabilistic information into geometric terms. Thus tools and results from convex geometry become available in the present setting. Hence, for any $K\in\mathcal{K}$, we obtain

$$\Theta(\mathcal{H}_K) = 2\hat{\gamma} \int_{S^{d-1}} h(K,u) S_{d-1}(B, \mathrm{d}u) = 2d\hat{\gamma} V_1(B,K).$$

At this point an application of **Minkowski's inequality** yields

$$\mathbb{P}(K \subset Z_0) = \exp\left(-2d\hat{\gamma} V_1(B,K)\right) \leq \exp\left(-2d\hat{\gamma} V_d(K)^{1/d} V_d(B)^{(d-1)/d}\right).$$

This shows that the inclusion probability is maximized precisely if K is homothetic to the direction body B and has the prescribed volume.

4.1 Large cells in Poisson hyperplane mosaics

We turn to the solution of Kendall's problem in a more general framework. Let $\hat{X}$ be a stationary hyperplane process with intensity $\hat{\gamma}$, direction distribution $\tilde{\mathbb{P}}$ and direction body B. Now we ask for the **limit shape** of the zero cell Z_0, given the volume of the zero cell is big. In view of the previous extremal problem, the shape of the direction body B is a natural candidate for the limit shape of Z_0, given "Z_0 is big".

An important initial step in the solution of the problem is to find a precise formulation of the type of result one is aiming at. For instance, we have to clarify what we mean by the shape of a convex body, and we have to specify the type of convergence which is used. In the following, we identify homothetic convex bodies and determine distances up to homotheties. In other words, we work in the shape space

$$\mathcal{S}^* := \{K \in \mathcal{K} : 0 \in K, R(K) = 1\},$$

where $R(K)$ is the radius of the circumscribed ball of K, and measure distances between (the shape of) a convex body K and the (shape of the) direction body B by means of the deviation measure

$$r_B(K) := \min\left\{\frac{r_2}{r_1} - 1 : r_1 B + z \subset K \subset r_2 B + z, 0 < r_1 \leq r_2, z \in \mathbb{R}^d\right\}.$$

Then we say that B^* is the limit shape of Z_0 with respect to the size functional V_d, which is used to measure the size of the zero cell, if

$$\mu_a := \mathbb{P}\left(Z_0^* \in \cdot \mid V_d(Z_0) \geq a\right) \to \delta_{B^*} \tag{6}$$

weakly on $\mathcal{S}^*$, as $a \to \infty$. By a compactness argument, (6) is equivalent to

$$\mathbb{P}(r_B(Z_0) \geq \epsilon \mid V_d(Z_0) \geq a) \to 0,$$

as $a \to \infty$, for all $\epsilon > 0$. The following theorem contains a much more explicit and quantitative result. As a very special case it yields a solution of Kendall's problem in generalized form, in arbitrary dimensions and, more importantly, without the assumption of isotropy.

Theorem 4.1. *Let $\hat{X}$ be a Poisson hyperplane process for the data $\hat{\gamma}$ and B. There is a constant $c_0 = c_0(B)$ such that the following is true. If $\epsilon \in (0,1)$ and $I = [a,b)$ with $a^{1/d}\hat{\gamma} \geq \sigma_0 > 0$, then*

$$\mathbb{P}(r_B(Z_0) \geq \epsilon \mid V_d(Z_0) \in I) \leq c \exp\left(-c_0\epsilon^{d+1}a^{1/d}\hat{\gamma}\right),$$

where $c = c(B, \epsilon, \sigma_0)$.

It turns out that Theorem 4.1 remains true if the zero cell is replaced by the typical cell.

Theorem 4.2. *The assertion of Theorem 4.1 remains true if the zero cell Z_0 is replaced by the typical cell Z.*

The proof of these two results is quite involved and technical. Instead of trying to give a sketch of proof, we have to content ourselves with providing some further motivation for the strategy of proof. The basic task consists in estimating the conditional probability

$$\mathbb{P}(r_B(Z_0) \geq \epsilon \mid V_d(Z_0) \geq a) = \frac{\mathbb{P}(r_B(Z_0) \geq \epsilon, V_d(Z_0) \geq a)}{\mathbb{P}(V_d(Z_0) \geq a)}.$$

We estimate numerator and denominator separately.

- To obtain a lower bound for the denominator, we define the convex body $K := (a/V_d(B))^{1/d}B$. Then

$$\begin{aligned}\mathbb{P}(V_d(Z_0) \geq a) &\geq \mathbb{P}(\hat{X}(\mathcal{H}_K) = 0) = \exp\left(-2dV_1(B,K)\hat{\gamma}\right)\\ &= \exp\left(-2dV_d(B)^{(d-1)/d}a^{1/d}\hat{\gamma}\right).\end{aligned}$$

  This simple argument has to be refined considerably to yield a *local* improvement.
- Next we consider an upper bound for the numerator. **Minkowski's inequality** states that

$$V_1(B,K) \geq V_d(B)^{(d-1)/d}V_d(K)^{1/d}$$

  with equality if and only if K and B are homothetic. If we know that $r_B(K) \geq \epsilon$, then in fact a stronger stability result

$$V_1(B,K) \geq (1 + f(\epsilon))V_d(B)^{(d-1)/d}V_d(K)^{1/d}$$

  is true involving a nonnegative and explicitly known stability function f. Hence, if the deterministic convex body K satisfies

$$r_B(K) \geq \epsilon \qquad \text{and} \qquad V_d(K) \geq a,$$

  then

$$\begin{aligned}\mathbb{P}(\hat{X}(\mathcal{H}_K) = 0) &= \exp\left(-2dV_1(B,K)\hat{\gamma}\right)\\ &\leq \exp\left(-2d(1 + f(\epsilon))V_d(B)^{(d-1)/d}a^{1/d}\hat{\gamma}\right).\end{aligned}$$

- In a bold analogy, we might hope that a similar result holds for the random cell Z_0 instead of a deterministic convex body K. In other words, we would like to have an estimate of the form
$$\mathbb{P}(r_B(Z_0) \geq \epsilon, V_d(Z_0) \geq a) \leq c \exp\left(-2d(1+g(\epsilon))V_d(B)^{(d-1)/d}a^{1/d}\hat{\gamma}\right)$$
with some explicitly given, nonnegative function g. Such an estimate and the lower bound from the first step would essentially imply the theorem in the special case considered.
- However, things are not so easy. The previous analogy cannot be justified in a straightforward way. We first consider $V_d(Z_0) \in a[1, 1+h]$ for small $h > 0$ and, in addition, we classify the random convex body Z_0 according to its degree of elongation.

The method of proof can be modified to yield results about asymptotic distributions.

Theorem 4.3. *For $\hat{X}$ and Z_0 as in Theorem 4.1,*
$$\lim_{r\to\infty} a^{-1/d} \ln \mathbb{P}(V_d(Z_0) \geq a) = -2dV_d(B)^{(d-1)/d}\hat{\gamma}.$$

The special case $d = 2$ and $B = B^2$ (isotropic case) had previously been established by Goldman [2] using different techniques.

4.2 Large cells in Poisson-Voronoi mosaics

We now describe an analogue of Kendall's problem for Poisson-Voronoi mosaics. Let $\tilde{X}$ be a stationary Poisson process with intensity $\tilde{\gamma}$. We write $\mathcal{K}_*$ for the space of convex bodies containing 0, and $X := \mathcal{V}(\tilde{X})$ for the Poisson-Voronoi mosaic induced by $\tilde{X}$. The **typical cell** of X is a random convex body Z having distribution $\mathbb{Q}$, where
$$\mathbb{Q}(\mathcal{A}) := \frac{1}{\tilde{\gamma}}\mathbb{E}\sum_{x\in\tilde{X}} \mathbf{1}\{x \in B\}\mathbf{1}\{C(x, \tilde{X}) - x \in \mathcal{A}\},$$
for measurable sets $B \subset \mathbb{R}^d$ with $\lambda_d(B) = 1$ and $\mathcal{A} \subset \mathcal{K}_*$. This mean value can be rewritten by means of the **Slivnyak-Mecke formula**. Thus we obtain
$$\begin{aligned}\mathbb{Q}(\mathcal{A}) &= \int_{\mathbb{R}^d} \mathbb{E}\mathbf{1}\{x \in B\}\mathbf{1}\{C(x, \tilde{X}\cup\{x\}) - x \in \mathcal{A}\}\lambda_d(\mathrm{d}x)\\ &= \int_{\mathbb{R}^d} \mathbb{E}\mathbf{1}\{x \in B\}\mathbf{1}\{C(0, \tilde{X}\cup\{0\}) \in \mathcal{A}\}\lambda_d(\mathrm{d}x)\\ &= \mathbb{P}(C(0, \tilde{X}\cup\{0\}) \in \mathcal{A}).\end{aligned}$$

Hence we can specify

$$Z = C(0, \tilde{X} \cup \{0\}) = \bigcap_{x \in \tilde{X}} H^-(x),$$

i.e. $Z = Z_0(X^*)$ with $X^* := \{H(x) : x \in \tilde{X}\}$. Here X^* is a nonstationary (but isotropic) Poisson hyperplane process with intensity measure

$$\mathbb{E}X^* = 2^d \tilde{\gamma} \int_{S^{d-1}} \int_0^\infty \mathbf{1}\{H(u,t) \in \cdot\} t^{d-1} \mathrm{d}t \, \sigma(\mathrm{d}u).$$

Hence, for $K \in \mathcal{K}_*$, we get $\mathbb{E}X^*(\mathcal{H}_K) = 2^d \tilde{\gamma} U(K)$ with

$$U(K) = \lambda_d(\Phi(K)) = \frac{1}{d} \int_{S^{d-1}} h(K,u)^d \sigma(\mathrm{d}u),$$

where

$$\Phi(K) = \{y \in \mathbb{R}^d : H(2y) \cap K \neq \emptyset\} = \bigcup \{B^d(x, \|x\|) : 2x \in K\}.$$

The strategy of proof for Theorem 4.1 can be adapted and extended to yield a corresponding result for large typical cells in Poisson-Voronoi mosaics. On the geometric side, however, we need a new stability result for the geometric functional U, which states that

$$U(K) \geq \left(1 + \gamma(d) \vartheta(K)^{(d+3)/2}\right) \kappa_d^{1-d/k} v_k(K)^{d/k}.$$

The much weaker estimate $U(K) \geq \kappa_d^{1-d/k} v_k(K)^{d/k}$ is easy to derive from results available in the literature. But again the main point is to obtain a quantitative improvement involving the nonnegative deviation measure ϑ which measures the deviation of the shape of K from the shape of a Euclidean ball. For details we refer to [5].

Theorem 4.4. *Let $\tilde{X}$ be a Poisson process with intensity $\tilde{\gamma}$. Let Z denote the typical cell of $X = \mathcal{V}(\tilde{X})$ and $k \in \{1, \ldots, d\}$. There is some $c_0 = c_0(d)$ such that the following is true. If $\epsilon \in (0,1)$ and $I = [a,b)$ with $a^{d/k}\tilde{\gamma} \geq \sigma_0 > 0$, then*

$$\mathbb{P}(\vartheta(Z) \geq \epsilon \mid v_k(Z) \in I) \leq c \exp\left(-c_0 \epsilon^{(d+3)/2} a^{d/k} \tilde{\gamma}\right),$$

where $c = c(d, \epsilon, \sigma_0)$.

The following result provides an analogue of Theorem 4.3.

Theorem 4.5. *For $\tilde{X}$, X and Z as in Theorem 4.4,*

$$\lim_{a \to \infty} a^{-d/k} \ln \mathbb{P}(v_k(Z) \geq a) = -2^d \kappa_d^{1-d/k} \tilde{\gamma}.$$

Similar results can be established for other size functionals as well, e.g., for the inradius of the typical cell.

4.3 Large cells in Poisson-Delaunay mosaics

To complete the picture we finally address a version of Kendall's problem for Poisson-Delaunay mosaics. Let $\tilde{X}$ be a stationary Poisson process with intensity $\tilde{\gamma}$. Let Z denote the typical cell of $Y = \mathcal{D}(\tilde{X})$ with distribution $\mathbb{Q}_0$. For d-simplices $S_1, S_2 \subset \mathbb{R}^d$, we write $\eta(S_1, S_2)$ for the smallest number η such that for any vertex p of one of the simplices, $B^d(p, \eta)$ contains a vertex q of the other simplex. Let $\mathcal{T}^d$ be the set of regular simplices inscribed to S^{d-1}. For a d-simplex inscribed to a ball with centre z and radius r, we put

$$\rho(S) := \min \left\{ \eta(r^{-1}(S - z), T) : T \in \mathcal{T}^d \right\}.$$

Then again we can state a large deviation result for the shape of the typical Poisson-Delaunay cell given it has large volume.

Theorem 4.6. *Let $\tilde{X}$, $\tilde{\gamma}$, Y and Z be as described above. There is some $c_0 = c_0(d)$ such that the following is true. If $\epsilon \in (0,1)$ and $I = [a,b)$ with $a\tilde{\gamma} \geq \sigma_0 > 0$, then*

$$\mathbb{P}(\rho(Z) \geq \epsilon \mid V_d(Z) \in I) \leq c \exp\left(-c_0 \epsilon^2 a \tilde{\gamma}\right),$$

where $c = c(d, \epsilon, \sigma_0)$.

Similar results hold for the zero cell Z_0 of Y and if $V_d(Z)$ is replaced by the inradius of Z. These results for Poisson-Delaunay mosaics can be obtained by more direct estimates than in the previous cases. This is mainly due to the explicit formula for the distribtion of the typical cell which is available in the present case (cf. Theorem 2.3). In addition, one needs sharp estimates of isoperimetric type. A simple and well known geometric extremal result states that a simplex S inscribed to S^{d-1} has maximal volume among all simplices inscribed to S^{d-1} if and only if S is regular. Let T^d be such a regular simplex with volume τ_d. In the proof of Theorem 4.6 we need an improved version of such a uniqueness result. It can be summarized in the estimate

$$V_d(S) \leq (1 - c\,\rho(S)) V_d(T^d).$$

It should be emphasized that corresponding uniqueness results are completely unknown in dimension $d \geq 3$ for such basic functionals as the mean width or the surface area.

There is also a result on the asymptotic distribution of the volume of the typical Poisson-Delaunay mosaic.

Theorem 4.7. *For $\tilde{X}$, $\tilde{\gamma}$, Y and Z as in Theorem 4.6,*

$$\lim_{a \to \infty} a^{-1} \ln \mathbb{P}(V_d(Z) \geq a) = -\frac{\kappa_d}{\tau_d} \tilde{\gamma}.$$

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