Submitted exclusively to the London Mathematical Society
DOI: 10.1112/S0000000000000000

HESSIAN MEASURES OF CONVEX FUNCTIONS AND APPLICATIONS TO AREA MEASURES

ANDREA COLESANTI AND DANIEL HUG

ABSTRACT

The Hessian measures of a (semi-)convex function can be introduced as coefficients of a local Steiner formula. We continue the investigation of Hessian measures by providing a geometric characterization of the support of these measures. Then we explore the Radon-Nikodym derivative and the absolute continuity of Hessian measures with respect to Lebesgue measure. As special cases of our results, we recover known results for surface area measures of convex bodies.

1. *Introduction*

A natural way to introduce the Hessian measures of a convex function is through a *local Steiner formula*. Let u be a convex function defined in an open convex subset Ω of the Euclidean space $\mathbb{R}^d$ ($d \geq 2$). For each point x in Ω, we denote by $\partial u(x)$ the subdifferential of u at x. If η is a Borel subset of Ω and $\rho > 0$, then the set

$$P_\rho(u,\eta) = \{x + \rho\zeta \,:\, x \in \eta\,, \zeta \in \partial u(x)\}$$

is measurable and its measure can be expressed as a polynomial of degree d in ρ, that is

$$\mathcal{H}^d(P_\rho(u,\eta)) = \sum_{k=0}^{d} \binom{d}{k} F_k(u,\eta)\rho^{d-k}\,, \tag{1.1}$$

where $\mathcal{H}^d$ denotes d-dimensional Hausdorff measure (Lebesgue measure). The coefficients $F_0(u,\cdot),\ldots,F_d(u,\cdot)$ are nonnegative Borel measures, which are called the Hessian measures of u. When $u \in C^2(\Omega)$, the Hessian measures are simply the integrals of the elementary symmetric functions of the eigenvalues of the second order differential D^2u of u; more precisely,

$$\binom{d}{k} F_k(u,\eta) = \int_\eta \sigma_{d-k}(D^2u(x))\mathcal{H}^d(dx)\,, \quad k = 0,\ldots,d\,, \tag{1.2}$$

where

$$\sigma_j(D^2u(x)) = \sum_{1\leq i_1<\cdots<i_j\leq d} \lambda_{i_1}\cdots\lambda_{i_j}\,, \quad j = 1,\ldots,d\,,$$

and $\lambda_1,\ldots,\lambda_d$ are the eigenvalues of the Hessian matrix $D^2u(x)$ (we also put $\sigma_0 = 1$).

Though in this paper we are concerned mainly with those aspects of Hessian measures which are related to the theory of convex bodies, let us recall that another interesting feature of these measures is their connection with fully non-linear elliptic partial differential

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2000 *Mathematics Subject Classification* 52A20 (primary), 26B25, 28A78 (secondary).

equations. For this topic we refer the reader to [**18**], [**8**] and the bibliographical references in these papers.

Formula (1.1) emphasizes the analogy between the Hessian measures of a convex function and the *curvature and (surface) area measures* of a convex body (see [**15**]). Indeed, curvature and area measures also admit a characterization as coefficients of a local Steiner formula. Another link is given by a pair of formulas proved in [**7**] that we recall here. For a convex body K (nonempty compact convex set in $\mathbb{R}^d$), let us denote by $C_j(K,\cdot)$ and $S_j(K,\cdot)$, $j=0,\dots,d-1$, the curvature and area measures of K, respectively. Let $\mathrm{bd}(K)$ denote the boundary of K. If d_K is the distance function of K, then

$$C_j(K,\eta) = dF_j(d_K, \eta\cap \mathrm{bd}(K))\,, \quad j=0,\dots,d-1\,, \tag{1.3}$$

for every Borel set $\eta\subset\mathbb{R}^d$; moreover, if h_K is the support function of K, then

$$S_j(K,\omega) = dF_{d-j}(h_K,\hat\omega)\,, \quad j=0,\dots,d-1\,, \tag{1.4}$$

where ω is a Borel subset of the unit sphere S^{d-1} and $\hat\omega := \{t\nu\,:\, t\in[0,1]\,,\,\nu\in\omega\}$. In some cases such relations allow one to deduce results regarding curvature and area measures from general properties of Hessian measures; examples are given in [**7**] and in the present paper.

Our first result is a geometric description of the support of the Hessian measures of convex functions. Such a result parallels corresponding characterizations of the support of curvature and area measures of convex bodies (see, for instance, Theorems 4.6.1 and 4.6.3 in [**15**]). For a convex function u, defined in an open convex set Ω and for $j\in\{0,\dots,d-1\}$, we introduce the set of its *j-extreme points*. A point $x\in\Omega$ is called j-extreme if there exists no $(j+1)$-dimensional ball centred at x and contained in Ω, on which u is affine. Thus, extreme points of convex functions can be seen in analogy to extreme boundary points of convex bodies. Our result, stated as Theorem 1, claims that the support of the Hessian measure $F_j(u,\cdot)$ is the closure, in Ω, of the set of j-extreme points of u. The proof of this fact is achieved by using a representation of Hessian measures, established in [**6**], and by an inspection of the proof for the characterization of the support of curvature measures of convex bodies.

The investigation of Hessian measures thus leads to a unified view on the characterization of the support of curvature and surface area measures of convex bodies. The latter results were first obtained in [**13**], [**14**]; see also [**17**].

In the second and main part of the paper, we investigate the *Radon-Nikodym derivative* and the *absolute continuity* of Hessian measures of (semi-)convex functions with respect to d-dimensional Hausdorff measure $\mathcal{H}^d$ in $\mathbb{R}^d$. First, we describe explicitly the absolutely continuous part (the Radon-Nikodym derivative) of the Hessian measures with respect to $\mathcal{H}^d$, in terms of pointwise second derivatives of the involved function. More precisely, consider a (semi-)convex function u defined in a convex open subset of $\mathbb{R}^d$. It is well-known that u is second order differentiable in Aleksandrov's sense at almost every point of Ω (see [**2**], [**3**], [**4**], [**9**]); hence, for $\mathcal{H}^d$-almost every x in Ω, the Hessian matrix $D^2u(x)$ is defined. In Theorem 2 we show that, for $j\in\{0,\dots,d\}$ and $\mathcal{H}^d$-almost everywhere, the Radon-Nikodym derivative $dF_{d-j}(u,\cdot)/d\mathcal{H}^d$ of $F_{d-j}(u,\cdot)$ with respect to d-dimensional Hausdorff measure satisfies

$$\binom{d}{j}\frac{dF_{d-j}}{d\mathcal{H}^d} = \sigma_j(D^2u). \tag{1.5}$$

This result has a natural counterpart for curvature and area measures of convex bodies; see

Theorems 3.2 and 3.5 in [**11**]. In fact, part of the information contained in Theorem 3.5 in [**11**] can be deduced from (1.5) via (1.4). An explicit description of the singular part of the Hessian measures with respect to $\mathcal{H}^d$ is implicitly contained in the proof of Theorem 2.

Our next result was inspired by a theorem which has been proved by Weil in [**19**] for area measures, and which has recently been established for curvature measures in [**12**]. Roughly speaking, for curvature measures the result states that the absolute continuity of the mean curvature measure of a convex set and some integrability assumption for the mean curvature (in Aleksandrov's sense) together imply the absolute continuity of further (lower order) curvature measures of the given set. In the case of Hessian measures, the result can be described as follows. Assume that the Hessian measure $F_{d-1}(u,\cdot)$ of a (semi-)convex function u, defined in Ω, is absolutely continuous with respect to $\mathcal{H}^d$ and that, for some $p \geq 2$,

$$\frac{dF_{d-1}}{d\mathcal{H}^d} \in L^p_{\text{loc}}(\Omega)\,. \tag{1.6}$$

By Theorem 2, the Radon-Nikodym derivative in (1.6) coincides with the Laplacian of u (up to the constant factor d), $\mathcal{H}^d$-almost everywhere. Theorem 3 states that for every $j \leq d-1$ such that $j \geq d-p$, $F_j(u,\cdot)$ is absolutely continuous with respect to $\mathcal{H}^d$ and

$$\frac{dF_j}{d\mathcal{H}^d} \in L^q_{\text{loc}}(\Omega)\,,$$

for every $q \geq 1$ such that $(d-j)q \leq p$. It is clear from the examples given in [**12**] that this result (in a sense) is best possible. As a consequence, we obtain Weil's theorem for area measures by applying our result in the special case of Hessian measures of support functions. Curiously, it does not seem to be possible to derive the corresponding result for curvature measures in a similar way.

2. *The support of Hessian measures*

In the following, we work in Euclidean spaces with scalar product $\langle\cdot,\cdot\rangle$ and with norm $\|\cdot\|$. A closed Euclidean ball in $\mathbb{R}^d$ with centre x and radius $r \geq 0$ will be denoted by $B^d(x,r)$. We write S^{d-1} for the Euclidean unit sphere. The *support of a Borel measure* μ will be denoted by supp μ. It is defined as the complement of the largest open set on which the measure vanishes. Let $\mathcal{K}^d_o$ denote the set of convex bodies with nonempty interiors. The geometric characterization of the support of the jth curvature measure of a convex body $K \in \mathcal{K}^d_o$ involves the closure of the set $\text{ext}_j(K)$ of j-extreme boundary points of K. A point $x \in \text{bd}(K)$ is called a j-extreme boundary point of K if x is not the centre of a $(j+1)$-dimensional ball contained in K; see [**15**, p. 65] for equivalent definitions. The following result is due to R. Schneider [**14**].

THEOREM (Schneider). *Let $K \in \mathcal{K}^d_o$ and $j \in \{0,\dots,d-1\}$. Then*

$$\text{supp}\ C_j(K,\cdot) = \text{cl}(\text{ext}_j(K))\,. \tag{2.1}$$

We now introduce the appropriate notion which is needed for the description of the support of a Hessian measure. Let $\Omega \subset \mathbb{R}^d$ be open and convex, and let $u : \Omega \to \mathbb{R}$ be a convex function. Then we say that $x \in \Omega$ is a *j-extreme point* of u if there is no $(j+1)$-dimensional ball centred at x and contained in Ω on which u is an affine function. The set of all j-extreme points of u is denoted by $\text{ext}_j(u)$. For a set $A \subset \Omega$, we write $\text{cl}_\Omega(A)$ for the closure of A with respect to Ω, that is $\text{cl}_\Omega(A) = \Omega \cap \text{cl}(A)$. All other definitions and notations are as in [**15**], [**6**], [**7**].

THEOREM 1. *Let $\Omega \subset \mathbb{R}^d$ be open and convex, let $u : \Omega \to \mathbb{R}$ be a convex function, and let $j \in \{0, \dots, d-1\}$. Then*

$$\text{supp } F_j(u, \cdot) = \text{cl}_\Omega(\text{ext}_j(u)) \,. \tag{2.2}$$

Proof. The generalized Hessian measure $\Theta_j(u, \cdot)$ of u is introduced in [**6**, Theorem 3.1] as coefficient measure of a Steiner formula. It is a measure on the Borel subsets of $\Omega \times \mathbb{R}^d$ which satisfies $F_j(u, \cdot) = \Theta_j(u, \cdot \times \mathbb{R}^d)$. Therefore [**6**, Theorem 3.1] especially yields an integral representation for $F_j(u, \cdot)$ (cf. (3.2)). A similar integral representation is available for the curvature measure $C_j(K, \cdot)$ of a closed convex set $K \subset \mathbb{R}^{d+1}$; see Theorem 3.2 in [**11**].

Now we compare the integral representations for $F_j(u, \cdot)$ and for $C_j(\text{epi}(u), \cdot)$, where $\text{epi}(u) := \{(x, z) \in \Omega \times \mathbb{R} : z \geq u(x)\}$ is the epigraph of u. Up to the homeomorphism $g_u : \Omega \to \text{graph}(u) \subset \mathbb{R}^{d+1}$, $x \mapsto (x, u(x))$, these two representations differ only by a positive and finite function under the integral. Therefore, we obtain

$$\text{supp } F_j(u, \cdot) = \text{supp } C_j(\text{epi}(u), g_u(\cdot)) \,. \tag{2.3}$$

An inspection of the proof of Theorem 4.6.1 in [**15**] shows that all arguments involved in that proof are of a local nature, and hence

$$\text{supp } C_j(\text{epi}(u), g_u(\cdot)) = g_u^{-1} \left(\text{cl}(\text{ext}_j(\text{epi}(u))) \right) \,. \tag{2.4}$$

Since obviously

$$\text{cl}(\text{ext}_j(\text{epi}(u))) = g_u \left(\text{cl}_\Omega(\text{ext}_j(u)) \right) \,, \tag{2.5}$$

the proof follows by combining (2.3) – (2.5). □

In [**14**] and [**15**], the proof of (2.1) is based on a combination of geometric arguments and on an integral-geometric mean value formula (Crofton formula). Instead of trying to repeat this argument by using a Crofton formula for convex functions (see [**7**, Theorem 2.1]), we applied the relationship between the Hessian measures of a convex function and the curvature measures of the epigraph of this function. Apart from providing a new result for Hessian measures, a major advantage of the present point of view is that we now obtain the characterization of the support of the surface area measures of convex bodies quite easily. Note that despite certain analogies, the proof of Theorem 4.6.3 in [**15**] for surface area measures (see also [**13**]) is slightly more involved than the proof of Theorem 4.6.1 in [**15**] concerning curvature measures.

To describe the support of the area measures of a convex body $K \in \mathcal{K}^d_o$, we recall that a unit vector $\nu \in S^{d-1}$ is a *j-extreme normal vector* of K if and only if there do not exist $j+2$ linearly independent normal vectors $\nu_1, \dots, \nu_{j+2}$ at one and the same boundary point of K such that $\nu = \nu_1 + \ldots + \nu_{j+2}$. An equivalent condition involving the support function $h(K, \cdot)$ of K is that $\nu = \nu_1 + \ldots + \nu_{j+2}$ with linearly independent vectors $\nu_1, \dots, \nu_{j+2}$ implies that $h(K, \nu) < h(K, \nu_1) + \ldots + h(K, \nu_{j+2})$; cf. [**16**, p. 278] or [**15**, Section 2.2]. Let $\text{extn}_j(K)$ denote the set of j-extreme normal vectors of $K \in \mathcal{K}^d_o$.

In view of another application in Section 3, the following lemma is stated in greater generality than needed for the proof of (2.9).

LEMMA 1. *Let $K \in \mathcal{K}^d_o$ and $j \in \{1, \dots, d\}$. Then*

$$dF_j(h_K, \cdot) = j \int_0^\infty \int_{S^{d-1}} \mathbf{1}\{r\nu \in \cdot\} r^{j-1} S_{d-j}(K, d\nu) dr \,.$$

Proof. Let $R : (0,\infty) \times S^{d-1} \to \mathbb{R}^d$ be defined by $R(r,\nu) := r\nu$. Then, as in the proof of Corollary 5.10 in [**7**], one obtains

$$dF_j(h_K, R((0,1] \times \omega)) = S_{d-j}(K,\omega) \tag{2.6}$$

for any Borel set $\omega \subset S^{d-1}$. Since $\partial h_K(t\nu) = \partial h_K(\nu)$ for $t > 0$ and $\nu \in S^{d-1}$, we find that

$$P_\rho(h_K, R((0,b] \times \omega)) = bP_{\rho/b}(h_K, R((0,1] \times \omega))$$

for any $\rho, b > 0$ and any Borel set $\omega \subset S^{d-1}$. Thus, applying the Steiner formula for Hessian measures and comparing coefficients, we get

$$F_j(h_K, R((0,b] \times \omega)) = b^j F_j(h_K, R((0,1] \times \omega))\,. \tag{2.7}$$

From (2.6) and (2.7) we deduce that

$$dF_j(h_K, R(\gamma)) = j\int_0^\infty \int_{S^{d-1}} \mathbf{1}\{(r,\nu) \in \gamma\} r^{j-1} S_{d-j}(K, d\nu)dr \tag{2.8}$$

with $\gamma = (0,b] \times \omega$. Since on both sides of (2.8), we have measures with respect to γ, the assertion follows by general measure theoretic extension arguments. □

THEOREM (Schneider). *Let $K \in \mathcal{K}_o^d$ and $j \in \{0,\dots,d-1\}$. Then*

$$\text{supp}\; S_{d-1-j}(K,\cdot) = \text{cl}(\text{extn}_j(K))\,. \tag{2.9}$$

Proof. For $\omega \subset S^{d-1}$, we put $\tilde\omega := \{t\nu : t \in (0,1), \nu \in \omega\}$. Lemma 1 implies that $\nu \notin \text{supp}\; S_{d-1-j}(K,\cdot)$ if and only if there is an open set $\omega \subset S^{d-1}$ with $\nu \in \omega$ such that $F_{j+1}(h_K, \tilde\omega) = 0$. By Theorem 1, this is equivalent to the following condition: for any $\zeta \in \tilde\omega$, there exist vectors $\nu_1,\dots,\nu_{j+1} \in \mathbb{R}^d$ and a number $\epsilon > 0$, all depending on ζ, such that $\zeta,\nu_1,\dots,\nu_{j+1}$ are linearly independent and h_K is an affine function on $\zeta + B^d(0,\epsilon) \cap \text{lin}\{\nu_1,\dots,\nu_{j+1}\}$. This is the same as requiring, for any $\zeta \in \tilde\omega$, the existence of vectors $\zeta_1,\dots,\zeta_{j+1} \in \mathbb{R}^d$ and of a number $\delta > 0$, all depending on ζ, such that $(\zeta,\zeta_1,\dots,\zeta_{j+1})$ is an orthonormal system and $h(K,\cdot)$ is a linear function on the convex cone

$$N := \left\{\lambda\left(\zeta + B^d(0,\delta) \cap \text{lin}\{\zeta_1,\dots\zeta_{j+1}\}\right) : \lambda \ge 0\right\}.$$

But for any $\zeta \in \tilde\omega$, this is equivalent to the existence of linearly independent vectors $w_1,\dots,w_{j+2}$ such that $\zeta/\|\zeta\| = w_1 + \ldots + w_{j+2}$ and

$$h(K,\zeta/\|\zeta\|) = h(K,w_1) + \ldots + h(K,w_{j+2}),$$

that is $\zeta/\|\zeta\| \notin \text{extn}_j(K)$. □

3. *Radon-Nikodym derivative and absolute continuity*

Let μ be a nonnegative Borel measure on $\Omega \subset \mathbb{R}^d$. If μ is absolutely continuous with respect to $\mathcal{H}^d$, then we write $\mu \ll \mathcal{H}^d$ and denote by $d\mu/d\mathcal{H}^d$ the density of μ with respect to $\mathcal{H}^d$. In general, μ can be written as the sum of two measures μ^a, μ^s, where $\mu^a \ll \mathcal{H}^d$ and μ^a, μ^s are mutually singular. The density $d\mu^a/d\mathcal{H}^d$ is called the Radon-Nikodym derivative of μ with respect to $\mathcal{H}^d$; see [**9**] for details.

In the following theorem, we determine the absolutely continuous part of the Hessian measures of convex functions with respect to $\mathcal{H}^d$. The singular part is explicitly described by equation (3.4) in the subsequent proof.

THEOREM 2. *Let $\Omega \subset \mathbb{R}^d$ be an open convex set, and let $u : \Omega \to \mathbb{R}$ be a convex function. Then, for $j \in \{0, \dots, d\}$ the Radon-Nikodym derivative of $\binom{d}{j} F_{d-j}(u, \cdot)$ with respect to $\mathcal{H}^d$ is given by $\sigma_j(D^2 u)$, where $D^2 u(x)$ is the Hessian matrix of u at x in Aleksandrov's sense, for $\mathcal{H}^d$-almost every $x \in \Omega$.*

Proof. In the following, we adopt the notation from [**6**], except that we write d for the dimension instead of n. So $\mathrm{Nor}(u) \subset \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$ is the normal bundle of u, $K_1(X,V), \dots, K_d(X,V)$ are the generalized curvatures on $\mathrm{Nor}(u)$, which are defined for $\mathcal{H}^d$-almost every $(X,V) \in \mathrm{Nor}(u)$, $U_1, \dots, U_d$ denote the corresponding eigenvectors (depending on (X,V)), and $E_1, \dots, E_{d+1}$ is the standard basis of $\mathbb{R}^{d+1}$. The span of $E_1, \dots, E_d$ is identified with $\mathbb{R}^d$. As in [**6**], $U_1, \dots, U_d$ are chosen in such a way that $(U_1, \dots, U_d, V)$ is an orthonormal basis which is negatively oriented with respect to the standard basis.

The generalized graph of u is

$$\Gamma(u) = \{(x,p) \in \Omega \times \mathbb{R}^d : p \in \partial u(x)\},$$

which is homeomorphic to $\mathrm{Nor}(u)$ via the locally bilipschitz map $T : \Gamma(u) \to \mathrm{Nor}(u)$ given by

$$T(x,p) = \Big((x, u(x)), (1 + \|p\|^2)^{-1/2}(p, -1)\Big) \tag{3.1}$$

with inverse

$$\tilde{T}(X,V) = \big(X - \langle X, E_{d+1}\rangle E_{d+1}, E_{d+1} - \langle V, E_{d+1}\rangle^{-1} V\big) .$$

The case $j = 0$ of Theorem 4.1 is covered by Corollary 3.2 in [**6**]. Therefore we suppose that $j \in \{1, \dots, d\}$ in the following. By a special case of Theorem 3.1 in [**6**],

$$\begin{aligned}\binom{d}{j} F_{d-j}(u, \cdot) = \int_{\mathrm{Nor}(u)} \mathbf{1}\{X - \langle X, E_{d+1}\rangle E_{d+1} \in \cdot\} \left(-\frac{1}{\langle V, E_{d+1}\rangle}\right)^j \\ \times \sum_{1 \le i_1 < \cdots < i_j \le d} \frac{K_{i_1}(X,V) \cdots K_{i_j}(X,V)}{\prod_{i=1}^d \sqrt{1 + K_i(X,V)^2}} \\ \times D_{i_1 \dots i_j}(X,V) \mathcal{H}^d(d(X,V)).\end{aligned} \tag{3.2}$$

For a definition of $D_{i_1 \dots i_j}$, we refer to [**6**, p. 3246–7]. We now split $\mathrm{Nor}(u)$ into the measurable sets

$$\mathrm{Nor}^a(u) := \{(X,V) \in \mathrm{Nor}(u) : K_i(X,V) < \infty \text{ for } i = 1, \dots, d\}$$

and

$$\mathrm{Nor}^s(u) := \mathrm{Nor}(u) \setminus \mathrm{Nor}^a(u) ;$$

cf. [**11**]. Let $\Pi_1 : \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ be defined by $\Pi_1(X,Y) := X$. The approximate Jacobian of the map $\Pi_1 \circ \tilde{T} : \mathrm{Nor}(u) \to \Omega$, $(X,V) \mapsto X - \langle X, E_{d+1}\rangle E_{d+1}$ satisfies

$$\begin{aligned}\mathrm{ap}J_d(\Pi_1 \circ \tilde{T})(X,V) &= \left\| \bigwedge_{i=1}^d \frac{1}{\sqrt{1 + K_i(X,V)^2}} A_i \right\| \\ &= (-\langle V, E_{d+1}\rangle) \prod_{i=1}^d \frac{1}{\sqrt{1 + K_i(X,V)^2}} ,\end{aligned}$$

where $A_i := U_i - \langle U_i, E_{d+1}\rangle E_{d+1}$ is defined as in [**6**]. Next we assert that

$$\binom{d}{j} F^a_{d-j}(u,\cdot) = \int_{\mathrm{Nor}^a(u)} \mathbf{1}\{\Pi_1 \circ \tilde{T}(X,V) \in \cdot\} \mathrm{ap}J_d(\Pi_1 \circ \tilde{T})(X,V) \times \left(-\frac{1}{\langle V, E_{d+1}\rangle}\right)^{j+1} \sum_{1\le i_1<\cdots<i_j\le d} K_{i_1}(X,V)\cdots K_{i_j}(X,V) \times D_{i_1\ldots i_j}(X,V)\mathcal{H}^d(d(X,V)) \tag{3.3}$$

and

$$\binom{d}{j} F^s_{d-j}(u,\cdot) = \int_{\mathrm{Nor}^s(u)} \mathbf{1}\{\Pi_1 \circ \tilde{T}(X,V) \in \cdot\} \left(-\frac{1}{\langle V, E_{d+1}\rangle}\right)^{j} \times \sum_{1\le i_1<\cdots<i_j\le d} \frac{K_{i_1}(X,V)\cdots K_{i_j}(X,V)}{\prod_{i=1}^d \sqrt{1+K_i(X,V)^2}} \times D_{i_1\ldots i_j}(X,V)\mathcal{H}^d(d(X,V))\,. \tag{3.4}$$

To verify the equations (3.3) and (3.4), we argue similarly as in [**11**]. Let $\mathcal{D}^2(u)$ denote the set of all points $x \in \Omega$ at which u is second order differentiable in the Aleksandrov sense. For $(X,V) := T(x, \nabla u(x))$ and $x \in \mathcal{D}^2(u)$, we get $K_j(X,V) < \infty$ for $j = 1,\ldots,d$, since then X is a normal boundary point of the epigraph of u. Therefore, $F^s_{d-j}(u,\cdot)$ vanishes on $\mathcal{D}^2(u)$. Moreover, since $\mathcal{H}^d(\Omega \setminus \mathcal{D}^2(u)) = 0$, the coarea formula yields that

$$\binom{d}{j} F^a_{d-j}(u,\cdot) = \int_\Omega \mathbf{1}\{x \in \cdot\} d_j(x)\mathcal{H}^d(dx),$$

where

$$d_j(x) = (-\langle V, E_{d+1}\rangle)^{-(j+1)} \sum_{1\le i_1<\cdots<i_j\le d} K_{i_1}(X,V)\cdots K_{i_j}(X,V) D_{i_1\ldots i_j}(X,V)$$

for $x \in \mathcal{D}^2(u)$ and $(X,V) = T(x,\nabla u(x))$. Thus $F^a_{d-j}(u,\cdot)$ is concentrated on $\mathcal{D}^2(u)$. This concludes the proof of (3.3) and (3.4).

It remains to show that

$$d_j(x) = \sigma_j(D^2u(x)) \tag{3.5}$$

for $\mathcal{H}^d$-almost every $x \in \mathcal{D}^2(u)$. The proof of (3.5) is split into two main steps.

Step 1. Let $x \in \mathcal{D}^2(u)$ and $(X,V) = T(x,\nabla u(x))$ be fixed. As in [**6**], we can define

$$\overline{B}_j := \frac{K_j(X,V)}{-\langle V, E_{d+1}\rangle} B_j := \frac{K_j(X,V)}{-\langle V, E_{d+1}\rangle}\left(U_j - \frac{\langle U_j, E_{d+1}\rangle}{\langle V, E_{d+1}\rangle} V\right),$$

where $K_j(X,V) < \infty$ is used. Since $A_1,\ldots,A_d$ is a basis of $\mathbb{R}^d$, there are coefficients β_{ij}, $i,j \in \{1,\ldots,d\}$, such that

$$\overline{B}_j = \sum_{i=1}^d \beta_{ij} A_i, \qquad j = 1,\ldots,d\,. \tag{3.6}$$

For $1 \le i_1 < \cdots < i_j \le d$ we put $I := (i_1,\ldots,i_j)$ and

$$\mathrm{sgn}(I) := \mathrm{sgn}(i_1 \ldots i_j i_{j+1} \ldots i_d)\,,$$

where $1 \le i_{j+1} < \cdots < i_d \le d$ is complementary to I with respect to $\{1,\ldots,d\}$.

Omitting the argument (X,V), we thus obtain

$$\begin{aligned}&\left(-\frac{1}{\langle V,E_{d+1}\rangle}\right)^j K_{i_1}\cdots K_{i_j} D_{i_1\dots i_j} E_1\wedge\cdots\wedge E_d\\ &\quad= \operatorname{sgn}(I)\overline{B}_{i_1}\wedge\cdots\wedge\overline{B}_{i_j}\wedge A_{i_{j+1}}\wedge\cdots\wedge A_{i_d}\\ &\quad= \operatorname{sgn}(I)\bigwedge_{\ell=1}^{j}\left[\sum_{k=1}^{d}\beta_{ki_\ell}A_k\right]\wedge\bigwedge_{\ell=j+1}^{d}A_{i_\ell}\\ &\quad= \operatorname{sgn}(I)\bigwedge_{\ell=1}^{j}\left[\sum_{r=1}^{j}\beta_{i_r i_\ell}A_{i_r}\right]\wedge\bigwedge_{\ell=j+1}^{d}A_{i_\ell}\\ &\quad= \det\left(\beta_{i_r i_\ell}\right)_{r,\ell=1}^{j}\bigwedge_{i=1}^{d}A_i\\ &\quad= \det\left(\beta_{i_r i_\ell}\right)_{r,\ell=1}^{j}(-\langle V,E_{d+1}\rangle)E_1\wedge\cdots\wedge E_d\,.\end{aligned}$$

This shows that

$$\binom{d}{j}F^a_{d-j}(u,\cdot)=\int_\Omega \mathbf{1}\{x\in\cdot\}\sum_{|I|=j}\det(\beta_I(x))\mathcal{H}^d(dx)\,, \tag{3.7}$$

where $\beta_I(x):=(\beta_{ij}(x))_{i,j\in I}$ for a subset $I\subset\{1,\dots,d\}$.

Step 2. We prove that

$$\sum_{|I|=j}\det(\beta_I(x))=\sigma_j(D^2u(x)) \tag{3.8}$$

for $\mathcal{H}^d$-almost every $x\in\mathcal{D}^2(u)$. Let $x_0\in\mathcal{D}^2(u)$ be fixed so that the approximate tangent space $\operatorname{Tan}^d(\mathcal{H}^d\llcorner\Gamma(u),(x_0,\nabla u(x_0)))$ is a d-dimensional subspace of $\mathbb{R}^d\times\mathbb{R}^d$ and

$$\operatorname{Tan}:=\operatorname{Tan}^d(\mathcal{H}^d\llcorner\Gamma(u),(x_0,\nabla u(x_0)))=\operatorname{lin}\{(A_i,\overline{B}_i):i=1,\dots,d\}\,.$$

Consider the linear map $\varphi:\mathbb{R}^d\to\mathbb{R}^d$ defined by $\varphi:=\pi_2\circ\pi_1^{-1}$, where $\pi_1:\operatorname{Tan}\to\mathbb{R}^d$, $(a,\bar b)\mapsto a$ and $\pi_2:\operatorname{Tan}\to\mathbb{R}^d$, $(a,\bar b)\mapsto\bar b$. Here we use that π_1 is surjective and hence also injective, since $\mathcal{A}:=(A_1,\dots,A_d)$ is a basis of $\mathbb{R}^d$. The matrix of φ with respect to $\mathcal{A}$ is given by $M_\varphi^{\mathcal{A}}=(\beta_{ij})_{i,j=1}^d$. Let $\lambda_1,\dots,\lambda_d$ denote the eigenvalues of $D^2u(x_0)$ with corresponding eigenvectors $e_1,\dots,e_d$, which form an orthonormal basis $\mathcal{E}:=(e_1,\dots,e_d)$ of $\mathbb{R}^d$. We will prove that

$$\operatorname{Tan}=\operatorname{lin}\{(e_i,\lambda_ie_i):i=1,\dots,d\}\,. \tag{3.9}$$

If this has been verified, then clearly $M_\varphi^{\mathcal{E}}=\operatorname{diag}(\lambda_1,\dots,\lambda_d)$. But then (3.8) easily follows from the fact that $M_\varphi^{\mathcal{E}}$ and $M_\varphi^{\mathcal{A}}$ have the same characteristic polynomial.

It remains to establish (3.9). It is sufficient to check that $(e_i,\lambda_ie_i)\in\operatorname{Tan}$ for $i=1,\dots,d$. Let ∇u denote a measurable choice of a subgradient field of u (cf. [**1**, Theorem 8.1.3]), and let $G(x):=(x,\nabla u(x))$, $x\in\Omega$. Then the inclusion $G(\Omega)\subset\Gamma(u)$ implies that

$$\operatorname{Tan}^d(\mathcal{H}^d\llcorner G(\Omega),G(x_0))\subset\operatorname{Tan}\,. \tag{3.10}$$

We will show that (e_1,λ_1e_1) (say) is contained in the set on the left-hand side of (3.10). This will follow once we have established that

$$\Theta^{*d}\left[\mathcal{H}^d\llcorner(G(\Omega)\cap E(G(x_0),(e_1,\lambda_1e_1),\epsilon)),G(x_0)\right]>0 \tag{3.11}$$

for any $\epsilon \in (0,1)$; cf. [**10**, p. 252] for a definition of the set $E(\cdot,\cdot,\cdot)$ and [**10**, p. 181] for a definition of the upper density $\Theta^{*d}[\cdot,\cdot]$. The proof of (3.11) requires some preparations.

Fix $\epsilon \in (0,1)$. Since $x_0 \in \mathcal{D}^2(u)$, there is a function $\omega : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\nabla u(x) - \nabla u(x_0) - D^2u(x_0)(x-x_0) = \|x-x_0\|\omega(x-x_0)\,, \qquad x \in \Omega\,,$$

and

$$\lim_{x\to x_0} \omega(x-x_0) = 0\,.$$

Let $\delta(\epsilon) > 0$ be chosen so that $x \in \Omega$ and

$$\|\omega(x-x_0)\| \le \frac{\epsilon}{8d}$$

whenever $\|x-x_0\| \le \delta(\epsilon)$. For $\rho \in (0,\delta(\epsilon)]$ we define the pyramid (with apex removed)

$$C_{\rho,\epsilon}(x_0,e_1) := \left\{ x_0 + t\left(e_1 + \sum_{i=2}^d \alpha_i e_i\right) : t \in \left(0, \frac{\rho}{4(1+\lambda)}\right], |\alpha_i| \le \frac{1}{1+\lambda}\frac{\epsilon}{2d} \text{ for } i = 2,\dots,d \right\}\,,$$

where $\lambda := \max\{\lambda_1,\dots,\lambda_d\}$. Let $x \in C_{\rho,\epsilon}(x_0,e_1)$, i.e.

$$x = x_0 + t\left(e_1 + \sum_{i=2}^d \alpha_i e_i\right) \quad \text{with} \quad t \in \big(0, \rho(4(1+\lambda))^{-1}\big]$$

and $|\alpha_i| \le (1+\lambda)^{-1}\epsilon/(2d)$ for $i = 2,\dots,d$. Then

$$\|x-x_0\| = t\left\{1 + \sum_{i=2}^d \alpha_i^2\right\}^{1/2} \le t\left\{1 + (d-1)\frac{\epsilon^2}{4d^2}\right\}^{1/2} \le 2t \le \frac{\rho}{2(1+\lambda)}\,,$$

and consequently

$$\begin{aligned}\|G(x)-G(x_0)\| &= \big(\|x-x_0\|^2 + \|\nabla u(x) - \nabla u(x_0)\|^2\big)^{1/2}\\ &= \big(\|x-x_0\|^2 + \|D^2u(x_0)(x-x_0) + \|x-x_0\|\omega(x-x_0)\|^2\big)^{1/2}\\ &\le \|x-x_0\|(1+(\lambda+1)^2)^{1/2}\\ &\le \|x-x_0\|2(1+\lambda) \le \rho\,.\end{aligned}$$

Moreover, choosing $r := t^{-1}$, we can estimate

$$\begin{aligned}&\|r(G(x)-G(x_0)) - (e_1,\lambda_1 e_1)\|^2\\ &\quad= \|(r(x-x_0)-e_1, rD^2u(x_0)(x-x_0) + r\|x-x_0\|\omega(x-x_0) - \lambda_1 e_1)\|^2\\ &\quad= \left\|\left(\sum_{i=2}^d \alpha_i e_i, \sum_{i=2}^d \lambda_i\alpha_i e_i + \left\{1+\sum_{i=2}^d \alpha_i^2\right\}^{1/2}\omega(x-x_0)\right)\right\|^2\\ &\quad\le (d-1)\frac{\epsilon^2}{4d^2} + \sum_{i=2}^d \left(|\lambda_i||\alpha_i| + \sqrt{2}\frac{\epsilon}{8d}\right)^2 + \left(\sqrt{2}\frac{\epsilon}{8d}\right)^2\\ &\quad\le \frac{\epsilon^2}{4} + \sum_{i=2}^d \left(\frac{\epsilon}{2d} + \frac{\epsilon}{4d}\right)^2 + \frac{\epsilon^2}{32d^2} \le \left[\frac{1}{4} + \left(\frac{3}{4}\right)^2\right]\epsilon^2 \le \epsilon^2\,.\end{aligned}$$

Thus we have shown that, for $\epsilon > 0$ and $\rho \in (0, \delta(\epsilon))$,

$$\begin{aligned} & C_{\rho,\epsilon}(x_0, e_1) \\ & \subset \bigcup_{r>0} \left\{ x \in \Omega : G(x) \in B^{2d}(G(x_0), \rho), \| r(G(x) - G(x_0)) - (e_1, \lambda_1 e_1) \| \le \epsilon \right\} \\ & = \pi_1 \left(G(\Omega) \cap B^{2d}(G(x_0), \rho) \cap E(G(x_0), (e_1, \lambda_1 e_1), \epsilon) \right) \end{aligned}$$

where $\pi_1 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is given by $\pi_1(x, y) := x$. Let κ_d denote the volume of the Euclidean unit ball in $\mathbb{R}^d$. Since π_1 is a contraction, we thus conclude that

$$\begin{aligned} & \Theta^{*d} \left[\mathcal{H}^d \llcorner (G(\Omega) \cap E(G(x_0), (e_1, \lambda_1 e_1), \epsilon)), G(x_0) \right] \\ & \ge \limsup_{\rho \downarrow 0} \frac{\mathcal{H}^d(C_{\rho,\epsilon}(x_0, e_1))}{\kappa_d \rho^d} > 0, \end{aligned}$$

which concludes the proof. □

Hessian measures can be defined not just for convex functions, but also for *semi-convex* functions. Locally, a semi-convex function equals the sum of a convex function and a smooth function. A more detailed description and references are contained in [**7**]. In the following corollary, we describe the extension of Theorem 2 to this more general framework.

COROLLARY 1. *Theorem 2 remains true if* $u : \Omega \to \mathbb{R}$ *is semi-convex.*

Proof. Let Ω' be an open convex set whose closure is a compact subset of Ω. Then there exists a constant $C \ge 0$ such that $v(x) := u(x) + (C/2)\|x\|^2$, $x \in \Omega'$, is convex. Hence, over Ω' we have

$$F_{d-j}(u, \cdot) = \sum_{i=0}^{j} \binom{j}{i} (-C)^i F_{d-j+i}(v, \cdot)\,; \tag{3.12}$$

see Proposition 1.6 in [**7**]. Applying Theorem 2 to v in (3.12) and a well-known differentiation result for measures, we thus get $\mathcal{H}^d$-almost everywhere in Ω'

$$\binom{d}{j} \frac{dF_{d-j}(u, \cdot)}{d\mathcal{H}^d} = \sum_{i=0}^{j} \binom{d-j+i}{i} (-C)^i \sigma_{j-i}(D^2 v) = \sigma_j(D^2 u)\,,$$

which proves the assertion. □

Theorem 2 will be used in the proof of our next result.

THEOREM 3. *Let* Ω *be an open convex subset of* $\mathbb{R}^d$*, and let* $u : \Omega \to \mathbb{R}$ *be a convex function. Assume that*

$$F_{d-1}(u, \cdot) \ll \mathcal{H}^d \quad \textit{and} \quad \Delta u \in L^p_{\mathrm{loc}}(\Omega)$$

for some $p \ge 2$*. Then, for every* $k \le d-1$ *such that* $d - k \le p$*,*

$$F_k(u, \cdot) \ll \mathcal{H}^d \quad \textit{and} \quad \sigma_{d-k}(D^2 u) \in L^q_{\mathrm{loc}}(\Omega)\,, \tag{3.13}$$

for every $q \ge 1$ *such that* $(d-k)q \le p$*.*

Proof. Let ϕ be a standard mollifier in $\mathbb{R}^d$, i.e. $\phi \in C^\infty(\mathbb{R}^d)$ has compact support in

$\{x \in \mathbb{R}^d : \|x\| \le 1\}$ and

$$\int_{\mathbb{R}^d} \phi(x)\mathcal{H}^d(dx) = 1\,.$$

For $\epsilon > 0$ we define

$$\phi_\epsilon(x) := \frac{1}{\epsilon^d}\,\phi\left(\frac{x}{\epsilon}\right)$$

and

$$u_\epsilon(x) := (u * \phi_\epsilon)(x) = \int_{\mathbb{R}^d} u(y)\phi_\epsilon(x-y)\mathcal{H}^d(dy)\,,$$

where $u(y) := 0$ for $y \notin \Omega$. The function u_ϵ is C^∞ and convex on compact subsets of Ω, if $\epsilon > 0$ is sufficiently small. Moreover, as $\epsilon \to 0^+$, u_ϵ converges uniformly to u on compact subsets of Ω and, for every $k \in \{0, 1, \ldots, d\}$, $F_k(u_\epsilon, \cdot)$ converges to $F_k(u, \cdot)$ in the sense of measures (in the vague topology); for a proof of the latter fact, see, for instance, [**7**, Theorem 1.1]. Let ψ be a test function of class $C^\infty(\Omega)$ with compact support in Ω. Using Theorem 2, we obtain

$$d \lim_{\epsilon\to 0^+} \int \psi(x)F_{d-1}(u_\epsilon, dx) = d\int \psi(x)F_{d-1}(u, dx) = \int \psi(x)\Delta u(x)\mathcal{H}^d(dx)\,.$$

On the other hand, by (1.2) and the divergence theorem,

$$d\int \psi(x)F_{d-1}(u_\epsilon, dx) = \int \psi(x)\Delta u_\epsilon(x)\mathcal{H}^d(dx) = \int u_\epsilon(x)\Delta\psi(x)\mathcal{H}^d(dx)$$

if $\epsilon > 0$ is small enough. Hence, by the uniform convergence of u_ϵ, we get

$$d \lim_{\epsilon\to 0^+} \int \psi(x)F_{d-1}(u_\epsilon, dx) = \int u(x)\Delta\psi(x)\mathcal{H}^d(dx)\,.$$

This shows that

$$\int \psi(x)\Delta u(x)\mathcal{H}^d(dx) = \int u(x)\Delta\psi(x)\mathcal{H}^d(dx)\,. \tag{3.14}$$

For an arbitrary point $y \in \Omega$, let $\psi(x) := \phi_\epsilon(y - x)$. Hence, if $\epsilon > 0$ is sufficiently small, then ψ has compact support in Ω and (3.14) implies that

$$(\Delta u)_\epsilon(y) = (\Delta u_\epsilon)(y). \tag{3.15}$$

Since by assumption $\Delta u \in L^p_{\mathrm{loc}}(\Omega)$, $(\Delta u)_\epsilon \to \Delta u$ in $L^p_{\mathrm{loc}}(\Omega)$ (see [**9**, p. 123, Theorem 1]), and thus $\Delta u_\epsilon \to \Delta u$ in $L^p_{\mathrm{loc}}(\Omega)$ as $\epsilon \to 0^+$.

Now we proceed to prove (3.13). Let $k \in \{0, \ldots, d-1\}$ satisfy $d - k \le p$. Let $\eta_0 \subset \Omega$ be a Borel set with $\mathcal{H}^d(\eta_0) = 0$. Let $\eta \subset \eta_0$ be a measurable subset with compact closure in Ω. Since $F_{d-1}(u, \cdot)$ is absolutely continuous, we obtain that $\Delta u(x) = 0$ for $\mathcal{H}^d$ almost every $x \in \eta$. Moreover, since $(\Delta u)^{d-k}$ is integrable, for any sufficiently small number $\delta > 0$, we can find an open set $\eta_\delta \supset \eta$ whose closure is a compact subset of Ω such that

$$\int_{\eta_\delta} (d^{-1}\Delta u(x))^{d-k}\mathcal{H}^d(dx) \le \delta\,.$$

Subsequently, we will use Newton's inequality in the form

$$\binom{d}{k}^{-1} \sigma_{d-k}(D^2 u(x)) \le (d^{-1}\Delta u(x))^{d-k} \tag{3.16}$$

for $k \in \{0, \ldots, d-1\}$, which holds for $\mathcal{H}^d$-almost every $x \in \Omega$. Applying the vague convergence of the Hessian measures and (3.16), we get

$$\begin{aligned} F_k(u,\eta) \le F_k(u,\eta_\delta) &\le \liminf_{\epsilon\to 0^+} F_k(u_\epsilon,\eta_\delta) \\ &= \liminf_{\epsilon\to 0^+} \int_{\eta_\delta} \binom{d}{k}^{-1} \sigma_{d-k}(D^2 u_\epsilon(x))\mathcal{H}^d(dx) \\ &\le \liminf_{\epsilon\to 0^+} \int_{\eta_\delta} \left(d^{-1}\Delta u_\epsilon(x)\right)^{d-k} \mathcal{H}^d(dx) \\ &= \int_{\eta_\delta} \left(d^{-1}\Delta u(x)\right)^{d-k} \mathcal{H}^d(dx) \le \delta\,. \end{aligned}$$

We conclude that $F_k(u,\eta) = 0$, and hence by the monotone convergence thereom we obtain $F_k(u,\eta_0) = 0$.

The second part of the assertion (3.13) is now an immediate consequence of Theorem 2, the first part of the assertion (3.13) and (3.16). □

As an easy consequence of Theorem 3 and its proof, we also have the following analogue of Satz 4.7 in [**19**].

COROLLARY 2. *Let Ω be an open convex subset of $\mathbb{R}^d$, and let $u:\Omega\to\mathbb{R}$ be a convex function. Then*

$$F_{d-1}(u,\cdot) \ll \mathcal{H}^d \quad \textit{and} \quad \Delta u \le C \text{ a.e. in } \Omega\,, \tag{3.17}$$

for some constant $C \ge 0$, if and only if there is a convex function $v:\Omega\to\mathbb{R}$ and a constant $C' \ge 0$ such that

$$u(x) + v(x) = C'\|x\|^2\,, \quad x\in\Omega\,. \tag{3.18}$$

If one of these conditions is satisfied, then $F_k(u,\cdot) \ll \mathcal{H}^d$ with bounded density for $k \in \{0,\dots,d-2\}$.

Proof. First, assume that (3.17) is satisfied. Let Ω' be an open convex set with compact closure in Ω. The assumptions and equation (3.15) yield that $\Delta u_\epsilon(x) \le C$ for $x\in\Omega'$ and sufficiently small $\epsilon > 0$. Hence the Hessian matrix of the function $v_\epsilon := (C/2)\|\cdot\|^2 - u_\epsilon$ on Ω' is positive semidefinite, and therefore v_ϵ is convex on Ω'. Since $u_\epsilon \to u$ as $\epsilon \to 0^+$, the function $v := (C/2)\|\cdot\|^2 - u$ is also convex.

For the converse, we observe that by approximation with smooth functions, we get

$$F_{d-1}(u+v,\cdot) = F_{d-1}(u,\cdot) + F_{d-1}(v,\cdot)$$

for any convex functions u, v on Ω. The assertion (3.17) now follows, since (3.18) implies that $F_{d-1}(u+v,\cdot) = 2C'\mathcal{H}^d$ and since $F_{d-1}(v,\cdot) \ge 0$. □

We can also derive Satz 4.7 in [**19**] by replacing the usual convolution employed above by the regularization of support functions described in Theorem 3.3.1 in [**15**]. A suitable modification of the proof of Corollary 2 then implies the following result which is equivalent to Weil's theorem.

COROLLARY 3. *Let $u:\mathbb{R}^d\to\mathbb{R}$ be a support function. Then*

$$F_{d-1}(u,\cdot) \ll \mathcal{H}^d \quad \textit{and} \quad \Delta u(x) \le C\|x\|^{-1} \text{ for a.e. } x \text{ in } \mathbb{R}^d \tag{3.19}$$

and some constant $C \ge 0$, if and only if there is a support function $v:\mathbb{R}^d\to\mathbb{R}$ and a *constant* $C' \geq 0$ *such that*

$$u(x) + v(x) = C'\|x\|, \quad x \in \mathbb{R}^d.$$

Proof. First, let (3.19) be satisfied. For $\epsilon > 0$, we denote by u_ϵ the regularization of u defined as in [**15**, Theorem 3.3.1]. Then u_ϵ is a support function of class C^∞ on $\mathbb{R}^d \setminus \{0\}$, $u_\epsilon \to u$ uniformly on compact subsets of $\mathbb{R}^d$, and thus as in the proof of Theorem 3 we obtain that $(\Delta u)_\epsilon(y) = (\Delta u_\epsilon)(y)$, $y \in \mathbb{R}^d$. Hence, by the particular form of the regularization and by (3.19), we get, for $x \neq 0$ and $\epsilon \leq 1/2$,

$$\begin{aligned}(\Delta u_\epsilon)(x) &= \int_{\mathbb{R}^d} \Delta u(x + \|x\|z)\varphi_\epsilon(\|z\|)\mathcal{H}^d(dz) \\ &\leq C \int_{\mathbb{R}^d} \|x + \|x\|z\|^{-1}\varphi_\epsilon(\|z\|)\mathcal{H}^d(dz)\end{aligned}$$

where φ_ϵ denotes the mollifier defined in [**15**, Theorem 3.3.1] for the parameter ϵ. If $\varphi_\epsilon(\|z\|) \neq 0$, then $\|z\| \leq \epsilon \leq 1/2$ and hence $\|x + \|x\|z\| \geq (1/2)\|x\|$, i.e.

$$(\Delta u_\epsilon)(x) \leq 2C\|x\|^{-1}, \qquad x \in \mathbb{R}^d \setminus \{0\}. \tag{3.20}$$

From (3.20) we can conclude that the function $v_\epsilon := 2C\|\cdot\| - u_\epsilon$ has a positive semidefinite Hessian matrix on $\mathbb{R}^d \setminus \{0\}$. Indeed, for $x \neq 0$ we have

$$D^2 v_\epsilon(x) = 2C\frac{1}{\|x\|}\left(\delta_{ij} - \frac{x_i x_j}{\|x\|^2}\right)_{i,j=1}^d - D^2 u_\epsilon(x).$$

Thus $D^2 v_\epsilon(x)$ has the eigenvalue 0 and further $d-1$ nonnegative eigenvalues. This implies that v_ϵ is convex on convex subsets of $\mathbb{R}^d \setminus \{0\}$, which yields the convexity of v_ϵ on $\mathbb{R}^d$ by a continuity argument. Hence $v := 2C\|\cdot\| - u$ is a support function.

For the converse, we can proceed as in the proof of Corollary 2, since we also have $F_{d-1}(u, \{0\}) = 0$. □

Similarly as Theorem 2, we can generalize Theorem 3 to semi-convex functions.

COROLLARY 4. *Theorem 3 remains true if* $u : \Omega \to \mathbb{R}$ *is semi-convex.*

Proof. We use the same notation as in the proof of Corollary 1. By assumption, we have $F_{d-1}(u, \cdot) \ll \mathcal{H}^d$ and $\Delta u \in L^p_{\mathrm{loc}}(\Omega)$. The following consideration can again be restricted to a relatively compact subset Ω' of Ω. A special case of (3.12) yields that

$$F_{d-1}(u, \cdot) = F_{d-1}(v, \cdot) - CF_d(v, \cdot) = F_{d-1}(v, \cdot) - C\mathcal{H}^d;$$

hence $F_{d-1}(v, \cdot) \ll \mathcal{H}^d$. Since $\Delta v = \Delta u + C \in L^p(\Omega')$, Theorem 3 implies that, for $k \leq d$ such that $d - k \leq p$,

$$F_k(v, \cdot) \ll \mathcal{H}^d \quad \text{and} \quad \sigma_{d-k}(D^2 v) \in L^q(\Omega')$$

for every $q \geq 1$ such that $(d-k)q \leq p$. For any $l \leq d$ such that $d - l \leq p$, we thus get

$$F_l(u, \cdot) = \sum_{i=0}^{d-l} \binom{d-l}{i} (-C)^i F_{l+i}(v, \cdot) \ll \mathcal{H}^d$$

on Ω'. The required integrability follows from the relation

$$\sigma_{d-l}(D^2 u) = \sum_{i=0}^{d-l} \binom{l+i}{i} (-C)^i \sigma_{d-l-i}(D^2 v),$$

which holds $\mathcal{H}^d$-almost everywhere in Ω', since $\sigma_{d-l-i}(D^2 v) \in L^q(\Omega')$ for $i \le d-l$ and $(d-l)q \le p$. □

Theorem 3 can be used to give a simplified proof (on the basis of the results for Hessian measures) of a theorem due to Weil [**19**], which concerns surface area measures of convex bodies. A corresponding result for curvature measures has been deduced recently in [**12**] from the one for surface area measures. However, it does not seem to be possible to derive the result for curvature measures directly from Theorem 3. In the following, we write $R_j(K,\cdot)$ for the Radon-Nikodym derivative of the jth surface area measure $S_j(K,\cdot)$ of a convex body $K \in \mathcal{K}^d_o$ with respect to the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}$ on S^{d-1}. We state the following result in a global form, but it is clear from the arguments presented that all statements can be localized as in [**19**].

THEOREM (Weil). *Let $K \in \mathcal{K}^d_o$ satisfy*

$$S_1(K,\cdot) \ll \mathcal{H}^{d-1} \quad \textit{and} \quad R_1(K,\cdot) \in L^p(S^{d-1})$$

for some $p \ge 2$. Then, for every $k \in \{2,\dots,d-1\}$ such that $k \le p$,

$$S_k(K,\cdot) \ll \mathcal{H}^{d-1} \quad \textit{and} \quad R_k(K;\cdot) \in L^q(S^{d-1})$$

for every $q \ge 1$ such that $qk \le p$.

Proof. The assertion of the theorem follows from Theorem 3, once we have shown that the conditions

(a) $F_j(h_k,\cdot) \ll \mathcal{H}^d$ and $\sigma_{d-j}(D^2 h_K) \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$

and

(b) $S_{d-j}(K,\cdot) \ll \mathcal{H}^{d-1}$ and $R_{d-j}(K,\cdot) \in L^p(S^{d-1})$

are equivalent for $j \in \{1,\dots,d-1\}$.

First, assume that (b) holds. Let $\eta \subset \mathbb{R}^d$ be a Borel set. Then, using Lemma 1 and introducing polar coordinates, we get

$$\begin{aligned} dF_j(h_K,\eta) &= j\int_0^\infty \int_{S^{d-1}} \mathbf{1}\{r\nu \in \eta\} r^{j-1} R_{d-j}(K,\nu)\mathcal{H}^{d-1}(d\nu)dr \\ &= j\int_{\mathbb{R}^d} \mathbf{1}\{x \in \eta\}\|x\|^{j-d} R_{d-j}(K, x/\|x\|)\mathcal{H}^d(dx), \end{aligned}$$

which shows that $F_j(h_K,\cdot) \ll \mathcal{H}^d$ and

$$\sigma_{d-j}(D^2 h_K(x)) = \|x\|^{j-d}\binom{d-1}{j} R_{d-j}(K, x/\|x\|)$$

for $\mathcal{H}^d$-almost every $x \in \mathbb{R}^d$. Now the required integrability property follows easily.

Conversely, assume that (a) is satisfied. Let $\omega \subset S^{d-1}$ be a Borel set and $0 < a < b < \infty$. Then, applying Lemma 1, introducing polar coordinates and using the fact that $D^2 h_K$ is positively homogeneous of degree -1, we get that

$$\begin{aligned} S_{d-j}(K,\omega) &= \frac{d}{b^j - a^j}\int_{\mathbb{R}^d} \mathbf{1}\{\|x\| \in [a,b], x/\|x\| \in \omega\}\binom{d}{j}^{-1} \sigma_{d-j}(D^2 h_K(x))\mathcal{H}^d(dx) \\ &= \binom{d-1}{j}^{-1}\int_\omega \sigma_{d-j}(D^2 h_K(\nu))\mathcal{H}^{d-1}(d\nu), \end{aligned}$$

from which (b) follows as above. □

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Andrea Colesanti
Dipartimento di Matematica
Università degli Studi di Firenze
Viale Morgagni 67/A
50134 Firenze
Italy

andrea.colesanti@bb.math.unifi.it

Daniel Hug
Mathematisches Institut
Albert-Ludwigs-Universität
Eckerstr. 1
D-79104 Freiburg
Germany

daniel.hug@math.uni-freiburg.de