LARGE TYPICAL CELLS IN POISSON–DELAUNAY MOSAICS

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Dedicated to Tudor Zamfirescu on the occasion of his sixtieth birthday

It is proved that the shape of the typical cell of a Poisson–Delaunay tessellation of $\mathbb{R}^d$ tends to the shape of a regular simplex, given that the surface area, or the inradius, or the minimal width, of the typical cell tends to infinity. Typical cells of large diameter tend to belong to a special class of simplices, distinct from the regular ones. In the plane, these are the right-angled triangles.

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1. INTRODUCTION

In Stochastic Geometry, random tessellations of $\mathbb{R}^d$, also known as random mosaics, are a frequently studied topic. This is due to their various practical applications in two and three dimensions, but also to their great geometric appeal. Introductions to random tessellations are found in [13, ch. 5], [15, ch. 10], [14, ch. 6], an important source is [12]. Particularly tractable are the random tessellations that are derived from stationary Poisson processes, either in the space of hyperplanes, leading to Poisson hyperplane tessellations, or in $\mathbb{R}^d$, where the corresponding Voronoï tessellations and their duals, the Delaunay tessellations, yield interesting classes of random mosaics. For such mosaics, the asymptotic shape of large cells has recently become an issue of investigation. The starting point was a conjecture of D.G. Kendall (see [15], foreword to the first edition), according to which

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the shape of the zero cell of a stationary, isotropic Poisson line tessellation in the plane, given that the area of the cell tends to infinity, should tend to circular shape. Contributions to the planar case in [2, 7, 8, 9, 11], including an affirmative answer by Kovalenko, were followed by higher dimensional versions and extensions in various directions, see [4, 6, 5]. Already in [11] and [5], the size of the zero cell was not only measured by the volume, but also by other functions, like intrinsic volumes or the inradius. In work in progress it has turned out that for stationary Poisson hyperplane tessellations (not necessarily isotropic) and for Poisson–Voronoï tessellations, asymptotic shapes of large zero cells can be determined for quite general interpretations of 'large', but that these shapes may depend on the function by which the size is measured. For example, large cells in the sense of diameter have shapes degenerating to segments. In the case of a Poisson–Delaunay mosaic, all cells are simplices, with probability one. It was proved in [6] that the asymptotic shapes of typical cells of large volume are regular simplices. In the present paper, we exhibit the same phenomenon for further functions, the surface area, the inradius, and the minimal width. If the size is measured by the diameter, the asymptotic shapes of large typical cells are no longer those of regular simplices, but of simplices for which one edge contains the centre of the circumscribed sphere. In the plane, this class consists of the right-angled triangles. Both types of results are special cases of a general theorem, where 'large' refers to an abstract size function. The paper concludes with a result on the asymptotic distribution of the size of the typical cell.

2. RESULTS

We refer to [14] for details about Poisson–Delaunay tessellations, but we recall here the basic definitions. Let X be a stationary Poisson point process in $\mathbb{R}^d$ with positive intensity. With probability one, any $d+1$ points $x_0, \dots, x_d$ of X lie on the boundary of a unique ball. If the interior of this ball contains no other point of X, then the simplex $\operatorname{conv}\{x_0, \dots, x_d\}$ is called a *cell*. The collection Y of all cells obtained in this way is a tessellation of $\mathbb{R}^d$, called the *Poisson–Delaunay tessellation* induced by X.

For a d-dimensional simplex $S \subset \mathbb{R}^d$, we denote by $z(S)$ the *circumcentre*, that is, the centre of the sphere through the vertices of S, and by $r(S)$ the radius of that sphere. Let Δ_0 be the space of all d-simplices in $\mathbb{R}^d$ with $z(S) = 0$, equipped with the Hausdorff metric δ.

Let Y be the Poisson–Delaunay tessellation induced by X, considered as a stationary particle process. As such, it has a grain distribution $\mathbb{Q}_0$. This is a probability measure on Δ_0 (that is, on the Borel σ-algebra of Δ_0 which is induced by the topology of the Hausdorff metric). Explicitly, choosing any convex body W with 0 in the interior,

$$\begin{aligned}\mathbb{Q}_0(\mathcal{A}) &= \frac{\mathbb{E}\,\mathrm{card}\{S \in Y : z(S) \in W,\, S - z(S) \in \mathcal{A}\}}{\mathbb{E}\,\mathrm{card}\{S \in Y : z(S) \in W\}} \\ &= \lim_{r\to\infty} \frac{\mathrm{card}\{S \in Y : z(S) \in rW,\, S - z(S) \in \mathcal{A}\}}{\mathrm{card}\{S \in Y : z(S) \in rW\}},\end{aligned}$$

for Borel sets $\mathcal{A} \subset \Delta_0$, where $\mathbb{E}$ denotes mathematical expectation and the second equality holds with probability one, due to the fact that the Poisson–Delaunay mosaic Y is ergodic. The *typical cell* of Y is, by definition, the random simplex with distribution $\mathbb{Q}_0$; thus, the typical cell is only determined up to stochastic equivalence. The intuitive heuristic idea behind this is that one takes a large region in a realization of Y, picks out at random one of the cells within this region (with equal chances), translates it so that its circumcentre becomes the origin, and thus obtains a realization of the typical cell.

We want to study asymptotic shapes of large typical cells. In order to be able to interpret 'large' in different ways, we introduce a class of abstract functions for measuring the size. By a *size function* we understand a positive function $\Sigma : \Delta_0 \to \mathbb{R}$ which has the following properties (V_d denotes volume):

- Σ is continuous.
- Σ is homogeneous of some degree $k > 0$.
- On the set of d-simplices inscribed to the unit sphere, Σ attains a maximum, and $V_d/\Sigma^{1/k}$ is bounded.

The maximum that Σ attains on the set of d-simplices inscribed to the unit sphere will be denoted by τ. By homogeneity,

$$\Sigma(S) \le r(S)^k \tau \qquad \text{for all } S \in \Delta_0. \tag{1}$$

Every $S \in \Delta_0$ yielding equality in (1) will be called an *extremal simplex* (for the given Σ).

As we want to estimate the probability for large deviations of the shape of large typical cells from the shapes of extremal simplices, we need a measure for that deviation. By a *deviation function* for Σ we understand a nonnegative function $\vartheta : \Delta_0 \to \mathbb{R}$ with the following properties:

- ϑ is continuous.
- ϑ is homogeneous of degree zero.
- $\vartheta(S) = 0$ if and only if S is an extremal simplex.

By a *stability function* for Σ and ϑ we understand a continuous function $f : [0,1) \to [0,1]$ with the properties $f(0) = 0$, $f(\epsilon) > 0$ for $\epsilon > 0$ and

$$\Sigma(S) \leq (1 - f(\epsilon)) r(S)^k \tau \qquad \text{for all } S \in \Delta_0 \text{ with } \vartheta(S) \geq \epsilon. \tag{2}$$

Now we can formulate our main result. By $\mathbb{P}$ we denote the underlying probability, and $\mathbb{P}(\cdot \mid \cdot)$ is a conditional probability.

THEOREM 1. *Let Z be the typical cell of the Poisson–Delaunay tessellation derived from a stationary Poisson process with intensity $\lambda > 0$ in $\mathbb{R}^d$. Suppose that functions Σ, ϑ, f with the properties listed above are given. There is a constant c_0 depending only on these functions and the dimension d such that the following is true. If $\epsilon \in (0,1)$ and $I = [a,b)$ is an interval ($b = \infty$ allowed) with $a^{d/k}\lambda \geq \sigma_0 > 0$ for some constant σ_0, then*

$$\mathbb{P}(\vartheta(Z) \geq \epsilon \mid \Sigma(Z) \in I) \leq c \exp\left\{-c_0 f(\epsilon) a^{d/k} \lambda\right\},$$

where c is a constant depending only on $d, \epsilon, \Sigma, \vartheta, f, \sigma_0$.

It follows from Theorem 1 that

$$\lim_{a \to \infty} \mathbb{P}(\vartheta(Z) < \epsilon \mid \Sigma(Z) \geq a) = 1$$

for every $\epsilon \in (0,1)$. Thus, the shapes of typical cells of large size have small deviation from the shape of extremal simplices, with high probability. Here, it does not matter how the deviation is measured. In this sense, the shapes of extremal simplices are the asymptotic shapes of typical cells of large size. In concrete cases, where the deviation function is explicit and has an intuitive geometric meaning, the estimate of Theorem 1 provides much stronger information, and even more so when a stability function is explicitly known. We consider some special cases of this type.

In [6] the case of the volume, $\Sigma = V_d$, was treated. The extremal simplices in that case are precisely the regular simplices. A measure of deviation of the shape of a simplex from the shape of a regular simplex can be defined as follows. Let $v_0, \dots, v_d \in S^{d-1}$ be points such that $T^d =$

$\text{conv}\{v_0, \dots, v_d\}$ is a regular simplex. Let S be a simplex in $\mathbb{R}^d$. We define $\vartheta_1(S)$ as the smallest number α with the following property. There are points $x_0, \dots, x_d \in S^{d-1}$ such that $\text{conv}\{x_0, \dots, x_d\}$ is similar to S and $\|x_i - v_i\| \leq \alpha$ for $i = 0, \dots, d$. Clearly, ϑ_1 has the properties that we require of a deviation function for V_d. The following stability estimate was proved in [6]. There is a positive constant $c_1(d)$ such that, for every $\epsilon \in [0,1]$ and for every simplex $S \subset \mathbb{R}^d$,

$$(3) \qquad V_d(S) \leq (1 - c_1(d)\epsilon^2) r(S)^d \tau \qquad \text{for all } S \in \Delta_0 \text{ with } \vartheta_1(S) \geq \epsilon$$

(recall that τ is always the maximum of Σ on the simplices inscribed to the unit sphere). With this, our present Theorem 1 reduces to Theorem 1 of [6].

The inclusion of any other concrete size functions requires the determination of the simplices inscribed to the unit sphere for which the size function attains its maximum. This is a purely geometric task, which may be difficult. For example, it seems to be unknown, for $d \geq 3$, whether the regular simplices yield the maximum for the mean width (see the discussion in Gritzmann and Klee [3, Section 9.10.2]). For the surface area, it follows from a more general result of Tanner [16] that the extremal simplices are the regular ones. For $d \geq 3$, we do not have an explicit stability estimate in this case. Nevertheless, the existence of a stability function (say, for the deviation function ϑ_1) can be shown, and then Theorem 1 is sufficient to ensure that typical cells of large surface area are asymptotically close to regular shape, with high probability. In the planar case, where the surface area reduces to the perimeter L (and also to π times the mean width), the following stability estimate will be proved in Section 4. Let $S \subset \mathbb{R}^2$ be a triangle inscribed to the unit circle, and let $\epsilon \in [0,1]$. Then

$$(4) \qquad L(S) \leq (1 - \epsilon^2/36)\tau \qquad \text{if } \vartheta_1(S) \geq \epsilon.$$

Tanner's [16] general result yields some other size functions for which the regular simplices are extremal, for example, the sum of the edge lengths.

Further functions which satisfy the requirements for a size function (as shown in Section 4) and for which the extremal simplices can be determined, are the inradius, the minimal width (or thickness), and the diameter. The inradius $\rho(S)$ of a simplex is the largest radius of a ball contained in S. The maximal simplices for the inradius are the regular ones, and the stability estimate

$$(5) \qquad \rho(S) \leq (1 - c_2(d)\epsilon^2) r(S) \tau \qquad \text{for all } S \in \Delta_0 \text{ with } \vartheta_1(S) \geq \epsilon$$

(with $c_2(d) = c_1(d)/d$) holds (see Section 4). Hence, in this case the estimate of Theorem 1 takes again an explicit form, with $\vartheta = \vartheta_1$ and $f(\epsilon)$ proportional to ϵ^2.

The minimal width $w(S)$ of a simplex is the minimal distance between any two parallel supporting hyperplanes of S. The maximal simplices are again the regular simplices, as shown by Alexander [1]. Also in this case, we must be satisfied with the mere existence of a stability function.

The situation changes if the size is measured by the diameter D. For a simplex S inscribed to the unit sphere we have $D(S) \leq 2$, and equality holds if and only if S has an edge of length 2. Generally, we say that a simplex S is *diametral* if its diameter is equal to the diameter of the circumscribed sphere. Thus, the maximal simplices for the size function $\Sigma = D$ are precisely the diametral simplices in Δ_0. We define a measure of deviation of the shape of a simplex from the shape of a diametral simplex. Let w_0, w_1 be two antipodal points of the unit sphere S^{d-1}. For a simplex in $S \subset \mathbb{R}^d$, let $\vartheta_2(S)$ be the smallest number α with the following property. There are points $x_0, \dots, x_d \in S^{d-1}$ such that $\operatorname{conv}\{x_0, \dots, x_d\}$ is similar to S, $x_0 = w_0$ and $\|x_1 - w_1\| \leq \alpha$. It is a trivial task to prove the stability estimate

$$(6) \qquad D(S) \leq (1 - \epsilon^2/8) r(S) \tau \qquad \text{for all } S \in \Delta_0 \text{ with } \vartheta_2(S) \geq \epsilon.$$

With this, Theorem 1 provides an explicit estimate for the deviation of the shape of typical cells of large diameter from the shape of diametral simplices.

3. PROOF OF THEOREM 1

For the proof of Theorem 1, we assume that a stationary Poisson process X in $\mathbb{R}^d$ with intensity $\lambda > 0$ and three functions Σ, ϑ, and f with the properties listed in Section 2 are given. We recall that the number k is the degree of homogeneity of Σ and the constant $\tau > 0$ is the maximal value of $\Sigma(S)$ for $S \in \Delta_0$ with $r(S) = 1$, according to (1). By σ we denote the spherical Lebesgue measure on the unit sphere S^{d-1}, and κ_d is the volume of the unit ball.

The proof to follow extends the one given in [6], for the case of the volume and for special explicit deviation and stability functions. We make use of the integral representation for the distribution $\mathbb{Q}_0$ due to Miles (see, e.g., [14, Satz 6.2.10]). For a Borel set $\mathcal{A} \in \Delta_0$,

$$\begin{aligned} \mathbb{Q}_0(\mathcal{A}) &= \alpha(d)\lambda^d \int_0^\infty \int_{S^{d-1}} \cdots \int_{S^{d-1}} \mathbf{1}_{\mathcal{A}}(\operatorname{conv}\{ru_0, \ldots, ru_d\}) \\ &\quad \times \mathrm{e}^{-\lambda\kappa_d r^d} r^{d^2-1} V_d(\operatorname{conv}\{u_0, \ldots, u_d\}) \, \mathrm{d}\sigma(u_0) \cdots \mathrm{d}\sigma(u_d) \, \mathrm{d}r \end{aligned}$$

with

$$\alpha(d) := \frac{d^2}{2^{d+1}\pi^{\frac{d-1}{2}}} \frac{\Gamma\left(\frac{d^2}{2}\right)}{\Gamma\left(\frac{d^2+1}{2}\right)} \left[\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}+1\right)}\right]^d .$$

For functions F defined on d-simplices, we use the abbreviation

$$F(\operatorname{conv}\{x_0, \ldots, x_d\}) =: F(x_0, \ldots, x_d).$$

In the following, $c_1, c_2, \ldots$ denote positive constants which may depend on d, the chosen functions Σ, ϑ, f, and the given number $\epsilon > 0$.

LEMMA 1. *Let $\epsilon > 0$ and define the number $h_0 = h_0(\epsilon)$ by $(1+h_0(\epsilon))^{d/k} = 1 + (\epsilon/(5+4\epsilon))$. Then there is a constant c_1 such that, for $0 < h \leq h_0$ and $a > 0$,*

$$\mathbb{P}(\Sigma(Z) \in a[1, 1+h]) \geq c_1 h (a^{d/k}\lambda)^d \exp\left\{-\frac{\kappa_d}{\tau^{d/k}}(1+\epsilon)a^{d/k}\lambda\right\}.$$

Proof. Let $\epsilon > 0$, $h \in (0, h_0]$ and $a > 0$ be given. Using the formula of Miles, we obtain

$$\begin{aligned} &\mathbb{P}(\Sigma(Z) \in a[1, 1+h]) \\ &= \alpha(d)\lambda^d \int_0^\infty \int_{S^{d-1}} \cdots \int_{S^{d-1}} \mathbf{1}\left\{r^k \Sigma(u_0, \ldots, u_d) \in a\,[1, (1+h)]\right\} \\ &\quad \times \mathrm{e}^{-\lambda\kappa_d r^d} r^{d^2-1} V_d(u_0, \ldots, u_d) \, \mathrm{d}\sigma(u_0) \cdots \mathrm{d}\sigma(u_d) \, \mathrm{d}r. \end{aligned}$$

Substituting $s = \lambda\kappa_d r^d$, we get

$$\begin{aligned} &\mathbb{P}\left(\Sigma(Z) \in a\,[1, 1+h]\right) \\ &= c_2 \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_0^\infty \mathbf{1}\left\{s \in \lambda\kappa_d (a/\Sigma(u_0, \ldots, u_d))^{d/k} [1, (1+h)^{d/k}]\right\} \\ &\quad \times \mathrm{e}^{-s} s^{d-1} \, \mathrm{d}s \, V_d(u_0, \ldots, u_d) \, \mathrm{d}\sigma(u_0) \cdots \mathrm{d}\sigma(u_d). \end{aligned}$$

For fixed $u_0, \dots, u_d \in S^{d-1}$ in general position, we apply to the inner integral the mean value theorem for integrals. This gives the existence of a number

$$\xi(u_0, \dots, u_d) \in \lambda\kappa_d(a/\Sigma(u_0, \dots, u_d))^{d/k}[1, (1+h)^{d/k}] \tag{7}$$

such that the inner integral is equal to

$$\begin{aligned} &\lambda\kappa_d(a/\Sigma(u_0, \dots, u_d))^{d/k}\left((1+h)^{d/k} - 1\right) \\ &\times \exp\{-\xi(u_0, \dots, u_d)\}\xi(u_0, \dots, u_d)^{d-1}. \end{aligned}$$

Using $(1+h)^{d/k} - 1 = (d/k)(1+\theta h)^{(d/k)-1}h$ with $0 \le \theta \le 1$ and $h \le h_0$ (in the case $d/k < 1$), we obtain

$$\begin{aligned} &\mathbb{P}\left(\Sigma(Z) \in a\,[1, 1+h]\right) \\ &= c_3\left((1+h)^{d/k} - 1\right) a^{d/k}\lambda \int_{S^{d-1}} \cdots \int_{S^{d-1}} \exp\{-\xi(u_0, \dots, u_d)\} \\ &\quad \times \xi(u_0, \dots, u_d)^{d-1} \frac{V_d(u_0, \dots, u_d)}{\Sigma(u_0, \dots, u_d)^{d/k}}\, \mathrm{d}\sigma(u_0) \cdots \mathrm{d}\sigma(u_d) \\ &\ge c_4 h a^{d/k}\lambda \underbrace{\int \cdots \int}_{R(d,\epsilon)} \exp\{-\xi(u_0, \dots, u_d)\}\xi(u_0, \dots, u_d)^{d-1} \\ &\quad \times \frac{V_d(u_0, \dots, u_d)}{\Sigma(u_0, \dots, u_d)^{d/k}}\, \mathrm{d}\sigma(u_0) \cdots \mathrm{d}\sigma(u_d), \end{aligned}$$

where we choose (up to a set of σ^{d+1}-measure zero)

$$\begin{aligned} R(d, \epsilon) := \Big\{(u_0, \dots, u_d) \in (S^{d-1})^{d+1} : \Sigma(u_0, \dots, u_d)^{d/k} \ge g(\epsilon)\tau^{d/k}, \\ V_d(u_0, \dots, u_d) \ge c_5\} \end{aligned}$$

with

$$g(\epsilon) := 1 - \frac{4\epsilon}{5 + 4\epsilon}$$

and a positive constant c_5 determined in the following way. The set of all $(u_0, \dots, u_d) \in (S^{d-1})^{d+1}$ with $\Sigma(u_0, \dots, u_d)^{d/k} > g(\epsilon)\tau^{d/k}$ is nonempty and open, hence we can choose c_5 in such a way that $\sigma^{d+1}(R(d, \epsilon)) =: c_6$ is positive.

For $(u_0,\dots,u_d)\in(S^{d-1})^{d+1}$ in general position we have $\Sigma(u_0,\dots,u_d)\le\tau$ and hence, by (7),

$$\xi(u_0,\dots,u_d)\ge\frac{\kappa_d}{\tau^{d/k}}a^{d/k}\lambda.$$

For $(u_0,\dots,u_d)\in R(d,\epsilon)$ in general position we can estimate

$$\xi(u_0,\dots,u_d)\le\frac{\kappa_d}{\tau^{d/k}}g(\epsilon)^{-1}(1+h_0)^{d/k}a^{d/k}\lambda$$

and

$$\frac{V_d(u_0,\dots,u_d)}{\Sigma(u_0,\dots,u_d)^{d/k}}\ge\frac{c_5}{\tau^{d/k}}.$$

Since $\sigma^{d+1}(R(d,\epsilon))=c_6$, we finally obtain

$$\begin{aligned}&\mathbb{P}\left(\Sigma(Z)\in a\,[1,1+h]\right)\\&\ge c_7h(a^{d/k}\lambda)^d\exp\left\{-\frac{\kappa_d}{\tau^{d/k}}g(\epsilon)^{-1}(1+h_0)^{d/k}a^{d/k}\lambda\right\}\\&=c_7h(a^{d/k}\lambda)^d\exp\left\{-\frac{\kappa_d}{\tau^{d/k}}(1+\epsilon)a^{d/k}\lambda\right\},\end{aligned}$$

where we used that

$$(1+h_0(\epsilon))^{d/k}=1+\frac{\epsilon}{5+4\epsilon}=\left(1-\frac{4\epsilon}{5+4\epsilon}\right)(1+\epsilon).$$

This completes the proof. □

LEMMA 2. *For each $\epsilon\in(0,1)$, there is a constant c_8 such that, for $h\in(0,1]$ and $a>0$,*

$$\begin{aligned}&\mathbb{P}(\Sigma(Z)\in a[1,1+h],\,\vartheta(Z)\ge\epsilon)\\&\le c_8h(a^{d/k}\lambda)^{1/d}\exp\left\{-\frac{\kappa_d}{\tau^{d/k}}(1+f(\epsilon)/2k)a^{d/k}\lambda\right\}.\end{aligned}$$

Proof. Similarly as in the proof of Lemma 1, we obtain

$$\begin{aligned}&\mathbb{P}(\Sigma(Z)\in a[1,1+h],\,\vartheta(Z)\ge\epsilon)\\&=\alpha(d)\lambda^d\int_{S^{d-1}}\cdots\int_{S^{d-1}}\int_0^\infty\\&\quad\mathbf{1}\left\{r\in(a/\Sigma(u_0,\dots,u_d))^{1/k}\left[1,(1+h)^{1/k}\right]\right\}\mathrm{e}^{-\lambda\kappa_dr^d}r^{d^2-1}\,\mathrm{d}r\\&\quad\times\mathbf{1}\{\vartheta(u_0,\dots,u_d)\ge\epsilon\}\,V_d(u_0,\dots,u_d)\,\mathrm{d}\sigma(u_0)\cdots\mathrm{d}\sigma(u_d).\end{aligned}$$

For fixed $u_0, \dots, u_d \in S^{d-1}$ in general position, there is a number

$$\xi(u_0, \dots, u_d) \in (a/\Sigma(u_0, \dots, u_d))^{1/k}[1, (1+h)^{1/k}] \tag{8}$$

such that the inner integral is equal to

$$(a/\Sigma(u_0, \dots, u_d))^{1/k} \left((1+h)^{1/k} - 1\right)$$
$$\times \exp\{-\lambda\kappa_d \xi(u_0, \dots, u_d)^d\} \xi(u_0, \dots, u_d)^{d^2-1}.$$

Estimating $(1+h)^{1/k} - 1 \le c_9 h$, we get

$$\mathbb{P}\left(\Sigma(Z) \in a\,[1, 1+h]\,,\, \vartheta(Z) \ge \epsilon\right)$$
$$\le c_{10} h a^{1/k} \lambda^d \int_{S^{d-1}} \cdots \int_{S^{d-1}} \exp\left\{-\lambda\kappa_d \xi(u_0, \dots, u_d)^d\right\} \xi(u_0, \dots, u_d)^{d^2-1}$$
$$\times\, \mathbf{1}\{\vartheta(u_0, \dots, u_d) \ge \epsilon\}\, \frac{V_d(u_0, \dots, u_d)}{\Sigma(u_0, \dots, u_d)^{1/k}}\, \mathrm{d}\sigma(u_0) \cdots \mathrm{d}\sigma(u_d).$$

By our assumptions on Σ,

$$\frac{V_d(u_0, \dots, u_d)}{\Sigma(u_0, \dots, u_d)^{1/k}} \le c_{11}. \tag{9}$$

If

$$\mathbf{1}\{\vartheta(u_0, \dots, u_d) \ge \epsilon\} \ne 0, \tag{10}$$

then, using (8) and (2),

$$\xi(u_0, \dots, u_d) \ge \frac{a^{1/k}}{\Sigma(u_0, \dots, u_d)^{1/k}} \ge \frac{a^{1/k}}{(1 - f(\epsilon))^{1/k} \tau^{1/k}} \ge \frac{a^{1/k}}{\tau^{1/k}} (1 + f(\epsilon))^{1/k}.$$

We determine $g(\epsilon)$ by

$$(1 - g(\epsilon))(1 + f(\epsilon))^{d/k} = 1 + f(\epsilon)/2k.$$

Since $(1 + f(\epsilon))^{d/k} > 1 + f(\epsilon)/2k$ (using $f(\epsilon) \le 1$ if $k > d$), we have $0 < g(\epsilon) < 1$. There exists a constant c_{12} such that, for $\xi \ge 0$,

$$\exp\left\{-\lambda\kappa_d \xi^d\right\} \xi^{d^2-1} \le c_{12} \lambda^{-(d^2-1)/d} \exp\left\{-(1 - g(\epsilon))\lambda\kappa_d \xi^d\right\}.$$

This shows that if (10) is satisfied, then

$$\begin{aligned}&\exp\{-\lambda\kappa_d\xi(u_0,\ldots,u_d)^d\}\xi(u_0,\ldots,u_d)^{d^2-1}\\&\leq c_{12}\lambda^{-(d^2-1)/d}\exp\left\{-\frac{\kappa_d}{\tau^{d/k}}(1-g(\epsilon))(1+f(\epsilon))^{d/k}a^{d/k}\lambda\right\}.\end{aligned}$$

Finally, this yields

$$\begin{aligned}&\mathbb{P}(\Sigma(Z)\in a[1,1+h],\,\vartheta(Z)\geq\epsilon)\\&\leq c_{13}h(a^{d/k}\lambda)^{1/d}\exp\left\{-\frac{\kappa_d}{\tau^{d/k}}(1+f(\epsilon)/2k)a^{d/k}\lambda\right\}.\end{aligned}$$

□

Let $\epsilon\in(0,1)$. In Lemma 1, we can replace ϵ by $f(\epsilon)/4k$. Then it follows that there exists a constant c_{14} (now depending also on f) such that, for $0<h\leq h_0:=h_0(f(\epsilon)/4k)$ and $a>0$,

$$\mathbb{P}(\Sigma(Z)\in a[1,1+h])\geq c_{14}h(a^{d/k}\lambda)^d\exp\left\{-\frac{\kappa_d}{\tau^{d/k}}(1+f(\epsilon)/4k)a^{d/k}\lambda\right\}.$$

From this result and Lemma 2, Theorem 1 is now deduced in precisely the same way as in [6] Theorem 1 was deduced from Lemmas 2 and 3.

4. SPECIAL CASES

For the special cases of size functionals Σ considered in Section 2, we have to show that they satisfy the requirements. Continuity and homogeneity are trivial, and also the existence of a maximum on the set of d-simplices inscribed to the unit sphere is clear in each case. Hence, it remains to show that $V_d/\Sigma^{1/k}$ is bounded on the d-simplices S inscribed to S^{d-1}. For the case of the surface area A, an estimate $V_d(S)/A(S)^{1/(d-1)}\leq c(d)$ follows from the isoperimetric inequality if $A(S)\leq 1$ and is trivial if $A(S)\geq 1$. Similarly one can argue for the diameter, using the isodiametric inequality. From this case, the corresponding assertion for the sum of the edge lengths is obtained. To treat the inradius, let $F_0,\ldots,F_d$ be the facets of the simplex S inscribed to the unit sphere, and let $|F_i|$ be the $(d-1)$-volume of F_i. Then $V_d(S)/\rho(S)=(|F_0|+\cdots+|F_d|)/d$ is bounded from above by a constant depending only on d. For the minimal width w, we have $V_d(S)/w(S)\leq\kappa_{d-1}$.

The stability estimate (3) was proved in [6]. From this, (5) is obtained as follows. Let S be a d-simplex inscribed to the unit sphere. Among all simplices of given inradius, the regular ones have the smallest volume (a proof is indicated, e.g., in [10, p. 318]). From this fact and from (3) we conclude for $\vartheta(S) \geq \epsilon$ that

$$\left(\frac{\rho(S)}{\rho(T^d)}\right)^d V_d(T^d) \leq V_d(S) \leq (1 - c_1(d)\epsilon^2)V_d(T^d),$$

where T^d is a regular simplex inscribed to the unit sphere. This yields (5). As mentioned, (6) is easily obtained. The remaining stability estimate (4) is proved in Lemma 3 below.

In the case where Σ is the surface area or the minimal width, we do not have an explicit stability estimate, but the existence of a stability function, say for the deviation function ϑ_1, is easy to see. For this, let $\epsilon \in (0,1)$ be given, and let A_ϵ be the closure of the set $\{S \in \Delta_0 : r(S) = 1,\, \vartheta_1(S) \geq \epsilon\}$. This set is compact, hence the continuous function Σ attains a maximum M_ϵ on this set. Since the maximum is not attained at a regular simplex, we have $M_\epsilon < \tau$ and can put $f(\epsilon) := 1 - M_\epsilon/\tau$. The function f defined in this way, together with $f(0) := 0$, has the required properties.

LEMMA 3. *Let S be a triangle inscribed to S^1, and let $\epsilon \in [0,1]$. Then*

$$L(S) \leq (1 - \epsilon^2/36)L(T^2) \qquad \text{if } \vartheta_1(S) \geq \epsilon.$$

Proof. Let S be a triangle inscribed to S^1 and satisfying the inequality $L(S) > (1 - \epsilon^2/36)L(T^2)$, where T^2 is a regular triangle inscribed to S^1, hence $L(T^2) = 3\sqrt{3}$. Then $0 \in \operatorname{int} S$. Let $2\alpha, 2\beta, 2\gamma$ be the angles at 0 spanned by the edges of S. Then

$$L(S) = 2(\sin\alpha + \sin\beta + \sin\gamma)$$

and $\alpha + \beta + \gamma = \pi$. We can choose the notation in such a way that the angles $\varphi := \alpha - \pi/3$ and $\psi := \beta - \pi/3$ are either both non-negative or both non-positive. We get

$$\begin{aligned} &L(S) - L(T^2) \\ &= 2\left[\sin\left(\frac{\pi}{3} + \varphi\right) + \sin\left(\frac{\pi}{3} + \psi\right) + \sin\left(\frac{2\pi}{3} + \varphi + \psi\right)\right] - 3\sqrt{3} \\ &= \sqrt{3}[\cos\varphi + \cos\psi + \cos(\varphi+\psi)] + \sin\varphi + \sin\psi - \sin(\varphi+\psi) - 3\sqrt{3}, \end{aligned}$$

hence

$$L(S) - L(T^2) = 2\sqrt{3}\left(\cos^2\frac{\varphi}{2} + \cos^2\frac{\psi}{2} - 2\right) + \Delta \tag{11}$$

with

$$\Delta := \sqrt{3}\cos(\varphi+\psi) - \sqrt{3} + \sin\varphi + \sin\psi - \sin(\varphi+\psi).$$

We show that $\Delta \le 0$. For this, we rewrite Δ as follows:

$$\begin{aligned}
\Delta &= \sqrt{3}\left(2\cos^2\frac{\varphi+\psi}{2} - 2\right) + \sin\varphi + \sin\psi - \sin(\varphi+\psi) \\
&= -2\sqrt{3}\sin^2\frac{\varphi+\psi}{2} + 2\sin\frac{\varphi+\psi}{2}\cos\frac{\varphi-\psi}{2} - 2\sin\frac{\varphi+\psi}{2}\cos\frac{\varphi+\psi}{2} \\
&= -2\sqrt{3}\sin^2\frac{\varphi+\psi}{2} + 2\sin\frac{\varphi+\psi}{2}\left(\cos\frac{\varphi-\psi}{2} - \cos\frac{\varphi+\psi}{2}\right) \\
&= -2\sqrt{3}\sin^2\frac{\varphi+\psi}{2} + 2\sin\frac{\varphi+\psi}{2} 2\sin\frac{\varphi}{2}\sin\frac{\psi}{2} \\
&= 4\sin\frac{\varphi+\psi}{2}\left(\sin\frac{\varphi}{2}\sin\frac{\psi}{2} - \frac{1}{2}\sqrt{3}\sin\frac{\varphi+\psi}{2}\right).
\end{aligned}$$

Here we have

$$\begin{aligned}
\sin\frac{\varphi}{2}\sin\frac{\psi}{2} - \frac{1}{2}\sqrt{3}\sin\frac{\varphi+\psi}{2} &= \sin\frac{\varphi}{2}\sin\left(\frac{\psi}{2} - \frac{\pi}{3}\right) + \sin\frac{\psi}{2}\sin\left(\frac{\varphi}{2} - \frac{\pi}{3}\right) \\
&= \sin\frac{\varphi}{2}\sin\left(\frac{\beta}{2} - \frac{\pi}{2}\right) + \sin\frac{\psi}{2}\sin\left(\frac{\alpha}{2} - \frac{\pi}{2}\right) \\
&= -\left(\sin\frac{\varphi}{2}\cos\frac{\beta}{2} + \sin\frac{\psi}{2}\cos\frac{\alpha}{2}\right)
\end{aligned}$$

and therefore

$$\Delta = -4\sin\frac{\varphi+\psi}{2}\left(\sin\frac{\varphi}{2}\cos\frac{\beta}{2} + \sin\frac{\psi}{2}\cos\frac{\alpha}{2}\right).$$

Since either $\varphi \ge 0$, $\psi \ge 0$ or $\varphi \le 0$, $\psi \le 0$ (and $|\varphi+\psi| < \pi$) and $\alpha/2, \beta/2 \in [0, \pi/2]$, we deduce that $\Delta \le 0$, as asserted. Thus (11) yields

$$-\frac{\epsilon^2}{36}3\sqrt{3} < L(S) - L(T^2) \le 2\sqrt{3}\left(\cos^2\frac{\varphi}{2} - 1\right) \le -2\sqrt{3}\cdot\frac{1}{6}\varphi^2,$$

where the latter inequality follows from $|\varphi/2| < \pi/6$ and the Taylor formula. This gives $|\varphi| < \epsilon/2$ and similarly $|\psi| < \epsilon/2$. From this, we deduce that $\vartheta_1(S) < \epsilon$. □

5. ON THE DISTRIBUTION OF THE SIZE OF LARGE CELLS

Lemmas 1 and 2 also permit us to obtain a limit relation for the probability $\mathbb{P}(\Sigma(Z) \geq a)$. The analogue for the area of the zero cell of a stationary, isotropic Poisson line process in the plane was first obtained by Goldman [2].

THEOREM 2. *Let Z be the typical cell of the Poisson–Delaunay tessellation derived from a stationary Poisson process in $\mathbb{R}^d$ with intensity $\lambda > 0$. Let Σ be a size function, of homogeneity k and with maximum τ on the simplices inscribed to the unit sphere. Then*

$$\lim_{a\to\infty} a^{-d/k} \ln \mathbb{P}(\Sigma(Z) \geq a) = -\frac{\kappa_d}{\tau^{d/k}}\lambda.$$

Proof. For $\epsilon > 0$ and $a > 0$, Lemma 1 with $h = h_0(\epsilon)$ gives

$$\mathbb{P}(\Sigma(Z) \geq a) \geq c_{15}(a^{d/k}\lambda)^d \exp\left\{-\frac{\kappa_d}{\tau^{d/k}}(1+\epsilon)a^{d/k}\lambda\right\},$$

hence

$$\liminf_{a\to\infty} a^{-d/k} \ln \mathbb{P}(\Sigma(Z) \geq a) \geq -\frac{\kappa_d}{\tau^{d/k}}(1+\epsilon)\lambda.$$

With $\epsilon \to 0$, this yields

$$\liminf_{a\to\infty} a^{-d/k} \ln \mathbb{P}(\Sigma(Z) \geq a) \geq -\frac{\kappa_d}{\tau^{d/k}}\lambda. \tag{12}$$

From the first part of the proof of Lemma 2, where we choose $h = 1$, omit the condition $\vartheta(Z) \geq \epsilon$, and use the estimate (9), we obtain

$$\begin{aligned}
&\mathbb{P}(\Sigma(Z) \in a[1,2]) \\
&\leq c_{16} a^{1/k}\lambda^d \int_{S^{d-1}} \cdots \int_{S^{d-1}} \exp\left\{-\lambda\kappa_d \xi(u_0,\dots,u_d)^d\right\} \xi(u_0,\dots,u_d)^{d^2-1} \\
&\quad \times \mathrm{d}\sigma(u_0)\cdots\mathrm{d}\sigma(u_d).
\end{aligned}$$

where $\xi(u_0,\dots,u_d)$ satisfies (8) (for $h = 1$).

For fixed $u_0,\dots,u_d \in S^{d-1}$ in general position, it follows from (8) and $\Sigma(u_0,\dots,u_d) \leq \tau$ that

$$\xi(u_0,\dots,u_d) \geq (a/\tau)^{1/k}.$$

Let $\epsilon \in (0, 1/3)$. There exists a constant c_{17} such that, for $\xi \geq 0$,

$$\exp\left\{-\lambda\kappa_d\xi^d\right\}\xi^{d^2-1} \leq c_{17}\lambda^{-(d^2-1)/d}\exp\left\{-(1-\epsilon)\lambda\kappa_d\xi^d\right\}.$$

These estimates together imply that, for $\xi = \xi(u_0, \ldots, u_d)$,

$$\exp\left\{-\lambda\kappa_d\xi^d\right\}\xi^{d^2-1} \leq c_{17}\lambda^{-(d^2-1)/d}\exp\left\{-\frac{\kappa_d}{\tau^{d/k}}(1-\epsilon)a^{d/k}\lambda\right\}.$$

Therefore,

$$\mathbb{P}(\Sigma(Z) \in [1,2]) \leq c_{18}(a^{d/k}\lambda)^{1/d}\exp\left\{-\frac{\kappa_d}{\tau^{d/k}}(1-\epsilon)a^{d/k}\lambda\right\}.$$

Assuming $a^{d/k}\lambda \geq \sigma_0 > 0$, we obtain

$$\mathbb{P}(\Sigma(Z) \in [1,2]) \leq c_{19}\exp\left\{-\frac{\kappa_d}{\tau^{d/k}}(1-2\epsilon)a^{d/k}\lambda\right\},$$

with a constant c_{19} depending also on σ_0. For $\epsilon \in (0, 1/3)$ and $a^{d/k}\lambda \geq \sigma_0$, we conclude that

$$\begin{aligned}
&\mathbb{P}(\Sigma(Z) \geq a)\\
&= \mathbb{P}\left(\Sigma(Z) \in \bigcup_{i=0}^{\infty} a2^i[1,2]\right)\\
&\leq \sum_{i=0}^{\infty}\mathbb{P}\left(\Sigma(Z) \in a2^i[1,2]\right)\\
&\leq \sum_{i=0}^{\infty} c_{19}\exp\left\{-\frac{\kappa_d}{\tau^{d/k}}(1-2\epsilon)(a2^i)^{d/k}\lambda\right\}\\
&\leq c_{19}\exp\left\{-\frac{\kappa_d}{\tau^{d/k}}(1-3\epsilon)a^{d/k}\lambda\right\}\sum_{i=0}^{\infty}\exp\left\{-\frac{\kappa_d}{\tau^{d/k}}\epsilon(a2^i)^{d/k}\lambda\right\}\\
&\leq c_{20}\exp\left\{-\frac{\kappa_d}{\tau^{d/k}}(1-3\epsilon)a^{d/k}\lambda\right\}.
\end{aligned}$$

From this result, we can conclude that

$$\limsup_{a\to\infty} a^{-d/k}\ln\mathbb{P}(\Sigma(Z) \geq a) \leq -\frac{\kappa_d}{\tau^{d/k}}(1-3\epsilon)\lambda$$

for any $\epsilon \in (0, 1/3)$, and therefore

$$\limsup_{a\to\infty} a^{-d/k}\ln\mathbb{P}(\Sigma(Z) \geq a) \leq -\frac{\kappa_d}{\tau^{d/k}}\lambda.$$

Together with (12), this proves Theorem 2. □

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