Asymptotic Shapes of Large Cells in Random Tessellations

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Abstract. We establish a close relationship between isoperimetric inequalities for convex bodies and asymptotic shapes of large random polytopes, which arise as cells in certain random mosaics in d-dimensional Euclidean space. These mosaics are generated by Poisson hyperplane processes satisfying a few natural assumptions (not necessarily stationarity or isotropy). The size of large cells is measured by a class of general functionals. The main result implies that the asymptotic shapes of large cells are completely determined by the extremal bodies of an inequality of isoperimetric type, which connects the size functional and the expected number of hyperplanes of the generating process hitting a given convex body. We obtain exponential estimates for the conditional probability of large deviations of zero cells from asymptotic or limit shapes, under the condition that the cells have large size.

1 Introduction

This paper studies a class of random polytopes and investigates their asymptotic shapes under the condition that the size of the polytopes becomes large. The polytopes are generated by Poisson processes in the space of hyperplanes of Euclidean space $\mathbb{R}^d$, by taking the cell of the induced tessellation of $\mathbb{R}^d$ that contains the origin. Our approach includes zero cells of stationary Poisson hyperplane tessellations as well as typical cells of Poisson–Voronoï tessellations, but goes beyond these special cases. The size of the random polytopes is measured by a size functional which is introduced axiomatically. All the common geometric measurements of size, like volume, diameter, inradius, and many others, are included. A shape of a convex body is defined as the orbit of the convex body under a subgroup of the group of similarities. If the conditional law for the shape of a random polytope, given the size of the random polytope, converges weakly, as the size tends to infinity, to the degenerate law concentrated at a fixed shape, then the latter is, by definition, the limit shape of the random polytope with respect to the chosen size functional. If limit shapes in this sense do not exist, it may still be possible to identify a class of asymptotic shapes, in a weaker sense.

Our main result reveals that the asymptotic shapes of the considered large random polytopes are completely determined by the extremal bodies of an inequality of isoperimetric type for convex bodies. It involves two functions on the space of convex bodies: the size functional and the expected number of hyperplanes of the defining process hitting a given convex body. If the extremal bodies of the isoperimetric inequality are unique up to transformations from some group of similarities, then large cells have a corresponding limit shape. Stability

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improvements of the isoperimetric inequality yield bounds for the probabilities of large deviations from the limit shape. These estimates also provide information on asymptotic shapes in cases where limit shapes do not exist. On the other hand, in a special case where the crucial isoperimetric inequality degenerates to a trivial equality, we show that size has no influence on shape and, hence, limit shapes do not exist.

The topic of shapes of large cells in random tessellations originated from what has become known as D.G. Kendall's conjecture (see [14], foreword to the first edition). For this, consider a stationary and isotropic Poisson line process in the plane and let Z_0 be the cell of the induced random tessellation that contains the origin. Kendall's conjecture stated that the conditional law for the shape of Z_0, given the area $A(Z_0)$ of Z_0, converges weakly, as $A(Z_0) \to \infty$, to the degenerate law concentrated at the circular shape. A proof was given by Kovalenko [6], [8], who also obtained in [7] an analogous result for the typical cell of a stationary Poisson–Voronoï mosaic in the plane. Already before that, Miles [10] had suggested, though not completely proved, similar results with the area replaced by other functionals, like perimeter, inradius, or width in a given direction. Higher-dimensional versions and analogues of Kendall's conjecture were investigated in [9], [4], [5]. Of these, [9] treats only a very special case, namely hyperplane mosaics where all cells are parallelepipeds. In [4], Kendall's problem was extended and solved for general stationary, not necessarily isotropic Poisson hyperplane processes, with size measured by the volume. Typical cells of higher-dimensional stationary Poisson–Voronoï mosaics were the topic of [5]; here the size was measured by an intrinsic volume.

As a byproduct, we obtain a result on the asymptotic distribution of the size functional, thus extending a result of Goldman [3] from the plane to higher dimensions and to general size functionals.

In the next section, we will describe the main results in detail.

2 Assumptions and Main Results

We work in d-dimensional real Euclidean vector space $\mathbb{R}^d$, with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. The unit ball $\{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| \leq 1\}$ is denoted by B^d; its boundary is the unit sphere S^{d-1}. By $\mathcal{H}^d$ we denote the space of hyperplanes in $\mathbb{R}^d$ not containing the origin $\boldsymbol{o}$, with its usual topology and Borel structure. For $\boldsymbol{u} \in S^{d-1}$ and $t \in \mathbb{R}$, we write

$$H(\boldsymbol{u}, t) := \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{x}, \boldsymbol{u} \rangle = t\}, \qquad H^-(\boldsymbol{u}, t) := \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{x}, \boldsymbol{u} \rangle \leq t\}.$$

Every hyperplane $H \in \mathcal{H}^d$ has a unique representation $H = H(\boldsymbol{u}, t)$ with $\boldsymbol{u} \in S^{d-1}$ and $t > 0$; thus $\boldsymbol{o} \in H^-(\boldsymbol{u}, t)$. We call this the *standard representation.* For $H \in \mathcal{H}^d$, we denote by H^- the closed halfspace bounded by H that contains $\boldsymbol{o}$. For a set $\mathcal{A} \subset \mathcal{H}^d$, we define

$$P(\mathcal{A}) := \bigcap_{H \in \mathcal{A}} H^-.$$

Let X be a Poisson hyperplane process in $\mathbb{R}^d$, that is, a Poisson point process in the space $\mathcal{H}^d$ (see [13], for example). We often identify a simple counting measure with its support, so that both notations, $X(\mathcal{A})$ and $\mathrm{card}(X \cap \mathcal{A})$, denote the number of elements of X in $\mathcal{A}$. We assume that the intensity measure $\Theta = \mathbb{E}X(\cdot)$ (where $\mathbb{E}$ denotes mathematical expectation) is of the form $\Theta = \lambda\mu$, with $\lambda > 0$ and a measure μ on $\mathcal{H}^d$ given by

$$\mu = \int_{S^{d-1}} \int_0^\infty \mathbf{1}\{H(\boldsymbol{u}, t) \in \cdot\} t^{r-1} \, \mathrm{d}t \, \varphi(\mathrm{d}\boldsymbol{u}). \tag{1}$$

Here $r \geq 1$, and φ is a Borel probability measure on S^{d-1} with the property that its support is not contained in some closed hemisphere. We call λ the *intensity* and φ the *directional distribution* of the hyperplane process X, and to the number r we refer as the *distance exponent.*

The random polytope

$$Z_0 := P(X) = \bigcap_{H \in X} H^-$$

is the *zero cell*, or *Crofton cell*, of the tessellation induced by X. The case of the zero cell of a stationary Poisson hyperplane process (of appropriate intensity) is included here, for $r = 1$ (observe that the hyperplanes of a stationary Poisson hyperplane process almost surely do not contain $\boldsymbol{o}$). Also included is the case of the typical cell of a stationary Poisson–Voronoï mosaic (of appropriate intensity), for $r = d$.

By $\mathcal{K}^d$ we denote the space of convex bodies (nonempty, compact, convex sets) in $\mathbb{R}^d$, equipped with the Hausdorff metric δ, and by $\mathcal{K}^d_o$ the subspace of all convex bodies containing the origin. Our investigation of asymptotic shapes of large zero cells will be governed by three continuous homogeneous functionals on the space $\mathcal{K}^d_o$: the parameter, size, and deviation functionals. We introduce them now.

For $K \in \mathcal{K}^d_o$, we define

$$\mathcal{H}_K := \{H \in \mathcal{H}^d : H \cap K \neq \emptyset\}.$$

We have

$$\mathbb{E} X(\mathcal{H}_K) = \lambda \Phi(K) \tag{2}$$

with

$$\Phi(K) := \mu(\mathcal{H}_K) = \frac{1}{r} \int_{S^{d-1}} h(K, \boldsymbol{u})^r \varphi(\mathrm{d}\boldsymbol{u}), \tag{3}$$

as follows from (1). Here, $h(K, \boldsymbol{u}) = \max\{\langle \boldsymbol{x}, \boldsymbol{u} \rangle : \boldsymbol{x} \in K\}$ is the value of the support function of K at $\boldsymbol{u}$. The function Φ is continuous on $\mathcal{K}^d_o$ and homogeneous of degree r, that is, it satisfies $\Phi(\alpha K) = \alpha^r \Phi(K)$ for $\alpha \geq 0$. We call Φ the *parameter functional* of the process X, since multiplied by the intensity λ, it gives the parameter of the Poisson distribution of the random variable $X(\mathcal{H}_K)$, for $K \in \mathcal{K}^d_o$:

$$\mathbb{P}(X(\mathcal{H}_K) = n) = \frac{[\Phi(K)\lambda]^n}{n!} \exp\{-\Phi(K)\lambda\}$$

for $n \in \mathbb{N}_0$; here $\mathbb{P}$ denotes the underlying probability.

The size of the zero cell can be measured by any real function Σ on $\mathcal{K}^d_o$ satisfying the following natural axioms:

(a) Σ is continuous,
(b) not identically zero,
(c) homogeneous of some degree $k > 0$,
(d) increasing under set inclusion ($K \subset M \Rightarrow \Sigma(K) \leq \Sigma(M)$).

Let a function Σ with these properties be given. We call it the *size functional.* Typical examples are volume, surface area, mean width, diameter, thickness, inradius, circumradius, width in a given direction.

It is easy to see (see Section 3) that Φ and Σ satisfy a sharp inequality of isoperimetric type,

$$\Phi(K) \geq \tau \, \Sigma(K)^{r/k} \qquad \text{for } K \in \mathcal{K}^d_o, \tag{4}$$

with a positive constant τ. That the inequality is sharp means that there exist convex bodies $K \in \mathcal{K}_o^d$ with more than one point for which equality holds; every such body is called an *extremal body* (for given Φ and Σ).

We remark that the extremal bodies have the following probabilistic characterization. *Among all convex bodies $K \in \mathcal{K}_o^d$ of size $\Sigma(K) = 1$, precisely the extremal bodies maximize the probability $\mathbb{P}(K \subset Z_0)$.* In fact, suppose that $K \in \mathcal{K}_o^d$ and $\Sigma(K) = 1$. Since $X(\mathcal{H}_K) = 0 \Rightarrow K \subset Z_0 \Rightarrow X(\mathcal{H}_{\alpha K}) = 0$ for all $\alpha < 1$ and $\mathbb{P}(X(\mathcal{H}_K) = 0) = \exp\{-\Phi(K)\lambda\}$, we get

$$\mathbb{P}(K \subset Z_0) = \exp\{-\Phi(K)\lambda\} \leq \exp\{-\tau\Sigma(K)^{r/k}\lambda\} = \mathrm{e}^{-\tau\lambda},$$

with equality if and only if equality holds in (4).

Our third functional measures the deviation of a convex body from the class of extremal bodies. Again, we introduce it axiomatically. We assume that Φ and Σ are given. A real function ϑ on $\{K \in \mathcal{K}_o^d : \Sigma(K) > 0\}$ is called a *deviation functional* if

(a) ϑ is continuous,
(b) nonnegative,
(c) homogeneous of degree zero,
(d) $\vartheta(K) = 0$ for some $K \in \mathcal{K}_o^d$ with $\Sigma(K) > 0$ holds if and only if K is an extremal body.

Such deviation functionals always exist. For example, one could take the *canonical deviation functional*

$$\vartheta(K) := \frac{\Phi(K)}{\tau\Sigma(K)^{r/k}} - 1. \tag{5}$$

However, in concrete examples, the deviation functional should be chosen in such a way that the deviation has a simple intuitive geometric meaning, and an inequality $\vartheta(K) < \epsilon$ should allow an explicit estimate of some geometric distance of K from an extremal body. One possibility is to use the Hausdorff metric δ and the diameter D and to define

$$\eta(K) := \min\{\delta(K, M) : M \text{ extremal body}\}/D(K) \tag{6}$$

for $K \in \mathcal{K}_o^d$ with $D(K) > 0$. The minimum is attained (see Section 3). Clearly η is a deviation functional; the homogeneity follows from the fact that together with M also αM for $\alpha > 0$ is an extremal body.

It follows from the properties of the involved functionals that the inequality (4) can be strengthened to a stability estimate: there exists a continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(\epsilon) > 0$ for $\epsilon > 0$ and $f(0) = 0$ such that

$$\Phi(K) \geq (1 + f(\epsilon))\tau\Sigma(K)^{r/k} \qquad \text{if} \quad \vartheta(K) \geq \epsilon, \tag{7}$$

for $K \in \mathcal{K}_o^d$ (see Section 3). Any such function f will be called a *stability function* for Φ, Σ, ϑ. In concrete cases, explicit stability functions are of interest.

In the following, it will be convenient to assume that every stability function f satisfies $f < 1$. This can always be achieved, since a given stability function f may be replaced by $\min\{f, 1/2\}$.

After these preparations, we can formulate our main result.

Theorem 1. *Suppose that a Poisson hyperplane process X with intensity λ, directional distribution φ and distance exponent r (which determine the parameter functional Φ), a size functional Σ, a deviation functional ϑ, and a stability function f for Φ, Σ, ϑ as explained are* *given. With a suitable constant* $c_0 > 0$ *(depending only on* τ*), the following holds. If* $\epsilon > 0$ *and* $0 < a < b \leq \infty$*, then*

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \in [a,b)) \leq c \exp\left\{-c_0 f(\epsilon) a^{r/k}\lambda\right\}, \tag{8}$$

where c *is a constant depending only on* $\varphi, r, \Sigma, f, \epsilon$*.*

The implications of this theorem on the existence and nature of limit shapes and asymptotic shapes in general and in various concrete examples will be discussed in Sections 4 and 5. The proof of Theorem 1 follows in Section 6.

Note that the intensity λ, not appearing explicitly on the left-hand side of (8), together with φ and r determines the distribution of the zero cell Z_0. We could restrict ourselves to processes with intensity one, but would then loose the information about the form of the essential parameter $a^{r/k}\lambda$.

In [5, Theorem 2], the special case of Theorem 1 was treated where the Poisson hyperplane process is stationary and isotropic and the size functional Σ is the kth intrinsic volume, for $k \in \{2, \dots, d\}$. It turned out that zero cells which are large in this sense tend to become spherical if the size functional tends to infinity. The case $k = 1$, where the size functional is a multiple of the mean width and thus also of the parameter functional, was excluded. Generally, our method breaks down if the size functional Σ is proportional to the parameter functional Φ. For this choice of size functional, every convex body $K \in \mathcal{K}_o^d$ is an extremal body, hence $\mathbb{P}(\vartheta(Z_0) \geq \epsilon) = 0$ for every $\epsilon > 0$. Theorem 1 holds trivially in this case, and does not distinguish between different shapes. In Section 7, we consider the special case where the directional distribution φ has finite support, and we prove that for the choice $\Sigma = \Phi$ a limit shape of the zero cell does, in fact, not exist. We are uncertain whether this example should lead one to speculate about a corresponding result for the uniform directional distribution. It is interesting, in this connection, to recall a statement of Miles [10] in his heuristic approach. He considered stationary, isotropic Poisson line processes in the plane, where the mean width is, up to a constant factor, the perimeter. Miles stated that the shape of typical cells with large perimeter tends to circular shape. It would be interesting to give a rigorous proof of this assertion (if true), to extend it to higher dimensions, and to see whether the analogue is true for zero cells.

In the formulation of Theorem 1, we have preferred the condition $\Sigma(Z_0) \in [a,b)$, which has positive probability. However, our results are strong enough to allow also a treatment of the conditional probability of the event $\vartheta(Z_0) \geq \epsilon$ under the condition $\Sigma(Z_0) = a$. Such conditional probabilities appeared in the original formulation of Kendall's problem. Under the assumptions of Theorem 1, we will prove in Section 9 that

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) = a) \leq c \exp\left\{-c_0 f(\epsilon) a^{r/k}\lambda\right\} \tag{9}$$

for almost all a, with suitable positive constants c_0, c.

In [3], Goldman described the asymptotic behaviour of the distribution function of the area of the zero cell of an isotropic, stationary Poisson line process in the plane. We will extend Goldman's result to the general situation to which Theorem 1 refers. While Goldman's method seems to be restricted to two dimensions, we can use some techniques developed for the proof of Theorem 1 to obtain the generalization stated in Theorem 2. This theorem will be proved in Section 8.

Theorem 2. *If X and Σ are as above, then*

$$\lim_{a\to\infty} a^{-r/k} \ln \mathbb{P}(\Sigma(Z_0) \geq a) = -\tau\lambda.$$

Recall that the constant τ is given by

$$\tau = \min\{\Phi(K) : K \in \mathcal{K}_o^d,\ \Sigma(K) = 1\},$$

and equivalently by

$$\mathrm{e}^{-\tau\lambda} = \max\{\mathbb{P}(K \subset Z_0) : K \in \mathcal{K}_o^d,\ \Sigma(K) = 1\}.$$

3 Auxiliary Results

The expository section 2 contained a few unproved assertions; first we give the proofs of these. We begin with the inequality (4).

We note that the set $\mathcal{K}(1) := \{K \in \mathcal{K}_o^d : \Phi(K) = 1\}$ is compact. In fact, let $K \in \mathcal{K}(1)$, let R be such that $K \subset RB^d$ and R is minimal. There exists $\boldsymbol{x} \in K$ with $\|\boldsymbol{x}\| = R$, hence

$$1 = \Phi(K) \geq \frac{1}{r} \int_{S^{d-1}} (\langle \boldsymbol{x}, \boldsymbol{u}\rangle^+)^r \, \varphi(\mathrm{d}\boldsymbol{u}) \geq c(\varphi, r) R^r \tag{10}$$

with a positive constant $c(\varphi, r)$, since φ is not concentrated on a closed hemisphere. Thus $K \subset (1/c(\varphi, r))^{1/r} B^d$. Since $\mathcal{K}_o^d$ is closed and Φ is continuous, $\mathcal{K}(1)$ is compact.

On $\mathcal{K}(1)$, the continuous function Σ attains a maximum. It is positive, since $\mathcal{K}(1)$ contains a dilate of any body different from $\{\boldsymbol{o}\}$ in $\mathcal{K}_o^d$, and the functional Σ is homogeneous and not identically zero; let $\tau^{-k/r}$ denote the value of the maximum. By homogeneity, we have

$$\Phi(K) \geq \tau\, \Sigma(K)^{r/k} \qquad \text{for } K \in \mathcal{K}_o^d. \tag{11}$$

As defined in Section 2, a convex body $K \in \mathcal{K}_o^d$ different from $\{\boldsymbol{o}\}$ for which equality holds in (11) is called an *extremal body.*

Next we show that the minimum in (6) is attained. The subset of $\mathcal{K}_o^d$ consisting of the extremal bodies together with $\{\boldsymbol{o}\}$ is closed, hence it follows from the continuity properties of the metric that $\delta(K, \cdot)$ attains a minimum δ_0 on this set. We have to show that the minimum is attained at an extremal body. Suppose the minimum is attained at $\{\boldsymbol{o}\}$. Then $K \subset \delta_0 B^d$. By homogeneity, there is an extremal body $M \subset \delta_0 B^d$. Since $\boldsymbol{o} \in K$, we have $M \subset K + \delta_0 B^d$, and from $\boldsymbol{o} \in M$ we get $K \subset M + \delta_0 B^d$. Thus, the minimum δ_0 is also attained at the extremal body M.

To establish the existence of the strengthening (7), let ϑ be a deviation functional. We may assume that there exist $K \in \mathcal{K}_o^d$ and $\epsilon_0 > 0$ with $\Sigma(K) > 0$ and $\vartheta(K) \geq \epsilon_0$, since otherwise the assertion is trivial. By homogeneity, there is $K \in \mathcal{K}_o^d$ with $\Sigma(K) > 0$, $\vartheta(K) \geq \epsilon_0$, and $\Phi(K) = 1$. Let $0 < \epsilon \leq \epsilon_0$. On the nonempty compact set $\{K \in \mathcal{K}_o^d : \Phi(K) = 1,\ \vartheta(K) \geq \epsilon\}$, the continuous function Σ attains a positive maximum, say at K_0, which we write as $\tau_\epsilon^{-k/r}$. We have $\tau_\epsilon > \tau$, since otherwise K_0 would be an extremal body and, hence, $\vartheta(K_0) = 0$. Thus, we can write $\tau_\epsilon = (1 + f(\epsilon))\tau$ with $f(\epsilon) > 0$. By homogeneity,

$$\Phi(K) \geq (1 + f(\epsilon))\tau \Sigma(K)^{r/k} \qquad \text{if } K \in \mathcal{K}_o^d \text{ and } \vartheta(K) \geq \epsilon. \tag{12}$$

We put $f(0) = 0$ and $f(\epsilon) := f(\epsilon_0)$ for $\epsilon > \epsilon_0$. The continuity of f is easy to check. Thus, f is a stability function for Φ, Σ, ϑ.

In the cases where the support of the directional distribution φ is not the whole unit sphere, the zero cells belong to a special class of convex bodies, which we now introduce.

Let $\Omega := \operatorname{supp}\varphi$, the support of the directional distribution φ. This is a closed set on S^{d-1}, not lying in a closed hemisphere. We say that a convex body $K \in \mathcal{K}^d$ is *φ-adapted* if

$$K = \bigcap_{\boldsymbol{u}\in\Omega} H^-(\boldsymbol{u}, h(K,\boldsymbol{u})),$$

that is, if K is the intersection of those of its supporting halfspaces that have an outer unit normal vector in the support of φ. The class of all φ-adapted convex bodies in $\mathcal{K}_o^d$ is denoted by $\mathcal{K}_\varphi$. For the inequality (4), there always exist extremal bodies belonging to $\mathcal{K}_\varphi$. In fact, let K be an extremal body. Then also

$$K' := \bigcap_{\boldsymbol{u}\in\Omega} H^-(\boldsymbol{u}, h(K,\boldsymbol{u}))$$

is an extremal body. This follows from $\Phi(K') = \Phi(K)$ (since the integrands in (3) for K' and K agree on the support of φ) and, by monotonicity, $\Sigma(K') \geq \Sigma(K)$ (which then implies $\Sigma(K') = \Sigma(K)$). Clearly, the body K' is φ-adapted.

There may also exist extremal bodies which are not in $\mathcal{K}_\varphi$. In the context of asymptotic shapes of large cells, these bodies are irrelevant.

4 Limit Shapes and Asymptotic Shapes

Before proving Theorem 1, we want to demonstrate which implications this theorem has for a very general version of Kendall's problem, that is, for the existence of limit shapes of large Crofton cells, and more generally for a study of asymptotic shapes. Immediately from (8) we get

$$\lim_{a\to\infty} \mathbb{P}(\vartheta(Z_0) < \epsilon \mid \Sigma(Z_0) \geq a) = 1$$

for every $\epsilon > 0$. Roughly, this shows that zero cells which are large in the sense of Σ have a small deviation from an extremal body, with high probability.

In order to draw precise conclusions about the existence of limit shapes, we introduce a suitable notion of shape. It is common to consider two convex bodies to be of the same shape if they are similar to each other. We need a more general notion. Let G be a subgroup of the group S of similarities of $\mathbb{R}^d$ which contains the group D of all positive dilatations. Every such group will be called an *admissible group*. A typical example is the group H of homotheties (dilatations followed by translations). Two convex bodies $K, M \in \mathcal{K}^d$ have the same G-*shape*, also written as $K \sim_{\mathsf{G}} M$, if $K = gM$ with some $g \in \mathsf{G}$. The quotient space $\mathcal{S}_{\mathsf{G}} := \mathcal{K}^d/\sim_{\mathsf{G}}$ is called the *space of* G-*shapes*. Let $s_{\mathsf{G}} : \mathcal{K}^d \to \mathcal{S}_{\mathsf{G}}$ be the canonical projection, thus $s_{\mathsf{G}}(K) = \{gK : g \in \mathsf{G}\}$ is the class of all convex bodies in $\mathcal{K}^d$ having the same G-shape as K.

Let the Poisson hyperplane process X, the zero cell Z_0 and the size functional Σ be as above.

Definition. The *conditional law of the* G*-shape* of Z_0, given the lower bound a for the size Σ, is the probability measure μ_a on $\mathcal{S}_\mathsf{G}$ defined by

$$\mu_a := \mathbb{P}(s_\mathsf{G}(Z_0) \in \cdot \mid \Sigma(Z_0) \geq a).$$

A shape $s_\mathsf{G}(B)$, where $B \in \mathcal{K}_o^d$, is the *limit shape of* Z_0 *with respect to* Σ if

$$\lim_{a\to\infty} \mu_a = \delta_{s_\mathsf{G}(B)} \quad \text{weakly},$$

where $\delta_{s_\mathsf{G}(B)}$ denotes the Dirac measure concentrated at $s_\mathsf{G}(B)$.

Now we can formulate a general theorem on the existence of limit shapes.

Theorem 3. *Let X, Z_0, Σ be as above. Suppose there exists an admissible group G such that the extremal bodies of the inequality (4) have a unique G-shape $s_\mathsf{G}(B)$. Then $s_\mathsf{G}(B)$ is the limit shape of Z_0 with respect to Σ.*

Proof. We deduce this from Theorem 1, assuming that all data are as given in that theorem and ϑ is chosen according to (5). For proving the asserted weak convergence of the measure μ_a, we have to show that

$$\limsup_{a\to\infty} \mu_a(\mathcal{C}) \leq \delta_{s_\mathsf{G}(B)}(\mathcal{C}) \tag{13}$$

for every closed set $\mathcal{C} \subset \mathcal{S}_\mathsf{G}$. Let $\mathcal{C}$ be closed and not empty, without loss of generality. Relation (13) holds trivially if $s_\mathsf{G}(B) \in \mathcal{C}$, hence we assume that $s_\mathsf{G}(B) \notin \mathcal{C}$. We set $\mathcal{K}^1 := \{K \in \mathcal{K}_o^d \cap s_\mathsf{G}^{-1}(\mathcal{C}) : \Sigma(K) = 1\}$, then $\vartheta > 0$ on $\mathcal{K}^1$. In fact, a body $K \in \mathcal{K}^1$ with $\vartheta(K) = 0$ would be an extremal body and hence satisfy $s_\mathsf{G}(K) = s_\mathsf{G}(B) \notin \mathcal{C}$.

We show that ϑ attains a minimum on $\mathcal{K}^1$. Put $\epsilon := \inf\{\vartheta(K) : K \in \mathcal{K}^1\}$. There is a sequence $(K_i)_{i\in\mathbb{N}}$ in $\mathcal{K}^1$ such that $\vartheta(K_i) \to \epsilon$ for $i \to \infty$. By the definition of ϑ and the estimate (10), this sequence is bounded and hence has a subsequence converging to some convex body $K_0 \in \mathcal{K}_o^d$, satisfying $s_\mathsf{G}(K_0) \in \mathcal{C}$ (since $\mathcal{C}$ is closed and s_G is continuous) and $\Sigma(K) = 1$, thus $K_0 \in \mathcal{K}^1$. It follows that $\epsilon > 0$. For any $K \in \mathcal{K}_o^d \cap s_\mathsf{G}^{-1}(\mathcal{C})$ with $\Sigma(K) > 0$, we can choose $\alpha > 0$ with $\Sigma(\alpha K) = 1$; then $\alpha K \in s_\mathsf{G}^{-1}(\mathcal{C})$ (since G contains the positive dilatations), and $\vartheta(\alpha K) = \vartheta(K)$. Thus $\vartheta \geq \epsilon$ on $\mathcal{K}_o^d \cap s_\mathsf{G}^{-1}(\mathcal{C}) \cap \{K : \Sigma(K) > 0\}$ and, hence, $\mathcal{K}_o^d \cap s_\mathsf{G}^{-1}(\mathcal{C}) \cap \{K : \Sigma(K) > 0\} \subset \{K \in \mathcal{K}_o^d : \vartheta(K) \geq \epsilon\}$. This gives

$$\mu_a(\mathcal{C}) = \mathbb{P}(Z_0 \in \mathcal{K}_o^d \cap s_\mathsf{G}^{-1}(\mathcal{C}) \mid \Sigma(Z_0) \geq a) \leq \mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \geq a) \to 0$$

for $a \to \infty$, by Theorem 1, and thus proves Theorem 3. □

In cases where limit shapes, as defined here, do not exist, one can still consider the extremal bodies of (4) as *asymptotic shapes* for large zero cells, since (8) estimates the deviation of the D-shapes of large zero cells from the D-shapes of the extremal bodies.

Closer to the original formulation of Kendall's problem, we can also consider the following variant.

Definition. The *conditional law of the* G*-shape* of Z_0, given the value a for the size Σ, is the probability measure ν_a on $\mathcal{S}_\mathsf{G}$ defined (for $\mathbb{P}_{\Sigma(Z_0)}$-almost all a) by

$$\nu_a := \mathbb{P}(s_\mathsf{G}(Z_0) \in \cdot \mid \Sigma(Z_0) = a).$$

A shape $s_{\mathsf{G}}(B)$, where $B \in \mathcal{K}_o^d$, is the *K-limit shape of* Z_0 *with respect to* Σ if

$$\lim_{a\to\infty} \nu_a = \delta_{s_{\mathsf{G}}(B)} \quad \text{weakly.}$$

To justify this definition, we remark that Z_0 is a random variable with values in the measurable space $(\mathcal{K}^d, \mathcal{B}(\mathcal{K}^d))$ ($\mathcal{B}$ denotes the Borel σ-algebra), and $\Sigma(Z_0)$ is a random variable with values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Since $\mathcal{K}^d$ is a Polish space, the regular conditional probability distribution of Z_0 with respect to $\Sigma(Z_0)$ exists, that is, there is a Markov kernel Q from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to $(\mathcal{K}^d, \mathcal{B}(\mathcal{K}^d))$ such that $Q(\cdot, B)$ is a version of the conditional probability $\mathbb{P}(Z_0 \in B \mid \Sigma(Z_0) = \cdot)$, for each $B \in \mathcal{B}(\mathcal{K}^d)$. Thus, for each $a \in \mathbb{R}$, the probability measure $Q(a, \cdot)$ is the conditional distribution of Z_0 under the hypothesis that $\Sigma(Z_0) = a$; for each $B \in \mathcal{B}(\mathcal{K}^d)$ we have

$$Q(a, B) = \mathbb{P}(Z_0 \in B \mid \Sigma(Z_0) = a) \qquad \text{for } \mathbb{P}_{\Sigma(Z_0)}\text{-almost all } a.$$

It is now clear that ν_a as defined above is a probability measure, and that with the aid of inequality (9) we can obtain an analogue of Theorem 3 for K-limit shapes, with verbally the same proof.

5 Special Examples

Before proving Theorem 1 in the next section, we want to present some special examples, where the size of the zero cells is measured by functionals of geometric interest.

In the following, we denote by σ the normalized spherical Lebesgue measure on the sphere S^{d-1} and by κ_k the volume of the k-dimensional unit ball, thus $\kappa_k = \pi^{k/2}/\Gamma(1 + k/2)$.

(1) *The zero cell of a stationary Poisson hyperplane mosaic; the size measured by the volume*

This higher-dimensional version of Kendall's problem, extended to the non-isotropic case, was treated in [4]. In this case, $r = 1$, $\Sigma = V_d$, and the parameter functional (with its present normalization) can be expressed as a mixed volume:

$$\Phi(K) = dV_1(B, K) = dV(B, \dots, B, K).$$

Here B is the convex body with centre $\boldsymbol{o}$ for which the directional distribution φ is the area measure; it exists by Minkowski's existence theorem. The isoperimetric inequality (4) is now Minkowski's classical inequality

$$V_1(B, K)^d \geq V_d(B)^{d-1} V_d(K).$$

Equality holds if and only if K is homothetic to B. Hence, $s_{\mathsf{H}}(B)$ (the set of convex bodies homothetic to B) is the limit shape of the zero cell with respect to the volume. If the hyperplane process is isotropic, then B is a ball, thus we get a higher dimensional version of Kendall's original problem.

A deviation functional with a simple intuitive meaning is given by

$$r_B(K) := \inf\{\beta/\alpha - 1 : \alpha B \subset K + \boldsymbol{z} \subset \beta B,\ \boldsymbol{z} \in \mathbb{R}^d,\ \alpha, \beta > 0\}. \tag{14}$$

A stability version of Minkowski's inequality due to Groemer then leads to the following deviation estimate:

$$\mathbb{P}(r_B(Z_0) \geq \epsilon \mid V(Z_0) \in [a,b)) \leq c \exp\left\{-c_0 \epsilon^{d+1} a^{1/d} \lambda\right\}.$$

(2) *The typical cell of a stationary Poisson–Voronoï mosaic; the size measured by the kth intrinsic volume V_k, $k \in \{1,\dots,n\}$*

In this case, which was treated in [5], $r = d$, $\Sigma = V_k$, the directional distribution φ is the rotation invariant probability measure σ, hence the parameter functional is given by

$$\Phi(K) = \frac{1}{d} \int_{S^{d-1}} h(K,\boldsymbol{u})^d \, \sigma(\mathrm{d}\boldsymbol{u}).$$

The isoperimetric inequality (4) now reads

$$\Phi(K) \geq \frac{1}{d} \left(\frac{\kappa_{d-k}}{\binom{d}{k}\kappa_d} \right)^{d/k} V_k(K)^{d/k}. \tag{15}$$

It is obtained by combining Hölder's inequality with the Aleksandrov-Fenchel inequalities. The extremal bodies are precisely the balls with centre $\boldsymbol{o}$, hence the set of centred balls is the limit shape of the typical cell Z with respect to V_k.

A convenient deviation functional is given by

$$\vartheta(K) := \frac{R_o(K) - \rho_o(K)}{R_o(K) + \rho_o(K)}, \tag{16}$$

where $R_o(K)$ (respectively $\rho_o(K)$) is the radius of the smallest (largest) ball with centre $\boldsymbol{o}$ containing K (contained in K). Using this deviation functional, a stability version of (15) can be proved, and the estimate

$$\mathbb{P}(\vartheta(Z) \geq \epsilon \mid V_k(Z) \in [a,b)) \leq c \exp\left\{-c_0 \epsilon^{(d+3)/2} a^{d/k} \lambda\right\}$$

is obtained.

Now we describe new cases, to which our extended general theorems can be applied. In contrast to the previously established results, we can now have non-spherical limit shapes even for zero cells of isotropic tessellations.

(3) *The zero cell of a stationary, isotropic Poisson hyperplane mosaic; the size measured by the diameter D*

If $K \in \mathcal{K}^d$, then K contains a segment of length $D(K)$, without loss of generality with centre at $\boldsymbol{o}$. We conclude that

$$\Phi(K) = \int_{S^{d-1}} h(K,\boldsymbol{u}) \, \sigma(\mathrm{d}\boldsymbol{u}) \geq \frac{\kappa_{d-1}}{d\kappa_d} D(K),$$

with equality if and only if K is a segment. Thus, the limit shape of Z_0 with respect to the diameter is the class of segments.

An intuitively natural deviation functional can be defined by

$$\vartheta(K) := \min\{\alpha \geq 0 : S \subset D(K)^{-1}K \subset S + \alpha B^d,\ S \text{ a unit segment}\}$$

for $K \in \mathcal{K}^d$ with $\dim K > 0$.

Suppose that $K \in \mathcal{K}^d$ is a convex body of positive dimension with $\vartheta(K) \geq \epsilon > 0$. There are a unit segment $S \subset D(K)^{-1}K$ and a point $\boldsymbol{x} \in K/D(K)$ on the boundary of $S + \epsilon B^d$. The (possibly degenerate) triangle $T := \operatorname{conv}(S \cup \{\boldsymbol{x}\})$ has perimeter $L(T) \geq 1 + \sqrt{1+4\epsilon^2}$. If w denotes the mean width in $\mathbb{R}^d$, we have $2\Phi = w$ and $w(T) = (\kappa_{d-1}/d\kappa_d)L(T)$. This gives

$$\Phi(D(K)^{-1}K) \geq \Phi(T) = \frac{1}{2}w(T) \geq \frac{\kappa_{d-1}}{2d\kappa_d}\left(1 + \sqrt{1+4\epsilon^2}\right)$$

and hence the stability estimate

$$\Phi(K) \geq \frac{\kappa_{d-1}}{d\kappa_d}(1 + \epsilon^2/2)D(K) \qquad \text{if } \vartheta(K) \geq \epsilon,$$

for $\epsilon \leq 1$. Hence, we obtain the deviation estimate

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid D(Z_0) \in [a,b)) \leq c \exp\left\{-c_0\epsilon^2 a\lambda\right\}.$$

(4) *The typical cell of a stationary Poisson–Voronoï mosaic; the size measured by the largest distance of a vertex from the nucleus*

The largest distance of a vertex of Z_0 from $\boldsymbol{o}$ is given by the centred circumradius $R_o(Z_0)$. If $K \in \mathcal{K}_o^d$, then K contains a segment of length $R_o(K)$ with one endpoint at $\boldsymbol{o}$. This gives

$$\Phi(K) = \frac{1}{d}\int_{S^{d-1}} h(K,\boldsymbol{u})^d\,\sigma(\mathrm{d}\boldsymbol{u}) \geq \frac{1}{2^{d-1}d}R_o^d.$$

Equality holds if and only if K coincides with the chosen segment. Thus, in this case the limit shape is the class of all segments with one endpoint at the origin.

(5) *The zero cell of a stationary, isotropic Poisson hyperplane mosaic; the size measured by the thickness θ*

The thickness $\theta(K)$ of the convex body K is the smallest distance between any two parallel supporting hyperplanes of K, thus $\theta(K) = \min\{h(K,\boldsymbol{u}) + h(K,-\boldsymbol{u}) : \boldsymbol{u} \in S^{d-1}\}$. This gives

$$\Phi(K) = \frac{1}{2}\int_{S^{d-1}} [h(K,\boldsymbol{u}) + h(K,-\boldsymbol{u})]\,\sigma(\mathrm{d}\boldsymbol{u}) \geq \frac{1}{2}\theta(K),$$

with equality if and only if K is a body of constant width. Hence, the asymptotic shape of Z_0 with respect to the thickness is the class of convex bodies of constant width.

(6) *The zero cell of a stationary, nonisotropic Poisson hyperplane process; the size measured by the inradius ρ*

For a convex body $K \in \mathcal{K}^d$, the inradius $\rho(K)$ is the radius of a largest ball contained in K. For the zero cell Z_0 of a stationary and isotropic Poisson hyperplane process X it was proved in [5, Theorem 3] that the limit shape with respect to the inradius is the class of balls. We are now in a position to treat the nonisotropic case, where other limit shapes appear. In this case, the consideration of φ-adapted convex bodies is essential.

Since the process X is stationary, the directional distribution φ can be chosen as an even measure, hence the parameter functional

$$\Phi(K) = \int_{S^{d-1}} h(K, \boldsymbol{u})\, \varphi(\mathrm{d}\boldsymbol{u}), \qquad K \in \mathcal{K}_o^d,$$

is translation invariant. We may therefore assume that $\boldsymbol{o}$ is the centre of a largest ball contained in K. Then $h(K, \boldsymbol{u}) \geq \rho(K)$, hence

$$\Phi(K) \geq \rho(K). \tag{17}$$

Equality holds if and only if $h(K, \boldsymbol{u}) = \rho(K)$ for all $\boldsymbol{u}$ in the support of the measure φ. Thus, equality in (17) holds for the convex body

$$B_\varphi := \bigcap_{\boldsymbol{u} \in \operatorname{supp} \varphi} H^-(\boldsymbol{u}, 1),$$

and for $K \in \mathcal{K}_\varphi$ equality in (17) holds if and only if K is homothetic to B_φ. We claim that $s_{\mathsf{H}}(B_\varphi)$ *is the limit shape of* Z_0 *with respect to* ρ. This cannot be deduced immediately from Theorem 3, since in general there are many extremal bodies not in $\mathcal{K}_\varphi$, and hence not homothetic to B_φ. Note, however, that the zero cell Z_0 belongs to $\mathcal{K}_\varphi$ with probability one.

To show that $s_{\mathsf{H}}(B_\varphi)$ is the limit shape of Z_0 with respect to ρ, we modify the proof of Theorem 3 (for $\mathsf{G} = \mathsf{H}$) in the following way. Let $\mathcal{C} \subset \mathcal{S}_{\mathsf{H}}$ be a closed set with $s_{\mathsf{H}}(B_\varphi) \notin \mathcal{C}$. Since $Z_0 \in \mathcal{K}_\varphi$ almost surely, we have

$$\mu_a(\mathcal{C}) = \mathbb{P}(s_{\mathsf{H}}(Z_0) \in \mathcal{C} \mid \rho(Z_0) \geq a) = \mu_a(\mathcal{C} \cap s_{\mathsf{H}}(\mathcal{K}_\varphi)).$$

In the proof of Theorem 3, we replace $\mathcal{K}_o^d$ by $\mathcal{K}_\varphi$ and Σ by ρ. The following must be shown: if a sequence $(K_i)_{i \in \mathbb{N}}$ in $\mathcal{K}_\varphi$ converges to a convex body K_0 with $\rho(K_0) = 1$, then $K_0 \in \mathcal{K}_\varphi$. Assume this were false. Then

$$K_0 \neq K_0^* := \bigcap_{\boldsymbol{u} \in \operatorname{supp} \varphi} H^-(\boldsymbol{u}, h(K_0, \boldsymbol{u})).$$

Since $K_0 \subset K_0^*$, there is a point $\boldsymbol{x} \in \operatorname{int} K_0^*$ such that $\boldsymbol{x} \notin K_0$ and hence a number $\alpha > 0$ such that the ball $B^d(\boldsymbol{x}, \alpha)$ with centre $\boldsymbol{x}$ and radius α satisfies $B^d(\boldsymbol{x}, \alpha) \subset K_0^*$ and $B^d(\boldsymbol{x}, \alpha) \cap K_0 = \emptyset$. For all sufficiently large i we have $|h(K_i, \boldsymbol{u}) - h(K_0, \boldsymbol{u}| \leq \alpha$ for $\boldsymbol{u} \in \operatorname{supp} \varphi$ and thus

$$\boldsymbol{x} \in \bigcap_{\boldsymbol{u} \in \operatorname{supp} \varphi} H^-(\boldsymbol{u}, h(K_i, \boldsymbol{u})) = K_i.$$

But this implies $\boldsymbol{x} \in K_0$, a contradiction. Thus, $K_0 \in \mathcal{K}_\varphi$. Now the proof can be completed as that of Theorem 3.

We remark that for the preceding proof it was essential that K_0 has nonempty interior. Otherwise, we cannot conclude from $K_i \in \mathcal{K}_\varphi$ and $K_i \to K_0$ that $K_0 \in \mathcal{K}_\varphi$. A counterexample is provided by [1, Example 20.6].

A stability improvement of (17) involving a simple geometrically reasonable deviation functional, like (14) or (16), can apparently not be achieved without special assumptions on the directional distribution φ.

(7) *The zero cell of a nonstationary, nonisotropic Poisson hyperplane process; the size measured by the centred inradius* ρ_0

In the nonstationary case, the parameter functional

$$\Phi(K) = \frac{1}{r} \int_{S^{d-1}} h(K, \boldsymbol{u})^r \, \varphi(\mathrm{d}\boldsymbol{u})$$

is not translation invariant, therefore we replace the inradius $\rho(K)$ by the centred inradius $\rho_o(K)$. As above, we obtain

$$\Phi(K) \geq \frac{1}{r} \rho_o(K)^r \qquad \text{for } K \in \mathcal{K}_o^d,$$

with equality for $K \in \mathcal{K}_\varphi$ if and only if K is a dilate of B_φ. Similarly as before, it can be shown that $s_{\mathsf{D}}(B_\varphi)$ is the limit shape of Z_0 with respect to the centred inradius ρ_o.

Up to now, we have only considered size functionals which are isotropic, that is, invariant under rotations. This is not required by our general result. Extending a study of Miles [10, Section 6] in the plane, we now measure the size by the width in a given direction.

(8) *The zero cell of a stationary, isotropic Poisson hyperplane mosaic; the size measured by the width* $w_{\boldsymbol{v}}$ *in a given direction* $\boldsymbol{v}$

The width $w_{\boldsymbol{v}}(K)$ of the convex body K in the given direction $\boldsymbol{v} \in S^{d-1}$ is the distance between the two supporting hyperplanes of K orthogonal to $\boldsymbol{v}$. Since K contains a segment of length at least $w_{\boldsymbol{v}}(K)$, one obtains, similarly as in Example **(3)** above, the estimate

$$\Phi(K) \geq \frac{\kappa_{d-1}}{d\kappa_d} w_{\boldsymbol{v}}(K),$$

with equality if and only if K is a segment parallel to $\boldsymbol{v}$. Hence, the limit shape of the zero cell with respect to $w_{\boldsymbol{v}}$ is the class of segments of direction $\boldsymbol{v}$.

Many more cases could be considered, but often they lead to unsolved geometric problems about the determination of the extremal bodies of the isoperimetric inequality (4). For example, the question for the asymptotic shape of the zero cell of a stationary, non-isotropic Poisson hyperplane process with respect to the kth intrinsic volume V_k leads to the problem of maximizing $V_k(K)$ under the condition $\int_{S^{d-1}} h(K, \boldsymbol{u}) \, \varphi(\mathrm{d}\boldsymbol{u}) = 1$, for a given even probability measure φ not concentrated on a great subsphere. Another example is the choice of the circumradius as a size functional. In the stationary and isotropic case, one would have to determine the convex bodies with given circumradius and minimal mean width. It is plausible that these are the segments, and for $d = 2$ this is easy to see, but in higher dimensions a proof seems lacking.

6 Proof of Theorem 1

As an introduction to the proof of Theorem 1, we extend a heuristic argument from [4], trying to make plausible why an estimate as in Theorem 1 can be expected. In these heuristics, we restrict ourselves to an interval $[a, \infty)$, with $a > 0$. We have to estimate the conditional probability

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \geq a) = \frac{\mathbb{P}(\vartheta(Z_0) \geq \epsilon,\ \Sigma(Z_0) \geq a)}{\mathbb{P}(\Sigma(Z_0) \geq a)}.$$

Estimation of the denominator is easy. There exists an extremal body $B \in \mathcal{K}_o^d$. Let B_a be the dilate of B with $\Sigma(B_a) = a$. Then, by the monotonicity of Σ,

$$\mathbb{P}(\Sigma(Z_0) \geq a) \geq \mathbb{P}(X(\mathcal{H}_{B_a}) = 0) = \exp\{-\Phi(B_a)\lambda\}.$$

Since B_a is an extremal body, we have

$$\Phi(B_a) = \tau\Sigma(B_a)^{r/k} = \tau a^{r/k}, \tag{18}$$

hence

$$\mathbb{P}(\Sigma(Z_0) \geq a) \geq \exp\{-\tau a^{r/k}\lambda\}. \tag{19}$$

For the estimation of the numerator, we try to compare the occurring zero cells with a deterministic convex body with similar properties, that is, not cut by hyperplanes of the process, with large size and large deviation from B. Suppose that $K \in \mathcal{K}_o^d$ is a convex body satisfying

$$\vartheta(K) \geq \epsilon > 0 \qquad \text{and} \qquad \Sigma(K) \geq a.$$

Then, by (7),

$$\mathbb{P}(X(\mathcal{H}_K) = 0) = \exp\{-\Phi(K)\lambda\} \leq \exp\{-(1 + f(\epsilon))\tau a^{r/k}\lambda\}.$$

Heuristically, we hope that here we may replace the deterministic convex body K satisfying

$$X(\mathcal{H}_K) = 0, \qquad \vartheta(K) \geq \epsilon, \qquad \Sigma(K) \geq a$$

by the random polytope Z_0 satisfying

$$X(\mathcal{H}_{\alpha Z_0}) = 0\ \ \forall\, \alpha \in (0, 1), \qquad \vartheta(Z_0) \geq \epsilon, \qquad \Sigma(Z_0) \geq a,$$

at the cost of only a slight weakening of the inequality, say

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon,\ \Sigma(Z_0) \geq a) \leq c'\ \exp\{-(1 + c'' f(\epsilon))\tau a^{r/k}\lambda\} \tag{20}$$

with $c', c'' > 0$. If (20) can be proved, then together with (19) this implies

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \geq a) \leq c' \exp\{-c'' f(\epsilon)\tau a^{r/k}\lambda\},$$

which is of the form asserted in Theorem 1.

The bulk of the proof will consist in replacing this heuristic argument by precise reasoning. The proof is an extension of the one given in [4] for the case of a stationary Poisson hyperplane process, the volume functional, and a special deviation functional. We merely quote those parts of the proof in [4] which require only obvious changes, but we give full proofs for the parts which require extended arguments or where simplifications have been possible.

To simplify the notation, we write

$$\mathcal{H}_K^n := \mathcal{H}_K \times \cdots \times \mathcal{H}_K \qquad (n \text{ times}),$$

$$\mu^n := \mu \otimes \cdots \otimes \mu \qquad (n \text{ times}),$$

and for given hyperplanes $H_1, \dots, H_n \in \mathcal{H}^d$, we write

$$H_1^- \cap \cdots \cap H_n^- =: P(H_{(n)}).$$

Convention about constants. In the following, $c_1, c_2, \dots$ denote positive constants which may depend on the dimension d, the directional distribution φ, the distance exponent r, and the size functional Σ. If a constant depends on additional data, these are either indicated as arguments or mentioned in the text.

For the proof of Theorem 1, we suppose that all data are given as assumed in that theorem and as explained in Section 2.

In contrast to the special case just considered heuristically, we will have to admit general intervals $(a, b) = a(1, 1 + h)$ as ranges for $\Sigma(Z_0)$. In a first stage, this is only possible for sufficiently small positive h.

The proof of Theorem 1 is preceded by a number of lemmas. Our first lemma is a counterpart to inequality (19) for certain bounded intervals. Its proof extends that of Lemma 2 in [5] (rather than that of Lemma 3.2 in [4]), but must be changed for the present situation, where extremal bodies may be of lower dimension and the support of the directional distribution φ need not be all of S^{d-1}.

Lemma 1. *For each $\beta > 0$, there are constants $h_0 > 0$, $N \in \mathbb{N}$ and $c > 0$, depending only on φ, r, Σ and β, such that for $a > 0$ and $0 < h < h_0$,*

$$\mathbb{P}(\Sigma(Z_0) \in a(1, 1 + h)) \geq c\, h (a^{r/k} \lambda)^N \exp\{-(1 + \beta)\tau a^{r/k} \lambda\}.$$

Proof. As shown in Section 3, there exist extremal bodies in the class $\mathcal{K}_\varphi$ of φ-adapted bodies. We choose such a body B with $\Sigma(B) = 1$, then $\Phi(B) = \tau$. For given data Φ and Σ, we make a definite choice of B, so that constants depending on B depend, in fact, on Φ and Σ.

For given numbers $a > 0$ and $\gamma > 0$, we set

$$B(a, \gamma) := a^{1/k}(B + \gamma B^d).$$

We choose γ so small that

$$\Phi(B(a, \gamma)) = \tau a^{r/k} \frac{\Phi(B + \gamma B^d)}{\Phi(B)} \leq (1 + \beta)\tau a^{r/k}, \tag{21}$$

which is possible by the continuity of Φ. The choice of γ depends on Φ, B and β.

Further, we choose a number $h_0 > 0$, depending only on B and γ, so that

$$(1 + h_0)^{1/k}[h(B, \boldsymbol{u}) + \gamma/2] \leq h(B, \boldsymbol{u}) + \gamma \qquad \text{for } \boldsymbol{u} \in S^{d-1}. \tag{22}$$

In the following, we assume that $0 < h < h_0$.

For a convex body $K \in \mathcal{K}_o^d$ we set

$$\mathcal{H}_{\partial K} := \{H \in \mathcal{H}^d : H \text{ touches } K\}.$$

By definition, H touches K if H meets K but not the interior of K. Thus, $H \in \mathcal{H}_{\partial K}$ if and only if $h(K, \boldsymbol{u}) > 0$ and $H = H(\boldsymbol{u}, h(K, \boldsymbol{u}))$ for some $\boldsymbol{u} \in S^{d-1}$. We define

$$\mathcal{M}_n := \left\{(H_1, \dots, H_n) \in \mathcal{H}^{n-1}_{B(1,\gamma/2)} \times \mathcal{H}_{\partial B} : P(H_{(n)}) \subset B(1, \gamma/2),\ \Sigma(P(H_{(n)})) \geq 1\right\}.$$

Let N be an arbitrary positive integer (it will be specified later). The crucial event $\Sigma(Z_0) \in a(1, 1+h)$ certainly occurs if the body $B(a, \gamma)$ is hit by precisely N hyperplanes $H_1, \dots, H_N$ of X and the polytope $P(H_{(N)})$ is contained in $B(a, \gamma)$ and satisfies $\Sigma(P(H_{(N)})) \in a(1, 1+h)$. By the Poisson property, under the condition that $X(\mathcal{H}_{B(a,\gamma)})$ $(= \operatorname{card}(X \cap \mathcal{H}_{B(a,\gamma)})) = N$, the process $X \cap \mathcal{H}_{B(a,\gamma)}$ is stochastically equivalent to the process defined by the set of N independent, identically distributed random hyperplanes with distribution $\Theta(\cdot \cap \mathcal{H}_{B(a,\gamma)})/\Theta(\mathcal{H}_{B(a,\gamma)})$. Recall that $\Theta = \lambda\mu$ and $\mu(\mathcal{H}_{B(a,\gamma)}) = \Phi(B(a, \gamma))$. Thus we get

$$\begin{aligned}
&\mathbb{P}(\Sigma(Z_0) \in a(1, 1+h)) \\
&\geq \mathbb{P}(X(\mathcal{H}_{B(a,\gamma)}) = N)\, \mathbb{P}(P(X \cap \mathcal{H}_{B(a,\gamma)}) \subset B(a, \gamma), \\
&\quad \Sigma(P(X \cap \mathcal{H}_{B(a,\gamma)})) \in a(1, 1+h) \mid X(\mathcal{H}_{B(a,\gamma)}) = N) \\
&= \frac{\lambda^N}{N!} \exp\{-\Phi(B(a, \gamma))\lambda\} \int_{\mathcal{H}^N_{B(a,\gamma)}} \mathbf{1}\{P(H_{(N)}) \subset B(a, \gamma)\} \\
&\quad \times \mathbf{1}\{\Sigma(P(H_{(N)})) \in a(1, 1+h)\}\, \mu^N(\mathrm{d}(H_1, \dots, H_N)). \qquad (23)
\end{aligned}$$

Let $H \in \mathcal{H}^d$. If the translate of H through $\boldsymbol{o}$ does not contain B, we denote by $b(H)$ the unique positive number for which $b(H)H \in \mathcal{H}_{\partial B}$; otherwise, we set $b(H) := 0$. Assume that $H_1, \dots, H_N \in \mathcal{H}^d$ satisfy the conditions

(i) $b(H_N)(H_1, \dots, H_N) \in \mathcal{M}_N$,

(ii) $\Sigma(P(H_{(N)})) \in a(1, 1+h)$.

We use the standard representations $H_i = H(\boldsymbol{v}_i, t_i)$, $i = 1, \dots, N$. Then, in particular,

$$b(H_N) = \frac{h(B, \boldsymbol{v}_N)}{t_N}$$

if $b(H_N) > 0$ (note that $b(H_N) > 0$ is implied by (i)). Using (i), (ii) and the definition of $\mathcal{M}_N$, we get

$$1 \leq \Sigma(b(H_N)P(H_{(N)})) = b(H_N)^k \Sigma(P(H_{(N)})) \leq \left(\frac{h(B, \boldsymbol{v}_N)}{t_N}\right)^k a(1+h),$$

thus

$$\frac{t_N}{h(B, \boldsymbol{v}_N)} \leq a^{1/k}(1+h)^{1/k}. \qquad (24)$$

Using (i) and the definition of $\mathcal{M}_N$, we find for $i = 1, \dots, N$ that

$$\frac{t_i h(B, \boldsymbol{v}_N)}{t_N} \leq h(B(1, \gamma/2), \boldsymbol{v}_i);$$

hence (24) and (22) give

$$t_i \le a^{1/k}(1+h)^{1/k}[h(B,\boldsymbol{v}_i)+\gamma/2] \le h(B(a,\gamma),\boldsymbol{v}_i),$$

thus

$$H_i \in \mathcal{H}_{B(a,\gamma)} \qquad \text{for } 1=1,\dots,N.$$

Finally, (i) and the definition of $\mathcal{M}_N$ together with (24) and (22) imply that

$$P(H_{(N)}) \subset \frac{t_N}{h(B,\boldsymbol{v}_N)} B(1,\gamma/2) \subset a^{1/k}(1+h)^{1/k} B(1,\gamma/2) \subset B(a,\gamma).$$

Thus, if $H_1,\dots,H_N$ satisfy (i), (ii), then the indicator functions in the integral (23) are equal to one. We deduce that

$$\mathbb{P}(\Sigma(Z_0) \in a(1,1+h)) \ge \frac{\lambda^N}{N!} \exp\{-\Phi(B(a,\gamma))\lambda\} \cdot I, \tag{25}$$

where

$$I := \int_{\mathcal{H}^N} \mathbf{1}\{b(H_N)(H_1,\dots,H_N) \in \mathcal{M}_N\} \mathbf{1}\{\Sigma(P(H_{(N)})) \in a(1,1+h)\}\, \mu^N(\mathrm{d}(H_1,\dots,H_N)).$$

With an arbitrary number $\eta > 0$, we can estimate

$$\begin{aligned} I \;\ge\; & \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_0^\infty \cdots \int_0^\infty \mathbf{1}\left\{(H(\boldsymbol{v}_1,t_1),\dots,H(\boldsymbol{v}_N,t_N)) \in (t_N/h(B,\boldsymbol{v}_N))\mathcal{M}_N\right\} \\ & \times \mathbf{1}\left\{\Sigma\left(H^-(\boldsymbol{v}_1,t_1)\cap\cdots\cap H^-(\boldsymbol{v}_N,t_N)\right) \in a(1,1+h)\right\} \mathbf{1}\{h(B,\boldsymbol{v}_N)\ge\eta\} \\ & \times (t_1\cdots t_N)^{r-1}\,\mathrm{d}t_1\dots\mathrm{d}t_N\,\varphi(\mathrm{d}\boldsymbol{v}_1)\dots\varphi(\mathrm{d}\boldsymbol{v}_N). \end{aligned}$$

In the inner integrals, we introduce new variables $\bar t_1,\dots,\bar t_{N-1},z$ by $t_i = z\bar t_i$ for $i=1,\dots,N-1$ and $t_N = zh(B,\boldsymbol{v}_N)$; then we first carry out the integration with respect to z. This gives

$$\begin{aligned} I \;\ge\; & \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_0^\infty \cdots \int_0^\infty \mathbf{1}\{h(B,\boldsymbol{v}_N)\ge\eta\} \\ & \times \mathbf{1}\left\{(H(\boldsymbol{v}_1,\bar t_1),\dots,H(\boldsymbol{v}_{N-1},\bar t_{N-1}),H(\boldsymbol{v}_N,h(B,\boldsymbol{v}_N))) \in \mathcal{M}_N\right\} \\ & \times \mathbf{1}\left\{z^k\Sigma\left(H^-(\boldsymbol{v}_1,\bar t_1)\cap\cdots\cap H^-(\boldsymbol{v}_{N-1},\bar t_{N-1})\cap H^-(\boldsymbol{v}_N,h(B,\boldsymbol{v}_N))\right) \in a(1,1+h)\right\} \\ & \times z^{rN-1}h(B,\boldsymbol{v}_N)^r(\bar t_1\cdots\bar t_{N-1})^{r-1}\,\mathrm{d}\bar t_1\dots\mathrm{d}\bar t_{N-1}\,\mathrm{d}z\,\varphi(\mathrm{d}\boldsymbol{v}_1)\dots\varphi(\mathrm{d}\boldsymbol{v}_N) \\ =\; & \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_0^\infty \cdots \int_0^\infty \mathbf{1}\{h(B,\boldsymbol{v}_N)\ge\eta\} \\ & \times \mathbf{1}\left\{(H(\boldsymbol{v}_1,\bar t_1),\dots,H(\boldsymbol{v}_{N-1},\bar t_{N-1}),H(\boldsymbol{v}_N,h(B,\boldsymbol{v}_N))) \in \mathcal{M}_N\right\} \\ & \times \frac{a^{rN/k}h(B,\boldsymbol{v}_N)^r}{rN}\Sigma\left(H^-(\boldsymbol{v}_1,\bar t_1)\cap\cdots\cap H^-(\boldsymbol{v}_{N-1},\bar t_{N-1})\cap H^-(\boldsymbol{v}_N,h(B,\boldsymbol{v}_N))\right)^{-rN/k} \\ & \times \left[(1+h)^{rN/k}-1\right](\bar t_1\cdots\bar t_{N-1})^{r-1}\,\mathrm{d}\bar t_1\dots\mathrm{d}\bar t_{N-1}\,\varphi(\mathrm{d}\boldsymbol{v}_1)\dots\varphi(\mathrm{d}\boldsymbol{v}_N) \\ \ge\; & \frac{a^{rN/k}\eta^r}{rN}\Sigma(B(1,\gamma/2))^{-rN/k}\frac{rN}{k}h\cdot J, \end{aligned}$$

where

$$J := \int_{\mathcal{H}^{N-1}} \int_{S^{d-1}} \mathbf{1}\{(H_1, \dots, H_{N-1}, H(\boldsymbol{u}, h(B, \boldsymbol{u}))) \in \mathcal{M}_N\} \mathbf{1}\{h(B, \boldsymbol{u}) \geq \eta\}$$
$$\varphi(\mathrm{d}\boldsymbol{u})\, \mu^{N-1}(\mathrm{d}(H_1, \dots, H_{N-1})).$$

So far, N and η were arbitrary. We must now show that there are numbers $N \in \mathbb{N}$ and $\eta > 0$ such that

$$J \geq c_1(\beta) > 0.$$

For this, we choose a vector $\boldsymbol{u}_0 \in \operatorname{supp} \varphi$ with

$$h(B, \boldsymbol{u}_0) > 0$$

(it exists since $B \neq \{\boldsymbol{o}\}$) and a countable dense subset $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots\}$ of $\operatorname{supp} \varphi$. Then

$$B = \bigcap_{i=1}^{\infty} H^-(\boldsymbol{u}_i, h(B, \boldsymbol{u}_i)). \tag{26}$$

In fact, denote the right-hand side of (26) by B', then $B \subset B'$ trivially. Let $\boldsymbol{x} \in \mathbb{R}^d \setminus B$. Since $B \in \mathcal{K}_\varphi$, there is a vector $\boldsymbol{u} \in \operatorname{supp} \varphi$ such that $\boldsymbol{x} \notin H^-(\boldsymbol{u}, h(B, \boldsymbol{u}))$. A vector $\boldsymbol{u}_i$ sufficiently close to $\boldsymbol{u}$ satisfies $x \notin H^-(\boldsymbol{u}_i, h(B, \boldsymbol{u}_i))$, hence $x \notin B'$. This proves (26).

For $n \in \mathbb{N}$, let

$$P_n := \bigcap_{i=1}^{n} H^-(\boldsymbol{u}_i, h(B, \boldsymbol{u}_i)),$$

then $P_n \downarrow B$ in the Hausdorff metric, as $n \to \infty$ ([12], Lemma 1.8.1). Hence, we can choose a number N for which $P'_N := P_{N-1} \cap H^-(\boldsymbol{u}_0, h(B, \boldsymbol{u}_0)) \subset B(1, \gamma/4)$. For given data Φ and Σ and accordingly chosen B, we make a definite choice of such a polytope P'_N satisfying $P'_N \subset B(1, \gamma/4)$; it then depends only on Φ, Σ and β.

By continuity, we can choose a neighbourhood U_N of $\boldsymbol{u}_0$ and a number $\eta > 0$ such that $h(B, \boldsymbol{v}_N) \geq \eta$ for $\boldsymbol{v}_N \in U_N$. Further, there exist neighbourhoods $U_i \subset S^{d-1}$ of $\boldsymbol{u}_i$, $i = 1, \dots, N-1$, and a number $\alpha > 0$ (depending on B, P'_N and γ) such that

$$\boldsymbol{v}_i \in U_i \quad \text{for } i = 1, \dots, N \quad \text{and} \quad 0 \leq t_i \leq \alpha \quad \text{for } i = 1, \dots, N-1$$

implies

$$H(\boldsymbol{v}_i, h(B, \boldsymbol{v}_i) + t_i) \cap B(1, \gamma/2) \neq \emptyset \quad \text{for } i = 1, \dots, N-1$$

and

$$P := \bigcap_{i=1}^{N-1} H^-(\boldsymbol{v}_i, h(B, \boldsymbol{v}_i) + t_i) \cap H^-(\boldsymbol{v}_N, h(B, \boldsymbol{v}_N)) \subset B(1, \gamma/2).$$

Since $B \subset P$, we have $\Sigma(P) \geq \Sigma(B) = 1$. This shows that

$$(H(\boldsymbol{v}_1, h(B, \boldsymbol{v}_1) + t_1), \dots, H(\boldsymbol{v}_{N-1}, h(B, \boldsymbol{v}_{N-1}) + t_{N-1}), H(\boldsymbol{v}_N, h(B, \boldsymbol{v}_N))) \in \mathcal{M}_N$$

and hence that

$$J \geq \left(\frac{1}{r}\alpha^r\right)^{N-1} \prod_{i=1}^{N} \varphi(U_i) =: c_1(\beta) > 0. \tag{27}$$

Here $\varphi(U_i) > 0$ follows from $\boldsymbol{u}_j \in \operatorname{supp} \varphi$ for $j = 0, \dots, N-1$.

Introducing (27) in the estimate for I and combining this with (25) and (21), we obtain the assertion of Lemma 1. $\square$

In the following, D denotes the diameter. On the compact set $\{K \in \mathcal{K}_o^d : D(K) = 1\}$, the continuous function Σ attains a (positive) maximum $1/c_2^k$. By homogeneity, it follows that $D(K) \geq c_2\Sigma(K)^{1/k}$ for $K \in \mathcal{K}_o^d$. For $K \in \mathcal{K}_o^d$ with $\Sigma(K) > 0$ we introduce the *relative diameter* Δ by

$$\Delta(K) := D(K)/c_2\Sigma(K)^{1/k}.$$

For $a > 0$, $\epsilon \geq 0$, $h > 0$ and $m \in \mathbb{N}$ we define

$$\mathcal{K}_{a,\epsilon,h}(m) := \{K \in \mathcal{K}_\varphi : \Sigma(K) \in a(1, 1 + h),\ \vartheta(K) \geq \epsilon,\ \Delta(K) \in [m, m + 1)\}$$

and

$$q_{a,\epsilon,h}(m) := \mathbb{P}(Z_0 \in \mathcal{K}_{a,\epsilon,h}(m)).$$

(Note that the condition $\vartheta(K) \geq \epsilon$ is trivially satisfied for $\epsilon = 0$.) We have

$$\sum_{m=1}^{\infty} q_{a,\epsilon,h}(m) = \mathbb{P}(\Sigma(Z_0) \in a(1, 1 + h),\ \vartheta(Z_0) \geq \epsilon). \tag{28}$$

For the latter probability, Lemma 9 will provide the upper estimate which is crucial for the proof of Theorem 1. The reason for introducing the additional restriction $\Delta(Z_0) \in [m, m+1)$ lies in the fact that it will allow us, thanks to Lemma 2, to consider in a first step only the zero cells lying in some fixed bounded set C. The delicate estimate of Lemma 5 obtained in this way will later be used for small numbers m, while for large m the estimate of Lemma 3 is sufficient. For technical reasons, we first consider only ranges for $\Sigma(Z_0)$ of the type $a(1, 2)$. Lemmas 6, 7, 8 are needed to extend this to intervals $a(1, 1+h)$, with sufficiently small $h > 0$. The completion of the proof then settles the case of general intervals.

By $\mathcal{P}_o^d$ we denote the set of polytopes in $\mathcal{K}_o^d$.

Lemma 2. *For each* $m \in \mathbb{N}$:

(a) $K \in \mathcal{K}_{a,0,1}(m)$ *implies* $K \subset c_3 m a^{1/k} B^d =: C$,

(b) *there exists a measurable map*

$$\mathcal{K}_{a,0,1}(m) \cap \mathcal{P}_o^d \ni P \mapsto \boldsymbol{v}(P)$$

such that $\boldsymbol{v}(P)$ *is a vertex of* P *with* $\|\boldsymbol{v}(P)\| \geq c_4 m a^{1/k}$.

Proof. Let $K \in \mathcal{K}_{a,0,1}(m)$. Then

$$D(K) < (m + 1)c_2\Sigma(K)^{1/k} \leq (m + 1)c_2(2a)^{1/k} \leq c_3 m a^{1/k}.$$

Since $\boldsymbol{o} \in K$, this proves (a).

Further,

$$D(K) \geq c_2 m\Sigma(K)^{1/k} \geq c_2 m a^{1/k}.$$

Hence, if P is a polytope in $\mathcal{K}_{a,0,1}(m)$, it has a vertex $\boldsymbol{v}$ such that $\|\boldsymbol{v}\| \geq (1/2)c_2 m a^{1/k}$. The existence of the measurable map follows as in [4], proof of Lemma 4.3. $\square$

Lemma 3. *For $a > 0$ and $m \in \mathbb{N}$,*

$$q_{a,0,1}(m) \le c_7 \exp\{-c_5 m^r a^{r/k} \lambda\}.$$

Proof. Let $m \in \mathbb{N}$ be given. Let C be the ball defined in Lemma 2. We will repeatedly make use of

$$q_{a,\epsilon,1}(m) = \sum_{N=d+1}^{\infty} \mathbb{P}(X(\mathcal{H}_C) = N)\mathbb{P}(Z_0 \in \mathcal{K}_{a,\epsilon,1}(m) \mid X(\mathcal{H}_C) = N), \tag{29}$$

where

$$\begin{aligned} p_N &:= \mathbb{P}(Z_0 \in \mathcal{K}_{a,\epsilon,1}(m) \mid X(\mathcal{H}_C) = N) \\ &= \frac{1}{\Phi(C)^N} \int_{\mathcal{H}_C^N} \mathbf{1}\{P(H_{(N)}) \in \mathcal{K}_{a,\epsilon,1}(m)\} \, \mu^N(\mathrm{d}(H_1, \dots, H_N)), \end{aligned} \tag{30}$$

since X is a Poisson process. In the present proof, $\epsilon = 0$. Suppose that $H_1, \dots, H_N \in \mathcal{H}_C$ are such that $P := P(H_{(N)}) \in \mathcal{K}_{a,0,1}(m)$. Let $\boldsymbol{v}(P)$ be the vertex according to Lemma 2. This vertex is the intersection of d facets of P. Hence, there exists an index set $J \subset \{1, \dots, N\}$ with d elements such that

$$\{\boldsymbol{v}(P)\} = \bigcap_{i \in J} H_i.$$

We denote the segment $[\boldsymbol{o}, \boldsymbol{v}(P)]$ by $S(H_i, \, i \in J)$. The segment $S = S(H_i, \, i \in J)$ satisfies

$$H_i \cap \operatorname{relint} S = \emptyset \qquad \text{for} \qquad i \in \{1, \dots, N\} \setminus J,$$

where relint denotes the relative interior. Since $S \subset C$, we have

$$\int_{\mathcal{H}_C} \mathbf{1}\{H \cap S = \emptyset\} \mu(\mathrm{d}H) = \Phi(C) - \Phi(S).$$

We denote by U_0 a closed segment of unit length and with one endpoint at $\boldsymbol{o}$ for which

$$\Phi(U_0) = \min\{\Phi([\boldsymbol{o}, \boldsymbol{u}]) : \boldsymbol{u} \in S^{d-1}\}.$$

Then

$$\Phi(C) - \Phi(S) \le \Phi(C) - |S|^r \Phi(U_0),$$

where $|S|$ is the length of S. Since φ is not concentrated on a closed hemisphere, we have $\Phi(U_0) > 0$. Thus, we obtain

$$\begin{aligned} p_N &\le \binom{N}{d} \frac{1}{\Phi(C)^N} \int_{\mathcal{H}_C^d} \int_{\mathcal{H}_C^{N-d}} \mathbf{1}\left\{|S(H_j, \, j \in \{1, \dots, d\})| \ge c_4 m a^{1/k}\right\} \\ &\quad \times \mathbf{1}\left\{S(H_j, \, j \in \{1, \dots, d\}) \cap H_i = \emptyset \text{ for } i = d+1, \dots, N\right\} \\ &\quad \mu^{N-d}(\mathrm{d}(H_{d+1}, \dots, H_N)) \, \mu^d(\mathrm{d}(H_1, \dots, H_d)) \\ &\le \binom{N}{d} \Phi(C)^{-N} \int_{\mathcal{H}_C^d} [\Phi(C) - (c_4 m a^{1/k})^r \Phi(U_0)]^{N-d} \, \mu^d(\mathrm{d}(H_1, \dots, H_d)) \\ &= \binom{N}{d} \Phi(C)^{d-N} \left[\Phi(C) - 2c_5 m^r a^{r/k}\right]^{N-d}, \end{aligned}$$

with $c_5 > 0$. This leads to the estimate

$$\begin{aligned}
q_{a,0,1}(m) &\leq \sum_{N=d+1}^{\infty} \frac{[\Phi(C)\lambda]^N}{N!} \exp\{-\Phi(C)\lambda\} \binom{N}{d} \Phi(C)^{d-N} \left[\Phi(C) - 2c_5 m^r a^{r/k}\right]^{N-d} \\
&= \frac{1}{d!}[\Phi(C)\lambda]^d \exp\{-\Phi(C)\lambda\} \sum_{N=d+1}^{\infty} \frac{1}{(N-d)!} \left[\Phi(C)\lambda - 2c_5 m^r a^{r/k}\lambda\right]^{N-d} \\
&\leq \frac{1}{d!}[\Phi(C)\lambda]^d \exp\left\{-2c_5 m^r a^{r/k}\lambda\right\} \\
&\leq c_6(m^r a^{r/k}\lambda)^d \exp\left\{-2c_5 m^r a^{r/k}\lambda\right\} \\
&\leq c_7 \exp\left\{-c_5 m^r a^{r/k}\lambda\right\}.
\end{aligned}$$

□

For a polytope P, we denote by $\operatorname{ext} P$ the set of vertices and by $f_0(P)$ the number of vertices of P. The following approximation result will allow us to essentially restrict ourselves to zero cells with a bounded number of vertices. The proof of the lemma is the same as that of Lemma 5 in [5], with the obvious changes.

Lemma 4. *Let $\alpha > 0$ be given. There is a number $\nu \in \mathbb{N}$ depending only on d, φ, r and α such that the following is true. For $P \in \mathcal{P}_o^d$ there exists a polytope $Q = Q(P) \in \mathcal{P}_o^d$ satisfying $\operatorname{ext} Q \subset \operatorname{ext} P$, $f_0(Q) \leq \nu$, and $\Phi(Q) \geq (1-\alpha)\Phi(P)$. Moreover, there exists a measurable selection $P \mapsto Q(P)$.*

From now on we assume that, for given Φ, Σ and ϑ, a stability function f according to (7) has been chosen.

Lemma 5. *For $a > 0$, $m \in \mathbb{N}$ and $\epsilon > 0$,*

$$q_{a,\epsilon,1}(m) \leq c_{10}(f,\epsilon) m^{rd\nu} \exp\left\{-(1 + f(\epsilon)/3)\tau a^{r/k}\lambda\right\},$$

where ν depends only on d, φ, r and ϵ.

Proof. Let B be the extremal body chosen in the proof of Lemma 1, and let B_a be the dilate of B with $\Sigma(B_a) = a$. For given $m \in \mathbb{N}$, we define C as in Lemma 2 and use (29) and (30). Suppose that $H_1, \dots, H_N \in \mathcal{H}_C$ are such that $P(H_{(N)}) \in \mathcal{K}_{a,\epsilon,1}(m)$. By (7) and (18),

$$\begin{aligned}
\Phi(P(H_{(N)})) &\geq (1 + f(\epsilon))\tau\Sigma(P(H_{(N)}))^{r/k} \geq (1 + f(\epsilon))\tau a^{r/k} \\
&= (1 + f(\epsilon))\Phi(B_a). \qquad (31)
\end{aligned}$$

Let $\alpha := f(\epsilon)/(2 + f(\epsilon))$, then $(1-\alpha)(1 + f(\epsilon)) = 1 + \alpha$.

We generalize the proof of Lemma 5.2 in [4]. By Lemma 4, there are $\nu = \nu(d, \varphi, r, \epsilon)$ vertices of $P(H_{(N)})$ such that the convex hull $Q(P(H_{(N)})) =: Q(H_{(N)}) =: Q$ of these vertices satisfies

$$\Phi(Q) \geq (1-\alpha)\Phi(P(H_{(N)})). \qquad (32)$$

The inequalities (31) and (32) imply that

$$\Phi(Q) \geq (1+\alpha)\Phi(B_a).$$

For each N-tuple $(H_1,\dots,H_N)$ such that $P(H_{(N)}) \in \mathcal{K}_{a,\epsilon,1}(m)$, we make a definite choice of $Q = Q(H_{(N)})$, in such a way that $Q(H_{(N)})$ is a measurable function of $(H_1,\dots,H_N)$.

Excluding a set of N-tuples $(H_1,\dots,H_N)$ of μ^N measure zero, we can assume that each of the vertices of Q lies in precisely d of the hyperplanes $H_1,\dots,H_N$, and the remaining hyperplanes are disjoint from Q. Hence, at most $d\nu$ of the hyperplanes $H_1,\dots,H_N$ meet Q; let $j \in \{d+1,\dots,d\nu\}$ denote their precise number. Suppose that $H_1,\dots,H_j$ are the hyperplanes meeting Q. Then there are subsets $J_1,\dots,J_{f_0(Q)} \subset \{1,\dots,j\}$, each of cardinality d, such that the intersections

$$\bigcap_{i\in J_r} H_i, \qquad r = 1,\dots,f_0(Q) \leq \nu,$$

yield the vertices of Q. In the following, the sum $\sum_{(J_1,\dots,J_\nu)}$ extends over all ν-tuples of d-element subsets of $\{1,\dots,j\}$. To estimate the inner ($(N-j)$-fold) integral below, we make use of the fact that for any convex body $K \subset C$,

$$\int_{\mathcal{H}_C} \mathbf{1}\{H \cap K = \emptyset\}\,\mu(\mathrm{d}H) = \Phi(C) - \Phi(K).$$

In this way, we obtain

$$\begin{aligned}
&\mathbb{P}(Z_0 \in \mathcal{K}_{a,\epsilon,1}(m) \mid X(\mathcal{H}_C) = N)\Phi(C)^N \\
&\leq \sum_{j=d+1}^{d\nu} \binom{N}{j} \int_{\mathcal{H}_C^N} \mathbf{1}\left\{P(H_{(N)}) \in \mathcal{K}_{a,\epsilon,1}(m)\right\} \mathbf{1}\{H_i \cap Q(H_{(N)}) \neq \emptyset \text{ for } i = 1,\dots,j\} \\
&\quad \times \mathbf{1}\{H_i \cap Q(H_{(N)}) = \emptyset \text{ for } i = j+1,\dots,N\}\mu^N(\mathrm{d}(H_1,\dots,H_N)) \\
&\leq \sum_{j=d+1}^{d\nu} \binom{N}{j} \sum_{(J_1,\dots,J_\nu)} \int_{\mathcal{H}_C^j} \int_{\mathcal{H}_C^{N-j}} \mathbf{1}\left\{\Phi\left(\operatorname{conv} \bigcup_{r=1}^{\nu} \bigcap_{i\in J_r} H_i\right) \geq (1+\alpha)\Phi(B_a)\right\} \\
&\quad \times \mathbf{1}\left\{H_s \cap \operatorname{conv} \bigcup_{r=1}^{\nu} \bigcap_{i\in J_r} H_i = \emptyset \text{ for } s = j+1,\dots,N\right\} \\
&\quad \mu^{N-j}(\mathrm{d}(H_{j+1},\dots,H_N))\,\mu^j(\mathrm{d}(H_1,\dots,H_j)) \\
&\leq \sum_{j=d+1}^{d\nu} \binom{N}{j} \binom{j}{d}^{\nu} [\Phi(C) - (1+\alpha)\Phi(B_a)]^{N-j}\Phi(C)^j.
\end{aligned}$$

Summation over N gives

$$\begin{aligned}
q_{a,\epsilon,1}(m) &\leq \sum_{N=d+1}^{\infty} \frac{[\Phi(C)\lambda]^N}{N!} \exp\{-\Phi(C)\lambda\} \sum_{j=d+1}^{d\nu} \binom{N}{j}\binom{j}{d}^{\nu} \frac{[\Phi(C) - (1+\alpha)\Phi(B_a)]^{N-j}}{\Phi(C)^{N-j}} \\
&= \sum_{j=d+1}^{d\nu} \binom{j}{d}^{\nu} \frac{[\Phi(C)\lambda]^j}{j!} \exp\{-\Phi(C)\lambda\} \sum_{N=j}^{\infty} \frac{1}{(N-j)!}[\Phi(C)\lambda - (1+\alpha)\Phi(B_a)\lambda]^{N-j} \\
&= \sum_{j=d+1}^{d\nu} \binom{j}{d}^{\nu} \frac{[\Phi(C)\lambda]^j}{j!} \exp\{-(1+\alpha)\Phi(B_a)\lambda\}.
\end{aligned}$$

Here $\Phi(B_a) = \tau a^{r/k}$ by (18), and by Lemma 2,

$$\Phi(C) = \Phi(c_3 m a^{1/k} B^d) = c_8 m^r a^{r/k}.$$

Thus we get

$$\begin{aligned} q_{a,\epsilon,1}(m) &\leq c_9(\epsilon)\left[(a^{r/k}\lambda)^{d\nu} + 1\right] m^{rd\nu} \exp\left\{-(1+\alpha)\tau a^{r/k}\lambda\right\} \\ &\leq c_{10}(\epsilon) m^{rd\nu} \exp\left\{-(1+f(\epsilon)/3)\tau a^{r/k}\lambda\right\}, \end{aligned}$$

since $f(\epsilon) < 1$. □

The next steps serve to extend the estimate for $q_{a,\epsilon,1}(m)$ to one for $q_{a,\epsilon,h}(m)$, for $h > 0$.

Lemma 6. *Let $w > 0$, $h \in (0, 1/2)$, $r \geq 1$ and $p \geq r - 1$. Then*

$$\int_1^{\sqrt[k]{1+h}} s^p \exp\{-ws^r\}\, \mathrm{d}s \leq b_1 h w [1 + (\exp\{b_2 w\} - 1)^{-1}] \int_1^{\sqrt[k]{2}} s^p \exp\{-ws^r\}\, \mathrm{d}s$$

with positive constants b_1, b_2 depending only on r and k.

Proof. Substituting $s = x^{1/r}$ and applying in turn the mean value theorems of integral calculus and differential calculus, we get

$$\int_1^{\sqrt[k]{1+h}} s^p \exp\{-ws^r\}\, \mathrm{d}s \leq b_1(r,k) h \frac{1}{r} \eta^{(p+1-r)/r} \exp\{-w\eta\}$$

with a suitable number $\eta \in (0, (1+h)^{r/k})$. As in the proof of Lemma 6.2 in [4], we obtain

$$\int_1^{\sqrt[k]{2}} s^p \exp\{-ws^r\}\, \mathrm{d}s \geq \frac{1}{r} \eta^{(p+1-r)/r} \exp\{-w\eta\} \frac{1}{w} \left[1 - \exp\{-b_2(r,k) w\}\right].$$

The assertion follows by combining these two inequalities. □

The next lemma is stated in a general version, since different specializations are needed. Here $f_{d-1}(P)$ denotes the number of facets of a polytope P.

Lemma 7. *For $n \in \mathbb{N}$, $n \geq d+1$ and a Borel set $\mathcal{B} \subset \mathcal{K}_o^d$, let*

$$R(\mathcal{B}, n) := \{(H_1, \dots, H_n) \in (\mathcal{H}^d)^n : P(H_{(n)}) \in \mathcal{B},\ f_{d-1}(P(H_{(n)})) = n\}.$$

Then

$$\mathbb{P}(Z_0 \in \mathcal{B},\ f_{d-1}(Z_0) = n) = \frac{\lambda^n}{n!} \int_{R(\mathcal{B},n)} \exp\left\{-\Phi(P(H_{(n)}))\lambda\right\} \mu^n(\mathrm{d}(H_1, \dots, H_n)).$$

Proof. Let $C \subset \mathbb{R}^d$ be a ball with centre $\boldsymbol{o}$ and put

$$R(\mathcal{B}, n, C) := \{(H_1, \dots, H_n) \in R(\mathcal{B}, n) : P(H_{(n)}) \subset C\}.$$

Then, for $N \geq n$,

$$
\begin{aligned}
&\mathbb{P}(Z_0 \in \mathcal{B},\ f_{d-1}(Z_0) = n,\ Z_0 \subset C \mid X(\mathcal{H}_C) = N) \\
&= \frac{1}{\Phi(C)^N} \int_{\mathcal{H}_C^N} \mathbf{1}\left\{P(H_{(N)}) \in \mathcal{B},\ f_{d-1}(P(H_{(N)})) = n,\ P(H_{(N)}) \subset C\right\} \mu^N(\mathrm{d}(H_1, \dots, H_N)) \\
&= \frac{\binom{N}{n}}{\Phi(C)^N} \int_{R(\mathcal{B},n,C)} \left[\Phi(C) - \Phi(P(H_{(n)}))\right]^{N-n} \mu^n(\mathrm{d}(H_1, \dots, H_n))
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\mathbb{P}(Z_0 \in \mathcal{B},\ f_{d-1}(Z_0) = n,\ Z_0 \subset C) \\
&= \sum_{N=n}^{\infty} \mathbb{P}(Z_0 \in \mathcal{B},\ f_{d-1}(Z_0) = n,\ Z_0 \subset C \mid X(\mathcal{H}_C) = N) \frac{[\Phi(C)\lambda]^N}{N!} \exp\{-\Phi(C)\lambda\} \\
&= \exp\{-\Phi(C)\lambda\} \\
&\quad \times \frac{\lambda^n}{n!} \int_{R(\mathcal{B},n,C)} \sum_{N=n}^{\infty} \frac{1}{(N-n)!} \left[\Phi(C)\lambda - \Phi(P(H_{(n)}))\lambda\right]^{N-n} \mu^n(\mathrm{d}(H_1, \dots, H_n)) \\
&= \frac{\lambda^n}{n!} \int_{R(\mathcal{B},n,C)} \exp\left\{-\Phi(P(H_{(n)}))\lambda\right\} \mu^n(\mathrm{d}(H_1, \dots, H_n)).
\end{aligned}
$$

We apply this to an increasing sequence of balls covering $\mathbb{R}^d$ and then obtain the assertion from the monotone convergence theorem. □

Now let $a > 0$, $\epsilon \geq 0$, $h > 0$ and $m \in \mathbb{N}$. We set

$$
q_{a,\epsilon,h}(m,n) := \mathbb{P}\left(Z_0 \in \mathcal{K}_{a,\epsilon,h}(m),\ f_{d-1}(Z_0) = n\right) \tag{33}
$$

for $n \in \mathbb{N}$, $n \geq d+1$. Next, we define

$$
\begin{aligned}
R_\epsilon(m,n) \;:=\; \Big\{&(H_1, \dots, H_n) \in (\mathcal{H}^d)^n : \vartheta(P(H_{(n)})) \geq \epsilon,\ \Delta(P(H_{(n)})) \in [m, m+1), \\
&f_{d-1}(P(H_{(n)})) = n\Big\}
\end{aligned}
$$

and

$$
R_{a,\epsilon,h}(m,n) := \left\{(H_1, \dots, H_n) \in R_\epsilon(m,n) : \Sigma(P(H_{(n)})) \in a(1, 1+h)\right\}.
$$

In the following two lemmas, we introduce an additional parameter σ_0. This can be put equal to 1 in the proofs of Theorems 1 and 2; it will only be needed for (41), which is applied in Section 9.

Lemma 8. *For $m \in \mathbb{N}$, $h \in (0, 1/2)$, $\epsilon \geq 0$ and $a^{r/k}\lambda \geq \sigma_0$, where $\sigma_0 > 0$ is a constant,*

$$
q_{a,\epsilon,h}(m) \leq c_{13}(\sigma_0) h\, a^{r/k} \lambda\, m^r q_{a,\epsilon,1}(m).
$$

Proof. We apply Lemma 7 with

$$
\mathcal{B} := \{K \in \mathcal{K}_{a,\epsilon,h}(m) : K \in \mathcal{P}_o^d,\ f_{d-1}(K) = n\}.
$$

Using standard representations $H_i = H(\boldsymbol{u}_i, t_i)$, we write the result as

$$
\begin{aligned}
&q_{a,\epsilon,h}(m,n) \\
&= \frac{\lambda^n}{n!}\int_{S^{d-1}}\cdots\int_{S^{d-1}}\int_0^\infty\cdots\int_0^\infty \mathbf{1}\left\{(H(\boldsymbol{u}_1,t_1),\ldots,H(\boldsymbol{u}_n,t_n))\in R_{a,\epsilon,h}(m,n)\right\} \\
&\quad\times \exp\left\{-\Phi\left(H^-(\boldsymbol{u}_1,t_1)\cap\cdots\cap H^-(\boldsymbol{u}_n,t_n)\right)\lambda\right\} \\
&\quad\times (t_1\cdots t_n)^{r-1}\,\mathrm{d}t_1\ldots\mathrm{d}t_n\,\varphi(\mathrm{d}\boldsymbol{u}_1)\ldots\varphi(\mathrm{d}\boldsymbol{u}_n).
\end{aligned}
$$

In the inner integrals, we introduce new variables $\bar{t}_1,\ldots,\bar{t}_{n-1},z$ by $t_i = z\bar{t}_i$ for $i=1,\ldots,n-1$ and $t_n = z$; then we first carry out the integration with respect to z. We exploit the fact that $R_\epsilon(m,n)$ is closed under dilatations. Writing $H(\boldsymbol{u}_i,\bar{t}_i) = H_i$ again for $i=1,\ldots,n-1$, we obtain

$$
\begin{aligned}
q_{a,\epsilon,h}(m,n) &= \frac{\lambda^n}{n!}\int_{(\mathcal{H}^d)^{n-1}}\int_{S^{d-1}} \mathbf{1}\left\{(H_1,\ldots,H_{n-1},H(\boldsymbol{u},1))\in R_\epsilon(m,n)\right\} \\
&\quad\times\int_0^\infty \mathbf{1}\left\{z^k\Sigma\left(H_1^-\cap\cdots\cap H_{n-1}^-\cap H^-(\boldsymbol{u},1)\right)\in a(1,1+h)\right\} \\
&\quad\times \exp\left\{-z^r\Phi\left(H_1^-\cap\cdots\cap H_{n-1}^-\cap H^-(\boldsymbol{u},1)\right)\lambda\right\} z^{rn-1}\,\mathrm{d}z \\
&\quad\times \varphi(\mathrm{d}\boldsymbol{u})\,\mu^{n-1}(\mathrm{d}(H_1,\ldots,H_{n-1})).
\end{aligned}
$$

For the computation and estimation of the inner integral $\int_0^\infty(\cdots)\,\mathrm{d}z$, we fix $H_1,\ldots,H_{n-1},\boldsymbol{u}$ with $(H_1,\ldots,H_{n-1},H(\boldsymbol{u},1))\in R_\epsilon(m,n)$, write

$$H_1^-\cap\cdots\cap H_{n-1}^-\cap H^-(\boldsymbol{u},1) =: P,$$

and define $z_a = z_a(H_1,\ldots,H_{n-1},\boldsymbol{u})$ by $\Sigma(z_aP)=a$. Then $\Sigma(z_a\sqrt[k]{1+h}P) = a(1+h)$. Since $z_aP\in\mathcal{K}_{a,0,1}(m)$, it follows from Lemma 2 that

$$c_4 m a^{1/k}[\boldsymbol{o},\boldsymbol{v}]\subset z_aP\subset c_3 m a^{1/k}B^d,$$

where $\boldsymbol{v}$ is a suitable unit vector, hence

$$c_{11}m^r a^{r/k}\le \Phi(z_aP)\le c_{12}m^r a^{r/k}. \tag{34}$$

In the subsequent computation, we substitute $z = z_a s$, then we apply Lemma 6, the inequalities (34), and reverse the substitution; this gives

$$
\begin{aligned}
\int_0^\infty(\cdots)\,\mathrm{d}z &= \int_{z_a}^{z_a\sqrt[k]{1+h}} \exp\{-\Phi(zP)\lambda\}z^{rn-1}\,\mathrm{d}z \\
&= z_a^{rn}\int_1^{\sqrt[k]{1+h}}\exp\{-s^r\Phi(z_aP)\lambda\}s^{rn-1}\,\mathrm{d}s \\
&\le b_1h\Phi(z_aP)\lambda\left[1+\left(\exp\{b_2\Phi(z_aP)\lambda\}-1\right)^{-1}\right] \\
&\quad\times z_a^{rn}\int_1^{\sqrt[k]{2}}\exp\{-s^r\Phi(z_aP)\lambda\}s^{rn-1}\,\mathrm{d}s \\
&\le c_{13}(\sigma_0)ha^{r/k}\lambda m^r\int_{z_a}^{z_a\sqrt[k]{2}}\exp\{-\Phi(zP)\lambda\}z^{rn-1}\,\mathrm{d}z.
\end{aligned}
$$

Inserting this in the multiple integral representing $q_{a,\epsilon,h}(m,n)$, we obtain the assertion of the lemma after a summation over $n \in \mathbb{N}$, $n \geq d+1$. $\square$

Our last lemma is the counterpart to Lemma 1. Recall that the stability function f was chosen before the formulation of Lemma 5.

Lemma 9. *Let $\epsilon > 0$, $h \in (0,1/2)$ and $a^{r/k}\lambda \geq \sigma_0$, where $\sigma_0 > 0$ is a constant. Then*

$$\mathbb{P}(\Sigma(Z_0) \in a(1,1+h),\ \vartheta(Z_0) \geq \epsilon) \leq c_{15}(f,\epsilon,\sigma_0)h \exp\left\{-(1+f(\epsilon)/4)\tau a^{r/k}\lambda\right\}.$$

Proof. With the constant c_5 appearing in Lemma 3, we set

$$c_{14}(f,\epsilon) := \max_{0\leq\epsilon\leq 1} [(2/c_5)(1+f(\epsilon)/3)\tau]^{1/r}$$

and $m_0 := \lfloor c_{14}(f,\epsilon) \rfloor$. Then

$$(c_5/2)m^r \geq (1+f(\epsilon)/3)\tau \qquad \text{for } m > m_0. \tag{35}$$

By (28) and Lemma 8, we have

$$\begin{aligned}\mathbb{P}(\Sigma(Z_0) \in a(1,1+h),\ \vartheta(Z_0) \geq \epsilon) &= \sum_{m\in\mathbb{N}} q_{a,\epsilon,h}(m) \\ &\leq c_{13}(\sigma_0)ha^{r/k}\lambda \left(\sum_{m=1}^{m_0} m^r q_{a,\epsilon,1}(m) + \sum_{m>m_0} m^r q_{a,\epsilon,1}(m)\right).\end{aligned}$$

For the estimation of $q_{a,\epsilon,1}(m)$ we use Lemma 5 for $m \leq m_0$ and Lemma 3 for $m > m_0$, observing that $q_{a,\epsilon,1}(m) \leq q_{a,0,1}(m)$. Then we can continue in the same way as in the proof of Proposition 7.1 in [4], where [4, (24)] is replaced by (35). $\square$

The proof of Theorem 1 can now be completed in exactly the same way as the proof of Theorem 1 in [4]. The latter used only Lemma 3.2 and Proposition 7.1 of [4], and our present Lemmas 1 and 9 have the same structure as those results; they differ only in some parameters. In Lemma 9, we may assume $\sigma_0 = 1$, then we obtain Theorem 1 under the assumption that $a^{r/k}\lambda \geq 1$. If we choose the constant c in (8) so that $c > \exp\{c_0 f(\epsilon)\}$, then (8) holds generally.

7 A Case of Large Cells with Indeterminate Shape

In this section, we want to study the situation of Theorem 1 in the special case where the size functional Σ is equal to the parameter functional Φ; we will then call $\Phi(K)$ the Φ-*content* of K. For a stationary and isotropic Poisson hyperplane process, this corresponds to the case where the size functional is essentially the mean width, the case excluded in [5, Theorem 2]. The inequality (4) is now a tautological equality. Hence, every convex body $K \in \mathcal{K}_o^d$ is an extremal body, and thus every deviation functional ϑ for Φ and Σ is identically zero. Therefore, $\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Phi(Z_0) \in [a,b)) = 0$ for every $\epsilon > 0$, so that inequality (8) is satisfied trivially, and provides no information.

That, in fact, the condition of large Φ-content need not influence the shape of the zero cell, can at least be seen in a very special case. Møller and Zuyev [11] have shown (extending a result of Miles), that under the condition $f_{d-1}(Z_0) = N$, the Φ-content and the shape of the zero cell are stochastically independent. Now let X be the stationary Poisson hyperplane process where the directional distribution is concentrated, with equal masses, at $\pm\boldsymbol{e}_1, \dots, \pm\boldsymbol{e}_d$, for an orthonormal basis $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_d)$ of $\mathbb{R}^d$ (a 'cuboid process', as studied in [2]). Then $f_{d-1}(Z_0) = 2d$ with probability 1. It follows that, in this case, the Φ-content of the zero cell and its shape are stochastically independent.

The latter independence of shape and size does not hold for more general Poisson hyperplane processes, but the following result on asymptotic independence can be proved. It shows that no limit shape of Z_0 with respect to Φ exists if the directional distribution has finite support.

Theorem 4. *Suppose that the directional distribution φ of the hyperplane process X has finite support consisting of N points. Then, for every Borel set $\mathcal{A} \subset s_{\mathsf{D}}(\mathcal{K}_o^d)$,*

$$\lim_{a\to\infty} \mathbb{P}(s_{\mathsf{D}}(Z_0) \in \mathcal{A} \mid \Phi(Z_0) \geq a) = \mathbb{P}(s_{\mathsf{D}}(Z_0) \in \mathcal{A} \mid f_{d-1}(Z_0) = N).$$

Proof. Let $\mathcal{A}$ be a Borel set in $s_{\mathsf{D}}(\mathcal{K}_o^d)$. In the following, we define $s_{\mathsf{D}}(A) := \{\alpha A : \alpha > 0\}$ also for unbounded convex sets $A \subset \mathbb{R}^d$, so that $s_{\mathsf{D}}(P(H_{(n)})) \in \mathcal{A}$ implies that $P(H_{(n)})$ is bounded. For $n \in \mathbb{N}$ and $a > 0$, we define

$$\begin{aligned}
R(\mathcal{A}, n) &:= \left\{(H_1, \dots, H_n) \in (\mathcal{H}^d)^n : s_{\mathsf{D}}(P(H_{(n)})) \in \mathcal{A},\ f_{d-1}(P(H_{(n)})) = n\right\},\\
R(\mathcal{A}, n, a) &:= \{(H_1, \dots, H_n) \in R(\mathcal{A}, n) : \Phi(P(H_{(n)})) \geq a\},\\
S(\mathcal{A}, n) &:= \left\{(H_1, \dots, H_{n-1}, \boldsymbol{u}) \in (\mathcal{H}^d)^{n-1} \times S^{d-1} : (H_1, \dots, H_{n-1}, H(\boldsymbol{u}, 1)) \in R(\mathcal{A}, n)\right\}.
\end{aligned}$$

From Lemma 7, with $\mathcal{B} := \{K \in \mathcal{K}_\varphi : s_{\mathsf{D}}(K) \in \mathcal{A},\ \Phi(K) \geq a\}$, we get, similarly as in the proof of Lemma 8,

$$\begin{aligned}
&\mathbb{P}(s_{\mathsf{D}}(Z_0) \in \mathcal{A},\ \Phi(Z_0) \geq a,\ f_{d-1}(Z_0) = n)\\
&= \frac{\lambda^n}{n!} \int_{R(\mathcal{A},n,a)} \exp\{-\Phi(P(H_{(n)}))\lambda\}\, \mu^n(\mathrm{d}(H_1, \dots, H_n))\\
&= \frac{\lambda^n}{n!} \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_0^\infty \cdots \int_0^\infty \mathbf{1}\{(H(\boldsymbol{u}_1, t_1), \dots, H(\boldsymbol{u}_n, t_n)) \in R(\mathcal{A}, n, a)\}\\
&\quad \times \exp\{-\Phi(H^-(\boldsymbol{u}_1, t_1) \cap \cdots \cap H^-(\boldsymbol{u}_n, t_n))\lambda\}\\
&\quad \times (t_1 \cdots t_n)^{r-1}\, \mathrm{d}t_1 \cdots \mathrm{d}t_n\, \varphi(\mathrm{d}\boldsymbol{u}_1) \cdots \varphi(\mathrm{d}\boldsymbol{u}_n)\\
&= \frac{\lambda^n}{n!} \int_{(\mathcal{H}^d)^{n-1}} \int_{S^{d-1}} \mathbf{1}\{(H_1, \dots, H_{n-1}, H(\boldsymbol{u}, 1)) \in R(\mathcal{A}, n)\}\\
&\quad \times \int_0^\infty \mathbf{1}\left\{z^r \Phi(H_1^- \cap \cdots \cap H_{n-1}^- \cap H^-(\boldsymbol{u}, 1)) \geq a\right\}\\
&\quad \times \exp\left\{-z^r \Phi(H_1^- \cap \cdots \cap H_{n-1}^- \cap H^-(\boldsymbol{u}, 1))\lambda\right\} z^{rn-1}\, \mathrm{d}z\\
&\quad \times \varphi(\mathrm{d}\boldsymbol{u})\, \mu^{n-1}(\mathrm{d}(H_1, \dots, H_{n-1})).
\end{aligned}$$

To compute the inner integral $\int_0^\infty(\cdots)\mathrm{d}z$, we fix $(H_1,\dots,H_{n-1},\boldsymbol{u})\in S(\mathcal{A},n)$, write

$$H_1^-\cap\cdots\cap H_{n-1}^-\cap H^-(\boldsymbol{u},1)=:P$$

and define $z_a=z_a(H_1,\dots,H_{n-1},\boldsymbol{u})$ by $\Phi(z_aP)=a$. Substituting $z=z_as$, we get

$$\begin{aligned}\int_0^\infty(\cdots)\,\mathrm{d}z &= \int_0^\infty \mathbf{1}\{\Phi(zP)\geq a\}\exp\{-\Phi(zP)\lambda\}z^{rn-1}\,\mathrm{d}z\\ &= \int_{z_a}^\infty \exp\{-\Phi(zP)\lambda\}z^{rn-1}\,\mathrm{d}z\\ &= \int_1^\infty \exp\{-s^ra\lambda\}z_a^{rn}s^{rn-1}\,\mathrm{d}s.\end{aligned}$$

This yields

$$\begin{aligned}&\mathbb{P}(s_{\mathsf{D}}(Z_0)\in\mathcal{A},\ \Phi(Z_0)\geq a,\ f_{d-1}(Z_0)=n)\\ &=\frac{\lambda^n}{n!}\int_{(\mathcal{H}^d)^{n-1}}\int_{S^{d-1}}\mathbf{1}\{(H_1,\dots,H_{n-1},\boldsymbol{u})\in S(\mathcal{A},n)\}\int_1^\infty\exp\{-s^ra\lambda\}z_a^{rn}s^{rn-1}\,\mathrm{d}s\\ &\quad\times\varphi(\mathrm{d}\boldsymbol{u})\,\mu^{n-1}(\mathrm{d}(H_1,\dots,H_{n-1}))\\ &=\frac{\lambda^n}{n!}\int_1^\infty\exp\{-s^ra\lambda\}s^{rn-1}\,\mathrm{d}s\\ &\quad\times\int_{S(\mathcal{A},n)}z_a(H_1,\dots,H_{n-1},\boldsymbol{u})^{rn}\,\varphi(\mathrm{d}\boldsymbol{u})\,\mu^{n-1}(\mathrm{d}(H_1,\dots,H_{n-1})).\end{aligned}$$

Here we substitute $s^ra\lambda=x$, to obtain

$$\int_1^\infty\exp\{-s^ra\lambda\}s^{rn-1}\,\mathrm{d}s=\frac{1}{r}(a\lambda)^{-n}\int_{a\lambda}^\infty e^{-x}x^{n-1}\,\mathrm{d}x.$$

Observing that $z_a^{rn}/a^n=z_1^{rn}$, we get

$$\mathbb{P}(s_{\mathsf{D}}(Z_0)\in\mathcal{A},\ \Phi(Z_0)\geq a,\ f_{d-1}(Z_0)=n)=\frac{1}{rn!}\int_{a\lambda}^\infty e^{-x}x^{n-1}\,\mathrm{d}x\cdot I(\mathcal{A},n)\tag{36}$$

with

$$I(\mathcal{A},n):=\int_{S(\mathcal{A},n)}z_1(H_1,\dots,H_{n-1},\boldsymbol{u})^{rn}\,\varphi(\mathrm{d}\boldsymbol{u})\,\mu^{n-1}(\mathrm{d}(H_1,\dots,H_{n-1})).$$

Letting a tend to zero, we see that (36) holds also for $a=0$.

We remark that a consequence of (36), namely

$$\mathbb{P}(\Phi(Z_0)\geq a\mid f_{d-1}(Z_0)=n,\ s_{\mathsf{D}}(Z_0)\in\mathcal{A})=\frac{1}{n!}\int_{a\lambda}^\infty e^{-x}x^{n-1}\,\mathrm{d}x$$

for the sets $\mathcal{A}$ with $\mathbb{P}(f_{d-1}(Z_0)=n,\ s_{\mathsf{D}}(Z_0)\in\mathcal{A})>0$, could also be deduced from the work of Møller and Zuyev [11]. We wanted to give here the explicit form of $I(\mathcal{A},n)$, in view of the remark at the end of this section.

By assumption, the support of φ has N points. Therefore,

$$\mathbb{P}(s_{\mathsf{D}}(Z_0)\in\mathcal{A}\mid\Phi(Z_0)\geq a)=\frac{\displaystyle\sum_{n=d+1}^N\frac{1}{n!}\int_{a\lambda}^\infty e^{-x}x^{n-1}\,\mathrm{d}x\cdot I(\mathcal{A},n)}{\displaystyle\sum_{n=d+1}^N\frac{1}{n!}\int_{a\lambda}^\infty e^{-x}x^{n-1}\,\mathrm{d}x\cdot I(s_{\mathsf{D}}(\mathcal{K}_o^d),n)}.$$

Since

$$\int_{a\lambda}^{\infty} e^{-x} x^{n-1}\,\mathrm{d}x = e^{-a\lambda}(a\lambda)^{n-1}(1+o(1)) \qquad \text{as } a \to \infty,$$

we get

$$\begin{aligned}\lim_{a\to\infty} \mathbb{P}(s_{\mathsf{D}}(Z_0) \in \mathcal{A} \mid \Phi(Z_0) \geq a) &= \frac{I(\mathcal{A}, N)}{I(s_{\mathsf{D}}(\mathcal{K}_o^d), N)} \\ &= \mathbb{P}(s_{\mathsf{D}}(Z_0) \in \mathcal{A} \mid f_{d-1}(Z_0) = N).\end{aligned}$$

□

Theorem 4 shows clearly that asymptotically the condition of large Φ-content has little influence on the shape of the zero cell. In fact, if $N > d+1$, we can construct quite differently looking Borel sets $\mathcal{A}$ of D-shapes in $\mathcal{K}_\varphi$ with N facets for which $I(\mathcal{A}, N) > 0$. Hence,

$$\lim_{a\to\infty} \mathbb{P}(s_{\mathsf{D}}(Z_0) \in \mathcal{A} \mid \Phi(Z_0) \geq a) > 0$$

can be satisfied for essentially different sets $\mathcal{A}$, which is in stark contrast to the existence of limit shapes.

8 Proof of Theorem 2

Let $\kappa \in (0,1)$ and $m \in \mathbb{N}$, and suppose that $a^{r/k}\lambda \geq \sigma_0$ with some constant $\sigma_0 > 0$. Let B_a and C be chosen as in the proof of Lemma 5. Suppose that $H_1, \dots, H_N \in \mathcal{H}_C$ are such that $P(H_{(N)}) \in \mathcal{K}_{a,0,1}(m)$; then

$$\Phi(P(H_{(N)})) \geq \Phi(B_a) \tag{37}$$

by (31) (which holds also for $\epsilon = 0$). As in the proof of Lemma 5, there are $\nu = \nu(d, \varphi, r, \kappa)$ vertices of $P(H_{(N)})$ such that the convex hull $Q = Q(P(H_{(N)}))$ of these vertices satisfies

$$\Phi(Q) \geq (1 - \kappa/8)\Phi(B_a), \tag{38}$$

by (32) and (37). The proof of Lemma 5 (using (38) instead of (32)) shows how this leads to

$$\begin{aligned} q_{a,0,1}(m) &\leq c_{16}(\kappa)\left[(a^{r/k}\lambda)^{d\nu} + 1\right] m^{rd\nu} \exp\left\{-(1-\kappa/8)\tau a^{r/k}\lambda\right\} \\ &\leq c_{17}(\kappa) m^{rd\nu} \exp\left\{-(1-\kappa/4)\tau a^{r/k}\lambda\right\}. \end{aligned} \tag{39}$$

Now we can argue as in the proof of Lemma 9. Let $h \in (0, 1/2)$. We set

$$c_{18}(\kappa) := [(2/c_5)(1-\kappa/4)\tau]^{1/r}$$

and $m_0 := \lfloor c_{18}(\kappa) \rfloor$. Then

$$(c_5/2)m^r \geq (1-\kappa/4)\tau \qquad \text{for } m > m_0. \tag{40}$$

By (28) and Lemma 8, both for $\epsilon = 0$, we have

$$\begin{aligned} &\mathbb{P}(\Sigma(Z_0) \in a(1, 1+h)) = \sum_{m\in\mathbb{N}} q_{a,0,h}(m) \\ &\leq c_{13}(\sigma_0) h a^{r/k}\lambda \left(\sum_{m=1}^{m_0} m^r q_{a,0,1}(m) + \sum_{m>m_0} m^r q_{a,0,1}(m)\right). \end{aligned}$$

For the estimation of $q_{a,0,1}(m)$ we use (39) for $m \le m_0$ and Lemma 3 for $m > m_0$. We continue as in the proof of Proposition 7.1 in [4], where [4, (24)] is now replaced by (40). We conclude that

$$\mathbb{P}(\Sigma(Z_0) \in a(1, 1+h)) \le c_{19}(\kappa, \sigma_0) h \exp\left\{-(1-\kappa/2)\tau a^{r/k}\lambda\right\}. \tag{41}$$

Here we replace a by $s^i a$ with $1 < s < 1+h$ and $i \in \mathbb{N}_0$ and estimate the exponent by

$$\begin{aligned}(1-\kappa/2)\tau s^{ir/k} a^{r/k}\lambda &= (1-\kappa)\tau s^{ir/k} a^{r/k}\lambda + (\kappa/2)\tau s^{ir/k} a^{r/k}\lambda \\ &\ge (1-\kappa)\tau a^{r/k}\lambda + (\kappa/2)\tau s^{ir/k}\sigma_0.\end{aligned}$$

Since $(a, \infty) = \bigcup_{i=0}^{\infty} s^i a(1, 1+h)$ (and $\mathbb{P}(\Sigma(Z_0) = a) = 0$), we get (using $h < 1/2$)

$$\begin{aligned}&\mathbb{P}(\Sigma(Z_0) \ge a) \le \sum_{i=0}^{\infty} \mathbb{P}(\Sigma(Z_0) \in s^i a(1, 1+h)) \\ &\le \sum_{i=0}^{\infty} c_{19}(\kappa, \sigma_0) \exp\left\{-(1-\kappa/2)\tau s^{ir/k} a^{r/k}\lambda\right\} \\ &\le c_{19}(\kappa, \sigma_0) \exp\left\{-(1-\kappa)\tau a^{r/k}\lambda\right\} \sum_{i=0}^{\infty} \exp\left\{-(\kappa/2)\tau s^{ir/k}\sigma_0\right\} \\ &\le c_{20}(\kappa, \sigma_0, s) \exp\left\{-(1-\kappa)\tau a^{r/k}\lambda\right\}.\end{aligned}$$

Thus, together with (19), we have

$$\exp\left\{-\tau a^{r/k}\lambda\right\} \le \mathbb{P}(\Sigma(Z_0) \ge a) \le c_{20}(\kappa, \sigma_0, s) \exp\left\{-(1-\kappa)\tau a^{r/k}\lambda\right\}.$$

This yields

$$\liminf_{a\to\infty} a^{-r/k} \ln \mathbb{P}(\Sigma(Z_0) \ge a) \ge -\tau\lambda$$

and

$$\limsup_{a\to\infty} a^{-r/k} \ln \mathbb{P}(\Sigma(Z_0) \ge a) \le -(1-\kappa)\tau\lambda.$$

Here the left-hand side is independent of κ, hence we conclude that

$$\lim_{a\to\infty} a^{-r/k} \ln \mathbb{P}(\Sigma(Z_0) \ge a) = -\tau\lambda.$$

This completes the proof of Theorem 2. □

9 Proof of Inequality (9)

Now we prove the inequality (9). From (41) we can deduce that the distribution of $\Sigma(Z_0)$ is absolutely continuous with respect to the Lebesgue measure λ_1 on $\mathbb{R}$. In fact, we have $\mathbb{P}(\Sigma(Z_0) = 0) = 0$. If a set $M \subset [a, \infty)$ with $a^{r/k}\lambda \ge \sigma_0$ for some $\sigma_0 > 0$ is covered by countably many intervals of total length ϵ, then it follows from (41) that the sum of the $\mathbb{P}_{\Sigma(Z_0)}$-measures of these intervals is at most $c_{21}\epsilon$, with a constant c_{21} not depending on ϵ. From this, the absolute continuity of $\mathbb{P}_{\Sigma(Z_0)}$ with respect to λ_1 follows. Moreover, Lemma

1 with $\beta = 1$, say, shows that the Radon–Nikodym derivative of $\mathbb{P}_{\Sigma(Z_0)}$ with respect to λ_1 satisfies

$$\begin{aligned}\frac{\mathrm{d}\mathbb{P}_{\Sigma(Z_0)}}{\mathrm{d}\lambda_1}(a) &= \lim_{h\downarrow 0}\frac{\mathbb{P}(\Sigma(Z_0)\in a(1,1+h))}{ah}\\ &\geq c\frac{1}{a}(a^{r/k}\lambda)^N\exp\{-2\tau a^{r/k}\lambda\} > 0,\end{aligned}$$

for λ_1-almost all $a > 0$. Hence $\mathbb{P}_{\Sigma(Z_0)}$ and λ_1 are equivalent measures. Since

$$\mathbb{P}(\vartheta(Z_0)\geq\epsilon,\ \Sigma(Z_0)\in B) = \int_B \mathbb{P}(\vartheta(Z_0)\geq\epsilon \mid \Sigma(Z_0)=a)\,\mathbb{P}_{\Sigma(Z_0)}(\mathrm{d}a)$$

for any Borel set $B\subset\mathbb{R}$, we obtain from Lebesgue's differentiation theorem that

$$\mathbb{P}(\vartheta(Z_0)\geq\epsilon \mid \Sigma(Z_0)=a) = \lim_{h\downarrow 0}\frac{\mathbb{P}(\vartheta(Z_0)\geq\epsilon,\ \Sigma(Z_0)\in a(1,1+h))}{\mathbb{P}(\Sigma(Z_0)\in a(1,1+h))}$$

for λ_1-almost all $a > 0$. Here we use Lemma 9, with $\sigma_0 = 1$, for the upper estimation of the numerator and Lemma 1 for the lower estimation of the denominator, and thus deduce inequality (9), for $a^{r/k}\lambda \geq 1$. Then we adapt the constant c as in the proof of Theorem 1 to obtain (9) for all $a > 0$.

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