A STABILITY RESULT FOR A VOLUME RATIO

By

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ABSTRACT

For a convex body K in $\mathbb{R}^n$, the volume quotient is the ratio of the smallest volume of the circumscribed ellipsoids to the largest volume of the inscribed ellipsoids, raised to power $1/n$. It attains its maximum if and only if K is a simplex. We improve this result by estimating the Banach–Mazur distance of K from a simplex if the volume quotient of K is close to the maximum.

Introduction and Result

We work in Euclidean space $\mathbb{R}^n$ ($n \geq 2$) with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $\mathcal{K}^n$ denote the set of compact, convex sets in $\mathbb{R}^n$ with nonempty interiors (convex bodies).

For $K \in \mathcal{K}^n$, let $\mathcal{E}_J(K)$ denote the ellipsoid of maximal volume contained in K (the John ellipsoid). By a result of John [1], the concentric homothetic ellipsoid $n(\mathcal{E}_J(K) - c) + c$, where c denotes the center of $\mathcal{E}_J(K)$, contains the body K. If K is a simplex, then the factor n cannot be decreased, but the simplex is not characterized by this extremal property. This changes if also shifts are allowed. The extended Banach–Mazur distance of not necessarily symmetric convex bodies $K, L \in \mathcal{K}^n$ is defined by

$$d_{BM}(K, L) := \inf\{\lambda \geq 1 : \exists\, \alpha \in \mathrm{Aff}(n)\, \exists\, x \in \mathbb{R}^n : L \subseteq \alpha K \subseteq \lambda L + x\}$$

where $\mathrm{Aff}(n)$ denotes the set of bijective affine transformations of $\mathbb{R}^n$. If B^n denotes the Euclidean unit ball, then John's result implies that

$$d_{BM}(K, B^n) \leq n \tag{1}$$

for $K \in \mathcal{K}^n$. Here, equality holds if and only if K is a simplex. This was proved by Leichtweiß [2] and was rediscovered by Palmon [3].

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As soon as one has uniqueness, the question for a stability improvement of the inequality can be raised. For the inequality (1), such a stability result seems to be unknown, but we will prove stability for a weaker version of (1), the inequality (2) below. Let $\mathcal{E}_L(K)$ be the ellipsoid of minimal volume containing K (the Löwner ellipsoid), and let

$$\mathrm{vq}(K) := \left(\frac{V(\mathcal{E}_L(K))}{V(\mathcal{E}_J(K))}\right)^{1/n},$$

where V denotes the volume (and vq stands for 'volume quotient'). Clearly,

$$\mathrm{vq}(K) \leq d_{BM}(K, B^n) \leq n.$$

For the inequality

$$\mathrm{vq}(K) \leq n, \tag{2}$$

mentioned by Leichtweiß [2] (Korollar), in which equality holds precisely for simplices, we establish an improvement in the form of a stability estimate. By T^n we denote an n-dimensional simplex in $\mathbb{R}^n$.

Theorem. *There exist constants $c_0(n), \epsilon_0(n) > 0$ depending only on the dimension n such that the following holds. If $0 \leq \epsilon \leq \epsilon_0(n)$ and*

$$\mathrm{vq}(K) \geq (1-\epsilon)n,$$

then

$$d_{BM}(K, T^n) \leq 1 + c_0(n)\epsilon^{1/4}.$$

Rough estimates yield that the constant $c_0(n)$ can be chosen of order n^7. We do not know whether the order of ϵ is optimal.

Proof of the Theorem

We write $S^{n-1} = \partial B^n$ for the boundary of the unit ball B^n with center 0. Suppose that $K \in \mathcal{K}^n$ is such that $\mathrm{vq}(K) \geq (1-\epsilon)n$, where $\epsilon \in [0, \epsilon_0]$ and $\epsilon_0 = \epsilon_0(n) \leq 1$ will be specified in the course of the proof. We assume, without loss of generality, that $\mathcal{E}_J(K) = n^{-1}B^n$.

Lemma 1. *Under the above assumptions, the support function $h(K,\cdot)$ of K satisfies*

$$h(K,u)h(K,-u) \geq \frac{1}{n} - 6\sqrt{\epsilon} \tag{3}$$

for all $u \in S^{n-1}$.

Proof. Since $\mathcal{E}_J(K) = n^{-1}B^n$, it follows from $\mathrm{vq}(K) \geq (1-\epsilon)n$ that

$$V(\mathcal{E}_L(K)) \geq [(1-\epsilon)n]^n n^{-n}\kappa_n = (1-\epsilon)^n\kappa_n, \tag{4}$$

where $\kappa_n := V(B^n)$, and it follows from John's theorem that $K \subseteq B^n$.

Let $u \in S^{n-1}$. We set $a := h(K,u)$ and $b := h(K,-u)$, hence $a,b \in [1/n,1]$. For $x \in K$, we have

$$\|x\|^2 - 1 \leq 0 \tag{5}$$

since $K \subseteq B^n$, and

$$\langle x,u\rangle - a \leq 0, \qquad \langle x,u\rangle + b \geq 0. \tag{6}$$

Combining (5) and (6), we obtain for $x \in K$ and all $\lambda \geq 0$ that

$$\|x\|^2 - 1 + \lambda(\langle x,u\rangle - a)(\langle x,u\rangle + b) \leq 0. \tag{7}$$

The set of all $x \in \mathbb{R}^n$ satisfying (7) is an ellipsoid $E_{a,b}(\lambda)$ with $V(E_{a,b}(\lambda)) = f_{a,b}(\lambda)\kappa_n$, where

$$f_{a,b}(\lambda) := \left[1 + (1+ab)\lambda + \left(\frac{a+b}{2}\right)^2 \lambda^2\right]^{\frac{n}{2}} (1+\lambda)^{-\frac{n+1}{2}}, \qquad \lambda \geq 0 \tag{8}$$

(so far, we essentially followed Leichtweiß [2]). From (4) and the fact that $K \subseteq E_{a,b}(\lambda)$, we have

$$f_{a,b}(\lambda) \geq (1-\epsilon)^n, \qquad \lambda \geq 0. \tag{9}$$

We derive a lower bound for ab. By (8) and (9),

$$1 + (1+ab)\lambda + \left(\frac{a+b}{2}\right)^2 \lambda^2 \geq (1+\lambda)^{\frac{n+1}{n}}(1-\epsilon)^2.$$

Put $(1-\epsilon)^2 =: \beta$. Since $a+b \leq 2$, we deduce that

$$1 + (1+ab)\lambda + \lambda^2 \geq [1 + (1+n^{-1})\lambda]\beta,$$

which is equivalent to

$$1 - \beta + [1 + ab - \beta(1+n^{-1})]\lambda + \lambda^2 \geq 0, \qquad \lambda \geq 0. \tag{10}$$

We remark that for $\epsilon = 0$ this yields $ab \geq 1/n$ (as also obtained in [2] and [3], in different ways). Now we assume $\epsilon \leq 1$ and assert that

$$ab \geq \frac{1}{n} - 6\sqrt{\epsilon}. \tag{11}$$

The polynomial of degree two in (10) has discriminant

$$D = [1 + ab - \beta(1+n^{-1})]^2 - 4(1-\beta).$$

If $D \leq 0$, then $-1 - ab + \beta(1+n^{-1}) \leq 2\sqrt{1-\beta}$ and hence

$$ab \geq (1-\epsilon)^2(1+n^{-1}) - 1 - 2\sqrt{1-(1-\epsilon)^2}. \tag{12}$$

If $D > 0$, the condition (10) implies that the linear term of the polynomial has a nonnegative coefficient, and this also implies (12). From (12) we get

$$ab \geq \frac{1}{n} - 3\epsilon - 2\sqrt{2\epsilon} \geq \frac{1}{n} - 6\sqrt{\epsilon},$$

which establishes (11) and thus proves Lemma 1. □

The ball $n^{-1}B^n$ is a maximal ball contained in K. It is known that this implies $0 \in \operatorname{conv}(\partial K \cap n^{-1}S^{n-1})$. (Otherwise, $\operatorname{conv}(\partial K \cap n^{-1}S^{n-1})$ and 0 can be strictly separated by a hyperplane. Hence, there is closed half ball of $n^{-1}B^n$ with positive distance from ∂K. The union of a suitable neighborhood of this half ball and of $n^{-1}B^n$ is contained in K and contains a ball larger than $n^{-1}B^n$, a contradiction.) Hence, by Carathéodory's theorem, there are mutually distinct vectors $u_1, \ldots, u_m \in S^{n-1}$ and numbers $\alpha_1, \ldots, \alpha_m > 0$ such that $m \leq n+1$,

$$\sum_{i=1}^{m} \alpha_i u_i = 0, \qquad \sum_{i=1}^{m} \alpha_i = 1 \tag{13}$$

and

$$n^{-1}u_i \in \partial K \cap n^{-1}S^{n-1}, \qquad i = 1, \ldots, m. \tag{14}$$

In the following, the unit vectors $u_1, \ldots, u_m$ and numbers $\alpha_1, \ldots, \alpha_m$ are fixed. Our aim is to show that, if ϵ is sufficiently small, then $m = n+1$ and $u_1, \ldots, u_{n+1}$ are close to the vertex vectors of a regular simplex with centroid 0. For this, we first use Lemma 1 to estimate how close the values of $\langle u_i, u_j \rangle$, $i \neq j$, and α_i are to those in the regular case.

Lemma 2. *There are positive constants $\epsilon_3 \leq 1$ and c_3 with the following properties. If $\epsilon \in [0, \epsilon_3]$, then $m = n+1$,*

$$\left| \langle u_i, u_j \rangle + \frac{1}{n} \right| \leq c_3 \epsilon^{1/4} \qquad \text{for } i, j = 1, \ldots, n+1,\ i \neq j, \tag{15}$$

and

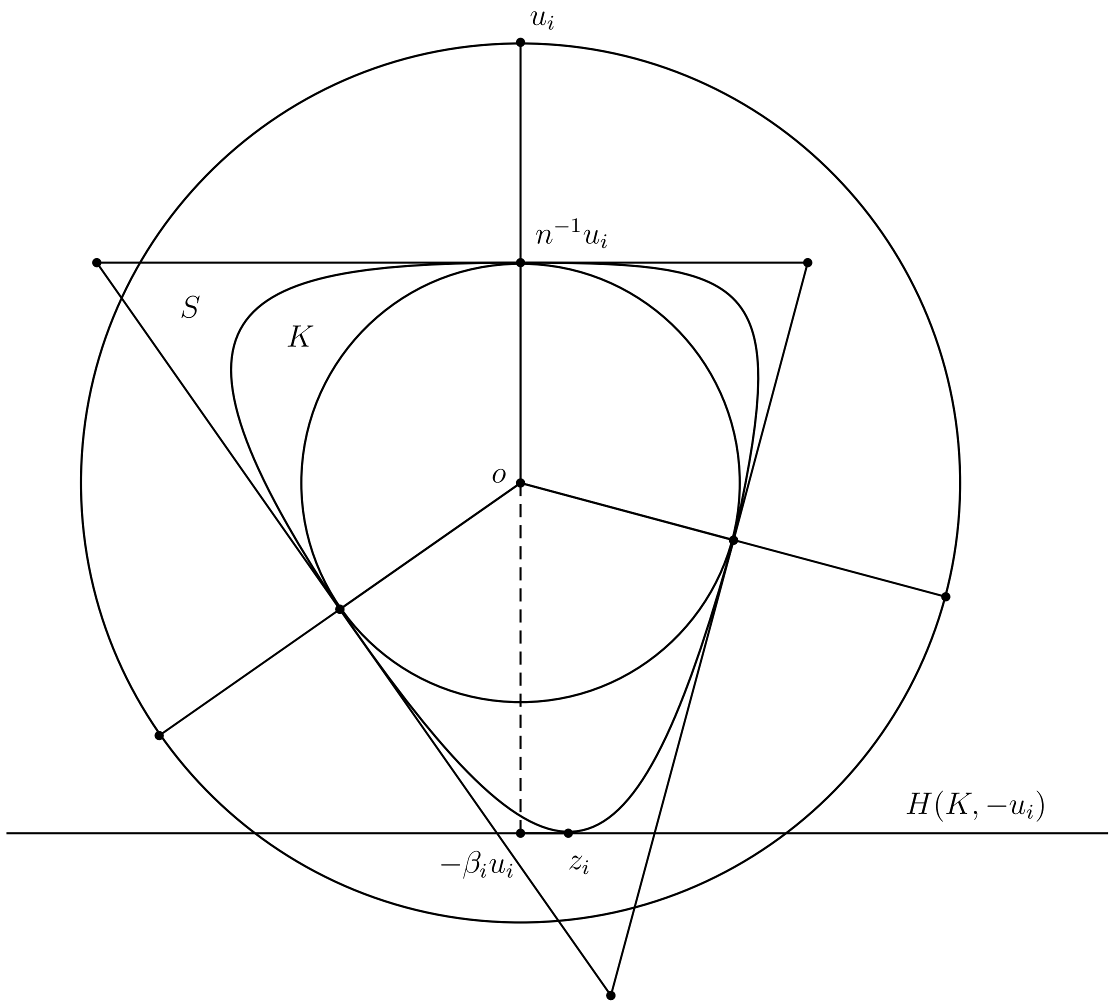
$$\left| \alpha_i - \frac{1}{n+1} \right| \leq c_3 \epsilon^{1/4} \qquad \text{for } i = 1, \ldots, n+1. \tag{16}$$

Proof. We set $\beta_i := h(K, -u_i)$ for $i = 1, \ldots, m$. Since $h(K, u_i) = 1/n$ by (14), the inequality (3) implies that

$$\beta_i \geq 1 - 6n\sqrt{\epsilon}, \qquad i = 1, \ldots, m.$$

We choose $z_i \in K \cap H(K, -u_i)$, where $H(K, -u_i)$ is the supporting hyperplane of K with outer normal vector $-u_i$. Since $K \subseteq B^n$, we can write $z_i = -\beta_i u_i + \gamma_i w_i$, where $w_i \in S^{n-1} \cap u_i^{\perp}$ and $\gamma_i \in \left[0, c_1 \epsilon^{1/4}\right]$ with $c_1 := \sqrt{12n}$.

The following figure illustrates the situation: For $\epsilon = 0$, the simplex S would be regular and inscribed to the outer ball B^n, and K would coincide with S. For small $\epsilon > 0$, we show that K is close to such a regular simplex.



For any $x \in K$ and $i \in \{1, \ldots, m\}$, we have $\langle u_i, x\rangle \leq h(K, u_i) = 1/n$. Since $z_j \in K$, we deduce that, for $i, j \in \{1, \ldots, m\}$ and $i \neq j$,

$$-\beta_j \langle u_i, u_j\rangle + \gamma_j \langle u_i, w_j\rangle \leq 1/n, \tag{17}$$

which implies

$$\langle u_i, u_j\rangle \geq -\frac{1}{n\beta_j} + \frac{\gamma_j}{\beta_j}\langle u_i, w_j\rangle \geq -\frac{1}{n(1-6n\epsilon^{1/2})} + \frac{\gamma_j}{\beta_j}\langle u_i, w_j\rangle.$$

We set $\epsilon_1 := (12n)^{-2}$ and assume $\epsilon \in [0, \epsilon_1]$. Then we can estimate

$$\frac{1}{1-6n\epsilon^{1/2}} \leq 1 + 12n\epsilon^{1/2}$$

and

$$\left|\frac{\gamma_j}{\beta_j}\langle u_i, w_j\rangle\right| \leq \frac{c_1\epsilon^{1/4}}{1-6n\epsilon^{1/2}} \leq 2c_1\epsilon^{1/4},$$

to obtain

$$\langle u_i, u_j\rangle \geq -\frac{1}{n} - c_2\epsilon^{1/4}, \qquad i \neq j \tag{18}$$

with $c_2 := 12 + 2c_1$.

Next, we introduce the auxiliary vector $s := u_1 + \ldots + u_m$. Then, for $i \in \{1, \ldots, m\}$,

$$\begin{aligned}
\langle s, u_i \rangle &= 1 + \sum_{\substack{j=1 \\ j \neq i}}^{m} \langle u_i, u_j \rangle \\
&\geq 1 + (m-1)\left(-\frac{1}{n} - c_2 \epsilon^{1/4}\right) \\
&= \frac{n+1-m}{n} - (m-1) c_2 \epsilon^{1/4}.
\end{aligned}$$

We set $\epsilon_2 := (n^2 c_2)^{-4}$ (which is $< \epsilon_1$) and require that $\epsilon \in [0, \epsilon_2]$. Then we can deduce that $m = n + 1$. In fact, otherwise $\langle s, u_i \rangle > 0$ for $i = 1, \ldots, m$, and by (13)

$$0 = \left\langle s, \sum_{i=1}^{m} \alpha_i u_i \right\rangle = \sum_{i=1}^{m} \alpha_i \langle s, u_i \rangle > 0,$$

a contradiction. As another consequence, we have

$$\langle s, u_i \rangle \geq -n c_2 \epsilon^{1/4}, \qquad i = 1, \ldots, n+1. \tag{19}$$

Our next purpose is to estimate $\langle u_i, u_j \rangle$ also from above for $i \neq j$. For this, we first establish upper and lower bounds for the coefficients $\alpha_1, \ldots, \alpha_{n+1}$ in (13):

$$\begin{aligned}
0 &= \left\langle u_i, \sum_{j=1}^{n+1} \alpha_j u_j \right\rangle = \alpha_i + \sum_{\substack{j=1 \\ j \neq i}}^{n+1} \alpha_j \langle u_i, u_j \rangle \\
&\geq \alpha_i + \sum_{\substack{j=1 \\ j \neq i}}^{n+1} \alpha_j \left(-\frac{1}{n} - c_2 \epsilon^{1/4}\right) \\
&= \alpha_i \left(1 + \frac{1}{n} + c_2 \epsilon^{1/4}\right) - \frac{1}{n} - c_2 \epsilon^{1/4},
\end{aligned}$$

hence

$$\alpha_i \leq \frac{1 + n c_2 \epsilon^{1/4}}{n + 1 + n c_2 \epsilon^{1/4}} \leq \frac{1}{n+1} + c_2 \epsilon^{1/4}, \qquad i = 1, \ldots, n+1. \tag{20}$$

On the other hand,

$$\alpha_i = 1 - \sum_{\substack{j=1 \\ j \neq i}}^{n+1} \alpha_j \geq 1 + n\left[-\frac{1}{n+1} - c_2 \epsilon^{1/4}\right] = \frac{1}{n+1} - n c_2 \epsilon^{1/4}. \tag{21}$$

Define $\epsilon_3 := (2n(n+1)c_2)^{-4}$ (which is $< \epsilon_2$). Then the obtained bounds for α_i imply that

$$\frac{1}{2(n+1)} \leq \alpha_i \leq \frac{2}{n+1}, \qquad i = 1, \ldots, n+1, \tag{22}$$

if $\epsilon \in [0, \epsilon_3]$, which we assume in the following.

Now we are in a position to assert that

$$\langle u_i, u_j \rangle \le -\frac{1}{n} + 2n^2 c_2 \epsilon^{1/4}, \qquad i \ne j. \tag{23}$$

To check this, we assume to the contrary that, for some pair $i, j \in \{1, \dots, n+1\}$ with $i \ne j$,

$$\langle u_i, u_j \rangle > -\frac{1}{n} + 2n^2 c_2 \epsilon^{1/4}.$$

Then, using (18), we get

$$\langle s, u_i \rangle > 1 + \left(-\frac{1}{n} + 2n^2 c_2 \epsilon^{1/4} \right) + (n-1) \left(-\frac{1}{n} - c_2 \epsilon^{1/4} \right) = (2n^2 - n + 1) c_2 \epsilon^{1/4},$$

and the same estimate is obtained for $\langle s, u_j \rangle$. From this we infer, using (19) and (22), that

$$\begin{aligned} 0 &= \left\langle s, \sum_{k=1}^{n+1} \alpha_k u_k \right\rangle = \sum_{k=1}^{n+1} \alpha_k \langle s, u_k \rangle \\ &\ge \alpha_i \langle s, u_i \rangle + \alpha_j \langle s, u_j \rangle - (n-1) \frac{2n}{n+1} c_2 \epsilon^{1/4} \\ &> c_2 \epsilon^{1/4} \ge 0, \end{aligned}$$

a contradiction. Setting $c_3 := 2n^2 c_2$, we obtain (15) from (18) and (23), and (16) from (20) and (21). This completes the proof of Lemma 2. $\square$

In the course of this proof, we have obtained points $z_1, \dots, z_{n+1}$ such that

$$\operatorname{conv}\{z_1, \dots, z_{n+1}\} \subseteq K \subseteq \bigcap_{i=1}^{n+1} H^-(u_i, 1/n) \tag{24}$$

with $H^-(u, t) := \{x \in \mathbb{R}^n : \langle x, u \rangle \le t\}$, and

$$\|z_i + u_i\| = \|(1 - \beta_i) u_i + \gamma_i w_i\| \le 1 - \beta_i + \gamma_i \le c_4 \epsilon^{1/4} \tag{25}$$

with $c_4 := \sqrt{12n} + 6n$. These points will be needed later.

By Lemma 2, the unit vectors $u_1, \dots, u_{n+1}$ have scalar products which are close to those of the vertex vectors of a regular simplex with centroid 0. From this, we must now deduce that $u_1, \dots, u_{n+1}$ are, in fact, close to the vertices of a suitable regular simplex.

Lemma 3. *There are vectors $v_1, \dots, v_{n+1} \in S^{n-1}$ satisfying $\langle v_i, v_j \rangle = -1/n$ for $i, j \in \{1, \dots, n+1\}$ with $i \ne j$ and $\|u_i - v_i\| \le c_5 \epsilon^{1/4}$ for $i = 1, \dots, n+1$, where $c_5 = 6(n+1)^{3/2} c_3$.*

Proof. In the Euclidean space $\mathbb{R}^n \times \mathbb{R}$ (with the standard scalar product, also denoted by $\langle \cdot, \cdot \rangle$) we define the vectors $U_i := \sqrt{n/(n+1)}(u_i, 1/\sqrt{n})$ for $i = 1, \dots, n+1$. They satisfy $\|U_i\| = 1$ and

$$\langle U_i, U_j \rangle = \frac{n}{n+1} \left(\langle u_i, u_j \rangle + \frac{1}{n} \right) \qquad \text{for } i \neq j.$$

By (15),

$$|\langle U_i, U_j \rangle| \leq c_3 \epsilon^{1/4} =: \eta \qquad \text{for } i \neq j. \tag{26}$$

We write the vectors of $\mathbb{R}^n \times \mathbb{R}$ as columns and define M as the matrix with columns $U_1, \dots, U_{n+1}$. Let $(E_1, \dots, E_{n+1})$ be an orthonormal basis of $\mathbb{R}^n \times \mathbb{R}$, and denote by $|||M|||$ the Hilbert-Schmidt norm of M, thus

$$|||M|||^2 = \sum_{i=1}^{n+1} \|ME_i\|^2.$$

By polar decomposition and diagonalization, the nonsingular matrix M has a representation $M = S_1 D S_2$ with orthogonal matrices S_1, S_2 and a diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_{n+1})$ with $d_i > 0$. Let I denote the $(n+1, n+1)$ unit matrix. By (26) and the invariance properties of the Hilbert-Schmidt norm, we have

$$(n+1)\eta \geq |||M^\top M - I||| = |||D^2 - I||| = \left(\sum_{i=1}^{n+1} (d_i^2 - 1)^2 \right)^{1/2} \geq \left(\sum_{i=1}^{n+1} (d_i - 1)^2 \right)^{1/2}.$$

Setting $D - I =: \tilde{D}$, we get

$$M = S_1(I + \tilde{D})S_2 = S + S_1 \tilde{D} S_2$$

with the orthogonal matrix $S = S_1 S_2$. The columns of S yield an orthonormal basis $(X_1, \dots, X_{n+1})$ of $\mathbb{R}^n \times \mathbb{R}$. For $i \in \{1, \dots, n+1\}$ we have, if E_i is the ith vector of the standard basis of $\mathbb{R}^n \times \mathbb{R}$,

$$\|U_i - X_i\| = \|ME_i - SE_i\| \leq |||M - S||| = |||S_1 \tilde{D} S_2||| = |||\tilde{D}||| = \left(\sum_{i=1}^{n+1} (d_i - 1)^2 \right)^{1/2},$$

hence

$$\|U_i - X_i\| \leq (n+1)\eta.$$

The hyperplane H through the orthonormal vectors $X_1, \dots, X_{n+1}$ has a normal vector

$$N_X := \frac{1}{n+1}(X_1 + \cdots + X_{n+1}) \qquad \text{of length } \|N_X\| = \frac{1}{\sqrt{n+1}}.$$

We compare this vector with the vectors

$$N_U := \frac{1}{n+1}(U_1 + \cdots + U_{n+1}) \quad \text{and} \quad N_0 := \sum_{i=1}^{n+1} \alpha_i U_i.$$

By (13), $N_0 = (0, 1/\sqrt{n+1})$. Further, $\|N_X - N_U\| \le (n+1)\eta$ and, by (16),

$$\|N_U - N_0\| = \left\| \sum_{i=1}^{n+1} \left(\frac{1}{n+1} - \alpha_i \right) U_i \right\| \le (n+1)\eta.$$

We deduce that $\|N_X - N_0\| \le 2(n+1)\eta$.

By the latter estimate, there exists a rotation ϑ of $\mathbb{R}^n \times \mathbb{R}$ for which $\vartheta N_X = N_0$ and

$$\|\vartheta Z - Z\| \le 2(n+1)\eta\sqrt{n+1}\|Z\| = 2(n+1)^{3/2}\eta\|Z\|$$

for each vector $Z \in \mathbb{R}^n \times \mathbb{R}$. The hyperplane H passes through N_X and is orthogonal to it, so ϑH passes through N_0 and is orthogonal to it, and it has the same distance from the origin as H. It follows that all points of ϑH have last coordinate $1/\sqrt{n+1}$. Therefore, we can define vectors $v_1, \dots, v_{n+1} \in \mathbb{R}^n$ by

$$\vartheta X_i = \sqrt{\frac{n}{n+1}} \left(v_i, \frac{1}{\sqrt{n}} \right).$$

From $\|\vartheta X_i\| = 1$ it follows that $\|v_i\| = 1$, and for $i \neq j$ it follows from $\langle \vartheta X_i, \vartheta X_j \rangle = 0$ that $\langle v_i, v_j \rangle = -1/n$. Finally,

$$\sqrt{\frac{n}{n+1}}\|v_i - u_i\| = \|\vartheta X_i - U_i\| \le \|\vartheta X_i - X_i\| + \|X_i - U_i\| \le 3(n+1)^{3/2}\eta,$$

hence

$$\|u_i - v_i\| \le c_5 \epsilon^{1/4} \tag{27}$$

with $c_5 := 6(n+1)^{3/2}c_3$. This proves Lemma 3. □

To complete the proof of the theorem, we now define the n-dimensional simplices

$$T := \bigcap_{i=1}^{n+1} H^-(v_i, 1/n) \qquad \text{and} \qquad S := \bigcap_{i=1}^{n+1} H^-(u_i, 1/n),$$

where T is regular and has vertices $-v_1, \dots, -v_{n+1}$. Our aim is to show that K has small Banach-Mazur distance from T. The definition of the Banach-Mazur distance involves inclusions of convex bodies, and these can be approached via estimates of support functions, or Hausdorff distances. Therefore, we first transfer the available information (27) on the unit normal vectors of the simplices T, S into estimates for their support functions, via the polar simplices. For the following, observe that the regular simplex T has inradius $1/n$ and hence circumradius 1; its polar then has circumradius n.

Let $\epsilon \in [0, \epsilon_3]$. The polar bodies S^o and T^o are simplices with vertices $nu_1, \dots, nu_{n+1}$ and $nv_1, \dots, nv_{n+1}$, respectively. If δ denotes the Hausdorff metric, we deduce from (27) that $\delta(S^o, T^o) \le nc_5\epsilon^{1/4}$. Assume that $\epsilon \in [0, \epsilon_4]$ with $\epsilon_4 := (2nc_5)^{-4}$ (which is $< \epsilon_3$). Then $\delta(S^o, T^o) \le 1/2$, and since $B^n \subset T^o$, this implies $2^{-1}B^n \subset S^o$. Clearly, $n^{-1}B^n \subset S, T$ and hence $S^o, T^o \subset nB^n$. For the radial function ρ and for $u \in S^{n-1}$ this gives

$$|\rho(T^o, u) - \rho(S^o, u)| \le 2n\delta(T^o, S^o). \tag{28}$$

To see this, assume that, say, $\rho(S^o,u)<\rho(T^o,u)$, and let H^- be a supporting halfspace of S^o at the boundary point $\rho(S^o,u)u$, with outer unit normal vector ν. The translated halfspace $H^-+\delta(T^o,S^o)\nu$ contains T^o, hence $\rho(T^o,u)-\rho(S^o,u)\le\delta(T^o,S^o)/\langle u,\nu\rangle$. On the other hand, $\langle u,\nu\rangle$ is bounded from below by the ratio of the radii of the inner and the outer ball. This proves (28).

From this we deduce

$$|h(T,u)-h(S,u)| \;=\; |\rho(T^o,u)^{-1}-\rho(S^o,u)^{-1}|\le 2|\rho(T^o,u)-\rho(S^o,u)|\le c_6\epsilon^{1/4}$$

with $c_6:=4n^2c_5$. But then

$$S\subseteq T+c_6\epsilon^{1/4}B^n\subseteq T+c_6\epsilon^{1/4}nT=\left(1+nc_6\epsilon^{1/4}\right)T,$$

and since $K\subseteq S$,

$$K\subseteq\left(1+nc_6\epsilon^{1/4}\right)T. \tag{29}$$

The points $z_1,\dots,z_{n+1}$ appearing in (24) can be written in the form

$$z_i=-v_i+p_i \qquad \text{with } \|p_i\|\le c_7\epsilon^{1/4},$$

for $i=1,\dots,n+1$, where $c_7:=c_4+c_5$, by (25) and (27). Assume that $\epsilon\in[0,\epsilon_5]$ with $\epsilon_5:=(2nc_7)^{-4}$ (which is $<\epsilon_4$). Then $\|p_i\|\le(2n)^{-1}$, hence the polytope $P:=\operatorname{conv}\{z_1,\dots,z_{n+1}\}$ satisfies $\delta(P,T)\le(2n)^{-1}$. Since $n^{-1}B^n\subseteq T$, this implies $(2n)^{-1}B^n\subseteq P$. We deduce that

$$\begin{aligned}\left(1-2nc_7\epsilon^{1/4}\right)(-v_i) &= \left(1-2nc_7\epsilon^{1/4}\right)(z_i-p_i)\in\left(1-2nc_7\epsilon^{1/4}\right)P+c_7\epsilon^{1/4}B^n\\ &\subseteq \left(1-2nc_7\epsilon^{1/4}\right)P+2nc_7\epsilon^{1/4}P=P\subseteq K.\end{aligned}$$

Since $-v_1,\dots,-v_{n+1}$ are the vertices of T, it follows that

$$\left(1-2nc_7\epsilon^{1/4}\right)T\subseteq K. \tag{30}$$

From (29) and (30) we obtain

$$\left(1-2nc_7\epsilon^{1/4}\right)T\subseteq K\subseteq\left(1+nc_6\epsilon^{1/4}\right)T.$$

This finally shows that

$$d_{BM}(K,T)\le\frac{1+nc_6\epsilon^{1/4}}{1-2nc_7\epsilon^{1/4}}\le 1+c_8\epsilon^{1/4}$$

for $\epsilon\in[0,\epsilon_6]$, where $\epsilon_6:=(4nc_7)^{-4}<\epsilon_5$ and $c_8:=2n(c_6+2c_7)$. □

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