

# NAKAJIMA'S PROBLEM: CONVEX BODIES OF CONSTANT WIDTH AND CONSTANT BRIGHTNESS

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*Dedicated to Rolf Schneider on the occasion of his 65th birthday*

ABSTRACT. For a convex body  $K \subset \mathbb{R}^n$ , the  $k$ th projection function of  $K$  assigns to any  $k$ -dimensional linear subspace of  $\mathbb{R}^n$  the  $k$ -volume of the orthogonal projection of  $K$  to that subspace. Let  $K$  and  $K_0$  be convex bodies in  $\mathbb{R}^n$ , and let  $K_0$  be centrally symmetric and satisfy a weak regularity and curvature condition (which includes all  $K_0$  with  $\partial K_0$  of class  $C^2$  with positive radii of curvature). Assume that  $K$  and  $K_0$  have proportional 1st projection functions (i.e., width functions) and proportional  $k$ th projection functions. For  $2 \leq k < (n+1)/2$  and for  $k=3, n=5$  we show that  $K$  and  $K_0$  are homothetic. In the special case where  $K_0$  is a Euclidean ball, we thus obtain characterizations of Euclidean balls as convex bodies of constant width and constant  $k$ -brightness.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let  $K$  be a convex body (a compact, convex set with nonempty interior) in  $\mathbb{R}^n$ ,  $n \geq 3$ . Assume that, for any line, the length of the projection of  $K$  to the line is independent of that line and, for any hyperplane, the volume of the projection of  $K$  to the hyperplane is independent of that hyperplane. Must  $K$  then be a Euclidean ball?

In dimension three, this problem has become known as Nakajima's problem [11]; see [1], [2], [3], [4], [5], [6]. It is easy to check that the answer to it is in the affirmative if  $K$  is a convex body in  $\mathbb{R}^3$  of class  $C^2$ . For general convex bodies in  $\mathbb{R}^3$ , the problem is much more difficult and a solution has only been found recently. Let  $\mathbb{G}(n, k)$  denote the Grassmannian of  $k$ -dimensional linear subspaces of  $\mathbb{R}^n$ . A convex body  $K$  in  $\mathbb{R}^n$  is said to have *constant  $k$ -brightness*,  $k \in \{1, \dots, n-1\}$ , if the  $k$ -volume  $V_k(K|U)$  of the orthogonal projection of  $K$  to the linear subspace  $U \in \mathbb{G}(n, k)$  is independent of that subspace. The map

$$\pi_k : \mathbb{G}(n, k) \rightarrow \mathbb{R}, \quad U \mapsto V_k(K|U),$$

is referred to as the  *$k$ th projection function* of  $K$ . Hence a convex body  $K$  has constant width (i.e. constant 1-brightness) if it has constant 1st projection function (width function).

**1.1. Theorem ([7]).** *Let  $K$  be a convex body in  $\mathbb{R}^n$  having constant width and constant 2-brightness. Then  $K$  is a Euclidean ball.*

This theorem provides a complete solution of the Nakajima problem in  $\mathbb{R}^3$  for general convex bodies. In the present paper, we continue this line of research. Our main result complements Theorem 1.1 by covering the cases of convex bodies of constant width and constant  $k$ -brightness with  $2 \leq k < (n+1)/2$  or  $k=3, n=5$ .

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**1.2. Theorem.** *Let  $K$  be a convex body in  $\mathbb{R}^n$  having constant width and constant  $k$ -brightness with  $2 \leq k < (n + 1)/2$ , or  $k = 3, n = 5$ . Then  $K$  is a Euclidean ball.*

The preceding two theorems can be generalized to pairs of convex bodies  $K, K_0$  having proportional projection functions, provided that  $K_0$  is centrally symmetric and has a minimal amount of regularity.

**1.3. Theorem.** *Let  $K, K_0$  be convex bodies in  $\mathbb{R}^n$ , and let  $K_0$  be centrally symmetric with positive principal radii of curvature on some Borel subset of the unit sphere of positive measure. Let  $2 \leq k < (n + 1)/2$ , or let  $k = 3, n = 5$  in which case assume the surface area measure  $S_4(K_0, \cdot)$  of  $K_0$  is absolutely continuous with positive density. Assume that there are constants  $\alpha, \beta > 0$  such that*

$$\pi_1(K) = \alpha \pi_1(K_0) \quad \text{and} \quad \pi_k(K) = \beta \pi_k(K_0).$$

*Then  $K$  and  $K_0$  are homothetic.*

As the natural measure on the unit sphere,  $\mathbb{S}^{n-1}$ , we use the invariant Haar probability measure (i.e. spherical Lebesgue measure), or what is the same thing the  $(n - 1)$ -dimensional Hausdorff measure,  $\mathcal{H}^{n-1}$ , normalized so that the total mass is one. We view the principal radii of curvature as functions of the unit normal, despite the fact that the unit normal map is in general a set valued function (cf. the beginning of Section 2 below). The assumption that the principal radii of curvature are positive on a set of positive measure means that there is a Borel subset of  $\mathbb{S}^{n-1}$  of positive measure such that on this set the reverse Gauss map is single valued, differentiable (in a generalized sense) and the eigenvalues of the differential are positive. Explicitly, this condition can be stated in terms of second order differentiability properties of the support function (again see Section 2). In particular, it is certainly satisfied if  $K_0$  is of class  $C_+^2$ , and therefore letting  $K_0$  be a Euclidean ball recovers Theorem 1.2. The required condition allows for parts of  $K_0$  to be quite irregular. For example if  $\partial K_0$  has a point that has a small neighborhood where  $\partial K_0$  is  $C^2$  with positive Gauss-Kronecker curvature, then the assumption will hold, regardless of how rough the rest of the boundary is. For example a ‘‘spherical polyhedron’’ constructed by intersecting a finite number of Euclidean balls in  $\mathbb{R}^n$  will satisfy the condition. More generally if the convex body  $K_0$  is an intersection of a finite collection of bodies of class  $C_+^2$ , it will satisfy the condition.

Theorem 1.3 extends the main results in [8] for the range of dimensions  $k, n$  where it applies by reducing the regularity assumption on  $K_0$  and doing away with any regularity assumptions on  $K$ . However, the classical Nakajima problem, which concerns the case  $n = 3$  and  $k = 2$ , is not covered by the present approach.

Despite recent progress on the Nakajima problem various questions remain open. For instance, can Euclidean balls be characterized as convex bodies having constant width and constant  $(n - 1)$ -brightness if  $n \geq 4$ ? This question is apparently unresolved even for smooth convex bodies. A positive answer is available for smooth convex bodies of revolution (cf. [8]). From the arguments of the present paper the following proposition is easy to check.

**1.4. Proposition.** *Let  $K, K_0 \subset \mathbb{R}^n$  be convex bodies that have a common axis of revolution. Let  $K_0$  be centrally symmetric with positive principal radii of curvature almost everywhere. Assume that  $K$  and  $K_0$  have proportional width functions and proportional  $k$ th projection functions for some  $k \in \{2, \dots, n - 2\}$ . Then  $K$  and  $K_0$  are homothetic.*

It is a pleasure for the authors to dedicate this paper to Rolf Schneider. Professor Rolf Schneider has been a large source of inspiration for countless students and colleagues all

over the world. His willingness to communicate and share his knowledge make contact with him a pleasurable and mathematically rewarding experience. The second named author has particularly been enjoying many years of support, personal interaction and joint research.

## 2. PRELIMINARIES

Let  $K$  be a convex body in  $\mathbb{R}^n$ , and let  $h_K: \mathbb{R}^n \rightarrow \mathbb{R}$  be the support function of  $K$ , which is a convex function. For  $x \in \mathbb{R}^n$  let  $\partial h_K(x)$  be the subdifferential of  $h_K$  at  $x$ . This is the set of vectors  $v \in \mathbb{R}^n$  such that the function  $h_K - \langle v, \cdot \rangle$  achieves its minimum at  $x$ . It is well known that, for all  $x \in \mathbb{R}^n$ ,  $\partial h_K(x)$  is a nonempty compact convex set and is a singleton precisely at those points where  $h_K$  is differentiable in the classical sense (cf. [13, pp. 30–31]). For  $u \in \mathbb{S}^{n-1}$  the set  $\partial h_K(u)$  is exactly the set of  $x \in \partial K$  such that  $u$  is an outward pointing normal to  $K$  at  $x$  (cf. [13, Thm 1.7.4]). But this is just the definition of the reverse Gauss map (which in general is not single valued, but a set valued function) and so the function  $u \mapsto \partial h_K(u)$  gives a formula for the reverse Gauss map in terms of the support function.

In the following, by “almost everywhere” on the unit sphere or by “for almost all unit vectors” we mean for all unit vectors with the possible exclusion of a set of spherical Lebesgue measure zero. A theorem of Aleksandrov states that a convex function has a generalized second derivative almost everywhere, which we will view as a positive semi-definite symmetric linear map rather than a symmetric bilinear form. This generalized derivative can either be defined in terms of a second order approximating Taylor polynomial at the point, or in terms of the set valued function  $x \mapsto \partial h_K(x)$  being differentiable in the sense of set valued functions (both these definitions are discussed in [13, p. 32]). At points where the Aleksandrov second derivative exists  $\partial h_K$  is single valued. Because  $h_K$  is positively homogeneous of degree one, if it is Aleksandrov differentiable at a point  $x$ , then it is Aleksandrov differentiable at all points  $\lambda x$  with  $\lambda > 0$ . Then Fubini’s theorem implies that not only is  $h_K$  Aleksandrov differentiable at  $\mathcal{H}^n$  almost all points of  $\mathbb{R}^n$ , but it is also Aleksandrov differentiable at  $\mathcal{H}^{n-1}$  almost all points of  $\mathbb{S}^{n-1}$ . For points  $u \in \mathbb{S}^{n-1}$  where it exists, let  $d^2 h_K(u)$  denote the Aleksandrov second derivative of  $h_K$ . Let  $u^\perp$  denote the orthogonal complement of  $u$ . Then the restriction  $d^2 h_K(u)|_{u^\perp}$  is the derivative of the reverse Gauss map at  $u$ . The eigenvalues of  $d^2 h_K(u)|_{u^\perp}$  are the principal radii of curvature at  $u$ . As the discussion above shows these exist at almost all points of  $\mathbb{S}^{n-1}$ .

A useful tool for the study of projection functions of convex bodies are the surface area measures. An introduction to these Borel measures on the unit sphere is given in [13], a more specialized reference (for the present purpose) is contained in the preceding work [8]. The top order surface area measure  $S_{n-1}(K, \cdot)$  of the convex body  $K \subset \mathbb{R}^n$  can be obtained as the  $(n-1)$ -dimensional Hausdorff measure  $\mathcal{H}^{n-1}$  of the reverse spherical image of Borel sets of the unit sphere  $\mathbb{S}^{n-1}$ . The Radon-Nikodym derivative of  $S_{n-1}(K, \cdot)$  with respect to the spherical Lebesgue measure is the product of the principal radii of curvature of  $K$ . Since for almost every  $u \in \mathbb{S}^{n-1}$ , the radii of curvature of  $K$  at  $u \in \mathbb{S}^{n-1}$  are the eigenvalues of  $d^2 h_K(u)|_{u^\perp}$ , the Radon-Nikodym derivative of  $S_{n-1}(K, \cdot)$  with respect to spherical Lebesgue measure is the function  $u \mapsto \det(d^2 h_K(u)|_{u^\perp})$ , which is defined almost everywhere on  $\mathbb{S}^{n-1}$ . In particular, if  $S_{n-1}(K, \cdot)$  is absolutely continuous with respect to spherical Lebesgue measure, the density function is just the Radon-Nikodym derivative. For explicit definitions of these and other basic notions of convex geometry needed here, we refer to [13] and [8].

The following lemma contains more precise information about the Radon-Nikodym derivative of the top order surface area measure. We denote the support function of a convex body  $K$  by  $h$ , if  $K$  is clear from the context. For a fixed unit vector  $u \in \mathbb{S}^{n-1}$  and  $i \in \mathbb{N}$ , we also put  $\omega_i := \{v \in \mathbb{S}^{n-1} : \langle v, u \rangle \geq 1 - (2i^2)^{-1}\}$ , whenever  $u$  is clear from the context. Hence  $\omega_i \downarrow \{u\}$ , as  $i \rightarrow \infty$ , in the sense of Hausdorff convergence of closed sets.

**2.1. Lemma.** *Let  $K \subset \mathbb{R}^n$  be a convex body. If  $u \in \mathbb{S}^{n-1}$  is a point of second order differentiability of the support function  $h$  of  $K$ , then*

$$\lim_{i \rightarrow \infty} \frac{S_{n-1}(K, \omega_i)}{\mathcal{H}^{n-1}(\omega_i)} = \det(d^2h(u)|u^\perp).$$

*Proof.* This is implicitly contained in the proof of Hilfssatz 2 in [10]. A similar argument, in a slightly more involved situation, can be found in [9].  $\square$

An analogue of Lemma 2.1 for curvature measures is provided in [12, (3.6) Hilfssatz].

As another ingredient in our approach to Nakajima's problem, we need two simple algebraic lemmas. Here we write  $|M|$  for the cardinality of a set  $M$ . If  $x_1, \dots, x_n$  are real numbers and  $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$  we set  $x_I := x_{i_1} \dots x_{i_k}$ . We also put  $x_\emptyset := 1$ .

**2.2. Lemma.** *Let  $b > 0$  be fixed. Let  $x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}$  be nonnegative real numbers satisfying*

$$x_i + y_i = 2 \quad \text{and} \quad x_I + y_I = 2b$$

*for all  $i = 1, \dots, n-1$  and all  $I \subset \{1, \dots, n-1\}$  with  $|I| = k$ , where  $k \in \{2, \dots, n-2\}$ . Then  $|\{x_1, \dots, x_{n-1}\}| \leq 2$  and  $|\{y_1, \dots, y_{n-1}\}| \leq 2$ .*

*Proof.* We can assume that  $x_1 \leq \dots \leq x_{n-1}$ . Then we have  $y_1 \geq \dots \geq y_{n-1}$ .

If  $x_1 = 0$ , then  $y_1 = 2$ . Further, for  $I' \subset \{2, \dots, n-1\}$  with  $|I'| = k-1$ , we have  $y_1 y_{I'} = 2b$ , hence  $y_{I'} = b$ . Since  $k \geq 2$ , we get  $y_2, \dots, y_{n-1} > 0$ . Moreover, since  $k-1 \leq n-3$ , we conclude that  $y_2 = \dots = y_{n-1}$ . This shows that also  $x_2 = \dots = x_{n-1}$ , and thus  $|\{x_1, \dots, x_{n-1}\}| \leq 2$  and  $|\{y_1, \dots, y_{n-1}\}| \leq 2$ .

If  $y_{n-1} = 0$ , the same conclusion is obtained by symmetry.

If  $x_1 > 0$  and  $y_{n-1} > 0$ , then  $x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1} > 0$ . Now we fix any set  $J \subseteq \{1, \dots, n-1\}$  with  $|J| = k+1$ . The argument at the beginning of the proof of Lemma 4.2 in [8] shows that  $|\{x_i : i \in J\}| \leq 2$ . Since  $k+1 \geq 3$ , we first obtain that  $|\{x_1, \dots, x_{n-1}\}| \leq 2$ , and then also  $|\{y_1, \dots, y_{n-1}\}| \leq 2$ .  $\square$

**2.3. Lemma.** *Let  $n \geq 4$ , and let  $b > 0$  be fixed. Let  $x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}$  be nonnegative real numbers satisfying*

$$x_i + y_i = 2 \quad \text{and} \quad x_I + y_I = 2b$$

*for all  $i = 1, \dots, n-1$  and all  $I \subset \{1, \dots, n-1\}$  with  $|I| = n-2$ . Then*

$$(2.1) \quad \prod_{l \neq i, j} x_l = \prod_{l \neq i, j} y_l = b$$

*whenever  $i, j \in \{1, \dots, n-1\}$  are such that  $x_i \neq x_j$ .*

*Proof.* For the proof, we may assume that  $i = 1$  and  $j = n - 1$ , to simplify the notation. Then we have

$$\begin{aligned} x_1 \cdots x_{n-2} + y_1 \cdots y_{n-2} &= 2b, \\ x_2 \cdots x_{n-1} + y_2 \cdots y_{n-1} &= 2b, \end{aligned}$$

which implies that

$$x_2 \cdots x_{n-2}(x_{n-1} - x_1) + y_2 \cdots y_{n-2}(y_{n-1} - y_1) = 0.$$

Moreover,  $x_1 + y_1 = 2 = x_{n-1} + y_{n-1}$  yields

$$x_{n-1} - x_1 = y_1 - y_{n-1} \neq 0,$$

and thus

$$x_2 \cdots x_{n-2} = y_2 \cdots y_{n-2}.$$

Hence

$$\begin{aligned} 2x_2 \cdots x_{n-2} &= (x_1 + y_1)x_2 \cdots x_{n-2} = x_1x_2 \cdots x_{n-2} + y_1x_2 \cdots x_{n-2} \\ &= x_1x_2 \cdots x_{n-2} + y_1y_2 \cdots y_{n-2} = 2b, \end{aligned}$$

and thus

$$b = x_2 \cdots x_{n-2} = y_2 \cdots y_{n-2}.$$

□

### 3. PROOFS

First, by possibly dilating  $K$ , we can assume that  $\alpha = 1$ . Hence the assumption can be stated as

$$(3.1) \quad \pi_1(K) = \pi_1(K_0) \quad \text{and} \quad \pi_k(K) = \beta \pi_k(K_0)$$

for some  $k \in \{2, \dots, n - 2\}$ . Let  $K^*$  denote the reflection of  $K$  in the origin. Then (3.1) yields that

$$K + K^* = 2K_0 \quad \text{and} \quad V_k(K|U) = \beta V_k(K_0|U)$$

for all  $U \in \mathbb{G}(n, k)$ . Minkowski's inequality (cf. [13]) then implies that

$$\begin{aligned} V_k(2K_0|U) &= V_k(K|U + K^*|U) \\ &\geq \left( V_k(K|U)^{\frac{1}{k}} + V_k(K^*|U)^{\frac{1}{k}} \right)^k \\ &= \left( 2V_k(K|U)^{\frac{1}{k}} \right)^k \\ &= \beta V_k(2K_0|U). \end{aligned}$$

Equality in Minkowski's inequality will hold if and only if  $K^*|U$  and  $K|U$  are homothetic. As they have the same volume this is equivalent to their being translates of each other, in which case  $K|U$  is centrally symmetric. Hence  $\beta \leq 1$  with equality if and only if  $K|U$  is centrally symmetric for all linear subspaces  $U \in \mathbb{G}(n, k)$ . Since  $k \geq 2$ , this is the case if and only if  $K$  is centrally symmetric (cf. [4, Thm. 3.1.3]). So if  $\beta = 1$ , then  $K$  and  $K_0$  must be homothetic.

In the following, we assume that  $\beta \in (0, 1)$ . This will lead to a contradiction and thus prove the theorem.

We write  $h, h_0$  for the support functions of  $K, K_0$ . Here and in the following, "almost all" or "almost every" refers to the natural Haar probability measure on  $\mathbb{S}^{n-1}$ . Moreover a linear subspace " $E$ " as an upper index indicates that the corresponding functional or

measure is considered with respect to  $E$  as the surrounding space. By assumption there is a Borel subset  $P \subseteq \mathbb{S}^{n-1}$  with positive measure such that for all  $u \in P$  all the radii of curvature of  $K_0$  in the direction  $u$  exist and are positive. As  $K_0$  is symmetric we can assume that  $u \in P$  if and only if  $-u \in P$ . Let  $N$  be the set of points  $u \in \mathbb{S}^{n-1}$  where the principal radii of curvature of  $K$  do not exist. Since  $N$  is the set of points where the Alexandrov second derivative of  $h$  does not exist, it is a set of measure zero. By replacing  $P$  by  $P \setminus (N \cup (-N))$  we can assume that the radii of curvature of both  $K_0$  and  $K$  exist at all points of  $P$ . As both  $N$  and  $-N$  have measure zero this set will still have positive measure.

Let  $u \in \mathbb{S}^{n-1}$  be such that  $h$  and  $h_0$  are second order differentiable at  $u$  and at  $-u$  and that the radii of curvature of  $K_0$  at  $u$  are positive. This is true of all points  $u \in P$ , which is not empty as it has positive measure. Let  $E \in \mathbb{G}(n, k+1)$  be such that  $u \in E$ . Then the assumption implies that also

$$\pi_k^E(K|E) = \beta \pi_k^E(K_0|E).$$

Hence we conclude as in [8] that

$$S_k^E(K|E, \cdot) + S_k^E(K^*|E, \cdot) = 2\beta S_k^E(K_0|E, \cdot).$$

Since  $h(K|E, \cdot) = h_K|E$  and  $h(K_0|E, \cdot) = h_{K_0}|E$  are second order differentiable at  $u$  and at  $-u$  with respect to  $E$ , Lemma 2.1 applied with respect to the subspace  $E$  implies that

$$\begin{aligned} \det(d^2h_{K|E}(u)|E \cap u^\perp) + \det(d^2h_{K^*|E}(u)|E \cap u^\perp) \\ = 2\beta \det(d^2h_{K_0|E}(u)|E \cap u^\perp). \end{aligned}$$

Since  $h$  and  $h_0$  are second order differentiable at  $u$  and at  $-u$ , the linear maps

$$\begin{aligned} L(h)(u): T_u\mathbb{S}^{n-1} &\rightarrow T_u\mathbb{S}^{n-1}, & v &\mapsto d^2h(u)(v), \\ L(h_0)(u): T_u\mathbb{S}^{n-1} &\rightarrow T_u\mathbb{S}^{n-1}, & v &\mapsto d^2h_0(u)(v), \end{aligned}$$

are well defined and positive semidefinite. Since the radii of curvature of  $K_0$  at  $u$  are positive, we can define

$$L_{h_0}(h)(u) := L(h_0)(u)^{-1/2} \circ L(h)(u) \circ L(h_0)(u)^{-1/2}$$

as in [8] in the smooth case.

In this situation, the arguments in [8] can be repeated to yield that

$$(3.2) \quad \begin{aligned} L_{h_0}(h)(u) + L_{h_0}(h)(-u) &= 2 \text{id} \\ \wedge^k L_{h_0}(h)(u) + \wedge^k L_{h_0}(h)(-u) &= 2\beta \wedge^k \text{id}, \end{aligned}$$

where  $\text{id}$  is the identity map on  $T_u\mathbb{S}^{n-1}$ . Lemma 3.4 in [8] shows that  $L_{h_0}(h)(u)$  and  $L_{h_0}(h)(-u)$  have a common orthonormal basis of eigenvectors  $e_1, \dots, e_{n-1}$ , with corresponding eigenvalues (relative principal radii of curvature)  $x_1, \dots, x_{n-1}$  at  $u$  and with eigenvalues  $y_1, \dots, y_{n-1}$  at  $-u$ . After a change of notation (if necessary), we can assume that  $0 \leq x_1 \leq x_2 \leq \dots \leq x_{n-1}$ . By (3.2) we thus obtain

$$(3.3) \quad x_i + y_i = 2 \quad \text{and} \quad x_I + y_I = 2\beta$$

for  $i = 1, \dots, n-1$  and  $I \subset \{1, \dots, n-1\}$  with  $|I| = k$ .

**Proof of Theorem 1.3 when  $2 \leq k < (n+1)/2$ .** From (3.3) and Lemma 2.2 we conclude that there is some  $\ell \in \{0, \dots, n-1\}$  such that

$$x_1 = \dots = x_\ell < x_{\ell+1} = \dots = x_{n-1} \quad \text{and} \quad y_1 = \dots = y_\ell > y_{\ell+1} = \dots = y_{n-1}.$$

(a) If  $k \leq \ell$ , then

$$x_1 + y_1 = 2 \quad \text{and} \quad x_1^k + y_1^k = 2\beta.$$

Hence

$$1 = \left( \frac{x_1 + y_1}{2} \right)^k \leq \frac{x_1^k + y_1^k}{2} = \beta,$$

contradicting the assumption that  $\beta < 1$ .

(b) Let  $k > \ell$ . Since  $k < (n+1)/2$  we have  $2k < n+1$  or  $k < n+1-k$ . Hence  $k \leq n-k < n-\ell$ , and thus  $k \leq n-1-\ell$ . But then

$$x_{\ell+1} + y_{\ell+1} = 2 \quad \text{and} \quad x_{\ell+1}^k + y_{\ell+1}^k = 2\beta,$$

and we arrive at a contradiction as before. This proves Theorem 1.3 when  $2 \leq k < (n+1)/2$   $\square$

**Proof of Theorem 1.3 when  $k = 3, n = 5$ .** In this case we are assuming that  $K_0$  has positive radii of curvature at almost all points of  $\mathbb{S}^{n-1}$ . As  $h$  has Alexandrov second derivatives at almost all points, for almost all  $u \in \mathbb{S}^{n-1}$  the radii of curvature of  $K$  exist at both  $u$  and  $-u$  and at these unit vectors  $K_0$  has positive radii of curvature. Recall that  $x_1 \leq \dots \leq x_4$  are the eigenvalues of  $L_{h_0}(h)(u)$ . We distinguish three cases each of which will lead to a contradiction.

(a)  $x_1 \neq x_2$ . Then Lemma 2.2 yields that  $x_1 < x_2 = x_3 = x_4$  and therefore also  $y_2 = y_3 = y_4$ . Hence

$$x_2^3 + y_2^3 = 2\beta \quad \text{and} \quad x_2 + y_2 = 2,$$

and thus

$$1 = \left( \frac{x_2 + y_2}{2} \right)^3 \leq \frac{x_2^3 + y_2^3}{2} = \beta,$$

contradicting that  $\beta < 1$ . So this case can not arise.

(b)  $x_1 = x_2$  and  $x_1 = x_3$ , i.e.  $x_1 = x_2 = x_3$ . Then also  $y_1 = y_2 = y_3$ , and we get

$$x_1^3 + y_1^3 = 2\beta \quad \text{and} \quad x_1 + y_1 = 2,$$

which, as before, leads to a contradiction and thus this case can not arise.

(c)  $x_1 = x_2$  and  $x_1 \neq x_3$ , i.e.  $x_1 = x_2 < x_3 = x_4$  by Lemma 2.2. Since  $x_1 \neq x_3$ , Lemma 2.3 implies that

$$(3.4) \quad x_2 x_4 = \beta = y_2 y_4.$$

In addition, we have

$$(3.5) \quad x_2 + y_2 = 2 = x_4 + y_4.$$

We show that these equations determine  $x_2, x_4, y_2, y_4$  as functions of  $\beta$ . Substituting (3.4) into (3.5), we get

$$\frac{\beta}{x_4} + y_2 = 2, \quad x_4 + \frac{\beta}{y_2} = 2.$$

Combining these two equations, we arrive at

$$y_2 + \frac{\beta}{2 - \frac{\beta}{y_2}} = 2,$$

where we used that  $x_4 = 2 - \frac{\beta}{y_2} \neq 0$ . This equation for  $y_2$  can be rewritten as

$$y_2^2 - 2y_2 + \beta = 0.$$

Hence, we find that (recall that  $0 < \beta < 1$ )

$$y_2 = 1 \pm \sqrt{1 - \beta}.$$

Consequently,

$$x_2 = 2 - y_2 = 1 \mp \sqrt{1 - \beta}.$$

From (3.4), we also get

$$x_4 = \frac{\beta}{x_2} = \frac{\beta}{1 \mp \sqrt{1 - \beta}} = 1 \pm \sqrt{1 - \beta},$$

and finally again by (3.4)

$$y_4 = \frac{\beta}{y_2} = \frac{\beta}{1 \pm \sqrt{1 - \beta}} = 1 \mp \sqrt{1 - \beta}.$$

Since  $x_1 = x_2 < x_3 = x_4$ , this shows that

$$(3.6) \quad x_1 = x_2 = 1 - \sqrt{1 - \beta}, \quad x_3 = x_4 = 1 + \sqrt{1 - \beta}.$$

By assumption the surface area measure  $S_4(K_0, \cdot)$  of  $K_0$  is absolutely continuous with density function  $u \mapsto \det(d^2 h_0(u)|u^\perp)$ . Since  $K + K^* = 2K_0$ , the non-negativity of the mixed surface area measures  $S(K[i], K^*[4-i], \cdot)$  and the multilinearity of the surface area measures yields that

$$\begin{aligned} S_4(K, \cdot) &\leq \sum_{i=0}^4 \binom{4}{i} S(K[i], K^*[4-i], \cdot) \\ &= S_4(K + K^*, \cdot) = 2^4 S_4(K_0, \cdot). \end{aligned}$$

This implies that  $S_4(K, \cdot)$  is absolutely continuous as well, with density function  $u \mapsto \det(d^2 h(u)|u^\perp)$ . Now observe that the cases (a) and (b) have already been excluded and therefore the present case (c) is the only remaining one. Hence, using the definition of  $L_{h_0}(h)(u)$ ,

$$\frac{\det(d^2 h(u)|u^\perp)}{\det(d^2 h_0(u)|u^\perp)} = \det(L_{h_0}(h)(u)) = x_1 x_2 x_3 x_4 = \beta^2,$$

for almost all  $u \in \mathbb{S}^4$ . Thus we deduce that

$$S_4(K, \cdot) = \beta^2 S_4(K_0, \cdot).$$

Minkowski's uniqueness theorem now implies that  $K$  and  $K_0$  are homothetic, hence  $K$  is centrally symmetric. Symmetric convex bodies with the same width function are translates of each other. But then again  $\beta = 1$ , a contradiction.

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