Integral Geometry of Tensor Valuations

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Abstract

We prove a complete set of integral geometric formulas of Crofton type (involving integrations over affine Grassmannians) for the Minkowski tensors of convex bodies. Minkowski tensors are the natural tensor valued valuations generalizing the intrinsic volumes (or Minkowski functionals) of convex bodies. By Hadwiger's general integral geometric theorem, the Crofton formulas yield also kinematic formulas for Minkowski tensors. The explicit calculations of integrals over affine Grassmannians require several integral geometric and combinatorial identities. The latter are derived with the help of Zeilberger's algorithm.

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1 Introduction

The kinematic formula, which goes back to Blaschke, Santaló and Chern, is a major classical result of integral geometry. When restricted to convex bodies in Euclidean

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space $\mathbb{R}^n$, it involves the intrinsic volumes (Minkowski functionals) $V_0, \ldots, V_n$. These are determined as the coefficients in the Steiner formula

$$V_n(K + \lambda B^n) = \sum_{i=0}^{n} \kappa_i V_{n-i}(K)\lambda^i, \qquad \lambda \geqslant 0.$$

Here K is in $\mathcal{K}^n$, the space of convex bodies in $\mathbb{R}^n$, B^n denotes the unit ball in $\mathbb{R}^n$, of volume κ_n, and V_n is the volume. In particular, V_{n-1} is half the surface area, V_1 is proportional to the mean width, and V_0 is the Euler characteristic.

Let $\mathbb{G}(n)$ denote the motion group of $\mathbb{R}^n$, and let μ be the Haar measure on $\mathbb{G}(n)$, normalized as in [20, p. 227]. Then the kinematic formula for convex bodies $K, L \in \mathcal{K}^n$ says that, for $j \in \{0, \ldots, n\}$,

$$\int_{\mathbb{G}(n)} V_j(K \cap gL)\, \mu(\mathrm{d}g) = \sum_{k=j}^{n} \alpha_{njk} V_k(K) V_{n+j-k}(L), \tag{1.1}$$

where

$$\alpha_{njk} = \frac{\binom{k}{j}\kappa_k \kappa_{n+j-k}}{\binom{n}{k-j}\kappa_j \kappa_n}.$$

Related to the kinematic formula is the Crofton formula, which involves an integration over $\mathcal{E}_k^n$, the affine Grassmannian of k-flats in $\mathbb{R}^n$. The motion invariant Haar measure μ_k^n on $\mathcal{E}_k^n$ will be normalized so that κ_{n-k} is the measure of the set of k-flats hitting B^n. Then, for $K \in \mathcal{K}^n$, $k \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, k\}$,

$$\int_{\mathcal{E}_k^n} V_j(K \cap E)\, \mu_k^n(\mathrm{d}E) = \alpha_{njk} V_{n+j-k}(K). \tag{1.2}$$

A basic result on intrinsic volumes is Hadwiger's characterization theorem. It says that the vector space of continuous and motion invariant real valuations on $\mathcal{K}^n$ is spanned by $V_0, \ldots, V_n$. This theorem can be used to prove (1.1) and (1.2). A deeper connection is exhibited by Hadwiger's general integral geometric theorem, stating that for any continuous valuation $\varphi : \mathcal{K}^n \to \mathbb{R}$ and for $K, L \in \mathcal{K}^n$ one has

$$\int_{\mathbb{G}(n)} \varphi(K \cap gL)\, \mu(\mathrm{d}g) = \sum_{q=0}^{n} \varphi_{n-q}(K) V_q(L) \tag{1.3}$$

with

$$\varphi_{n-q}(K) := \int_{\mathcal{E}_q^n} \varphi(K \cap E)\, \mu_q(\mathrm{d}E). \tag{1.4}$$

A modern presentation of the proof is found in [3, p. 132-4]. In particular, choosing $\varphi = V_j$ in (1.3), we obtain (1.1) via (1.2). The remarkable fact is that for a general continuous valuation φ, the kinematic integral on the left-hand side of (1.3) is known once the Crofton type integrals on the right-hand side of (1.4) have been determined.

During the last 30 years, the integral geometry of convex bodies has been generalized considerably and extended in several directions; this includes local versions of classical results, more general translative formulas, and applications to stochastic geometry. A partial survey is given in [9].

One line of extension concerns the natural tensor valued generalizations of the intrinsic volumes. The rank one case, vector valued valuations, was already investigated in the 1970s by Hadwiger and Schneider, see [7], [19]. Characterizations and integral geometric formulas turned out to be very similar to the scalar case. The systematic investigation of tensor valuations of higher rank, and in particular of Minkowski tensors, began only with a paper by P. McMullen [11], and here the situation turned out to be quite different.

To explain the Minkowski tensors, we denote by $\mathbb{T}^p$ the vector space of symmetric tensors of rank p over $\mathbb{R}^n$. We use the scalar product to identify $\mathbb{R}^n$ with its dual space; then $\mathbb{T}^p$ can be viewed as the vector space of symmetric p-linear functionals on $\mathbb{R}^n$. The symmetric tensor product of symmetric tensors a, b is denoted by ab, and x^r is the r-fold symmetric tensor product of $x \in \mathbb{R}^n$ (and this is equal to the r-fold tensor product of x). We need the support measures (generalized curvature measures) $\Lambda_0(K,\cdot), \dots, \Lambda_{n-1}(K,\cdot)$ of a convex body $K \in \mathcal{K}^n$, which are defined by a localized Steiner formula. Let $\langle \cdot\,,\cdot\rangle$ be the scalar product and $\|\cdot\|$ the norm in $\mathbb{R}^n$. The unit sphere of $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1}$. For $x \in \mathbb{R}^n$, let $p(K,x)$ denote the metric projection of x to K, and put $u(K,x) := (x - p(K,x))/\|x - p(K,x)\|$ for $x \notin K$. Then, for any $\epsilon > 0$ and Borel set $\eta \subset \Sigma := \mathbb{R}^n \times \mathbb{S}^{n-1}$, the n-dimensional Hausdorff measure (volume) $\mathcal{H}^n$ of the local parallel set

$$M_\epsilon(K,\eta) := \{x \in (K + \epsilon B^n) \setminus K : (p(K,x), u(K,x)) \in \eta\}$$

is a polynomial in ϵ,

$$\mathcal{H}^n(M_\epsilon(K,\eta)) = \sum_{k=0}^{n-1} \epsilon^{n-k} \kappa_{n-k} \Lambda_k(K,\eta);$$

see [20], [21] for further information. Clearly, we have $V_i(K) = \Lambda_i(K,\Sigma)$ for $i = 0,\dots,n-1$. In addition, we define $\Lambda_n(K,\cdot)$ as the restriction of $\mathcal{H}^n$ to K.

For $K \in \mathcal{K}^n$ and integers $r, s \geqslant 0$, $0 \le j \le n-1$, the *Minkowski tensors* are defined by

$$\Phi_{j,r,s}(K) := \frac{1}{r!s!} \frac{\omega_{n-j}}{\omega_{n-j+s}} \int_\Sigma x^r u^s \Lambda_j(K, \mathrm{d}(x,u))$$

and

$$\Phi_{n,r,0}(K) := \frac{1}{r!} \int_{\mathbb{R}^n} x^r \Lambda_n(K, \mathrm{d}x),$$

where $\omega_m := m\kappa_m$. In view of later use, we extend the definition by $\Phi_{j,r,s} := 0$ if $j \notin \{0,\dots,n\}$ or if r or s is not in $\mathbb{N}_0$ or if $j = n$ and $s \neq 0$. Vector valuations are obtained for $r = 1$ and $s = 0$; the case $r = 0$ and $s = 1$ leads to valuations which are identically zero. All functions $\Phi_{j,r,s} : \mathcal{K}^n \to \mathbb{T}^{r+s}$ are continuous (with respect to the Hausdorff metric on $\mathcal{K}^n$ and the standard topology on $\mathbb{T}^{r+s}$) and isometry covariant valuations (we refer to [11], [21], [10] for explicit definitions). These properties are still shared by the tensor functions $Q^m\Phi_{j,r,s}$, $m \in \mathbb{N}_0$, where $Q \in \mathbb{T}^2$ denotes the metric tensor, defined by $Q(x, y) = \langle x, y\rangle$ for $x, y \in \mathbb{R}^n$. We call the functions $Q^m\Phi_{j,r,s}$ with $r, s \in \mathbb{N}_0$ and either $j \in \{0, \dots, n-1\}$ or $(j, s) = (n, 0)$, the *basic tensor valuations* (the terminology here and in [10] is different from the one in [21]).

The characterization theorems for rank zero by Hadwiger [6] and for rank one by Hadwiger and Schneider [7] were extended in remarkable work [1], [2] of Alesker, who obtained the following result.

Theorem 1.1 (Alesker) *Let $p \in \mathbb{N}_0$, and let $\varphi : \mathcal{K}^n \to \mathbb{T}^p$ be a continuous, isometry covariant valuation. Then φ is a linear combination, with constant real coefficients, of the basic tensor valuations $Q^m\Phi_{k,r,s}$, where $m, k, r, s \in \mathbb{N}_0$ are such that $2m + r + s = p$.*

For general $p \in \mathbb{N}_0$, the basic tensor valuations of rank p are not linearly independent. McMullen [11] found that, for $r \geqslant 2$ and $k \in \{0, \dots, n + r - 2\}$,

$$2\pi \sum_s s\,\Phi_{k-r+s,r-s,s} = Q \sum_s \Phi_{k-r+s,r-s,s-2}. \tag{1.5}$$

It was proved in [10] that these are essentially, that is, up to multiplication by powers of Q and linear combinations, the only non-trivial linear relations between the basic tensor valuations. This led to the explicit determination of dimensions and bases for the vector spaces of continuous, isometry covariant tensor valuations of rank p. For example, for rank two the dimension is $3n + 1$ and a basis is given by

- QV_j, $j = 0, \dots, n$,
- $\Phi_{j,2,0}$, $j = 0, \dots, n$,
- $\Phi_{j,0,2}$, $j = 1, \dots, n-1$.

Since

$$\Phi_{k,0,2} = \frac{1}{4\pi} Q\Phi_{k,0,0} - \frac{1}{2}\Phi_{k-1,1,1},$$

for $k = 1, \dots, n-1$, we can replace $\Phi_{j,0,2}$, $j = 1, \dots, n-1$, by $\Phi_{j,1,1}$, $j = 0, \dots, n-2$, in the preceding basis. While the rank two case is still easy to treat, for tensor valuations of higher rank the situation is considerably more complicated.

The existence of non-trivial linear relations between the basic tensor valuations of rank $p \geqslant 2$ is one reason for the fact that, in contrast to the cases $p = 0, 1$, the characterization theorem seems to be of no help in obtaining integral geometric formulas for Minkowski tensors of higher rank. Another reason lies in the difficulty of explicitly calculating Minkowski tensors for special convex bodies.

We mention that interest in the integral geometry of intrinsic volumes and their generalizations comes also from applied sciences. We refer to the work of K. Mecke [12], [13], [14] in statistical physics and point out that Minkowski tensors up to rank two are used in [5], [4], as tools in the morphometry of spatial patterns.

Basic integral geometry for Minkowski tensors requires to determine the kinematic integrals

$$\int_{\mathbb{G}(n)} \Phi_{j,r,s}(K \cap gL)\, \mu(\mathrm{d}g) \tag{1.6}$$

and the Crofton integrals

$$\int_{\mathcal{E}_k^n} \Phi_{j,r,s}(K \cap E)\, \mu_k^n(\mathrm{d}E). \tag{1.7}$$

By Hadwiger's general integral geometric theorem, which can be applied to each coordinate of $\Phi_{j,r,s}$ with respect to some basis of $\mathbb{T}^{r+s}$, it is sufficient to determine the Crofton integrals (1.7). In some special cases, this has been done in [21]. For $s = 0$, for example, the required formulas follow immediately from the known corresponding formulas for curvature measures. The new cases settled in [21] are those where $j = n - 1$ or $j = n - 2$. Hence, all formulas of types (1.7) and (1.6) were obtained in dimensions $n = 2$ and $n = 3$. Further progress requires more sophisticated methods and calculations. The complete determination of all integrals (1.7) is the subject of this paper.

2 Results

For the statement of our main results, we distinguish two cases. This corresponds to the distinction between the cases $k < n$ and $k = n$ in the definition of the support measures. We start with the latter case which is easier to state and to prove.

Theorem 2.1 *For $K \in \mathcal{K}^n$, $r, s \in \mathbb{N}_0$ and $0 \leqslant k \leqslant n - 1$,*

$$\int_{\mathcal{E}_k^n} \Phi_{k,r,s}(K \cap E)\, \mu_k^n(\mathrm{d}E) = \begin{cases} \tilde{\alpha}_{n,k,s}\, Q^{\frac{s}{2}} \Phi_{n,r,0}(K), & \text{if } s \text{ is even,} \\ 0, & \text{if } s \text{ is odd,} \end{cases}$$

where

$$\tilde{\alpha}_{n,k,s} = \frac{1}{(4\pi)^{\frac{s}{2}} \left(\frac{s}{2}\right)!} \frac{\Gamma\left(\frac{n-k+s}{2}\right) \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+s}{2}\right) \Gamma\left(\frac{n-k}{2}\right)}.$$

Theorem 2.2 *For $K \in \mathcal{K}^n$ and $k, j, r, s \in \mathbb{N}_0$ with $0 \leqslant j < k \leqslant n-1$,*

$$\int_{\mathcal{E}_k^n} \Phi_{j,r,s}(K \cap E)\, \mu_k^n(\mathrm{d}E)$$
$$= \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \chi_{n,j,k,s,z}^{(1)} Q^z \Phi_{n+j-k,r,s-2z}(K) + \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor -1} \chi_{n,j,k,s,z}^{(2)} Q^z \times$$
$$\sum_{l=0}^{s-2z-1} \left(2\pi l \Phi_{n+j-k-s+2z+l,r+s-2z-l,l}(K) - Q\Phi_{n+j-k-s+2z+l,r+s-2z-l,l-2}(K)\right),$$

where the constants $\chi_{n,j,k,s,z}^{(1)}$ and $\chi_{n,j,k,s,z}^{(2)}$ are given by (5.16) *and* (5.17)*.*

The result can be given an alternative form by using the McMullen relations (1.5).

Theorem 2.3 *For $K \in \mathcal{K}^n$ and $k, j, r, s \in \mathbb{N}_0$ with $0 \leqslant j < k \leqslant n-1$,*

$$\int_{\mathcal{E}_k^n} \Phi_{j,r,s}(K \cap E)\, \mu_k^n(\mathrm{d}E)$$
$$= \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \chi_{n,j,k,s,z}^{(1)} Q^z \Phi_{n+j-k,r,s-2z}(K) + \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor -1} \chi_{n,j,k,s,z}^{(2)} Q^z \times$$
$$\sum_{l \geqslant s-2z} \left(Q\Phi_{n+j-k-s+2z+l,r+s-2z-l,l-2}(K) - 2\pi l \Phi_{n+j-k-s+2z+l,r+s-2z-l,l}(K)\right),$$

with the same constants constants $\chi_{n,j,k,s,z}^{(1)}$ and $\chi_{n,j,k,s,z}^{(2)}$ as in Theorem 2.2.

In the preceding formulas, the Minkowski tensor $\Phi_{j,r,s}(K \cap E)$ of the lower dimensional set $K \cap E$ contained in E is computed in $\mathbb{R}^n$. For convex bodies lying in an affine subspace E, there is an alternative version $\Phi_{j,r,s}^{(E)}$ of the Minkowski tensors, involving only support measures corresponding to that subspace; see the next section for a more detailed explanation. Formulas similar to those in the preceding theorems hold also if $\Phi_{j,r,s}(K \cap E)$ is replaced by $\Phi_{j,r,s}^{(E)}(K \cap E)$.

Theorem 2.4 *Let $K \in \mathcal{K}^n$ and $k, r, s \in \mathbb{N}_0$ with $0 \leqslant k \leqslant n-1$. Then*

$$\int_{\mathcal{E}_k^n} \Phi_{k,r,s}^{(E)}(K \cap E)\, \mu_k^n(\mathrm{d}E) = \begin{cases} \Phi_{n,r,0}(K), & \text{if } s = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.5 *Let $K \in \mathcal{K}^n$ and $k, j, r, s \in \mathbb{N}_0$ with $0 \leqslant j < k \leqslant n-1$. Then*

$$\int_{\mathcal{E}_k^n} \Phi_{j,r,s}^{(E)}(K \cap E)\, \mu_k^n(\mathrm{d}E)$$

$$= \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \bar{\chi}_{n,j,k,s,z}^{(1)} Q^z \Phi_{n+j-k,r,s-2z}(K) + \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor - 1} \bar{\chi}_{n,j,k,s,z}^{(2)} Q^z \times$$

$$\sum_{l=0}^{s-2z-1} \left(2\pi l \Phi_{n+j-k-s+2z+l,r+s-2z-l,l}(K) - Q\Phi_{n+j-k-s+2z+l,r+s-2z-l,l-2}(K)\right),$$

where the constants $\bar{\chi}_{n,j,k,s,z}^{(1)}$ and $\bar{\chi}_{n,j,k,s,z}^{(2)}$ are given by (5.18) *and* (5.19).

Theorem 2.6 *Let $K \in \mathcal{K}^n$ and $k, j, r, s \in \mathbb{N}_0$ with $0 \leqslant j < k \leqslant n-1$. Then*

$$\int_{\mathcal{E}_k^n} \Phi_{j,r,s}^{(E)}(K \cap E)\, \mu_k^n(\mathrm{d}E)$$

$$= \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \bar{\chi}_{n,j,k,s,z}^{(1)} Q^z \Phi_{n+j-k,r,s-2z}(K) + \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor - 1} \bar{\chi}_{n,j,k,s,z}^{(2)} Q^z \times$$

$$\sum_{l \geqslant s-2z} \left(Q\Phi_{n+j-k-s+2z+l,r+s-2z-l,l-2}(K) - 2\pi l \Phi_{n+j-k-s+2z+l,r+s-2z-l,l}(K)\right),$$

with the same constants constants $\bar{\chi}_{n,j,k,s,z}^{(1)}$ and $\bar{\chi}_{n,j,k,s,z}^{(2)}$ as in Theorem 2.5.

3 Geometric identities for tensor valuations

In the proofs of our main results we shall use two identities for tensor valuations, which are due to McMullen [11]. The first of these relates the Minkowski tensors of convex bodies lying in some affine subspace to the Minkowski tensors computed in this subspace. Before stating these results, we introduce some notation.

By $\mathcal{L}_k^n$ we denote the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^n$. For $L \in \mathcal{L}_k^n$, $k \in \{0, \dots, n\}$, let $p_L : \mathbb{R}^n \to L$ denote the orthogonal projection, and define $\pi_L : \mathbb{S}^{n-1} \setminus L^\perp \to L \cap \mathbb{S}^{n-1}$ by

$$\pi_L(u) := \frac{p_L(u)}{\|p_L(u)\|}.$$

For $k = n$ this is the identity map on $\mathbb{S}^{n-1}$, for $k = 0$ we obtain the empty map.

The tensor $Q(L) \in \mathbb{T}^2$ is defined by $Q(L)(x, y) := \langle p_L(x), p_L(y) \rangle$ for $x, y \in \mathbb{R}^n$. If $(e_1, \dots, e_n)$ is an orthonormal basis of $\mathbb{R}^n$, we have $Q = \sum_{i=1}^n e_i^2$. If now the basis is such that $(e_1, \dots, e_k)$ is a basis of L, then $Q(L) = \sum_{i=1}^k e_i^2$.

For $E \in \mathcal{E}_k^n$, we denote by E^0 the linear subspace which is a translate of E. The linear subspace $E^\perp := (E^0)^\perp$ is the orthogonal complement of E^0.

Let $E \in \mathcal{E}_k^n$, $k \in \{1, \dots, n\}$, and a convex body $K \in \mathcal{K}^n$ with $K \subset E$ be given. For any $j \in \{0, \dots, k-1\}$, let $\Lambda_j^{(E)}(K, \cdot)$ denote the jth support measure of K with respect to E. It is defined as the image measure of the restriction of $\Lambda_j(K, \cdot)$ to $\mathbb{R}^n \times (\mathbb{S}^{n-1} \backslash E^\perp)$ under the map

$$\mathbb{R}^n \times (\mathbb{S}^{n-1} \backslash E^\perp) \to \mathbb{R}^n \times (E^0 \cap \mathbb{S}^{n-1}), \qquad (x, u) \mapsto (x, \pi_{E^0}(u)).$$

Thus the measure $\Lambda_j^{(E)}(K, \cdot)$ is defined on the Borel subsets of $\mathbb{R}^n \times (E^0 \cap \mathbb{S}^{n-1})$, and it is concentrated on $\Sigma^{(E)} := E \times (E^0 \cap \mathbb{S}^{n-1})$ since $K \subset E$. If $L \in \mathcal{L}_k^n$, which can be identified with $\mathbb{R}^k$, then the restriction of $\Lambda_j^{(L)}(K, \cdot)$ to $\Sigma^{(L)}$ is just the jth support measure of $K \subset L$ as defined in L. For a translation vector $t \in \mathbb{R}^n$, one has $\Lambda_j^{(L+t)}(K + t, \eta + t) := \Lambda^{(L)}(K, \eta)$, where $\eta + t := \{(x + t, u) : (x, u) \in \eta\}$.

Next we define tensor valuations of convex bodies K contained in the affine subspace $E \in \mathcal{E}_k^n$ by

$$\Phi_{j,r,s}^{(E)}(K) := \frac{\omega_{k-j}}{r!s!\omega_{k-j+s}} \int_{\Sigma^{(E)}} x^r u^s \, \Lambda_j^{(E)}(K, \mathrm{d}(x, u)), \tag{3.8}$$

for $j \in \{0, \dots, k-1\}$, and

$$\Phi_{k,r,0}^{(E)}(K) := \frac{1}{r!} \int_K x^r \, \mathcal{H}^k(\mathrm{d}x); \tag{3.9}$$

in all other cases, $\Phi_{j,r,s}^{(E)}$ is defined as the zero function. For polytopes, these tensor valuations can be described in a simple way. We denote by $\mathcal{P}^n$ the set of polytopes in $\mathcal{K}^n$, and for $P \in \mathcal{P}^n$ and $j \in \{0, \dots, n\}$, $\mathcal{F}^j(P)$ is the set of j-dimensional faces of P. The normal cone of the polytope P at its face F is denoted by $N(P, F)$, and if P is contained in the affine subspace E, then $N_E(P, F) := N(P, F) \cap E^0$. The proof of the following lemma is an immediate consequence of corresponding properties of the support measures (see [20, Section 4.2]).

Lemma 3.1 *Let $P \in \mathcal{P}^n$, $0 \leqslant j < n$ and $r, s \in \mathbb{N}_0$. Then*

$$\Phi_{j,r,s}(P) = \sum_{F \in \mathcal{F}^j(P)} \frac{1}{r!s!\omega_{n-j+s}} \int_F x^r \, \mathcal{H}^j(\mathrm{d}x) \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^s \, \mathcal{H}^{n-j-1}(\mathrm{d}u).$$

For $E \in \mathcal{E}_k^n$, $k \in \{1, \dots, n\}$, a polytope $P \subset E$, and for $0 \leqslant j < k$,

$$\Phi_{j,r,s}^{(E)}(P) = \sum_{F \in \mathcal{F}^j(P)} \frac{1}{r!s!\omega_{k-j+s}} \int_F x^r \, \mathcal{H}^j(\mathrm{d}x) \int_{N_E(P,F) \cap \mathbb{S}^{n-1}} u^s \, \mathcal{H}^{k-j-1}(\mathrm{d}u).$$

The relation between these two types of tensors is not so simple as one might expect.

Theorem 3.2 (McMullen) *Let $E \in \mathcal{E}_k^n$, $k \in \{0, \dots, n-1\}$, and $K \in \mathcal{K}^n$ with $K \subset E$. Let $j \in \mathbb{N}_0$ with $0 \leqslant j \leqslant k$. Then, for all $r, s \in \mathbb{N}_0$,*

$$\Phi_{j,r,s}(K) = \sum_{m \geqslant 0} \frac{Q(E^\perp)^m}{(4\pi)^m m!} \Phi_{j,r,s-2m}^{(E)}(K).$$

In the case of a linear subspace, this is Theorem 5.1 of McMullen [11]. From this case, the result for an affine subspace is obtained by developing $(x+t)^r$, for a fixed translation vector t, in the integrands on both sides.

To formulate another result by McMullen [11, p. 269], we need a bit more notation. Let $P \in \mathcal{P}^n$ be a polytope and $F \in \mathcal{F}^k(P)$ a k-face of P, $k \in \{0, \dots, n\}$. For integers $r, s \in \mathbb{N}_0$, define

$$\Upsilon_r(F) := \frac{1}{r!} \int_F x^r \, \mathcal{H}^k(\mathrm{d}x)$$

and

$$\Theta_s(P,F) := \frac{1}{s!} \int_{N(P,F)} x^s e^{-\pi \|x\|^2} \, \mathcal{H}^{n-k}(\mathrm{d}x).$$

Furthermore, put $\Upsilon_r(F) := 0$ whenever $r < 0$, and $\Theta_s(P,F) := 0$ whenever $s < 0$ or if $k = n$ and $s \neq 0$. If $k < n$ and $s \geqslant 0$, then

$$\begin{aligned}
\Theta_s(P,F) &= \frac{1}{s!} \int_0^\infty \lambda^{s+n-k-1} e^{-\pi\lambda^2} \, \mathrm{d}\lambda \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^s \, \mathcal{H}^{n-k-1}(\mathrm{d}u) \\
&= \frac{\Gamma(\frac{s+n-k}{2})}{s! 2\pi^{\frac{s+n-k}{2}}} \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^s \, \mathcal{H}^{n-k-1}(\mathrm{d}u) \\
&= \frac{1}{s! \omega_{s+n-k}} \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^s \, \mathcal{H}^{n-k-1}(\mathrm{d}u).
\end{aligned}$$

This yields a nice generalization of a well-known representation of intrinsic volumes, namely

$$\Phi_{k,r,s}(P) = \sum_{F \in \mathcal{F}^k(P)} \Upsilon_r(F) \Theta_s(P,F)$$

for $r, s \in \mathbb{N}_0$ and $k \in \{0, \dots, n\}$; the case $k = n$ can be checked separately.

Lemma 3.3 (McMullen) *Let $P \in \mathcal{P}^n$ be a polytope. Then, for $r, s \in \mathbb{N}_0$ and $k \in \{0, \dots, n\}$,*

$$\begin{aligned}
&2\pi s \Phi_{k,r,s}(P) \\
&= \sum_{F \in \mathcal{F}^k(P)} Q(N(P,F)) \Upsilon_r(F) \Theta_{s-2}(P,F) + \sum_{G \in \mathcal{F}^{k+1}(P)} Q(G) \Upsilon_{r-1}(G) \Theta_{s-1}(P,G),
\end{aligned}$$

where $Q(N(P,F)) := Q(\operatorname{lin} N(P,F))$ and $Q(G) := Q((\operatorname{aff} G)^0)$.

4 Integration over Grassmannians

In this preparatory section, we obtain some auxiliary results concerning integrations over Grassmannians, in particular transformation formulas of integral geometric type and the values of some special integrals.

We need some notation. The bracket or generalized sine function of two linear subspaces L, L' of $\mathbb{R}^n$ is defined as follows. One chooses an orthonormal basis of $L \cap L'$ and extends it to an orthonormal basis of L and to one of L'. Then $[L, L']$ is the volume of the parallelepiped spanned by the obtained vectors. This depends only on the subspaces L and L'.

For $q, k \in \{0, \dots, n\}$ and a fixed subspace $F \in \mathcal{L}_k^n$ let

$$\mathcal{L}_q^F := \begin{cases} \{L \in \mathcal{L}_q^n : L \subset F\}, \text{ if } q \leqslant k, \\ \{L \in \mathcal{L}_q^n : L \supset F\}, \text{ if } q > k. \end{cases}$$

There exists a unique (Borel) probability measure ν_q^F on $\mathcal{L}_q^n$ with the following properties: it is concentrated on $\mathcal{L}_q^F$, invariant under $SO(F)$ and $SO(F^\perp)$, where $SO(F)$ denotes the group of all rotations of $\mathbb{R}^n$ mapping F into itself and leaving $F^\perp$ pointwise fixed, and $\nu_q^{\vartheta F}(\vartheta A) = \nu_q^F(A)$ for every rotation ϑ of $\mathbb{R}^n$ and every Borel set $A \subset \mathcal{L}_q^n$ (see [22, Section 6.1]).

The transformation formula of the following lemma will later be used to simplify some integrations. It adapts an integration over all k-subspaces to a fixed r-subspace F with $r + k \geqslant n$, in the following way. One first integrates over all k-subspaces containing a fixed $(r + k - n)$-subspace of F, and then over all $(r + k - n)$-subspaces of F. The formula was proved, in a different but equivalent formulation, by Petkantschin [15, formula (48)]; it appears also in [18, formula (14.40)]. The case $r = n - 1$ is Lemma 5.6 in [8], with a different proof. The general case can also be obtained by a straightforward extension of the argument used there.

Lemma 4.1 *Let $k, r \in \{0, \dots, n\}$ with $r + k \geqslant n$, let $h : \mathcal{L}_k^n \to \mathbb{R}$ be integrable and $F \in \mathcal{L}_r^n$. Then*

$$\int_{\mathcal{L}_k^n} h(L) \nu_k^n(\mathrm{d}L) = c \int_{\mathcal{L}_{r+k-n}^F} \int_{\mathcal{L}_k^U} h(L)[F, L]^{r+k-n} \, \nu_k^U(\mathrm{d}L) \, \nu_{r+k-n}^F(\mathrm{d}U),$$

where

$$c := \left(\prod_{j=1}^{n-r} \frac{\Gamma(\frac{j}{2})}{\Gamma(\frac{2n-r-k-j+1}{2})} \right) \left(\prod_{j=1}^{r+k-n} \frac{\Gamma(\frac{j}{2})}{\Gamma(\frac{r-j+1}{2})} \right) \left(\prod_{j=1}^{k} \frac{\Gamma(\frac{j}{2})}{\Gamma(\frac{n-j+1}{2})} \right)^{-1}.$$

We denote by $\mathrm{lin}\{U, v\}$ the linear hull of $U \subset \mathbb{R}^n$ and $v \in \mathbb{R}^n$.

Corollary 4.2 *Let $u \in \mathbb{S}^{n-1}$, $1 \leqslant k \leqslant n-1$, and let $h : \mathcal{L}_k^n \to \mathbb{T}$ be a tensor valued integrable function. Then*

$$\begin{aligned}\int_{\mathcal{L}_k^n} h(L)\,\nu_k^n(\mathrm{d}L) =& \frac{\omega_k}{2\omega_n}\int_{\mathcal{L}_{k-1}^{u^\perp}}\int_{-1}^{1}\int_{U^\perp\cap u^\perp\cap\mathbb{S}^{n-1}} |t|^{k-1}(1-t^2)^{\frac{n-k-2}{2}} \\ &\times h(\mathrm{lin}\{U, tu+\sqrt{1-t^2}w\})\mathcal{H}^{n-k-1}(\mathrm{d}w)\,\mathrm{d}t\,\nu_{k-1}^{u^\perp}(\mathrm{d}U).\end{aligned}$$

Proof. This follows from Lemma 4.1 by an application of the coarea formula. □

We will have to calculate integrals of the form

$$\int_{\mathcal{L}_k^n} [F, L]^a Q(L)^m\,\nu_k^n(\mathrm{d}L)$$

where $F \in \mathcal{L}_r^n$ is fixed, $m \in \mathbb{N}_0$ and $a \geqslant 0$. As a first step, we consider the two special cases $a = 0$ and $m = 0$.

Lemma 4.3 *For $m \in \mathbb{N}_0$ and $k \in \{1, \dots, n\}$,*

$$\int_{\mathcal{L}_k^n} Q(L)^m\,\nu_k^n(\mathrm{d}L) = \prod_{j=0}^{m-1}\frac{k+2j}{n+2j}Q^m = \frac{\Gamma(\frac{k}{2}+m)\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}+m)\Gamma(\frac{k}{2})}Q^m.$$

Proof. The assertion is true for $k = n$. Suppose that $k \leqslant n-1$. Since the tensor on the left-hand side is rotation invariant, it is a multiple of Q^m. To determine the factor, fix any $u \in \mathbb{S}^{n-1}$ and apply u^{2m} to the integral. By Lemma 4.2 we thus obtain

$$\begin{aligned}&\int_{\mathcal{L}_k^n}\langle Q(L)^m, u^{2m}\rangle\nu_k^n(\mathrm{d}L)\\ &= \frac{\omega_k}{2\omega_n}\int_{\mathcal{L}_{k-1}^{u^\perp}}\int_{-1}^{1}\int_{U^\perp\cap u^\perp\cap\mathbb{S}^{n-1}} |t|^{k-1}(1-t^2)^{\frac{n-k-2}{2}}\\ &\quad\times(\langle Q(U), u^2\rangle + \langle tu+\sqrt{1-t^2}w, u\rangle^2)^m\,\mathcal{H}^{n-k-1}(\mathrm{d}w)\,\mathrm{d}t\,\nu_{k-1}^{u^\perp}(\mathrm{d}U)\\ &= \frac{\omega_k}{2\omega_n}\int_{\mathcal{L}_{k-1}^{u^\perp}}\int_{-1}^{1}\int_{U^\perp\cap u^\perp\cap\mathbb{S}^{n-1}} |t|^{k+2m-1}(1-t^2)^{\frac{n-k-2}{2}}\,\mathcal{H}^{n-k-1}(\mathrm{d}w)\,\mathrm{d}t\,\nu_{k-1}^{u^\perp}(\mathrm{d}U)\\ &= \frac{\omega_k}{2\omega_n}\frac{\Gamma(\frac{k}{2}+m)\Gamma(\frac{n-k}{2})}{\Gamma(\frac{n}{2}+m)}\omega_{n-k} = \frac{\pi^{\frac{k}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{k}{2}+m)2\pi^{\frac{n-k}{2}}}{2\pi^{\frac{n}{2}}\Gamma(\frac{k}{2})\Gamma(\frac{n}{2}+m)}\\ &= \frac{\Gamma(\frac{k}{2}+m)\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}+m)\Gamma(\frac{k}{2})},\end{aligned}$$

which yields the assertion. □

Lemma 4.4 *Let $a \geqslant 0$, $r, k \in \{1, \ldots, n\}$ with $k + r \geqslant n$ and $F \in \mathcal{L}_r^n$. Then*

$$A(n,k,r,a) := \int_{\mathcal{L}_k^n} [F, L]^a \, \nu_k^n(\mathrm{d}L) = \prod_{i=0}^{n-r-1} \frac{\Gamma(\frac{n-i}{2})\Gamma(\frac{k-i+a}{2})}{\Gamma(\frac{n-i+a}{2})\Gamma(\frac{k-i}{2})},$$

where for $r = n$ the right-hand side is defined as 1.

Proof. For $r = n$ or $k = n$ the assertion holds. Suppose that $r, k \in \{1, \ldots, n-1\}$ with $k + r \geqslant n$. We fix some $u_1 \in \mathbb{S}^{n-1} \cap F^\perp$. Then Lemma 4.2 yields

$$\begin{aligned}
&\int_{\mathcal{L}_k^n} [F, L]^a \, \nu_k^n(\mathrm{d}L) \\
&= \frac{\omega_k}{2\omega_n} \int_{\mathcal{L}_{k-1}^{u_1^\perp}} \int_{-1}^{1} \int_{U^\perp \cap u_1^\perp \cap \mathbb{S}^{n-1}} |t|^{k-1} (1-t^2)^{\frac{n-k-2}{2}} \left([F, \operatorname{lin}\{U, tu_1 + \sqrt{1-t^2}w\}] \right)^a \\
&\quad \times \mathcal{H}^{n-k-1}(\mathrm{d}w) \, \mathrm{d}t \, \nu_{k-1}^{u_1^\perp}(\mathrm{d}U). \qquad (4.10)
\end{aligned}$$

Let $L \in \mathcal{L}_k^n$ and F be in general relative position and $u_1 \notin L, L^\perp$. Denoting by $[F, L \cap u_1^\perp]^{(u_1^\perp)}$ the bracket of F and $L \cap u_1^\perp$ calculated with respect to $u_1^\perp$, we have

$$[F, L] = [F, L \cap u_1^\perp]^{(u_1^\perp)} \|p_L(u_1)\|. \qquad (4.11)$$

We now apply (4.11) with the linear subspace $L := \operatorname{lin}\{U, tu_1 + \sqrt{1-t^2}w\}$ and $t \in (-1, 1)$ (the assumption of general relative position is almost surely satisfied), and thus get

$$[F, \operatorname{lin}\{U, tu_1 + \sqrt{1-t^2}w\}] = [F, U]^{(u_1^\perp)} |t|.$$

Substituting the result into (4.10), we obtain

$$\begin{aligned}
&\int_{\mathcal{L}_k^n} [F, L]^a \, \nu_k^n(\mathrm{d}L) \\
&= \frac{\omega_k}{2\omega_n} \int_{\mathcal{L}_{k-1}^{u_1^\perp}} \int_{-1}^{1} \int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} |t|^{k-1+a} (1-t^2)^{\frac{n-k-2}{2}} \left([F, U]^{(u_1^\perp)} \right)^a \\
&\quad \times \mathcal{H}^{n-k-1}(\mathrm{d}w) \, \mathrm{d}t \, \nu_{k-1}^{u_1^\perp}(\mathrm{d}U) \\
&= \frac{\omega_k}{2\omega_n} \frac{\Gamma(\frac{k+a}{2})\Gamma(\frac{n-k}{2})}{\Gamma(\frac{n+a}{2})} \omega_{n-k} \int_{\mathcal{L}_{k-1}^{u_1^\perp}} \left([F, U]^{(u_1^\perp)} \right)^a \nu_{k-1}^{u_1^\perp}(\mathrm{d}U) \\
&= \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+a}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n+a}{2})} \int_{\mathcal{L}_{k-1}^{u_1^\perp}} \left([F, U]^{(u_1^\perp)} \right)^a \nu_{k-1}^{u_1^\perp}(\mathrm{d}U).
\end{aligned}$$

This is a recursive formula. In the next step, we choose $u_2 \in \mathbb{S}^{n-1} \cap F^\perp \cap u_1^\perp$ and obtain

$$\begin{aligned}
&\int_{\mathcal{L}_k^n}[F,L]^a\nu_k^n(\mathrm{d}L)\\
&=\frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+a}{2})\Gamma(\frac{n-1}{2})\Gamma(\frac{k-1+a}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n+a}{2})\Gamma(\frac{k-1}{2})\Gamma(\frac{n-1+a}{2})}\int_{\mathcal{L}_{k-2}^{u_1^\perp\cap u_2^\perp}}\left([F,U]^{(u_1^\perp\cap u_2^\perp)}\right)^a\nu_{k-2}^{u_1^\perp\cap u_2^\perp}(\mathrm{d}U).
\end{aligned}$$

The procedure terminates after $n-r$ steps with the intgral

$$\int_{\mathcal{L}_{r+k-n}^{u_1^\perp\cap\ldots\cap u_{n-r}^\perp}}\left([F,U]^{(u_1^\perp\cap\cdots\cap u_{n-r}^\perp)}\right)^a\nu_{r+k-n}^{u_1^\perp\cap\cdots\cap u_{n-r}^\perp}(\mathrm{d}U)=1,$$

since $\dim(u_1^\perp\cap\ldots\cap u_{n-r}^\perp)=n-(n-r)=r=\dim F$. $\square$

The following proposition is a common generalization of the preceding two lemmas. However, the proof of this extension is essentially based on the previous special cases.

Proposition 4.5 *Let $a\geqslant 0$, $m\in\mathbb{N}_0$, $k,r\in\{1,\ldots,n\}$ with $k+r\geqslant n$ and $F\in\mathcal{L}_r^n$. Then*

$$\begin{aligned}
\int_{\mathcal{L}_k^n}[F,L]^aQ(L)^m\,\nu_k^n(\mathrm{d}L)&=\frac{A(n,k,r,a)\Gamma(\frac{n+a}{2})}{\Gamma(\frac{k+a}{2})\Gamma(\frac{n+a}{2}+m)}\sum_{i=0}^m\binom{m}{i}\Gamma\left(\tfrac{k+a}{2}+m-i\right)\\
&\quad\times\frac{\Gamma(\frac{n-k}{2}+i)\Gamma(\frac{a}{2}+1)\Gamma(\frac{r}{2})}{\Gamma(\frac{n-k}{2})\Gamma(\frac{a}{2}+1-i)\Gamma(\frac{r}{2}+i)}(-1)^iQ^{m-i}Q(F)^i,
\end{aligned}$$

where $\Gamma(-p)^{-1}$ for $p\in\mathbb{N}_0$ has to be read as 0.

Proof. The case $a=0$ is covered by Lemma 4.3. Hence we consider the case $a>0$. We prove the assertion by induction with respect to the dimension of $F^\perp$. For $r=n$ we have $[F,L]=1$, and the assertion of the lemma follows from Lemma 4.3 and Lemma 6.3. This settles the case where F has codimension 0.

Let us assume that the assertion of the lemma is proved for $\dim(F^\perp)=n-r-1$, $r\in\{1,\ldots,n-1\}$. We now establish the lemma for $\dim(F^\perp)=n-r$. Fix $F\in\mathcal{L}_r^n$ and choose $u\in\mathbb{S}^{n-1}\cap F^\perp$. Using Lemma 4.2 we get, as in the proof of Lemma 4.4,

$$\begin{aligned}
&\int_{\mathcal{L}_k^n}[F,L]^aQ(L)^m\,\nu_k^n(\mathrm{d}L)\\
&=\frac{\omega_k}{2\omega_n}\int_{\mathcal{L}_{k-1}^{u^\perp}}\int_{-1}^1\int_{U^\perp\cap u^\perp\cap\mathbb{S}^{n-1}}|t|^{k-1}(1-t^2)^{\frac{n-k-2}{2}}\left([F,U]^{(u^\perp)}\right)^a|t|^a\\
&\quad\times\left(Q(U)+(tu+\sqrt{1-t^2}w)^2\right)^m\mathcal{H}^{n-k-1}(\mathrm{d}w)\,\mathrm{d}t\,\nu_{k-1}^{u^\perp}(\mathrm{d}U).
\end{aligned}$$

Now we expand the m-th power. Since the integrals of odd powers of w vanish, we get

$$\begin{aligned}
&\int_{\mathcal{L}_k^n}[F,L]^a Q(L)^m\,\nu_k^n(\mathrm{d}L)\\
&=\frac{\omega_k}{2\omega_n}\sum_{p=0}^{m}\int_{\mathcal{L}_{k-1}^{u^\perp}}\int_{-1}^{1}\int_{U^\perp\cap u^\perp\cap\mathbb{S}^{n-1}}|t|^{k+a-1}(1-t^2)^{\frac{n-k-2}{2}}\left([F,U]^{(u^\perp)}\right)^a\\
&\quad\times\binom{m}{p}Q(U)^{m-p}(tu+\sqrt{1-t^2}w)^{2p}\,\mathcal{H}^{n-k-1}(\mathrm{d}w)\,\mathrm{d}t\,\nu_{k-1}^{u^\perp}(\mathrm{d}U)\\
&=\frac{\omega_k}{2\omega_n}\sum_{p=0}^{m}\sum_{q=0}^{p}\int_{\mathcal{L}_{k-1}^{u^\perp}}\int_{-1}^{1}\int_{U^\perp\cap u^\perp\cap\mathbb{S}^{n-1}}\binom{m}{p}\binom{2p}{2q}|t|^{k+a-1}(1-t^2)^{\frac{n-k-2}{2}}\\
&\quad\times\left([F,U]^{(u^\perp)}\right)^a Q(U)^{m-p}t^{2p-2q}u^{2p-2q}(1-t^2)^q w^{2q}\,\mathcal{H}^{n-k-1}(\mathrm{d}w)\,\mathrm{d}t\,\nu_{k-1}^{u^\perp}(\mathrm{d}U)\\
&=\frac{\omega_k}{2\omega_n}\sum_{p=0}^{m}\sum_{q=0}^{p}\int_{\mathcal{L}_{k-1}^{u^\perp}}\int_{-1}^{1}\int_{U^\perp\cap u^\perp\cap\mathbb{S}^{n-1}}\binom{m}{p}\binom{2p}{2q}|t|^{k+a-1+2p-2q}(1-t^2)^{\frac{n-k}{2}-1+q}\\
&\quad\times u^{2p-2q}w^{2q}\left([F,U]^{(u^\perp)}\right)^a Q(U)^{m-p}\mathcal{H}^{n-k-1}(\mathrm{d}w)\,\mathrm{d}t\,\nu_{k-1}^{u^\perp}(\mathrm{d}U).
\end{aligned}$$

The integration with respect to w can be carried out, since

$$\int_{\mathbb{S}^{n-1}}w^{2q}\,\mathcal{H}^{n-1}(\mathrm{d}w)=2\frac{\omega_{2q+n}}{\omega_{2q+1}}Q^q.$$

Integrating also with respect to t, we arrive at

$$\begin{aligned}
&\int_{\mathcal{L}_k^n}[F,L]^a Q(L)^m\,\nu_k^n(\mathrm{d}L)\\
&=\frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2})}\sum_{p=0}^{m}\sum_{q=0}^{p}\frac{\Gamma(\frac{k+a}{2}+p-q)}{\Gamma(\frac{n+a}{2}+p)}\Gamma\left(q+\tfrac{1}{2}\right)\binom{m}{p}\binom{2p}{2q}\\
&\quad\times\int_{\mathcal{L}_{k-1}^{u^\perp}}\left([F,U]^{(u^\perp)}\right)^a Q(U)^{m-p}(Q(u^\perp)-Q(U))^q\,\nu_{k-1}^{u^\perp}(\mathrm{d}U)\,u^{2p-2q}.
\end{aligned}$$

Next we expand $(Q(u^\perp)-Q(U))^q$ and change the order of the summation with respect to q to get

$$\begin{aligned}
&\int_{\mathcal{L}_k^n}[F,L]^a Q(L)^m\,\nu_k^n(\mathrm{d}L)\\
&=\frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2})}\sum_{p=0}^{m}\sum_{q=0}^{p}\sum_{j=0}^{p-q}\frac{\Gamma(\frac{k+a}{2}+q)}{\Gamma(\frac{n+a}{2}+p)}\Gamma\left(p-q+\tfrac{1}{2}\right)\binom{m}{p}\binom{2p}{2q}\binom{p-q}{j}\\
&\quad\times(-1)^{p-q-j}\int_{\mathcal{L}_{k-1}^{u^\perp}}\left([F,U]^{(u^\perp)}\right)^a Q(U)^{m-q-j}\,\nu_{k-1}^{u^\perp}(\mathrm{d}U)\,Q(u^\perp)^j u^{2q}.
\end{aligned}$$

Since F has codimension $n-1-r$ with respect to $u^\perp$, we can apply the induction hypothesis. This yields

$$
\begin{aligned}
&\int_{\mathcal{L}_k^n} [F,L]^a Q(L)^m \,\nu_k^n(\mathrm{d}L)\\
&= \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2})} \sum_{p=0}^{m} \sum_{q=0}^{p} \sum_{j=0}^{p-q} \frac{\Gamma(\frac{k+a}{2}+q)}{\Gamma(\frac{n+a}{2}+p)} \Gamma\left(p-q+\tfrac{1}{2}\right) \binom{m}{p}\binom{2p}{2q}\binom{p-q}{j}\\
&\quad \times (-1)^{p-q-j} Q(u^\perp)^j u^{2q} \frac{A(n-1,k-1,r,a)\Gamma(\frac{n-1+a}{2})}{\Gamma(\frac{k-1+a}{2})\Gamma(\frac{n-1+a}{2}+m-q-j)} \sum_{i=0}^{m-q-j} \binom{m-q-j}{i}\\
&\quad \times \frac{\Gamma(\frac{k-1+a}{2}+m-q-j-i)\Gamma(\frac{n-k}{2}+i)\Gamma(\frac{a}{2}+1)\Gamma(\frac{r}{2})}{\Gamma(\frac{n-k}{2})\Gamma(\frac{a}{2}+1-i)\Gamma(\frac{r}{2}+i)}\\
&\quad \times (-1)^i Q(u^\perp)^{m-q-j-i} Q(F)^i.
\end{aligned}
$$

We change the order of summation and do some rearrangements to get

$$
\begin{aligned}
&\int_{\mathcal{L}_k^n} [F,L]^a Q(L)^m \,\nu_k^n(\mathrm{d}L)\\
&= \frac{A(n-1,k-1,r,a)}{\sqrt{\pi}} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n-1+a}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{k-1+a}{2})} \sum_{i=0}^{m} \frac{\Gamma(\frac{n-k}{2}+i)\Gamma(\frac{a}{2}+1)\Gamma(\frac{r}{2})}{\Gamma(\frac{n-k}{2})\Gamma(\frac{a}{2}+1-i)\Gamma(\frac{r}{2}+i)}\\
&\quad \times (-1)^i Q(F)^i \left(\sum_{q=0}^{m-i} u^{2q} Q(u^\perp)^{m-q-i} \Gamma\left(\tfrac{k+a}{2}+q\right) \left(\sum_{j=0}^{m-q-i} \binom{m-q-j}{i} \right.\right.\\
&\quad \times \frac{\Gamma(\frac{k-1+a}{2}+m-q-j-i)}{\Gamma(\frac{n-1+a}{2}+m-q-j)} \left(\sum_{p=j+q}^{m} \frac{\Gamma(p-q+\frac{1}{2})}{\Gamma(\frac{n+a}{2}+p)} \binom{m}{p}\binom{2p}{2q} \right.\\
&\quad \times \left.\left.\left. \binom{p-q}{j} (-1)^{p-q-j} \right)\right)\right).
\end{aligned}
$$

Then we first apply Lemma 6.1 to simplify the summation over p, and subsequently use Lemma 6.2 to simplify the summation over j. Thus we arrive at

$$
\begin{aligned}
&\int_{\mathcal{L}_k^n} [F,L]^a Q(L)^m \,\nu_k^n(\mathrm{d}L)\\
&= \frac{A(n-1,k-1,r,a)\Gamma(\frac{n}{2})\Gamma(\frac{a}{2}+1)\Gamma(\frac{r}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n+a}{2}+m)\Gamma(\frac{n-k}{2})} \sum_{i=0}^{m} \frac{\Gamma(\frac{n-k}{2}+i)}{\Gamma(\frac{a}{2}+1-i)\Gamma(\frac{r}{2}+i)}\\
&\quad \times (-1)^i Q(F)^i \binom{m}{i} \Gamma\left(\tfrac{k+a}{2}+m-i\right) \left(\sum_{q=0}^{m-i} u^{2q} Q(u^\perp)^{m-q-i} \binom{m-i}{q} \right).
\end{aligned}
$$

The expression in brackets is just $(Q(u^\perp) + u^2)^{m-i} = Q^{m-i}$. Since also

$$A(n,k,r,a) = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+a}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n+a}{2})} A(n-1,k-1,r,a),$$

the induction is finished. $\square$

Here we are mainly interested in the special case $a = 2$.

Corollary 4.6 *Let $k, r \in \{1, \dots, n\}$ with $k + r \geqslant n$, $F \in \mathcal{L}_r^n$ and $m \in \mathbb{N}_0$. Then*

$$\int_{\mathcal{L}_k^n} [F,L]^2 Q(L)^m \, \nu_k^n(\mathrm{d}L) = \frac{k!r!}{(k-n+r)!n!} \frac{\Gamma(\frac{n}{2}+1)\Gamma(\frac{k}{2}+m)}{\Gamma(\frac{n}{2}+m+1)\Gamma(\frac{k}{2}+1)} \times \left(\left(\tfrac{k}{2}+m\right) Q^m + \tfrac{m}{r}(k-n) Q^{m-1} Q(F) \right).$$

Proof. The proof follows from Lemma 4.4 and Proposition 4.5. $\square$

The mean value formula of Corollary 4.6 has to be refined further. We introduce the following constants, which are needed in the subsequent proposition and its proof:

$$\beta_{n,j,k} := \frac{(k-1)!(n+j-k)!}{\sqrt{\pi} j!(n-1)!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{k+1}{2})},$$

$$\gamma^{(1)}_{n,k,l,p,q} := \sum_{y=0}^{q} (-1)^{l+y} \binom{q}{y} \frac{\Gamma(\frac{k-1}{2}+l-p+y)}{\Gamma(\frac{n+1}{2}+l-p+y)} \left(\tfrac{k-1}{2}+l-p+y\right),$$

$$\gamma^{(2)}_{n,k,l,p,q} := \sum_{y=0}^{q} (-1)^{l+y} \binom{q}{y} \frac{\Gamma(\frac{k-1}{2}+l-p+y)}{\Gamma(\frac{n+1}{2}+l-p+y)} (l-p+y)$$

with $\gamma^{(2)}_{n,k,l,p,q} = 0$ if $l - p + q = 0$. Moreover, we define

$$\zeta^{(1)}_{n,j,k,s,z,m} := \sum_{l=\max\{0,m-z\}}^{m} \sum_{p=0}^{l} \sum_{q=\max\{0,z-m+p\}}^{\lfloor \frac{s}{2} \rfloor - m + p} (-1)^{m-p+q-z} \gamma^{(1)}_{n,k,l,p,q} \binom{m}{l} \binom{l}{p} \times \binom{s-2m+2p}{2q} \binom{l-p+q}{z-m+l} \frac{\Gamma(\frac{s+j}{2}-m+p-q+1)\Gamma(q+\frac{1}{2})}{\Gamma(\frac{s+n-k+j}{2}-m+p+1)}$$

and

$$\zeta^{(2)}_{n,j,k,s,z,m} := \sum_{l=\max\{0,m-z\}}^{m} \sum_{p=0}^{l} \sum_{q=\max\{0,z-m+p+1\}}^{\lfloor \frac{s}{2}\rfloor-m+p} (-1)^{m-p+q-z-1}\gamma^{(2)}_{n,k,l,p,q}\binom{m}{l}\binom{l}{p}$$

$$\times \binom{s-2m+2p}{2q}\binom{l-p+q-1}{z-m+l}\frac{\Gamma(\frac{s+j}{2}-m+p-q+1)\Gamma(q+\frac{1}{2})}{\Gamma(\frac{s+n-k+j}{2}-m+p+1)}.$$

Proposition 4.7 *Let $k, r \in \{1, \dots, n\}$ with $k + r \geq n$, and $m, s \in \mathbb{N}_0$ with $m \leq \lfloor \frac{s}{2} \rfloor$. Let $F \in \mathcal{L}^n_r$ and $u \in F^\perp$. Then, for $j \in \{0, \dots, k-1\}$,*

$$\int_{\mathcal{L}^n_k} Q(L^\perp)^m \pi_L(u)^{s-2m}\|p_L(u)\|^{j-k}[F, L]^2\, \nu^n_k(\mathrm{d}L)$$

$$= \beta_{n,j,k}\sum_{z=0}^{\lfloor \frac{s}{2}\rfloor} \zeta^{(1)}_{n,j,k,s,z,m} Q^z u^{s-2z} + \beta_{n,j,k}\frac{k-n}{n+j-k}\sum_{z=0}^{\lfloor \frac{s}{2}\rfloor-1} \zeta^{(2)}_{n,j,k,s,z,m} Q^z u^{s-2z-2} Q(F).$$

Proof. Corollary 4.2 yields

$$\begin{aligned}
&\int_{\mathcal{L}^n_k} Q(L)^l \pi_L(u)^{s-2m}\|p_L(u)\|^{j-k}[F, L]^2\, \nu^n_k(\mathrm{d}L)\\
&= \frac{\omega_k}{2\omega_n}\int_{\mathcal{L}^{u^\perp}_{k-1}}\int_{-1}^{1}\int_{U^\perp\cap u^\perp\cap\mathbb{S}^{n-1}} |t|^{k-1}(1-t^2)^{\frac{n-k-2}{2}} Q(\mathrm{lin}\{U, tu+\sqrt{1-t^2}w\})^l\\
&\quad\times \pi_{\mathrm{lin}\{U,tu+\sqrt{1-t^2}w\}}(u)^{s-2m}\|p_{\mathrm{lin}\{U,tu+\sqrt{1-t^2}w\}}(u)\|^{j-k}\\
&\quad\times \left[F, \mathrm{lin}\{U, tu+\sqrt{1-t^2}w\}\right]^2 \mathcal{H}^{n-k-1}(\mathrm{d}w)\,\mathrm{d}t\,\nu^{u^\perp}_{k-1}(\mathrm{d}U). \qquad (4.12)
\end{aligned}$$

For $-1 < t < 1$ and $t \neq 0$, we have

$$\begin{aligned}
\left[F, \mathrm{lin}\{U, tu+\sqrt{1-t^2}w\}\right] &= \left([F, U]^{(u^\perp)}\right)|t|,\\
\pi_{\mathrm{lin}\{U,tu+\sqrt{1-t^2}w\}}(u) &= |t|u + \sqrt{1-t^2}(\mathrm{sign}(t))w,\\
\mathrm{lin}\{U, tu+\sqrt{1-t^2}w\} &= \mathrm{lin}\{U, |t|u+\sqrt{1-t^2}(\mathrm{sign}(t))w\},\\
\|p_{\mathrm{lin}\{U,tu+\sqrt{1-t^2}w\}}(u)\| &= |\langle u, tu+\sqrt{1-t^2}w\rangle| = |t|.
\end{aligned}$$

Combining these relations with (4.12) and observing that the integration with respect to w is invariant under reflection in the origin, we deduce that

$$
\begin{aligned}
&\int_{\mathcal{L}_k^n} Q(L)^l \pi_L(u)^{s-2m} \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k^n(\mathrm{d}L) \\
&= \frac{\omega_k}{2\omega_n} \int_{\mathcal{L}_{k-1}^{u^\perp}} \int_{-1}^{1} \int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} |t|^{j+1} (1-t^2)^{\frac{n-k-2}{2}} ((|t|u + \sqrt{1-t^2}w)^2 + Q(U))^l \\
&\quad \times (|t|u + \sqrt{1-t^2}w)^{s-2m} \left([F,U]^{(u^\perp)}\right)^2 \mathcal{H}^{n-k-1}(\mathrm{d}w) \, \mathrm{d}t \, \nu_{k-1}^{u^\perp}(\mathrm{d}U).
\end{aligned}
$$

From this equation it is easy to check the required integrability assertions. Next we expand $((|t|u+\sqrt{1-t^2}w)^2+Q(U))^l$ and $(|t|u+\sqrt{1-t^2}w)^{s-2m}$, use the fact that the integration with respect to w yields zero for odd powers of w, and carry out the integration with respect to t and w. Thus we obtain

$$
\begin{aligned}
&\int_{\mathcal{L}_k^n} Q(L)^l \pi_L(u)^{s-2m} \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k^n(\mathrm{d}L) \\
&= \frac{\omega_k}{2\omega_n} \sum_{p=0}^{l} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor - m + p} \binom{l}{p} \binom{s-2m+2p}{2q} \\
&\quad \times \frac{\Gamma(\frac{s+j}{2} - m + p - q + 1) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{s+n-k+j}{2} - m + p + 1)} \frac{2\omega_{2q+n-k}}{\omega_{2q+1}} \\
&\quad \times u^{s-2m+2p-2q} \int_{\mathcal{L}_{k-1}^{u^\perp}} Q(U^\perp \cap u^\perp)^q Q(U)^{l-p} \left([F,U]^{(u^\perp)}\right)^2 \nu_{k-1}^{u^\perp}(\mathrm{d}U) \\
&= \sum_{p=0}^{l} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor - m + p} \binom{l}{p} \binom{s-2m+2p}{2q} \\
&\quad \times \frac{\Gamma(\frac{s+j}{2} - m + p - q + 1) \Gamma(q + \frac{1}{2}) \Gamma(\frac{n}{2})}{\sqrt{\pi}\,\Gamma(\frac{s+n-k+j}{2} - m + p + 1) \Gamma(\frac{k}{2})} u^{s-2m+2p-2q} \\
&\quad \times \sum_{y=0}^{q} (-1)^y \binom{q}{y} (Q-u^2)^{q-y} \int_{\mathcal{L}_{k-1}^{u^\perp}} Q(U)^{l-p+y} \left([F,U]^{(u^\perp)}\right)^2 \nu_{k-1}^{u^\perp}(\mathrm{d}U),
\end{aligned}
$$

since $Q(U^\perp \cap u^\perp) = (Q-u^2) - Q(U)$. In the next step, we apply Corollary 4.6 in

$u^\perp$ to the integral on the right-hand side. This yields

$$\begin{aligned}
&\int_{\mathcal{L}_k^n} Q(L)^l \pi_L(u)^{s-2m} \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k^n(\mathrm{d}L) \\
&= \sum_{p=0}^{l} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor - m + p} \sum_{y=0}^{q} (-1)^y \binom{l}{p} \binom{s-2m+2p}{2q} \binom{q}{y} \\
&\quad \times \frac{\Gamma(\frac{s+j}{2} - m + p - q + 1)\Gamma(q + \frac{1}{2})\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{s+n-k+j}{2} - m + p + 1)\Gamma(\frac{k}{2})} u^{s-2m+2p-2q} (Q - u^2)^{q-y} \\
&\quad \times \frac{(k-1)!(n+j-k)!}{j!(n-1)!} \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{k-1}{2} + l - p + y)}{\Gamma(\frac{n+1}{2} + l - p + y)\Gamma(\frac{k+1}{2})} \\
&\quad \times \Bigg(\left(\tfrac{k-1}{2} + l - p + y\right) (Q - u^2)^{l-p+y} \\
&\quad + \frac{(l-p+y)(k-n)}{n+j-k} (Q - u^2)^{l-p+y-1} Q(F) \Bigg).
\end{aligned}$$

Substituting the basic relation $Q(L^\perp) = Q - Q(L)$, we arrive at

$$\begin{aligned}
&\int_{\mathcal{L}_k^n} Q(L^\perp)^m \pi_L(u)^{s-2m} \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k^n(\mathrm{d}L) \\
&= \beta_{n,j,k} \sum_{l=0}^{m} \sum_{p=0}^{l} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor - m + p} \binom{m}{l} \binom{l}{p} \binom{s-2m+2p}{2q} \\
&\quad \times \frac{\Gamma(\frac{s+j}{2} - m + p - q + 1)\Gamma(q + \frac{1}{2})}{\Gamma(\frac{s+n-k+j}{2} - m + p + 1)} Q^{m-l} u^{s-2m+2p-2q} \\
&\quad \times \left(\gamma^{(1)}_{n,k,l,p,q} (Q - u^2)^{l-p+q} + \frac{k-n}{n+j-k} \gamma^{(2)}_{n,k,l,p,q} (Q - u^2)^{l-p+q-1} Q(F). \right)
\end{aligned}$$

Using the expansion

$$(Q - u^2)^{l-p+q} = \sum_{z=0}^{l-p+q} \binom{l-p+q}{z} Q^z (-1)^{l-p+q-z} u^{2l-2p+2q-2z}$$

and changing the bounds for the summation with respect to z, we get

$$\int_{\mathcal{L}_k^n} Q(L^\perp)^m \pi_L(u)^{s-2m} \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k^n(\mathrm{d}L)$$

$$
\begin{aligned}
&= \beta_{n,j,k} \sum_{l=0}^{m} \sum_{p=0}^{l} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor - m + p} \sum_{z=m-l}^{m-p+q} (-1)^{m-p+q-z} \gamma^{(1)}_{n,k,l,p,q} \binom{m}{l} \binom{l}{p} \binom{s-2m+2p}{2q} \\
&\quad \times \binom{l-p+q}{z-m+l} \frac{\Gamma(\frac{s+j}{2} - m + p - q + 1)\Gamma(q+\frac{1}{2})}{\Gamma(\frac{s+n-k+j}{2} - m + p + 1)} Q^z u^{s-2z} \\
&\quad + \beta_{n,j,k} \frac{k-n}{n+j-k} \sum_{l=0}^{m} \sum_{p=0}^{l} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor - m + p} \sum_{z=m-l}^{m-p+q-1} (-1)^{m-p+q-z-1} \gamma^{(2)}_{n,k,l,p,q} \\
&\quad \times \binom{m}{l} \binom{l}{p} \binom{s-2m+2p}{2q} \binom{l-p+q-1}{z-m+l} \\
&\quad \times \frac{\Gamma(\frac{s+j}{2} - m + p - q + 1)\Gamma(q+\frac{1}{2})}{\Gamma(\frac{s+n-k+j}{2} - m + p + 1)} Q^z u^{s-2z-2} Q(F).
\end{aligned}
$$

Now we change the order of summation (we first sum with respect to z) and thus obtain

$$
\begin{aligned}
&\int_{\mathcal{L}^n_k} Q(L^\perp)^m \pi_L(u)^{s-2m} \|p_L(u)\|^{j-k} [F,L]^2 \, \nu^n_k(\mathrm{d}L) \\
&= \beta_{n,j,k} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \left(\sum_{l=\max\{0,m-z\}}^{m} \sum_{p=0}^{l} \sum_{q=\max\{0,z-m+p\}}^{\lfloor \frac{s}{2} \rfloor - m + p} (-1)^{m-p+q-z} \gamma^{(1)}_{n,k,l,p,q} \binom{m}{l} \binom{l}{p} \right. \\
&\quad \left. \times \binom{s-2m+2p}{2q} \binom{l-p+q}{z-m+l} \frac{\Gamma(\frac{s+j}{2} - m + p - q + 1)\Gamma(q+\frac{1}{2})}{\Gamma(\frac{s+n-k+j}{2} - m + p + 1)} \right) Q^z u^{s-2z} \\
&\quad + \beta_{n,j,k} \frac{k-n}{n+j-k} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor - 1} \left(\sum_{l=\max\{0,m-z\}}^{m} \sum_{p=0}^{l} \sum_{q=\max\{0,z-m+p+1\}}^{\lfloor \frac{s}{2} \rfloor - m + p} (-1)^{m-p+q-z-1} \right. \\
&\quad \times \gamma^{(2)}_{n,k,l,p,q} \binom{m}{l} \binom{l}{p} \binom{s-2m+2p}{2q} \binom{l-p+q-1}{z-m+l} \\
&\quad \left. \times \frac{\Gamma(\frac{s+j}{2} - m + p - q + 1)}{\Gamma(\frac{s+n-k+j}{2} - m + p + 1)} \Gamma\left(q+\frac{1}{2}\right) \right) Q^z u^{s-2z-2} Q(F) \\
&= \beta_{n,j,k} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \zeta^{(1)}_{n,j,k,s,z,m} Q^z u^{s-2z} + \beta_{n,j,k} \frac{k-n}{n+j-k} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor - 1} \zeta^{(2)}_{n,j,k,s,z,m} Q^z u^{s-2z-2} Q(F),
\end{aligned}
$$

which completes the proof. $\square$

5 Crofton formulas

Now we are in a position to prove the theorems stated in Section 2, in particular, to express the mean values

$$\int_{\mathcal{E}_k^n} \Phi_{j,r,s}(K \cap E)\, \mu_k^n(\mathrm{d}E),$$

for $K \in \mathcal{K}^n$, $r, s \in \mathbb{N}_0$ and $0 \leqslant j \leqslant k \leqslant n-1$, in terms of basic tensor valuations.

5.1 The case $j = k$

For $L \in \mathcal{L}_k^n$ and $t \in L^\perp$ we put $L_t := L + t$. Let $t \in L^\perp$ be fixed. By Theorem 3.2 we have

$$\Phi_{k,r,s}(K \cap L_t) = \sum_{m \geqslant 0} \frac{Q(L^\perp)^m}{(4\pi)^m m!} \Phi_{k,r,s-2m}^{(L_t)}(K \cap L_t). \tag{5.13}$$

Since $\Phi_{k,r,s-2m}^{(L_t)}(K \cap L_t) = 0$ if $s - 2m \neq 0$, (5.13) shows that if s is odd, then $\Phi_{k,r,s}(K \cap L_t) = 0$, and if s is even, then

$$\Phi_{k,r,s}(K \cap L_t) = \frac{Q(L^\perp)^{\frac{s}{2}}}{(4\pi)^{\frac{s}{2}} (\frac{s}{2})! r!} \int_{K \cap L_t} x^r\, \mathcal{H}^k(\mathrm{d}x).$$

Hence, if s is even, we obtain

$$\begin{aligned}
&\int_{\mathcal{E}_k^n} \Phi_{k,r,s}(K \cap E)\, \mu_k^n(\mathrm{d}E) \\
&= \int_{\mathcal{L}_k^n} \int_{L^\perp} \Phi_{k,r,s}(K \cap L_t)\, \mathcal{H}^{n-k}(\mathrm{d}t)\, \nu_k^n(\mathrm{d}L) \\
&= \frac{1}{(4\pi)^{\frac{s}{2}} (\frac{s}{2})! r!} \int_{\mathcal{L}_k^n} Q(L^\perp)^{\frac{s}{2}} \int_{L^\perp} \int_{K \cap L_t} x^r\, \mathcal{H}^k(\mathrm{d}x)\, \mathcal{H}^{n-k}(\mathrm{d}t)\, \nu_k^n(\mathrm{d}L).
\end{aligned}$$

By Fubini's theorem we get

$$\begin{aligned}
\int_{\mathcal{E}_k^n} \Phi_{k,r,s}(K \cap E)\, \mu_k^n(\mathrm{d}E) &= \frac{1}{(4\pi)^{\frac{s}{2}} (\frac{s}{2})! r!} \int_{\mathcal{L}_k^n} Q(L^\perp)^{\frac{s}{2}} \int_K x^r\, \mathcal{H}^n(\mathrm{d}x)\, \nu_k^n(\mathrm{d}L) \\
&= \frac{1}{(4\pi)^{\frac{s}{2}} (\frac{s}{2})!} \Phi_{n,r,0}(K) \int_{\mathcal{L}_k^n} Q(L^\perp)^{\frac{s}{2}}\, \nu_k^n(\mathrm{d}L).
\end{aligned}$$

Since Lemma 4.3 yields (for $k \leq n-1$)

$$\int_{\mathcal{E}_k^n} Q(L^\perp)^{\frac{s}{2}} \nu_k^n(\mathrm{d}L) = \int_{\mathcal{E}_{n-k}^n} Q(U)^{\frac{s}{2}} \nu_{n-k}^n(\mathrm{d}U) = \frac{\Gamma\left(\frac{n-k+s}{2}\right) \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+s}{2}\right) \Gamma\left(\frac{n-k}{2}\right)} Q^{\frac{s}{2}},$$

we finally get (if s is even)

$$\int_{\mathcal{E}_k^n} \Phi_{k,r,s}(K \cap E)\, \mu_k^n(\mathrm{d}E) = \tilde{\alpha}_{n,k,s}\, Q^{\frac{s}{2}} \Phi_{n,r,0}(K),$$

where $\tilde{\alpha}_{n,k,s}$ is as stated in Theorem 2.1. This proves the Crofton formula for $j = k$.

5.2 The cases $0 \leqslant j \leqslant k-1$

Throughout the main part of this subsection we will assume that $P \in \mathcal{P}^n$ is a polytope. Once the Crofton formula has been established in this special case, the general result follows by approximation. Theorem 3.2 shows that, for fixed $L \in \mathcal{L}_k^n$ and $t \in L^\perp$,

$$\begin{aligned}
&\Phi_{j,r,s}(P \cap L_t)\\
&= \sum_{m \geqslant 0} \frac{Q(L^\perp)^m}{(4\pi)^m m!} \Phi^{(L_t)}_{j,r,s-2m}(P \cap L_t)\\
&= \sum_{m \geqslant 0} \alpha_{j,k,s,m} \frac{1}{r!} Q(L^\perp)^m \omega_{k-j} \int_{L_t \times (\mathbb{S}^{n-1} \cap L)} x^r u^{s-2m}\, \Lambda_j^{(L_t)}(P \cap L_t, \mathrm{d}(x,u)),
\end{aligned}$$

where

$$\alpha_{j,k,s,m} := \left((4\pi)^m m! (s-2m)! \omega_{k-j+s-2m}\right)^{-1}.$$

To proceed further, we need a translative Crofton formula for support measures, which is due to Rataj. By a special case of [17, Theorem 3.1], for any measurable bounded function $g : \mathbb{R}^n \times (\mathbb{S}^{n-1} \cap L) \to \mathbb{R}$ we have

$$\begin{aligned}
&\int_{L^\perp} \int_{L_t \times (\mathbb{S}^{n-1} \cap L)} g(x,v)\, \Lambda_j^{(L_t)}(P \cap L_t, \mathrm{d}(x,v))\, \mathcal{H}^{n-k}(\mathrm{d}t)\\
&= \frac{1}{\omega_{k-j}} \sum_{F \in \mathcal{F}^{n+j-k}(P)} \int_{F \times (N(P,F) \cap \mathbb{S}^{n-1})} g(x, \pi_L(u))\\
&\quad \times \|p_L(u)\|^{j-k} [F, L]^2\, \mathcal{H}^{n-1}(\mathrm{d}(x,u)).
\end{aligned}$$

For $u \in L^\perp$ the integrand on the right-hand side is interpreted as zero. Hence,

$$\begin{aligned}
&\int_{L^\perp} \Phi_{j,r,s}(P \cap L_t)\, \mathcal{H}^{n-k}(\mathrm{d}t)\\
&= \sum_{m \geqslant 0} \alpha_{j,k,s,m} \frac{1}{r!} Q(L^\perp)^m \sum_{F \in \mathcal{F}^{n+j-k}(P)} \int_F x^r\, \mathcal{H}^{n+j-k}(\mathrm{d}x) [F, L]^2\\
&\quad \times \int_{N(P,F) \cap \mathbb{S}^{n-1}} \pi_L(u)^{s-2m} \|p_L(u)\|^{j-k}\, \mathcal{H}^{k-j-1}(\mathrm{d}u).
\end{aligned}$$

Integrating this equation over $\mathcal{L}_k^n$ and applying Fubini's theorem (the required integrability is easy to check), we get

$$\begin{aligned}
&\int_{\mathcal{L}_k^n}\int_{L^\perp}\Phi_{j,r,s}(P\cap L_t)\,\mathcal{H}^{n-k}(\mathrm{d}t)\,\nu_k^n(\mathrm{d}L)\\
&=\sum_{m\geqslant 0}\sum_{F\in\mathcal{F}^{n+j-k}(P)}\alpha_{j,k,s,m}\frac{1}{r!}\int_F x^r\,\mathcal{H}^{n+j-k}(\mathrm{d}x)\int_{N(P,F)\cap\mathbb{S}^{n-1}}\\
&\quad\times\int_{\mathcal{L}_k^n}Q(L^\perp)^m\pi_L(u)^{s-2m}\|p_L(u)\|^{j-k}[F,L]^2\,\nu_k^n(\mathrm{d}L). \qquad (5.14)
\end{aligned}$$

Now we can apply Proposition 4.7 to (5.14). Using the notation introduced in Section 3, we obtain

$$\begin{aligned}
&\int_{\mathcal{L}_k^n}\int_{L^\perp}\Phi_{j,r,s}(P\cap L_t)\,\mathcal{H}^{n-k}(\mathrm{d}t)\,\nu_k^n(\mathrm{d}L)\\
&=\beta_{n,j,k}\sum_{z=0}^{\lfloor\frac{s}{2}\rfloor}\left(\sum_{m\geqslant 0}\alpha_{j,k,s,m}\zeta^{(1)}_{n,j,k,s,z,m}\right)Q^z\sum_{F\in\mathcal{F}^{n+j-k}(P)}\Upsilon_r(F)\\
&\quad\times\int_{N(P,F)\cap\mathbb{S}^{n-1}}u^{s-2z}\,\mathcal{H}^{k-j-1}(\mathrm{d}u)\\
&\quad+\beta_{n,j,k}\frac{k-n}{n+j-k}\sum_{z=0}^{\lfloor\frac{s}{2}\rfloor-1}\left(\sum_{m\geqslant 0}\alpha_{j,k,s,m}\zeta^{(2)}_{n,j,k,s,z,m}\right)Q^z\sum_{F\in\mathcal{F}^{n+j-k}(P)}\Upsilon_r(F)\\
&\quad\times\int_{N(P,F)\cap\mathbb{S}^{n-1}}u^{s-2z-2}Q(F)\,\mathcal{H}^{k-j-1}(\mathrm{d}u).
\end{aligned}$$

The integrations over $N(P,F)\cap\mathbb{S}^{n-1}$ can be transformed into expressions of the form $\Theta_{s-2z}(P,F)$ and $\Theta_{s-2z-2}(P,F)$, respectively, by introducing polar coordinates. Using also the comments accompanying Lemma 3.3, we obtain

$$\begin{aligned}
&\int_{\mathcal{L}_k^n}\int_{L^\perp}\Phi_{j,r,s}(P\cap L_t)\,\mathcal{H}^{n-k}(\mathrm{d}t)\,\nu_k^n(\mathrm{d}L)\\
&=\beta_{n,j,k}\sum_{z=0}^{\lfloor\frac{s}{2}\rfloor}\xi^{(1)}_{n,j,k,s,z}(s-2z)!\omega_{s-2z-j+k}Q^z\Phi_{n+j-k,r,s-2z}(P)\\
&\quad+\beta_{n,j,k}\frac{k-n}{n+j-k}\sum_{z=0}^{\lfloor\frac{s}{2}\rfloor-1}\xi^{(2)}_{n,j,k,s,z}(s-2z-2)!\omega_{s-2z-2-j+k}\\
&\quad\times Q^z\sum_{F\in\mathcal{F}^{n+j-k}(P)}Q(F)\Upsilon_r(F)\Theta_{s-2z-2}(P,F), \qquad (5.15)
\end{aligned}$$

where

$$\xi^{(i)}_{n,j,k,s,z}:=\sum_{m\geqslant 0}\alpha_{j,k,s,m}\zeta^{(i)}_{n,j,k,s,z,m}\qquad (i=1,2).$$

The remaining sum

$$\sum_{F\in\mathcal{F}^{n+j-k}(P)} Q(F)\Upsilon_r(F)\Theta_{s-2z-2}(P,F)$$

will now be rewritten in terms of the basic tensor valuations. In the present situation, using the relation $Q = Q(F)+Q(N(P,F))$, we can express Lemma 3.3 in the form

$$\begin{aligned}
&\sum_{F\in\mathcal{F}^{n+j-k}(P)} Q(F)\Upsilon_r(F)\Theta_{s-2z-2}(P,F)\\
&= Q\Phi_{n+j-k,r,s-2z-2}(P) - 2\pi(s-2z)\Phi_{n+j-k,r,s-2z}(P)\\
&\quad + \sum_{G\in\mathcal{F}^{n+j-k+1}(P)} Q(G)\Upsilon_{r-1}(G)\Theta_{s-2z-1}(P,G).
\end{aligned}$$

Finitely many iterations of this formula finally lead to

$$\begin{aligned}
&\sum_{F\in\mathcal{F}^{n+j-k}(P)} Q(F)\Upsilon_r(F)\Theta_{s-2z-2}(P,F)\\
&= \sum_{l\geqslant s-2z} Q\Phi_{n+j-k-s+2z+l,r+s-2z-l,l-2}(P)\\
&\quad - 2\pi \sum_{l\geqslant s-2z} l\Phi_{n+j-k-s+2z+l,r+s-2z-l,l}(P).
\end{aligned}$$

By McMullen's relations (1.5) the right-hand side can be rewritten, which leads to

$$\begin{aligned}
&\sum_{F\in\mathcal{F}^{n+j-k}(P)} Q(F)\Upsilon_r(F)\Theta_{s-2z-2}(P,F)\\
&= 2\pi \sum_{l=0}^{s-2z-1} l\Phi_{n+j-k-s+2z+l,r+s-2z-l,l}(P)\\
&\quad - \sum_{l=0}^{s-2z-1} Q\Phi_{n+j-k-s+2z+l,r+s-2z-l,l-2}(P).
\end{aligned}$$

Substituting either of these relation into (5.15) and introducing the abbreviations

$$\chi^{(1)}_{n,j,k,s,z} := \beta_{n,j,k}\xi^{(1)}_{n,j,k,s,z}(s-2z)!\omega_{k-j+s-2z}, \tag{5.16}$$

$$\chi^{(2)}_{n,j,k,s,z} := \beta_{n,j,k}\frac{k-n}{n+j-k}\xi^{(2)}_{n,j,k,s,z}(s-2z-2)!\omega_{k-j+s-2z-2}, \tag{5.17}$$

we complete the proof of Theorems 2.2 and 2.3 in the case of polytopes. □

Theorems 2.4 to 2.6 are obtained in the same way (with $m = 0$); the appearing constants turn out to be

$$\bar{\chi}^{(1)}_{n,j,k,s,z} := \beta_{n,j,k}\alpha_{j,k,s,0}\zeta^{(1)}_{n,j,k,s,z,0}(s-2z)!\omega_{k-j+s-2z}, \tag{5.18}$$

$$\bar{\chi}^{(2)}_{n,j,k,s,z} := \beta_{n,j,k}\frac{k-n}{n+j-k}\alpha_{j,k,s,0}\zeta^{(2)}_{n,j,k,s,z,0}(s-2z-2)!\omega_{k-j+s-2z-2}. \tag{5.19}$$

6 Appendix: Identities obtained by Zeilberger's algorithm

In this section, we prove three identities for sums involving binomial coefficients and Gamma functions, which were needed in Section 4. They were found by using Zeilberger's algorithm and the available software, as described in [16]. Below we write down, in each case, the function G and the recurrence produced by the algorithm. Once this recurrence is there, it can be verified directly by elementary calculations, and it immediately yields the proof of the identity.

Lemma 6.1 *For $n \in \mathbb{N}$, $m, q, j \in \mathbb{N}_0$ with $q + j \leqslant m$ and $a \geqslant 0$,*

$$\sum_p (-1)^{p-q-j}\frac{\Gamma(p-q+\frac{1}{2})}{\Gamma(\frac{n+a}{2}+p)}\binom{m}{p}\binom{2p}{2q}\binom{p-q}{j}$$
$$= \binom{m}{q+j}\binom{2j+2q}{2q}\frac{\Gamma(j+\frac{1}{2})\Gamma(\frac{n-1+a}{2}+m-q-j)}{\Gamma(\frac{n-1+a}{2})\Gamma(\frac{n+a}{2}+m)}.$$

Proof. A continuity argument shows that we can restrict ourselves to the case where $n + a \neq 1$. For $m, p \in \mathbb{N}_0$ we define

$$F(m,p) := (-1)^{p-q-j}\frac{\Gamma(p-q+\frac{1}{2})\Gamma(\frac{n+a}{2}+m)\binom{m}{p}\binom{2p}{2q}\binom{p-q}{j}}{\Gamma(\frac{n+a}{2}+p)\Gamma(\frac{n-1+a}{2}+m-q-j)\binom{m}{q+j}}.$$

Note that $F(m,p) \neq 0$ only for $q + j \leqslant p \leqslant m$. Thus we have to show that

$$f(m) := \sum_{p=q+j}^{m} F(m,p) = \binom{2j+2q}{2q}\frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{n-1+a}{2})}.$$

Zeilberger's algorithm provides a function G defined by

$$G(m,p) := \frac{(p-q-j)(n+a+2p-2)}{(n+a-2q-2j+2m-1)(p-m-1)}F(m,p)$$

for $p \in \{q+j, \ldots, m\}$, by

$$G(m, m+1) := F(m+1, m) + G(m,m) - F(m,m)$$

and by $G(m,p) := 0$ in all other cases. The twofold recurrence relation

$$F(m+1,p) - F(m,p) = G(m,p+1) - G(m,p) \tag{6.20}$$

is satisfied; this can be verified by a direct calculation. Summation of (6.20) over p from $q+j$ to $m+1$ yields

$$f(m+1)-f(m)=G(m,m+2)-G(m,q+j)=0-0=0.$$

Thus we obtain $f(m+1)=f(m)$ and therefore

$$f(m)=f(q+j)=F(q+j,q+j)=\binom{2j+2q}{2q}\frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{n-1+a}{2})},$$

which completes the proof. □

Lemma 6.2 *Let $m\in\mathbb{N}_0$, $i\in\{0,\dots,m\}$, $q\in\{0,\dots,m-i\}$, $k\in\mathbb{N}$ and $a\geqslant 0$ with $k+a\neq 1$. Then*

$$\sum_j \binom{m-q-j}{i}\binom{m}{q+j}\binom{2j+2q}{2q}\Gamma\left(\tfrac{k-1+a}{2}+m-q-j-i\right)\Gamma\left(j+\frac{1}{2}\right)$$
$$=\sqrt{\pi}\binom{m}{i}\binom{m-i}{q}\frac{\Gamma(\frac{k-1+a}{2})\Gamma(\frac{k+a}{2}+m-i)}{\Gamma(\frac{k+a}{2}+q)}.$$

Proof. For $\bar{m},p\in\mathbb{N}_0$ we define

$$F(\bar{m},j):=\binom{\bar{m}+i-j}{i}\binom{\bar{m}+q+i}{q+j}\binom{2j+2q}{2q}\frac{\Gamma(\frac{k-1+a}{2}+\bar{m}-j)\Gamma(j+\frac{1}{2})}{\binom{\bar{m}+q+i}{i}\binom{\bar{m}+q}{q}\Gamma(\frac{k+a}{2}+\bar{m}+q)}.$$

Note that $F(\bar{m},j)\neq 0$ only for $j\in\{0,\dots,\bar{m}\}$. With $\bar{m}:=m-q-i$, the assertion of the lemma is equivalent to

$$f(\bar{m}):=\sum_{j=0}^{\bar{m}}F(\bar{m},j)=\sqrt{\pi}\frac{\Gamma(\frac{k-1+a}{2})}{\Gamma(\frac{k+a}{2}+q)}.$$

Zeilberger's algorithm yields a function G defined by

$$G(\bar{m},j):=\frac{(-j)(k-1+a+2\bar{m}-2j)}{(k+a+2\bar{m}+2q)(\bar{m}+1-j)}F(\bar{m},j)$$

for $j\in\{0,\dots,\bar{m}\}$, by

$$G(\bar{m},\bar{m}+1):=F(\bar{m}+1,\bar{m})+G(\bar{m},\bar{m})-F(\bar{m},\bar{m})$$

and by $G(\bar{m},j):=0$ in all other cases. With these definitions, we have

$$F(\bar{m}+1,j)-F(\bar{m},j)=G(\bar{m},j+1)-G(\bar{m},j).$$

This can be verified by a direct calculation. Summation over $j=0,\dots,\bar{m}+1$ then yields

$$f(\bar{m}+1)-f(\bar{m})=G(\bar{m},\bar{m}+2)-G(\bar{m},0)=0-0=0$$

for all $\bar{m} \geqslant 0$. Hence we get

$$f(\bar{m}) = f(0) = F(0,0) = \frac{\Gamma(\frac{k-1+a}{2})\Gamma(\frac{1}{2})\binom{i+q}{q}}{\Gamma(\frac{k+a}{2}+q)\binom{i+q}{i}} = \sqrt{\pi}\frac{\Gamma(\frac{k-1+a}{2})}{\Gamma(\frac{k+a}{2}+q)},$$

as required. $\square$

Lemma 6.3 *For $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $k \in \{1,\dots,n-1\}$ and $a \geqslant 0$,*

$$\sum_i (-1)^i \binom{m}{i} \Gamma\left(\tfrac{k+a}{2} + m - i\right) \frac{\Gamma(\frac{n-k}{2}+i)\Gamma(\frac{a}{2}+1)\Gamma(\frac{n}{2})}{\Gamma(\frac{n-k}{2})\Gamma(\frac{a}{2}+1-i)\Gamma(\frac{n}{2}+i)}$$
$$= \frac{\Gamma(\frac{k}{2}+m)\Gamma(\frac{n+a}{2}+m)\Gamma(\frac{n}{2})\Gamma(\frac{k+a}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n+a}{2})\Gamma(\frac{n}{2}+m)}.$$

Proof For $m, i \in \mathbb{N}_0$ we define

$$F(m,i) := (-1)^i \frac{\binom{m}{i}\Gamma\left(\frac{k+a}{2}+m-i\right)\Gamma(\frac{n-k}{2}+i)\Gamma(\frac{a}{2}+1)\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}+m)}{\Gamma(\frac{n-k}{2})\Gamma(\frac{a}{2}+1-i)\Gamma(\frac{n}{2}+i)\Gamma(\frac{k}{2}+m)\Gamma(\frac{n+a}{2}+m)},$$

then $F(m,i) \neq 0$ only for $i \in \{0,\dots,m\}$. We have to show that

$$f(m) := \sum_{i=0}^{m} F(m,i) = \frac{\Gamma(\frac{k+a}{2})\Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n+a}{2})}.$$

Zeilberger's algorithm yields the function defined by

$$G(m,i) := -\frac{(n+2i-2)(k+a+2m-2i)i}{(n+a+2m)(k+2m)(m+1-i)}F(m,i)$$

for $i \in \{0,\dots,m\}$, by

$$G(m,m+1) := F(m+1,m) + G(m,m) - F(m,m)$$

and by $G(m,i) := 0$ in all other cases. Then the relation

$$F(m+1,i) - F(m,i) = G(m,i+1) - G(m,i)$$

is satisfied. Again this can be checked directly. Summation over $i = 0,\dots,m+1$ shows that

$$f(m+1) - f(m) = G(m,m+2) - G(m,0) = 0 - 0 = 0$$

for all $m \geqslant 0$. Hence

$$f(m) = f(0) = F(0,0) = \frac{\Gamma(\frac{k+a}{2})\Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n+a}{2})},$$

which proves the lemma. $\square$

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