The mean width of random polytopes circumscribed around a convex body

Károly J. Böröczky, Ferenc Fodor and Daniel Hug

ABSTRACT

Let K be a d-dimensional convex body, and let $K^{(n)}$ be the intersection of n halfspaces containing K whose bounding hyperplanes are independent and identically distributed. Under suitable distributional assumptions, we prove an asymptotic formula for the expectation of the difference of the mean widths of $K^{(n)}$ and K, and another asymptotic formula for the expectation of the number of facets of $K^{(n)}$. These results are achieved by establishing an asymptotic result on weighted volume approximation of K and by "dualizing" it using polarity.

1. *Introduction*

Let K be a convex body (compact convex set with nonempty interior) in d-dimensional Euclidean space $\mathbb{R}^d$. The convex hull $K_{(n)}$ of n independent random points in K chosen according to the uniform distribution is a common model of a random polytope contained in K. The famous four-point problem of Sylvester [**40**] is the starting point of an extensive investigation of random polytopes of this type. Beside specific probabilities as in Sylvester's problem, important objects of study are expectations, variances and distributions of various geometric functionals associated with $K_{(n)}$. Typical examples of such functionals are volume, other intrinsic volumes, and the number of i-dimensional faces. In their ground-breaking papers [**30**] and [**31**], Rényi and Sulanke considered random polytopes in the Euclidean plane and proved asymptotic results for the expectations of basic functionals of random polytopes in a convex domain K in the cases where K is sufficiently smooth or a convex polygon. Since then most results have been in the form of asymptotic formulae as the number n of random points tends to infinity.

In the last three decades, much effort has been devoted to exploring the properties of this particular model of a random polytope contained in a d-dimensional convex body K. For instance, for a sufficiently smooth convex body K, asymptotic formulae were proved for the expectation of the mean width difference $W(K) - W(K_{(n)})$ by Schneider and Wieacker [**37**], and for the volume difference $V(K) - V(K_{(n)})$ by Bárány [**1**]. The assumption of smoothness was relaxed in the case of the mean width by Böröczky, Fodor, Reitzner and Vígh [**6**], and removed by Schütt [**38**] in the case of the volume, general intrinsic volumes are treated in [**7**] under a weak smoothness assumption. Recently, even variance estimates, laws of large numbers, and central limit theorems have been proved in different settings in a sequence of contributions, for instance by Bárány, Reitzner, and Vu [**3**], [**4**], [**28**], [**29**], [**41**],

---

2000 *Mathematics Subject Classification* 52A22 (primary), 60D05, 52A27 (secondary)..

The first author was supported by OTKA grants 068398 and 049301 and by the EU Marie Curie projects BudAlgGeo (MTKD-CT-2004-002988) and DiscConvGeo (MTKD-CT-2005-014333), and by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

The second author was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences and by OTKA grant 068398.

The third author was supported by the European Network PHD, FP6 Marie Curie Actions, RTN, Contract MCRN-CT-2004-511953.

[**42**]. For more details on the current state-of-the-art of this line of research, see the survey papers by Weil and Wieacker [**43**], Gruber [**15**] and Schneider [**35**], and the recent monograph of Schneider and Weil [**36**].

In a third paper, Rényi and Sulanke [**32**] suggested a model which is "dual" to the model of a random polytope contained in a given convex body K (a random inscribed polytope), that is, they considered a random polytope containing a given convex body (a random circumscribed polytope). Subsequently, this approach has not received as much attention as the "inscribed case", although it is closely related to linear optimization (cf. [**10**], [**33**, §6]). There are various ways of producing circumscribed random polytopes containing a given convex body. In this paper, we consider a model in which the circumscribed polytope arises as an intersection of closed halfspaces whose bounding hyperplanes are randomly chosen hyperplanes. The rough description of the probability model is the following, it is described more precisely in Section 2, a more general setting is provided in Section 5. In Euclidean space $\mathbb{R}^d$, we consider hyperplanes that intersect the radius one parallel domain of a given convex body K but miss the interior of K, and we use the restriction of the (suitably normalized) Haar measure on the set of hyperplanes in $\mathbb{R}^d$ to provide an associated probability measure. For n independent random hyperplanes $H_1, \dots, H_n$ chosen according to this distribution, the intersection of the closed halfspaces bounded by $H_1, \dots, H_n$ and containing K determines a circumscribed random polyhedral set containing K (which might be unbounded). The main goal of this article is to find asymptotic formulae for the expectation of the difference of the mean widths of a random circumscribed polytope and the given convex body K, and for the expectation of the number of facets of a circumscribed random polyhedral set. These (and more general) results will be achieved by establishing general results on weighted volume approximation of a given convex body by inscribed random polytopes. In all these results, no regularity or curvature assumptions on K are requird.

As for earlier results, we mention the paper [**47**] by Ziezold who investigated circumscribed polygons in the plane, and the doctoral dissertation [**21**] of Kaltenbach who proved asymptotic formulae for the expectation of the volume difference and for the expectation of the number of vertices of circumscribed random polytopes around a convex body, under the assumption that the boundary of the reference body K is sufficiently smooth. Recently, Böröczky and Schneider [**9**] established upper and lower bounds for the expectation of the mean width difference for a general convex body K. Furthermore, they also proved asymptotic formulae for the expected number of vertices and facets of $K^{(n)}$, and an asymptotic formula for the expectation of the mean width difference, under the assumption that the reference body K is a simplicial polytope with r facets.

In [**8**], Böröczky and Reitzner discuss a different model of a random circumscribed polytope where n independent random points are chosen from the boundary of a given smooth convex body K, and the intersection of the supporting halfspaces of K at these points is the random polyhedral set under consideration. This framework is again dual to the one considered by Schütt and Werner (see [**36**]) who study the expected volume of the convex hull of n independent random points chosen from the boundary of a convex body satisfying a weak regularity assumption.

2. *The probability space and the main goal*

Let us first describe the setting for stating our results on circumscribed random polyhedral sets. Throughout this article, K will denote a compact convex set with interior points (a convex body) in d-dimensional Euclidean space $\mathbb{R}^d$ ($d \geq 2$). We write $\langle \cdot , \cdot \rangle$ for the scalar product and $\| \cdot \|$ for the norm in $\mathbb{R}^d$. For background on convexity, we refer to the monographs by Schneider [**34**] or by Gruber [**16**]. Let V denote volume, and let $\mathcal{H}^j$ denote the j-dimensional Hausdorff measure. The unit ball of $\mathbb{R}^d$ with center at the origin o is denoted by B^d, and S^{d-1} is its boundary. We put $\alpha_d := V(B^d)$ and $\omega_d := \mathcal{H}^{d-1}(S^{d-1}) = d\alpha_d$. The parallel body of K of radius 1 is $K_1 := K + B^d$. Let $\mathcal{H}$ denote the space (with its usual topology) of hyperplanes in $\mathbb{R}^d$, and let $\mathcal{H}_K$ be the subspace of hyperplanes meeting K_1 but not the interior of K. For $H \in \mathcal{H}_K$, the closed halfspace bounded by H that contains

K is denoted by H^-. Let μ denote the motion invariant Borel measure on $\mathcal{H}$, normalized so that $\mu(\{H \in \mathcal{H} : H \cap M \neq \emptyset\})$ is the mean width $W(M)$ of M, for every convex body $M \subset \mathbb{R}^d$. Let $2\mu_K$ be the restriction of μ to $\mathcal{H}_K$. Since $\mu(\mathcal{H}_K) = W(K + B^d) - W(K) = W(B^d) = 2$, the measure μ_K is a probability measure. For $n \in \mathbb{N}$, let $H_1, \dots, H_n$ be independent random hyperplanes in $\mathbb{R}^d$, i.e. independent $\mathcal{H}$-valued random variables on some probability space $(\Omega, \mathbf{A}, \mathbb{P})$, each with distribution μ_K. The possibly unbounded intersection

$$K^{(n)} := \bigcap_{i=1}^{n} H_i^-$$

of the halfspaces H_i^-, with $H_i \in \mathcal{H}_K$ for $i = 1, \dots, n$, is a random polyhedral set. A major aim of the present work is to investigate $\mathbb{E}W(K^{(n)} \cap K_1)$, where $\mathbb{E}$ denotes mathematical expectation. The intersection with K_1 is considered, since $K^{(n)}$ is unbounded with positive probability. Instead of $\mathbb{E}W(K^{(n)} \cap K_1)$, we could consider $\mathbb{E}_1 W(K^{(n)})$, the conditional expectation of $W(K^{(n)})$ under the condition that $K^{(n)} \subset K_1$. Since $\mathbb{E}W(K^{(n)} \cap K_1) = \mathbb{E}_1 W(K^{(n)}) + O(\gamma^n)$ with $\gamma \in (0, 1)$ (cf. [**9**]), there is no difference in the asymptotic behaviors of both quantities, as $n \to \infty$. We also remark that, for the asymptotic results, the parallel body K_1 could be replaced by any other convex body containing K in its interior; this would only affect some normalization constants.

Let ∂K denote the boundary of K. We call ∂K twice differentiable in the generalized sense at a boundary point $x \in \partial K$ if there exists a quadratic form Q on $\mathbb{R}^{d-1}$, the second fundamental form of K at x, with the following property: If K is positioned in such a way that $x = o$ and $\mathbb{R}^{d-1}$ is a support hyperplane of K at o, then in a neighborhood of o, ∂K is the graph of a convex function f defined on a $(d-1)$-dimensional ball around o in $\mathbb{R}^{d-1}$ satisfying

$$f(z) = \tfrac{1}{2}\, Q(z) + o(\|z\|^2), \tag{2.1}$$

as $z \to o$. Alternatively, we call x a normal boundary point of K. If this is the case, we write $\kappa(x) = \det(Q)$ to denote the generalized Gaussian curvature of K at x. Writing $\kappa(x)$, we always assume that ∂K is twice differentiable in the generalized sense at $x \in \partial K$. According to a classical result of Alexandrov (see [**34**], [**16**]), ∂K is twice differentiable in the generalized sense almost everywhere with respect to the boundary measure of K ($\mathcal{H}^{d-1}$-almost all boundary points are normal boundary points). Finally, we define the constant

$$c_d = \frac{(d^2+d+2)(d^2+1)}{2(d+3)\cdot(d+1)!}\Gamma\left(\frac{d^2+1}{d+1}\right)\left(\frac{d+1}{\alpha_{d-1}}\right)^{2/(d+1)} \tag{2.2}$$

(cf. J.A. Wieacker) [**46**], which will appear in the statements of our main results. In the following, we simply write dx instead of $\mathcal{H}^d(dx)$.

The main asymptotic result concerning the expected difference of the mean widths of $K^{(n)}$ and K is the following theorem. Generalizations of Theorem 2.1, and also of Theorem 2.2 below, which hold under more general distributional assumptions, are provided in Section 5. There we also indicate the connection to the p-affine surface area of a convex body.

THEOREM 2.1. *If K is a convex body in $\mathbb{R}^d$, then*

$$\lim_{n\to\infty} n^{\frac{2}{d+1}}\, \mathbb{E}(W(K^{(n)} \cap K_1) - W(K)) = 2\, c_d\, \omega_d^{-\frac{d-1}{d+1}} \int_{\partial K} \kappa(x)^{\frac{d}{d+1}}\, \mathcal{H}^{d-1}(dx).$$

Let $f_i(P)$, $i \in \{0, \dots, d-1\}$, denote the number of i-dimensional faces of a polyhedral set P. In the statement of the following theorem, $K^{(n)}$ could be replaced by the intersection of $K^{(n)}$ with a fixed polytope containing K in its interior without changing the right-hand side. Alternatively, instead of $\mathbb{E}(f_{d-1}(K^{(n)}))$ we could consider the conditional expectation of $f_{d-1}(K^{(n)})$ under the assumption that $K^{(n)}$ is contained in K_1.

THEOREM 2.2. *If K is a convex body in $\mathbb{R}^d$, then*

$$\lim_{n\to\infty} n^{-\frac{d-1}{d+1}}\,\mathbb{E}(f_{d-1}(K^{(n)})) = c_d\,\omega_d^{-\frac{d-1}{d+1}}\int_{\partial K}\kappa(x)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(dx).$$

Both theorems will be deduced from a "dual" result on weighted volume approximation of convex bodies by inscribed random polytopes which is stated in the subsequent section. The usefulness of duality in random or best approximation has previously been observed e.g. in [**21**], [**14**], [**11**].

3. *Weighted volume approximation by inscribed polytopes*

For a given convex body, we introduce a class of inscribed random polytopes. Let C be a convex body in $\mathbb{R}^d$, let ϱ be a bounded, nonnegative, measurable function on C, and let $\mathcal{H}^d\llcorner C$ denote the restriction of $\mathcal{H}^d$ to C. Assuming that $\int_C \varrho(x)\,\mathcal{H}^d(dx) > 0$, we choose random points from C according to the probability measure

$$\mathbb{P}_{\varrho,C} := \left(\int_C \varrho(x)\,dx\right)^{-1}\varrho\,\mathcal{H}^d\llcorner C.$$

Expectation with respect to $\mathbb{P}_{\varrho,C}$ is denoted by $\mathbb{E}_{\varrho,C}$. The convex hull of n independent and identically distributed random points with distribution $\mathbb{P}_{\varrho,C}$ is denoted by $C_{(n)}$ if ϱ is clear from the context. This yields a general model of an inscribed random polytope.

Generalizing a result by C. Schütt [**38**], we prove the following theorem.

THEOREM 3.1. *For a convex body K in $\mathbb{R}^d$, a probability density function ϱ on K, and an integrable function $\lambda : K \to \mathbb{R}$ such that, on a neighborhood of ∂K relative to K, λ and ϱ are continuous and ϱ is positive,*

$$\lim_{n\to\infty} n^{\frac{2}{d+1}}\,\mathbb{E}_{\varrho,K}\int_{K\setminus K_{(n)}}\lambda(x)\,dx = c_d\int_{\partial K}\varrho(x)^{\frac{-2}{d+1}}\lambda(x)\kappa(x)^{\frac{1}{d+1}}\,\mathcal{H}^{d-1}(dx) \qquad (3.1)$$

where c_d is defined in (2.2).

The limit on the right-hand side of (3.1) depends only on the values of ϱ and λ on the boundary of K. In particular, we may prescribe any continuous, positive function ϱ on ∂K. Then any continuous extension of ϱ to a probability density on K (there always exists such an extension) will satisfy Theorem 3.1 with the prescribed values of ϱ on the right-hand side.

Our proof of Theorem 3.1 is inspired by the argument in C. Schütt [**38**] who considered the special case $\varrho \equiv \lambda \equiv 1$. We note that for Lemma 2 in [**38**], which is crucial for the proof in [**38**], no explicit proof is provided, but reference is given to an analogous result in an unpublished note by M. Schmuckenschläger. Besides a missing factor $\frac{1}{2}$, Lemma 2 does not hold in the generality stated in [**38**]. For instance, it is not true for simplices. Most probably, this gap can be overcome, but still our approach to prove Theorem 3.1, where Lemma 2 in [**38**] is replaced by the elementary Lemma 4.2, might be of some interest.

The present partially new approach to Theorem 3.1 involves also some other interesting new features. In particular, we do not need the concept of a Macbeath region. An outline of the proof is given below. It should also be emphasized that the generality of Theorem 3.1 is needed for our study of circumscribed random polyhedral sets via duality.

A classical argument going back to Efron shows that

$$\mathbb{E}_{\varrho,K}\left(f_0(K_{(n)})\right) = n\cdot\mathbb{E}_{\varrho,K}\int_{K\setminus K_{(n-1)}}\varrho(x)\,dx,$$

which yields the following consequence of Theorem 3.1.

COROLLARY 3.2. *For a convex body K in $\mathbb{R}^d$, and for a probability density function ϱ on K which is continuous and positive in a neighborhood of ∂K relative to K,*

$$\lim_{n\to\infty} n^{-\frac{d-1}{d+1}}\,\mathbb{E}_{\varrho,K}(f_0(K_{(n)})) = c_d \int_{\partial K} \varrho(x)^{\frac{d-1}{d+1}}\,\kappa(x)^{\frac{1}{d+1}}\,\mathcal{H}^{d-1}(dx)$$

where c_d is defined in (2.2).

The proof of Theorem 3.1 is obtained through the following intermediate steps. Details are provided in Section 4. Since the convex body K is fixed, we write $\mathbb{E}_\varrho$ and $\mathbb{P}_\varrho$ instead of $\mathbb{E}_{\varrho,K}$ and $\mathbb{P}_{\varrho,K}$, respectively. The basic observation to prove Theorem 3.1 is that

$$\mathbb{E}_\varrho \int_{K\setminus K_{(n)}} \lambda(x)\,dx = \int_K \mathbb{P}_\varrho\left(x \notin K_{(n)}\right)\lambda(x)\,dx, \tag{3.2}$$

which is an immediate consequence of Fubini's theorem. Throughout the proof, we may assume that $o \in \operatorname{int}(K)$. The asymptotic behavior, as $n \to \infty$, of the right-hand side of (3.2) is determined by points $x \in K$ which are sufficiently close to the boundary of K. In order to give this statement a precise meaning, scaled copies of K are introduced as follows. For $t \in (0,1)$, we define $K_t := (1-t)K$ and $y_t := (1-t)y$ for $y \in \partial K$. In Lemma 4.3, we show that

$$\lim_{n\to\infty} n^{\frac{2}{d+1}} \int_{K_{n^{\frac{-1}{d+1}}}} \mathbb{P}_\varrho\left(x \notin K_{(n)}\right)\lambda(x)\,dx = 0.$$

This limit relation is based on a geometric estimate of $\mathbb{P}_\varrho\left(x \notin K_{(n)}\right)$, provided in Lemma 4.1, and on a disintegration result stated as Lemma 4.2.

For $y \in \partial K$, we write $u(y)$ for some exterior unit normal of K at y. This exterior unit normal is uniquely determined for $\mathcal{H}^{d-1}$-almost all boundary points of K. Applying the disintegration result again and using Lebesgue's dominated convergence result, we finally get

$$\lim_{n\to\infty} n^{\frac{2}{d+1}}\mathbb{E}_\varrho \int_{K\setminus K_{(n)}} \lambda(x)\,dx = \int_{\partial K} \lambda(y)J_\varrho(y)\,\mathcal{H}^{d-1}(dy),$$

where

$$J_\varrho(y) = \lim_{n\to\infty} \int_0^{n^{\frac{-1}{d+1}}} n^{\frac{2}{d+1}}\langle y, u(y)\rangle \mathbb{P}_\varrho\left(y_t \notin K_{(n)}\right)\,dt$$

for $\mathcal{H}^{d-1}$-almost all $y \in \partial K$. For the subsequent analysis, it is sufficient to consider a small cap of K at a normal boundary point $y \in \partial K$. The case $\kappa(y) = 0$ is treated in Lemma 4.4. The main case is $\kappa(y) > 0$. Here we reparametrize y_t as $\tilde{y}_s$, in terms of the probability content of a small cap of K whose bounding hyperplane passes through y_t. This implies that

$$J_\varrho(y) = (d+1)^{-\frac{d-1}{d+1}}\alpha_{d-1}^{-\frac{2}{d+1}}\varrho(y)^{\frac{-2}{d+1}}\kappa(y)^{\frac{1}{d+1}} \lim_{n\to\infty}\int_0^{n^{-1/2}} n^{\frac{2}{d+1}}\mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right)s^{-\frac{d-1}{d+1}}\,ds,$$

cf. (4.26). It is then a crucial step in the proof to show that the remaining integral asymptotically is independent of the particular convex body K, and thus the limit of the integral is the same as for a Euclidean ball (see Lemma 4.6). To achieve this, the integral is first approximated, up to a prescribed error of order $\varepsilon > 0$, by replacing $\mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right)$ by the probability of an event that depends only on a small cap of K at y and on a small number of random points. This important step is accomplished in Lemma 4.5. For the proofs of Lemmas 4.5 and 4.6 it is essential that the boundary of K near the normal boundary point y can be suitably approximated by the osculating paraboloid of K at y.

4. *Proof of Theorem 3.1*

To start with the actual proof, we fix some further notation. For $y \in \partial K$ and $t \in (0,1)$, we define the cap $C(y,t) := \{x \in K : \langle u(y), x\rangle \geq \langle u(y), y_t\rangle\}$ whose bounding hyperplane passes through y_t and has normal $u(y)$. For $u \in \mathbb{R}^d \setminus \{o\}$ and $t \in \mathbb{R}$, we define the hyperplane $H(u,t) := \{x \in \mathbb{R}^d : \langle x, u\rangle = t\}$, and the closed halfspaces $H^+(u,t) := \{x \in \mathbb{R}^d : \langle x, u\rangle \geq t\}$ and $H^-(u,t) := \{x \in \mathbb{R}^d : \langle x, u\rangle \leq t\}$ bounded by $H(u,t)$. We denote by $h(K,\cdot) = h_K$ the support function of K, that is $h(K,u) := \max\{\langle x, u\rangle : x \in K\}$ for $u \in \mathbb{R}^d$.

For $y \in \partial K$, the maximal number $r \geq 0$ such that $y - ru(y) + rB^d \subset K$ is denoted by $r(y)$. This number is called the interior reach of the boundary point y. It is well known that $r(y) > 0$ for $\mathcal{H}^{d-1}$-almost all $y \in \partial K$. If $r(y) > 0$, there is a unique tangent plane of K at y. In particular, $r(y) \leq r(K)$ where $r(K)$ is the inradius of K. The convex hull of subsets $X_1, \ldots, X_r \subset \mathbb{R}^d$ and points $z_1, \ldots, z_s \in \mathbb{R}^d$ is denoted by $[X_1, \ldots, X_r, z_1, \ldots, z_s]$.

For real functions f and g defined on the same space I, we write $f \ll g$ or $f = O(g)$ if there exists a positive constant γ, depending only on K, ϱ and λ, such that $|f| \leq \gamma \cdot g$ on I. In general, we write $\gamma_0, \gamma_1, \ldots$ to denote positive constants depending only on K, ϱ and λ. The Landau symbol $o(\cdot)$ is defined as usual. We further put $\mathbb{R}^+ := [0, \infty)$.

Finally, we observe that there exists a constant $\gamma_0 \in (0,1)$ such that for $y \in \partial K$, we have

$$|\langle y, u(y)\rangle| \geq \gamma_0 \|y\|, \text{ and hence } \|y|u(y)^\perp\| \leq \sqrt{1-\gamma_0^2} \cdot \|y\|, \tag{4.1}$$

where $y|u^\perp$ denotes the orthogonal projection of y onto the orthogonal complement of the vector $u \in \mathbb{R}^d \setminus \{o\}$. Subsequently, we always assume that $n \in \mathbb{N}$.

LEMMA 4.1. *There exists a constant $\delta > 0$, depending on K and ϱ, such that if $y \in \partial K$ and $t \in (0, \delta)$, then*

$$\mathbb{P}_\varrho\left(y_t \notin K_{(n)}\right) \ll \left(1 - \gamma_1 r(y)^{\frac{d-1}{2}} t^{\frac{d+1}{2}}\right)^n.$$

REMARKS.

(i) In addition, we may assume that on $K \setminus \mathrm{int}(K_\delta)$, both functions ϱ, λ are continuous, ϱ is positive and $\gamma_1 r(K)^{\frac{d-1}{2}} \delta^{\frac{d+1}{2}} < 1$.

(ii) In the following, we will use the notion of a "coordinate corner". Given an orthonormal basis in a linear i-dimensional subspace L, the corresponding $(i-1)$-dimensional coordinate planes cut L into 2^i convex cones, which we call coordinate corners (with respect to L and the given basis).

Proof of Lemma 4.1. If $r(y) = 0$, then there is nothing to prove. So let $r(y) > 0$, thence $u(y)$ is uniquely determined. Choose an orthonormal basis in $u(y)^\perp$, and let $\Theta'_1, \ldots, \Theta'_{2^{d-1}}$ be the corresponding coordinate corners in $u(y)^\perp$. For $i = 1, \ldots, 2^{d-1}$ and $t \in [0,1]$, we define

$$\Theta_{i,t} := C(y,t) \cap \left(y_t + \left[\Theta'_i, \mathbb{R}^+ y\right]\right).$$

If $\delta > 0$ is small enough to ensure that $\varrho > 0$ is positive and continuous in a neighborhood (relative to K) of ∂K, then

$$\int_{\Theta_{i,t}} \varrho(x)\, dx \geq \gamma_2\, V(\Theta_{i,t}).$$

If $y_t \notin K_{(n)}$ and $o \in K_{(n)}$, then there exists a hyperplane H through y_t, bounding the halfspaces H^- and H^+, for which $K_{(n)} \subset H^-$. Moreover, there is some $i \in \{1, \ldots, 2^{d-1}\}$ such that $\Theta_{i,t} \subset H^+$.

Therefore

$$\mathbb{P}_\varrho\left(y_t \notin K_{(n)}, o \in K_{(n)}\right) \ll \sum_{i=1}^{2^{d-1}} \left(1 - \gamma_2 V(\Theta_{i,t})\right)^n . \tag{4.2}$$

Finally, we prove

$$V(\Theta_{i,t}) \gg r(y)^{\frac{d-1}{2}} t^{\frac{d+1}{2}}, \tag{4.3}$$

for $i = 1, \ldots, 2^{d-1}$. According to (4.1), there exist positive constants γ_3, γ_4 with $\gamma_3 \leq 1$ such that if $t \leq \gamma_3 r(y)$, then $(y_t + \Theta_i') \cap K$ contains a $(d-1)$-ball of radius at least

$$\gamma_4 \sqrt{r(y)^2 - (r(y) - t)^2} \geq \gamma_4 \sqrt{r(y) t},$$

and we are done. On the other hand, if $t \geq \gamma_3 r(y)$, then

$$V(\Theta_{i,t}) \gg t^d \gg r(y)^{\frac{d-1}{2}} t^{\frac{d+1}{2}} .$$

To deal with the case $o \notin K_{(n)}$, we observe that there exists a positive constant $\gamma_5 \in (0,1)$ such that the probability measure of each of the 2^d coordinate corners of $\mathbb{R}^d$ is at least γ_5. If $o \notin K_{(n)}$, then $\{x_1, \ldots, x_n\}$ is disjoint from one of these coordinate corners, and hence

$$\mathbb{P}_\varrho(o \notin K_{(n)}) \leq 2^d (1 - \gamma_5)^n. \tag{4.4}$$

Now the assertion follows from (4.2), (4.3) and (4.4). □

Subsequently, the estimate of Lemma 4.1 will be used, for instance, to restrict the domain of integration on the right-hand side of (3.2) (cf. Lemma 4.3) and to justify an application of Lebesgue's dominated convergence theorem (see (4.9)). For these applications, we also need that if $c > 0$ is such that $\omega := c\,\delta^{\frac{d+1}{2}} < 1$, then

$$\int_0^\delta \left(1 - c\,t^{\frac{d+1}{2}}\right)^n dt = \frac{2}{d+1} c^{\frac{-2}{d+1}} \int_0^\omega s^{\frac{2}{d+1}-1} (1-s)^n \, ds \ll c^{\frac{-2}{d+1}} \cdot n^{\frac{-2}{d+1}}, \tag{4.5}$$

where we use that $(1-s)^n \leq e^{-ns}$ for $s \in [0,1]$ and $n \in \mathbb{N}$.

The next lemma will allow us to decompose integrals in a suitable way.

LEMMA 4.2. *If* $0 \leq t_0 \leq t_1 < \delta$ *and* $h : K \to [0, \infty]$ *is a measurable function, then*

$$\int_{K_{t_0} \setminus K_{t_1}} h(x)\, dx = \int_{\partial K} \int_{t_0}^{t_1} (1-t)^{d-1} \langle y, u(y) \rangle h(y_t)\, dt\, \mathcal{H}^{d-1}(dy).$$

Proof. The map $T : \partial K \times [t_0, t_1] \to K_{t_0} \setminus K_{t_1}$, $(y,t) \mapsto (1-t)y$, provides a bilipschitz parametrization of $K_{t_0} \setminus K_{t_1}$ with $(1-t)y = y_t \in \partial K_t$. The Jacobian of T, for $\mathcal{H}^{d-1}$-almost all $y \in \partial K$ and $t \in [t_0, t_1]$, is given by $JT(y,t) = (1-t)^{d-1} \langle y, u(y) \rangle$, where $u(y)$ is the ($\mathcal{H}^{d-1}$-almost everywhere) unique exterior unit normal of ∂K at y. The assertion now follows from Federer's area/coarea theorem (see [**13**]). □

In the following, we will use the important fact that, for $\alpha > -1$,

$$\int_{\partial K} r(y)^\alpha \, \mathcal{H}^{d-1}(dy) < \infty, \tag{4.6}$$

which is a result due to C. Schütt and E. Werner [**39**].

By decomposing λ in its positive and its negative part, we can henceforth assume that λ is a nonnegative, integrable function.

LEMMA 4.3. *As n tends to infinity,*

$$\int_{K_{n^{\frac{-1}{d+1}}}} \mathbb{P}_\varrho\left(x \notin K_{(n)}\right) \lambda(x)\, dx = o\left(n^{\frac{-2}{d+1}}\right).$$

Proof. Let $\delta > 0$ be chosen as in Lemma 4.1 and the subsequent remark. First, we consider a point x in K_δ. Let ω be the minimal distance between the points of ∂K and K_δ, and let $z_1, \ldots, z_k$ be a maximal family of points in $K \setminus \operatorname{int}(K_\delta)$ such that $\|z_i - z_j\| \geq \frac{\omega}{4}$ for $i \neq j$. We define $p_0 > 0$ by

$$p_0 := \min\left\{\mathbb{P}_\varrho\left(z_i + \tfrac{\omega}{4} B^d\right) : i = 1, \ldots, k\right\}.$$

Let $x \in K_\delta$. If $x \notin K_{(n)}$, then there exists a hyperplane $H(u,t)$ such that $x \in \operatorname{int}(H^+(u,t))$ and $K_{(n)} \subset H^-(u,t)$. Since $x \in K_\delta$, there exists a supporting hyperplane $H(u, h(K_\delta, u))$ of K_δ for which $K_{(n)} \subset \operatorname{int}(H^-(u, h(K_\delta, u)))$. If $z \in H(u, h(K_\delta, u)) \cap \partial K_\delta$, then

$$z + \frac{\omega}{2} u + \frac{\omega}{2} B^d \subset K \cap H^+(u, h(K_\delta, u)).$$

By the maximality of the set $\{z_1, \ldots, z_k\}$, we have

$$\{z_1, \ldots, z_k\} \cap \left(z + \frac{\omega}{2} u + \frac{\omega}{4} B^d\right) \neq \emptyset.$$

Let z_j lie in the intersection. Then $z_j + \frac{\omega}{4} B^d \subset H^+(u, h(K_\delta, u))$, and hence $x_i \notin z_j + \frac{\omega}{4} B^d$ for $i = 1, \ldots, n$. This implies that, for $x \in K_\delta$,

$$\mathbb{P}_\varrho\left(x \notin K_{(n)}\right) \leq k(1 - p_0)^n. \tag{4.7}$$

Put $\varepsilon := (2(d^2-1))^{-1}$ and let $n \geq \delta^{-(d+1)}$. For $y \in \partial K$ we show that

$$\int_{n^{\frac{-1}{d+1}}}^{\delta} \mathbb{P}_\varrho\left(y_t \notin K_{(n)}\right)\, dt \ll r(y)^{-\frac{d}{d+1}} n^{\frac{-2}{d+1} - \varepsilon}. \tag{4.8}$$

In fact, if $r(y) \leq n^{-(d+1)\varepsilon}$, then Lemma 4.1 and (4.5) yield

$$\begin{aligned}\int_{n^{\frac{-1}{d+1}}}^{\delta} \mathbb{P}_\varrho\left(y_t \notin K_{(n)}\right)\, dt &\leq \int_0^{\delta} \left(1 - \gamma_1 r(y)^{\frac{d-1}{2}} t^{\frac{d+1}{2}}\right)^n dt \\ &\ll r(y)^{-\frac{d-1}{d+1}} n^{-\frac{2}{d+1}} \\ &\leq r(y)^{-\frac{d}{d+1}} n^{-\frac{2}{d+1} - \varepsilon},\end{aligned}$$

where the assumption on $r(y)$ is used for the last estimate.

If $r(y) \geq n^{-(d+1)\varepsilon}$ and $n \geq n_0$, where n_0 depends on K, ϱ and λ, then Lemma 4.1 implies for all $t \in (n^{\frac{-1}{d+1}}, \delta)$ that

$$\mathbb{P}_\varrho\left(y_t \notin K_{(n)}\right) \ll \left(1 - \gamma_1 n^{-\frac{d^2-1}{2}\varepsilon - \frac{1}{2}}\right)^n = (1 - \gamma_1 n^{-3/4})^n \leq e^{-\gamma_1 n^{1/4}} \leq r(K)^{-\frac{d}{d+1}} n^{\frac{-2}{d+1} - \varepsilon},$$

which again yields (4.8). In particular, writing I to denote the integral in Lemma 4.3, we obtain from Lemma 4.2, (4.7), (4.8) and (4.6) that

$$\begin{aligned} I &\ll \int_{K_\delta} \mathbb{P}_\varrho\left(x \notin K_{(n)}\right) \lambda(x)\, dx + \int_{\partial K} \int_{n^{\frac{-1}{d+1}}}^{\delta} \mathbb{P}_\varrho\left(y_t \notin K_{(n)}\right)\, dt\, \mathcal{H}^{d-1}(dy) \\ &\ll k(1-p)^n + \int_{\partial K} r(y)^{-\frac{d}{d+1}} n^{\frac{-2}{d+1} - \varepsilon}\, \mathcal{H}^{d-1}(dy) \ll n^{\frac{-2}{d+1} - \varepsilon},\end{aligned}$$

where we also used that λ is integrable on K and bounded on $K \setminus K_\delta$. This is the required estimate. □

It follows from (3.2), Lemma 4.3 and Lemma 4.2 that

$$\begin{aligned}&\lim_{n\to\infty} n^{\frac{2}{d+1}}\,\mathbb{E}_\varrho \int_{K\setminus K_{(n)}} \lambda(x)\,dx\\ &\quad= \lim_{n\to\infty} n^{\frac{2}{d+1}} \int_K \mathbb{P}_\varrho\left(x\notin K_{(n)}\right)\lambda(x)\,dx\\ &\quad= \lim_{n\to\infty}\int_{\partial K}\int_0^{n^{\frac{-1}{d+1}}} n^{\frac{2}{d+1}}(1-t)^{d-1}\langle y,u(y)\rangle \mathbb{P}_\varrho\left(y_t\notin K_{(n)}\right)\lambda(y_t)\,dt\,\mathcal{H}^{d-1}(dy).\end{aligned}$$

Lemma 4.1 and (4.5) imply that if $y\in\partial K$ and $r(y)>0$, then

$$\int_0^{n^{\frac{-1}{d+1}}} n^{\frac{2}{d+1}}\mathbb{P}_\varrho\left(y_t\notin K_{(n)}\right)\langle y,u(y)\rangle\lambda(y_t)\,dt \ll r(y)^{-\frac{d-1}{d+1}}.$$

Therefore, by (4.6) and since λ is bounded and continuous in a neighborhood of ∂K we may apply Lebesgue's dominated convergence theorem, and thus we conclude

$$\lim_{n\to\infty} n^{\frac{2}{d+1}}\mathbb{E}_\varrho\int_{K\setminus K_{(n)}}\lambda(x)\,dx = \int_{\partial K}\lambda(y)J_\varrho(y)\,\mathcal{H}^{d-1}(dy), \tag{4.9}$$

where

$$J_\varrho(y) := \lim_{n\to\infty}\int_0^{n^{\frac{-1}{d+1}}} n^{\frac{2}{d+1}}\langle y,u(y)\rangle\mathbb{P}_\varrho\left(y_t\notin K_{(n)}\right)\,dt,$$

for $\mathcal{H}^{d-1}$-almost all $y\in\partial K$.

LEMMA 4.4. *If $y\in\partial K$ is a normal boundary point of K with $\kappa(y)=0$, then $J_\varrho(y)=0$.*

Proof. In view of the estimate (4.4), it is sufficient to prove that for any given $\varepsilon>0$,

$$\int_0^{n^{\frac{-1}{d+1}}} n^{\frac{2}{d+1}}\mathbb{P}_\varrho\left(y_t\notin K_{(n)}, o\in K_{(n)}\right)\,dt \ll \varepsilon, \tag{4.10}$$

if n is sufficiently large. We choose the coordinate axes in $u(y)^\perp$ parallel to the principal curvature directions of K at y, and denote by $\Theta'_1,\dots,\Theta'_{2^{d-1}}$ the corresponding coordinate corners. For $i=1,\dots,2^{d-1}$ and $t\in(0,n^{\frac{-1}{d+1}})$, let

$$\Theta_{i,t} := C(y,t)\cap\left(y_t+\left[\Theta'_i,\mathbb{R}^+y\right]\right),$$

and hence, if n is large enough, then

$$\int_{\Theta_{i,t}}\varrho(x)\,dx \gg V(\Theta_{i,t}),$$

since ϱ is continuous and positive near ∂K. If $y_t\notin K_n$ and $o\in K_{(n)}$, then there exists a halfspace H^- which contains $K_{(n)}$ and for which $y_t\in\partial H^-$. Moreover, for some $i\in\{1,\dots,2^{d-1}\}$ the interior of H^- is disjoint from $\Theta_{i,t}$. Hence, as in the proof of Lemma 4.1,

$$\mathbb{P}_\varrho\left(y_t\notin K_{(n)}, o\in K_{(n)}\right) \ll \sum_{i=1}^{2^{d-1}}\left(1-\gamma_6 V(\Theta_{i,t})\right)^n. \tag{4.11}$$

Since ∂K is twice differentiable in the generalized sense at y, we have $r(y)>0$. By assumption, $\kappa(y)=0$, therefore one principal curvature at y is zero, and hence less than $\varepsilon^{d+1}r(y)^{d-2}$. In particular, there exists $\delta'\in(0,\delta)$, which by (4.1) depends only on y and ε, such that if $i\in\{1,\dots,2^{d-1}\}$ and

$t \in (0, \delta')$, then

$$\mathcal{H}^{d-1}\left((y_t + \Theta_i') \cap K\right) \gg \sqrt{t\varepsilon^{-(d+1)} r(y)^{-(d-2)}} \cdot \sqrt{tr(y)}^{d-2}.$$

We deduce $V(\Theta_{i,t}) \gg \varepsilon^{-\frac{d+1}{2}} t^{\frac{d+1}{2}}$. Therefore (4.10) follows from (4.5) and (4.11). □

Next we consider the case of a normal boundary point $y \in \partial K$ with $\kappa(y) > 0$. First, we prove that $J_\varrho(y)$ depends only on the random points near y (see Lemma 4.5). In a second step, we compare the simplified expression obtained for $J_\varrho(y)$ with the corresponding expression which is obtained if K is a ball.

We start by reparametrizing y_t in terms of the probability measure of the corresponding cap. For $t \in (0, n^{\frac{-1}{d+1}})$, where $n \geq n_0$ is sufficiently large so that ϱ is positive and continuous on $C(y, t)$, for all $y \in \partial K$, we put

$$\tilde{y}_s := y_t$$

where for given $s > 0$ (sufficiently small) the corresponding $t = t(s)$ is determined by the relation

$$s = \int_{C(y,t)} \varrho(x)\, dx. \tag{4.12}$$

It is easy to see that the right-hand side of (4.12) is a continuous and strictly increasing function $s = s(t)$ of t, if $t > 0$ is sufficiently small. This implies that for a given $s > 0$ (sufficiently small) there is a unique $t(s)$ such that (4.12) is satisfied.

Moreover, observe that

$$\frac{ds}{dt} = \langle u(y), y\rangle \int_{H(y,t)\cap K} \varrho(x)\, \mathcal{H}^{d-1}(dx) \tag{4.13}$$

for $t \in (0, n^{\frac{-1}{d+1}})$. We further define

$$\widetilde{C}(y, s) := C(y, t) \qquad \text{and} \qquad \widetilde{H}(y, s) := \{x \in \mathbb{R}^d : \langle u(y), x\rangle = \langle u(y), \tilde{y}_s\rangle\},$$

where $t = t(s)$.

Let Q denote the second fundamental form of ∂K at y (cf. (2.1)), considered as a function on $u(y)^\perp$. We define

$$E := \{z \in u(y)^\perp : Q(z) \leq 1\}.$$

and put $u := u(y)$. Choosing a suitable orthonormal basis $v_1, \ldots, v_{d-1}$ of $u(y)^\perp$, we have

$$Q(z) = \sum_{i=1}^{d-1} k_i(y) z_i^2,$$

where $k_i(y)$, $i = 1, \ldots, d-1$, are the generalized principal curvatures of K at y and where $z = z_1 v_1 + \ldots + z_{d-1} v_{d-1}$. Since y is a normal boundary point of K, there is a nondecreasing function $\mu : (0, \infty) \to \mathbb{R}$ with $\lim_{r\to 0^+} \mu(t) = 1$ such that

$$\frac{\mu(r)^{-1}}{\sqrt{2r}}(K(u, r) + ru - y) \subset E \subset \frac{\mu(r)}{\sqrt{2r}}(K(u, r) + ru - y), \tag{4.14}$$

where $K(u, r) := K \cap H(u, h(K, u) - r)$. In the following, $\mu_i : (0, \infty) \to \mathbb{R}$, $i = 1, 2, \ldots$, always denote nondecreasing functions with $\lim_{r\to 0^+} \mu(t) = 1$. Applying (4.14) and Fubini's theorem, we get

$$V(K \cap H^+(u, h(K, u) - r)) = \mu_1(r) \frac{(2r)^{\frac{d+1}{2}}}{d+1} \alpha_{d-1} \kappa(y)^{-\frac{1}{2}},$$

which yields that

$$s(t) = \mu_2(t) \frac{(2t\langle y, u\rangle)^{\frac{d+1}{2}}}{d+1} \alpha_{d-1} \kappa(y)^{-\frac{1}{2}} \varrho(y), \tag{4.15}$$

since ϱ is continuous at y. Moreover, defining

$$\eta := (d+1)^{\frac{1}{d+1}} \alpha_{d-1}^{-\frac{1}{d+1}} \varrho(y)^{\frac{-1}{d+1}} \kappa(y)^{\frac{1}{2(d+1)}},$$

we obtain

$$\lim_{s \to 0^+} s^{\frac{-1}{d+1}} [(\widetilde{H}(y,s) \cap K) - \tilde{y}_s] = \eta \cdot E \tag{4.16}$$

in the sense of the Hausdorff metric on compact convex sets (see Schneider [**34**] or Gruber [**16**]). Here we also use that

$$\lim_{s \to 0^+} s^{-\frac{1}{d+1}} (\tilde{y}_s - \langle \tilde{y}_s, u \rangle u) = o. \tag{4.17}$$

Now it follows from (4.13) and (4.16) that (4.9) turns into

$$J_\varrho(y) = (d+1)^{-\frac{d-1}{d+1}} \alpha_{d-1}^{-\frac{2}{d+1}} \varrho(y)^{\frac{-2}{d+1}} \kappa(y)^{\frac{1}{d+1}} \lim_{n \to \infty} \int_0^{g(n,y)} n^{\frac{2}{d+1}} \mathbb{P}_\varrho \left(\tilde{y}_s \notin K_{(n)} \right) s^{-\frac{d-1}{d+1}} \, ds,$$

where

$$\lim_{n \to \infty} n^{\frac{1}{2}} g(n,y) = (d+1)^{-1} \alpha_{d-1} \varrho(y) (2 \langle u(y), y \rangle)^{\frac{d+1}{2}} \kappa(y)^{-\frac{1}{2}}.$$

The rest of the proof is devoted to identifying the asymptotic behavior of the integral. First, we adjust the domain of integration and the integrand in a suitable way. In a second step, the resulting expression is compared to the case where K is the unit ball. We recall that $x_1, \ldots, x_n$ are random points in K, and we put $\Xi_n := \{x_1, \ldots, x_n\}$, and hence $K_{(n)} = [\Xi_n]$. Let $\#X$ denote the cardinality of a finite set $X \subset \mathbb{R}^d$.

LEMMA 4.5. *For $\varepsilon \in (0,1)$, there exist $\alpha, \beta > 1$ and an integer $k > 1$, depending only on ε and d, with the following property. If $y \in \partial K$ is a normal boundary point of K with $\kappa(y) > 0$ and if $n > n_0$, where n_0 depends on $\varepsilon, y, K, \varrho$, then*

$$\int_0^{g(n,y)} \mathbb{P}_\varrho \left(\tilde{y}_s \notin K_{(n)} \right) s^{-\frac{d-1}{d+1}} \, ds = \int_{\frac{\varepsilon^{(d+1)/2}}{n}}^{\frac{\alpha}{n}} \varphi(K, y, \varrho, \varepsilon, s) s^{-\frac{d-1}{d+1}} \, ds + O\left(\frac{\varepsilon}{n^{\frac{2}{d+1}}} \right),$$

where

$$\varphi(K, y, \varrho, \varepsilon, s) = \mathbb{P}_\varrho \left(\left(\tilde{y}_s \notin [\widetilde{C}(y, \beta s) \cap \Xi_n] \right) \text{and } \left(\#(\widetilde{C}(y, \beta s) \cap \Xi_n) \leq k \right) \right).$$

Proof. Let Q be the second fundamental form of ∂K at the normal boundary point y, and let $v_1, \ldots, v_{d-1}$ be an orthonormal basis of $u(y)^\perp$ with respect to Q, as described above. Let $\Theta'_1, \ldots, \Theta'_{2^{d-1}}$ be the corresponding coordinate corners, and, for $i = 1, \ldots, 2^{d-1}$ and for $s \in (0, n^{-1/2})$, put

$$\widetilde{\Theta}_{i,s} := \widetilde{C}(y,s) \cap \left(\tilde{y}_s + \left[\Theta'_i, \mathbb{R}^+ y \right] \right).$$

Let A_s, $s > 0$, be the affine map of $\mathbb{R}^d$ with $A_s(y) = y$ for which the associated linear map $\widetilde{A}_s$ is determined by $\widetilde{A}_s(v) = s^{\frac{1}{d+1}} v$, for $v \in u^\perp$, and $\widetilde{A}_s(u) = s^{\frac{2}{d+1}} u$. Then $\det(\widetilde{A}_s) = s$ and $A_{s^{-1}}(\widetilde{C}(y,s))$ converges in the Hausdorff metric as $s \to 0^+$ to the cap $\widetilde{C}(y)$ of the osculating paraboloid of K at y having volume $\varrho(y)^{-1}$. Here we use that ϱ is continuous at y, $\varrho(y) > 0$ and relation (4.12). Let $\lambda > 0$ be such that $\tilde{y} := y - \lambda u \in \partial \widetilde{C}(y)$. Then $A_{s^{-1}}(\widetilde{\Theta}_{i,s})$ converges in the Hausdorff metric as $s \to 0^+$ to $\widetilde{C}(y) \cap (\tilde{y} + [\Theta'_i, \mathbb{R}^+ u])$, since (4.17) is satisfied. Using again that ϱ is continuous and positive at y, we deduce that

$$
\begin{aligned}
\lim_{s\to 0^+} s^{-1}\int_{\widetilde{\Theta}_{i,s}} \varrho(x)\,dx &= \lim_{s\to 0^+} s^{-1}V(\widetilde{\Theta}_{i,s})\varrho(y)\\
&= \lim_{s\to 0^+} V(A_{s^{-1}}(\widetilde{\Theta}_{i,s}))\varrho(y)\\
&= V(\widetilde{C}(y)\cap(\tilde{y}+[\Theta_i',\mathbb{R}^+u]))\varrho(y)\\
&= 2^{-(d-1)}V(\widetilde{C}(y))\varrho(y)\\
&= 2^{-(d-1)}\lim_{s\to 0^+} V(A_{s^{-1}}(\widetilde{C}(y,s))\varrho(y)\\
&= 2^{-(d-1)}\lim_{s\to 0^+} s^{-1}V(\widetilde{C}(y,s))\varrho(y)\\
&= 2^{-(d-1)}\lim_{s\to 0^+} s^{-1}\int_{\widetilde{C}(y,s)} \varrho(x)\,dx\\
&= 2^{-(d-1)},
\end{aligned}
$$

that is

$$
\lim_{s\to 0^+} s^{-1}\int_{\widetilde{\Theta}_{i,s}} \varrho(x)\,dx = 2^{-(d-1)}. \tag{4.18}
$$

Let $\alpha > 1$ be chosen such that

$$
2^{d-1+2d/(d+1)}\int_{2^{-d}\alpha}^{\infty} e^{-x}x^{\frac{2}{d+1}-1}\,dx \le \varepsilon.
$$

Then we first choose $\beta \ge (16(d-1))^{d+1}$ such that

$$
2^{d-1}e^{-d^{-1}2^{-(d+3)}\beta^{\frac{1}{d+1}}\varepsilon^{\frac{d+1}{2}}} \le \frac{\varepsilon}{\alpha^{\frac{2}{d+1}}},
$$

and then we fix an integer $k > 1$ such that

$$
\frac{(\alpha\beta)^k}{k!} \le \frac{\varepsilon}{\alpha^{\frac{2}{d+1}}}.
$$

Lemma 4.5 follows from the following three statements, which we will prove assuming that n is sufficiently large.

(i)

$$
\int_0^{g(n,y)} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\,ds = \int_{\frac{\varepsilon^{(d+1)/2}}{n}}^{\frac{\alpha}{n}} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\,ds + O\left(\frac{\varepsilon}{n^{\frac{2}{d+1}}}\right).
$$

(ii) If $\frac{\varepsilon^{(d+1)/2}}{n} < s < \frac{\alpha}{n}$, then

$$
\mathbb{P}_\varrho\left(\#\left(\widetilde{C}(y,\beta s)\cap\Xi_n\right)\ge k\right) = O\left(\frac{\varepsilon}{\alpha^{\frac{2}{d+1}}}\right).
$$

(iii) If $\frac{\varepsilon^{\frac{d+1}{2}}}{n} < s < \frac{\alpha}{n}$, then

$$
\mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) = \mathbb{P}_\varrho\left(\tilde{y}_s \notin \left[\widetilde{C}(y,\beta s)\cap\Xi_n\right]\right) + O\left(\frac{\varepsilon}{\alpha^{\frac{2}{d+1}}}\right).
$$

To prove (i), we first observe that

$$
\int_0^{\frac{\varepsilon^{(d+1)/2}}{n}} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\,ds \le \int_0^{\frac{\varepsilon^{(d+1)/2}}{n}} s^{-\frac{d-1}{d+1}}\,ds \ll \frac{\varepsilon}{n^{\frac{2}{d+1}}}.
$$

If $\frac{\alpha}{n} < s < g(n,y)$, $o \in K_{(n)}$, $\tilde{y}_s \notin K_{(n)}$ and if n is sufficiently large, then there is some $i \in \{1,\dots,2^{d-1}\}$ such that $\widetilde{\Theta}_{i,s} \cap K_{(n)} = \emptyset$, and hence (4.4) and (4.18) yield

$$\mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) \ll 2^{d-1}(1-2^{-d}s)^n \leq 2^{d-1}e^{-2^{-d}ns}. \tag{4.19}$$

Therefore, by the definition of α, we get

$$\begin{aligned}\int_{\frac{\alpha}{n}}^{g(n,y)} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\,ds &\ll 2^{d-1}\int_{\frac{\alpha}{n}}^{\infty} e^{-2^{-d}ns} s^{\frac{2}{d+1}-1}\,ds\\ &= 2^{d-1}2^{2d/(d+1)}n^{-\frac{2}{d+1}}\int_{2^{-d}\alpha}^{\infty} e^{-x}x^{\frac{2}{d+1}-1}\,dx\\ &\leq \varepsilon\, n^{-\frac{2}{d+1}},\end{aligned}$$

which verifies (i).

Next (ii) simply follows from (4.12) as if $s < \frac{\alpha}{n}$, then

$$\mathbb{P}_\varrho\left(\#\left(\widetilde{C}(y,\beta s)\cap \Xi_n\right) \geq k\right) = \binom{n}{k}(\beta s)^k \leq \binom{n}{k}\left(\frac{\alpha\beta}{n}\right)^k < \frac{(\alpha\beta)^k}{k!} \leq \frac{\varepsilon}{\alpha^{\frac{2}{d+1}}}.$$

Now we prove (iii). To this end, for s in the given range, our plan is to construct sets $\widetilde{\Omega}_{1,s},\dots,\widetilde{\Omega}_{2^{d-1},s} \subset K$ such that

$$\int_{\widetilde{\Omega}_{i,s}} \varrho(x)\,dx \geq d^{-1}2^{-(d+3)}\beta^{\frac{1}{d+1}}s, \qquad \text{for } i = 1,\dots,2^{d-1}, \tag{4.20}$$

and if $\tilde{y}_s \in K_{(n)}$ but $\tilde{y}_s \notin \left[\widetilde{C}(y,\beta s)\cap \Xi_n\right]$, then $\Xi_n \cap \widetilde{\Omega}_{i,s} = \emptyset$ for some $i \in \{1,\dots,2^{d-1}\}$.

For $i = 1,\dots,2^{d-1}$, let $w_i \in \Theta_i'$ be the vector whose coordinates (up to sign) in the basis $v_1,\dots,v_{d-1}$ are

$$w_i := \left(\sqrt{\beta}s\right)^{\frac{1}{d+1}} \frac{\eta}{2\sqrt{d-1}}\left(\pm\frac{1}{\sqrt{k_1(y)}},\dots,\pm\frac{1}{\sqrt{k_{d-1}(y)}}\right).$$

Further, for $i = 1,\dots,2^{d-1}$ we define

$$\widetilde{\Omega}_{i,s} = [\tilde{y}_{\sqrt{\beta}\,s} + w_i, K \cap (\tilde{y}_s + \Theta_i')].$$

Then, if $s > 0$ is small enough, $\tilde{y}_{\sqrt{\beta}\,s} + w_i \in K$, and hence $\widetilde{\Omega}_{i,s} \subset K$. Here we use that

$$w_i \in (\sqrt{\beta}s)^{\frac{1}{d+1}}\frac{1}{2}\eta E$$

and therefore by (4.16)

$$\tilde{y}_{\sqrt{\beta}s} + w_i \in \widetilde{H}(y,\sqrt{\beta}s)\cap K \subset K.$$

Using that $\tilde{y}_s = (1-t)y$, where s and t are related by (4.15), and if $s,t > 0$ are sufficiently small, we obtain

$$\langle u(y), \tilde{y}_s - \tilde{y}_{\sqrt{\beta}\,s}\rangle > \frac{\beta^{\frac{1}{d+1}}-1}{2}\langle u(y), y-\tilde{y}_s\rangle > \frac{\beta^{\frac{1}{d+1}}}{4}\langle u(y), y-\tilde{y}_s\rangle, \tag{4.21}$$

since $\beta \geq 2^{d+1}$. Moreover, we have

$$\langle u(y), y-\tilde{y}_s\rangle \cdot \mathcal{H}^{d-1}\left(K\cap(\tilde{y}_s+\Theta_i')\right) \geq V(\widetilde{\Theta}_{i,s}). \tag{4.22}$$

Combining (4.21), (4.22), (4.18) and the continuity of ϱ at y with $\varrho(y) > 0$, we deduce (4.20), that is

$$\begin{aligned}\int_{\widetilde{\Omega}_{i,s}} \varrho(x)\,dx &\geq \frac{1}{\sqrt{2}}\frac{1}{d}\varrho(y)\langle u(y), \tilde{y}_s - \tilde{y}_{\sqrt{\beta}\,s}\rangle \mathcal{H}^{d-1}\left(K \cap (\tilde{y}_s + \Theta_i')\right)\\ &\geq \frac{\beta^{\frac{1}{d+1}}}{4}\frac{1}{\sqrt{2}d}V(\widetilde{\Theta}_{i,s})\\ &\geq \frac{\beta^{\frac{1}{d+1}}}{4}\frac{1}{2d}\int_{\widetilde{\Theta}_{i,s}} \varrho(x)\,dx\\ &\geq \frac{\beta^{\frac{1}{d+1}}s}{8d\,2^d}.\end{aligned}$$

It is still left to prove that if $\tilde{y}_s \in K_{(n)}$ but $\tilde{y}_s \notin \left[\widetilde{C}(y,\beta s) \cap \Xi_n\right]$, then $\Xi_n \cap \widetilde{\Omega}_{i,s} = \emptyset$ for some $i \in \{1, \ldots, 2^{d-1}\}$. So we assume that $\tilde{y}_s \in K_{(n)}$ but $\tilde{y}_s \notin \left[\widetilde{C}(y,\beta s) \cap \Xi_n\right]$. Then there exist $a \in \left[\widetilde{C}(y,\beta s) \cap \Xi_n\right]$ and $b \in K_{(n)} \setminus \widetilde{C}(y,\beta s)$ such that $\tilde{y}_s \in [a,b]$, and hence there exists a hyperplane H containing $\tilde{y}_s$ bounding the halfspaces H^+ and H^- such that $\widetilde{C}(y,\beta s) \cap \Xi_n \subset \mathrm{int}(H^+)$ and $b \in \mathrm{int}(H^-)$.

Next we show that there exists $q \in [\tilde{y}_s, b]$ such that

$$q \in H^- \cap \left(\tilde{y}_{\sqrt{\beta}\,s} + \frac{\eta}{2\sqrt{d-1}}(\sqrt{\beta}s)^{\frac{1}{d+1}}E\right). \tag{4.23}$$

In fact, define $q := [\tilde{y}_s, b] \cap \widetilde{H}(y, \sqrt{\beta}s)$ and $q' := [\tilde{y}_s, b] \cap \widetilde{H}(y, \beta s)$. Since $a \in H^+$ and $\tilde{y}_s \in H$, it follows that $q \in H^-$. From (4.16) we get

$$\widetilde{H}(y,\beta s) \cap K \subset \tilde{y}_{\beta s} + 2\beta^{\frac{1}{d+1}}s^{\frac{1}{d+1}}\eta E. \tag{4.24}$$

Applying (4.15), we deduce

$$\begin{aligned}\langle u(y), \tilde{y}_s - \tilde{y}_{\beta\,s}\rangle &< \frac{\beta^{\frac{2}{d+1}}}{\beta^{\frac{2}{d+1}}-1}\cdot\frac{\beta^{\frac{2}{d+1}}-1}{\beta^{\frac{2}{d+1}}-\beta^{\frac{1}{d+1}}}\langle u(y), \tilde{y}_{\sqrt{\beta}\,s} - \tilde{y}_{\beta s}\rangle\\ &< \frac{\beta^{\frac{1}{d+1}}}{\beta^{\frac{1}{d+1}}-1}\langle u(y), \tilde{y}_{\sqrt{\beta}\,s} - \tilde{y}_{\beta s}\rangle.\end{aligned} \tag{4.25}$$

Furthermore, elementary geometry yields

$$\frac{\|q - \tilde{y}_{\sqrt{\beta}s}\|}{\|q' - \tilde{y}_{\beta s}\|} = \frac{\langle u, \tilde{y}_s - \tilde{y}_{\sqrt{\beta}s}\rangle}{\langle u, \tilde{y}_s - \tilde{y}_{\beta s}\rangle}.$$

Then (4.24) and (4.25) imply that

$$\begin{aligned}q &\in \tilde{y}_{\sqrt{\beta}s} + \frac{\langle u, \tilde{y}_s - \tilde{y}_{\sqrt{\beta}s}\rangle}{\langle u, \tilde{y}_s - \tilde{y}_{\beta s}\rangle}\cdot 2(\beta s)^{\frac{1}{d+1}}\eta E\\ &\subset \tilde{y}_{\sqrt{\beta}s} + \left(1 - \frac{\langle u, \tilde{y}_{\sqrt{\beta}s} - \tilde{y}_{\beta s}\rangle}{\langle u, \tilde{y}_s - \tilde{y}_{\beta s}\rangle}\right)\cdot 2\beta^{\frac{1}{d+1}}s^{\frac{1}{d+1}}\eta E\\ &\subset \tilde{y}_{\sqrt{\beta}s} + 2s^{\frac{1}{d+1}}\eta E\\ &\subset \tilde{y}_{\sqrt{\beta}s} + \frac{1}{2\sqrt{d-1}}(\sqrt{\beta}s)^{\frac{1}{d+1}}\eta E,\end{aligned}$$

where $\beta \geq (16(d-1))^{d+1}$ is used for the last inclusion. Now there exists some $i \in \{1, \ldots, 2^{d-1}\}$ such that $\tilde{y}_s + \Theta_i' \subset H^-$, and hence $q + \Theta_i' \subset H^-$. By (4.23) this finally yields

$$\tilde{y}_{\sqrt{\beta}s} + w_i \subset q + \Theta_i' \subset H^-.$$

Therefore we obtain $\widetilde{\Omega}_{i,s} \cap \Xi_n = \emptyset$.

Finally, (iii) follows as if $\frac{\varepsilon^{\frac{d+1}{2}}}{n} < s < \frac{\alpha}{n}$, then

$$\begin{aligned}
0 \le \mathbb{P}_\varrho\left(\tilde{y}_s \notin \left[\widetilde{C}(y,\beta s)\cap \Xi_n\right]\right) - \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) \\
&\le \sum_{i=1}^{2^{d-1}} (1 - \int_{\widetilde{\Omega}_{i,s}} \varrho(x)\,dx)^n \\
&\le \sum_{i=1}^{2^{d-1}} e^{-n\int_{\widetilde{\Omega}_{i,s}}\varrho(x)\,dx} \\
&\le 2^{d-1} e^{-d^{-1}2^{-(d+3)}\beta^{\frac{1}{d+1}}\varepsilon^{\frac{d+1}{2}}} \\
&\le \varepsilon\,\alpha^{-\frac{2}{d+1}},
\end{aligned}$$

by the choice of β. $\square$

REMARK. As a consequence of the proof of Lemma 4.5, it follows that

$$J_\varrho(y) = (d+1)^{-\frac{d-1}{d+1}} \alpha_{d-1}^{-\frac{2}{d+1}} \varrho(y)^{\frac{-2}{d+1}} \kappa(y)^{\frac{1}{d+1}} \lim_{n\to\infty} \int_0^{n^{-1/2}} n^{\frac{2}{d+1}} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\,ds. \tag{4.26}$$

In fact, since $g(n,y) \ll n^{-1/2}$, it is sufficient to show that

$$\lim_{n\to\infty} n^{\frac{2}{d+1}} \int_{c_1 n^{-1/2}}^{c_2 n^{-1/2}} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\,ds = 0$$

for any two constants $0 < c_1 \le c_2 < \infty$. Since the estimate (4.19) can be applied, we get

$$\begin{aligned}
n^{\frac{2}{d+1}} \int_{c_1 n^{-1/2}}^{c_2 n^{-1/2}} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\,ds &\ll n^{\frac{2}{d+1}} \int_{c_1 n^{-1/2}}^{c_2 n^{-1/2}} e^{-2^{-d}ns} s^{\frac{2}{d+1}-1}\,ds \\
&\ll \int_{2^{-d}c_1 n^{1/2}}^{2^{-d}c_2 n^{1/2}} e^{-r} r^{\frac{2}{d+1}-1}\,dr,
\end{aligned}$$

from which the conclusion follows.

Subsequently, we write $\mathbf{1}$ to denote the constant one function on $\mathbb{R}^d$. For the unit ball B^d, we recall that $B^d_{(n)}$ denotes the convex hull of n random points distributed uniformly and independently in B^d. We fix a point $w \in \partial B^d$, and for $s \in (0, \frac{1}{2})$, define $\tilde{w}_s := t \cdot w$, where $t \in (0,1)$ is chosen such that

$$s = \alpha_d^{-1} \cdot V(\{x \in B^d : \langle x, w\rangle \ge \langle \tilde{w}_s, w\rangle\}).$$

A classical result due to J.A. Wieacker [**46**] is that

$$\lim_{n\to\infty} n^{\frac{2}{d+1}} \mathbb{E}_{\mathbf{1},B^d} V(B^d \setminus B^d_{(n)}) = c_d\,\omega_d\,\alpha_d^{\frac{2}{d+1}},$$

where the constant c_d is given in (2.2). It follows from (4.9), (4.26) and the preceding remark that

$$\lim_{n\to\infty} \int_0^{n^{-1/2}} n^{\frac{2}{d+1}} \mathbb{P}_{\mathbf{1},B^d}\left(\tilde{w}_s \notin B^d_{(n)}\right) s^{-\frac{d-1}{d+1}}\,ds = c_d\,(d+1)^{\frac{d-1}{d+1}} \alpha_{d-1}^{\frac{2}{d+1}}. \tag{4.27}$$

We are now going to show that the same limit is obtained if B^d is replaced by the convex body K and if a normal boundary point y of K with positive Gauss curvature is considered instead of $w \in \partial B^d$.

LEMMA 4.6. *If $y \in \partial K$ is a normal boundary point of K satisfying $\kappa(y) > 0$, then*

$$\lim_{n\to\infty} \int_0^{n^{-1/2}} n^{\frac{2}{d+1}} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\,ds = c_d(d+1)^{\frac{d-1}{d+1}} \alpha_{d-1}^{\frac{2}{d+1}}.$$

Proof. Let $\varepsilon \in (0,1)$ be arbitrarily chosen. According to Lemma 4.5 and its notation and by the preceding remark, if n is sufficiently large, we have

$$\int_0^{n^{-1/2}} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\, ds = O\left(\frac{\varepsilon}{n^{\frac{2}{d+1}}}\right) + \sum_{i=0}^{k}\binom{n}{i}\int_{\frac{\varepsilon(d+1)/2}{n}}^{\frac{\alpha}{n}} (\beta s)^i(1-\beta s)^{n-i} \times \mathbb{P}_{\varrho,\widetilde{C}(y,\beta s)}\left(\tilde{y}_s \notin \widetilde{C}(y,\beta s)_{(i)}\right) s^{-\frac{d-1}{d+1}}\, ds. \tag{4.28}$$

We fix a unit vector p, and consider the reference paraboloid Ψ which is the graph of $z \mapsto \|z\|^2$ on $p^\perp$. For $\tau > 0$, define

$$C(\tau) := \left\{z + tp : z \in p^\perp \text{ and } \|z\|^2 \le t \le \tau^{\frac{2}{d+1}}\right\},$$

that is a cap of Ψ of height $\tau^{\frac{2}{d+1}}$. It is easy to check that $V(C(\tau)) = \tau V(C(1))$. We define

$$\tilde{s}(\beta,s) := \frac{V(\widetilde{C}(y,\beta s))}{V(C(\beta))}.$$

Then (4.12) implies that

$$\tilde{s}(\beta,s) = \frac{\beta s}{\mu(\beta,s)\varrho(y)\beta V(C(1))} = \frac{s}{\mu(\beta,s)\varrho(y)V(C(1))},$$

where $\mu(\beta,s) \to 1$ as $s \to 0^+$. Let A_s, $s > 0$, denote the affinity of $\mathbb{R}^d$ with $A_s(y) = y$ for which the associated linear map $\tilde{A}_s$ satisfies $\tilde{A}_s(v) = s^{\frac{1}{d+1}} v$ for $v \in u^\perp$ and $\tilde{A}_s(u) = s^{\frac{2}{d+1}} u$. Then the image under $A_{s^{-1}}$ of a cap von K at y converges in the Hausdorff metric as $s \to 0^+$ to a cap of the osculating paraboloid of K at y. For a more explicit statement, let A be a volume preserving affinity of $\mathbb{R}^d$ such that $A(y) = o$ and $A(y-u) = p$, which maps the osculating paraboloid of K at y to Ψ. Then $\Phi_{s,\beta} := A \circ A_{\tilde{s}(\beta,s)^{-1}}$ is an affinity satisfying

$$\Phi_{s,\beta}(y) = o, \quad \det(\Phi_{s,\beta}) = \tilde{s}(\beta,s)^{-1} = \frac{V(C(\beta))}{V(\widetilde{C}(y,\beta s))},$$

and, consequently, $\Phi_{s,\beta}(\widetilde{C}(y,\beta s)) \to C(\beta)$ in the Hausdorff metric as $s \to 0^+$. Moreover, we have

$$\lim_{s\to 0^+} \Phi_{s,\beta}(\tilde{y}_s) = \lim_{s\to 0^+} \Phi_{s,1}(\tilde{y}_s) = p,$$

since $\mu(\beta,s) \to 1$ and $\mu(1,s) \to 1$ as $s \to 0^+$, $\tilde{y}_s \in \partial\widetilde{C}(y,s)$ and $\Phi_{s,1}(\tilde{y}_s) \in \partial C(1)$, and by (4.17). Since ϱ is continuous at y, the properties of $\Phi_{s,\beta}$ imply that, for $i = 0,\ldots,k$,

$$\lim_{s\to 0^+} \mathbb{P}_{\varrho,\widetilde{C}(y,\beta s)}\left(\tilde{y}_s \notin \widetilde{C}(y,\beta s)_{(i)}\right) = \mathbb{P}_{\mathbf{1},C(\beta)}\left(p \notin C(\beta)_{(i)}\right). \tag{4.29}$$

We conclude from (4.28) and (4.29) that

$$\int_0^{n^{-1/2}} \mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\, ds = O\left(\frac{\varepsilon}{n^{\frac{2}{d+1}}}\right) + \sum_{i=0}^{k}\binom{n}{i}\int_{\frac{\varepsilon(d+1)/2}{n}}^{\frac{\alpha}{n}} (\beta s)^i(1-\beta s)^{n-i} \times \mathbb{P}_{\mathbf{1},C(\beta)}\left(p \notin C(\beta)_{(i)}\right) s^{-\frac{d-1}{d+1}}\, ds.$$

The same formula is obtained for

$$\int_0^{n^{-1/2}} \mathbb{P}_{\mathbf{1},B^d}\left(\tilde{w}_s \notin B^d_{(n)}\right) s^{-\frac{d-1}{d+1}}\, ds,$$

since $C(\beta)$ is independent of K. Since $\varepsilon \in (0,1)$ was arbitrary, we conclude

$$\lim_{n\to\infty}\int_0^{n^{-1/2}} n^{\frac{2}{d+1}}\mathbb{P}_\varrho\left(\tilde{y}_s \notin K_{(n)}\right) s^{-\frac{d-1}{d+1}}\, ds = \lim_{n\to\infty}\int_0^{n^{-1/2}} n^{\frac{2}{d+1}}\mathbb{P}_{\mathbf{1},B^d}\left(\tilde{w}_s \notin B^d_{(n)}\right) s^{-\frac{d-1}{d+1}}\, ds.$$

Now (4.27) yields Lemma 4.6. □

Proof of Theorem 3.1. Let $y \in \partial K$ be a normal boundary point of K. Combining Lemma 4.4, Lemma 4.6 and (4.26), we obtain

$$J_\varrho(y) = c_d\, \varrho(y)^{\frac{-2}{d+1}} \kappa(y)^{\frac{1}{d+1}}.$$

Therefore Theorem 3.1 is implied by (4.9). □

5. *Polarity and the proof of Theorem 2.1*

In this section, we deduce Theorem 2.1 and Theorem 2.2 from Theorem 3.1 and Corollary 3.2, respectively. In order to obtain more general results, for not necessarily homogeneous or isotropic hyperplane distributions, we start with a description of the basic setting.

Let $K \subset \mathbb{R}^d$ be a convex body with $o \in \operatorname{int}(K)$, as usual let $K^* := \{z \in \mathbb{R}^d : \langle x, z\rangle \le 1 \text{ for all } x \in K\}$ denote the polar body of K, and put $K_1 := K + B^d$. Let $\mathcal{H}_K$ denote the set of all hyperplanes H in $\mathbb{R}^d$ for which $H \cap \operatorname{int}(K) = \emptyset$ and $H \cap K_1 \neq \emptyset$. The motion invariant locally finite measure μ on the space $A(d, d-1)$ of hyperplanes, which satisfies $\mu(\mathcal{H}_K) = 2$, is explicitly given by

$$\mu = 2 \int_{S^{d-1}} \int_0^\infty \mathbf{1}\{H(u,t) \in \cdot\}\, dt\, \sigma(du),$$

where σ is the rotation invariant probability measure on the unit sphere S^{d-1}. The model of a random polytope (random polyhedral set) described in the introduction is based on random hyperplanes with distribution $\mu_K := 2^{-1}(\mu \llcorner \mathcal{H}_K)$. More generally, we now consider random hyperplanes with distribution

$$\mu_q := \int_{S^{d-1}} \int_0^\infty \mathbf{1}\{H(u,t) \in \cdot\} q(t,u)\, dt\, \sigma(du), \tag{5.1}$$

where $q : [0,\infty) \times S^{d-1} \to [0,\infty)$ is a measurable function which is

(q1) concentrated on $D_K := \{(t,u) \in [0,\infty) \times S^{d-1} : h(K,u) \le t \le h(K_1,u)\}$,

(q2) positive and continuous in a neighborhood of $\{(t,u) \in [0,\infty) \times S^{d-1} : t = h(K,u)\}$ relative to D_K,

(q3) and satisfies $\mu_q(\mathcal{H}_K) = 1$.

The intersection of n halfspaces H_i^- containing the origin o and bounded by n independent random hyperplanes H_i with distribution μ_q is denoted by $K^{(n)} := \bigcap_{i=1}^n H_i^-$. Probabilities and expectations with respect to μ_q are denoted by $\mathbb{P}_{\mu_q}$ and $\mathbb{E}_{\mu_q}$, respectively. The special example $q \equiv \mathbf{1}_{D_K}$ (q is the characteristic function of D_K) covers the situation discussed in the introduction.

In the following, beside the support function, we will also need the radial function $\rho(L,\cdot)$ of a convex body L with $o \in \operatorname{int}(L)$. Let F be a nonnegative measurable functional on convex polyhedral sets in $\mathbb{R}^d$. Using (5.1) and Fubini's theorem, we get

$$\begin{aligned}
\mathbb{E}_{\mu_q}(F(K^{(n)})) &= \int_{A(d,d-1)^n} F\left(\bigcap_{i=1}^n H_i^-\right) \mu_q^{\otimes n}(d(H_1,\ldots,H_n)) \\
&= \int_{(S^{d-1})^n} \int_{h(K,u_1)}^{h(K_1,u_1)} \cdots \int_{h(K,u_n)}^{h(K_1,u_n)} F\left(\bigcap_{i=1}^n H_i^-(u_i,t_i)\right) \prod_{i=1}^n q(t_i,u_i) \\
&\qquad \times dt_n \ldots dt_1\, \sigma^{\otimes n}(d(u_1,\ldots,u_n)).
\end{aligned}$$

For $t_1,\ldots,t_n > 0$, we have

$$\bigcap_{i=1}^n H_i^-(u_i,t_i) = \left[t_1{}^{-1}u_1, \ldots, t_n{}^{-1}u_n\right]^*.$$

Using the substitution $s_i = 1/t_i$, $\rho(L^*, u_i) = h(L, u_i)^{-1}$ for $L \in \mathcal{K}^n$ with $o \in \operatorname{int}(L)$, and polar coordinates, we obtain

$$\mathbb{E}_{\mu_q}(F(K^{(n)})) = \frac{1}{\omega_d^n} \int_{(K^* \setminus K_1^*)^n} F([x_1, \ldots, x_n]^*) \prod_{i=1}^n \left(\tilde{q}(x_i) \|x_i\|^{-(d+1)} \right) d(x_1, \ldots, x_n)$$

with $K_1^* := (K_1)^*$ and

$$\tilde{q}(x) := q\left(\frac{1}{\|x\|}, \frac{x}{\|x\|} \right), \qquad x \in K^* \setminus \{o\}.$$

The case $n = 1$ and $F \equiv 1$ yields

$$\frac{1}{\omega_d} \int_{K^* \setminus K_1^*} \tilde{q}(x) \|x\|^{-(d+1)} \, dx = 1,$$

hence

$$\varrho(x) := \begin{cases} {\omega_d}^{-1} \tilde{q}(x) \|x\|^{-(d+1)}, & x \in K^* \setminus K_1^*, \\ 0, & x \in K_1^*, \end{cases}$$

is a probability density with respect to $\mathcal{H}^d \llcorner K^*$ which is positive and continuous in a neighborhood of ∂K^* relative to K^*. Thus we conclude that

$$\begin{aligned} \mathbb{E}_{\mu_q}(F(K^{(n)})) &= \int_{(K^*)^n} F([x_1, \ldots, x_n]^*) \prod_{i=1}^n \varrho(x_i) \, d(x_1, \ldots, x_n) \\ &= \mathbb{E}_{\varrho, K^*} \left(F((K^*_{(n)})^*) \right), \end{aligned}$$

where $K^*_{(n)} := (K^*)_{(n)}$.

PROPOSITION 5.1. *Let $K \subset \mathbb{R}^d$ be a convex body with $o \in \operatorname{int}(K)$, and let q and ϱ be defined as above. Then the random polyhedral sets $K^{(n)}$ and $(K^*_{(n)})^*$ are equal in distribution.*

For a first application, let

$$F(P) := \mathbf{1}\{P \subset K_1\} \left(W(P) - W(K) \right),$$

for a polyhedral set $P \subset \mathbb{R}^d$, with the convention $0 \cdot \infty := 0$. For $x_1, \ldots, x_n \in K^* \setminus K_1^*$, we have $K \subset [x_1, \ldots, x_n]^*$ and, arguing as before,

$$\begin{aligned} F([x_1, \ldots, x_n]^*) &= \mathbf{1}\{[x_1, \ldots, x_n]^* \subset K_1\} \left(W([x_1, \ldots, x_n]^*) - W(K) \right) \\ &= 2 \cdot \mathbf{1}\{[x_1, \ldots, x_n]^* \subset K_1\} \int_{K^* \setminus [x_1, \ldots, x_n]} \lambda(x) \, dx, \end{aligned}$$

where

$$\lambda(x) := \begin{cases} {\omega_d}^{-1} \|x\|^{-(d+1)}, & x \in K^* \setminus K_1^*, \\ 0, & x \in K_1^*. \end{cases}$$

Note that if $[x_1, \ldots, x_n]^* \subset K_1$, then the set $[x_1, \ldots, x_n]^*$ is bounded, hence $o \in \operatorname{int}([x_1, \ldots, x_n])$, and therefore $K_1^* \subset [x_1, \ldots, x_n]^{**} = [x_1, \ldots, x_n]$.

As in [**9**], it can be shown that $\mathbb{P}_{\mu_q}(K^{(n)} \not\subset K_1) \ll \alpha^n$, for some $\alpha \in (0, 1)$ depending on K and q. By Proposition 5.1, we also get

$$\mathbb{P}_{\varrho, K^*} \left((K^*_{(n)})^* \not\subset K_1 \right) = \mathbb{P}_{\mu_q} \left(K^{(n)} \not\subset K_1 \right) \ll \alpha^n.$$

Hence

$$\begin{aligned}
&\mathbb{E}_{\mu_q}\Big(W(K^{(n)}\cap K_1)-W(K)\Big)\\
&\quad=\mathbb{E}_{\mu_q}\Big(\mathbf{1}\{K^{(n)}\subset K_1\}\Big(W(K^{(n)})-W(K)\Big)\Big)+O(\alpha^n)\\
&\quad=2\cdot\mathbb{E}_{\varrho,K^*}\left(\mathbf{1}\{(K^*_{(n)})^*\subset K_1\}\int_{K^*\setminus K^*_{(n)}}\lambda(x)\,dx\right)+O(\alpha^n)\\
&\quad=2\cdot\mathbb{E}_{\varrho,K^*}\left(\int_{K^*\setminus K^*_{(n)}}\lambda(x)\,dx\right)+O(\alpha^n),
\end{aligned}$$

where we used that λ is integrable. Therefore, Theorem 3.1 implies

$$\begin{aligned}
&\lim_{n\to\infty}n^{\frac{2}{d+1}}\,\mathbb{E}_{\mu_q}(W(K^{(n)}\cap K_1)-W(K))\\
&\quad=2\cdot\lim_{n\to\infty}n^{\frac{2}{d+1}}\,\mathbb{E}_{\varrho,K^*}\int_{K^*\setminus K^*_{(n)}}\lambda(x)\,dx\\
&\quad=2\,c_d\int_{\partial K^*}\varrho(x)^{-\frac{2}{d+1}}\lambda(x)\kappa^*(x)^{\frac{1}{d+1}}\,\mathcal{H}^{d-1}(dx)\\
&\quad=2\,c_d\,\omega_d{}^{-\frac{d-1}{d+1}}\int_{\partial K^*}\tilde{q}(x)^{-\frac{2}{d+1}}\|x\|^{-d+1}\kappa^*(x)^{\frac{1}{d+1}}\,\mathcal{H}^{d-1}(dx),
\end{aligned}$$

where κ^* denotes the generalized Gauss curvature of K^*. In the following, for $x\in\partial K$, let $\sigma_K(x)$ denote an exterior unit normal vector of K at x. It is unique for $\mathcal{H}^{d-1}$-almost all $x\in\partial K$.

THEOREM 5.2. *Let $K\subset\mathbb{R}^d$ be a convex body with $o\in\operatorname{int}(K)$, and let $q:[0,\infty)\times S^{d-1}\to[0,\infty)$ be a measurable function satisfying (q1)–(q3). Then*

$$\begin{aligned}
&\lim_{n\to\infty}n^{\frac{2}{d+1}}\,\mathbb{E}_{\mu_q}(W(K^{(n)}\cap K_1)-W(K))\\
&\quad=2\,c_d\,\omega_d{}^{-\frac{d-1}{d+1}}\int_{\partial K}q(h(K,\sigma_K(x)),\sigma_K(x))^{-\frac{2}{d+1}}\kappa(x)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(dx). \qquad (5.2)
\end{aligned}$$

The proof is completed in Section 6 by providing Lemma 6.2.

EXAMPLE. Observe that if $q:\{(h(K,u),u)\in(0,\infty)\times S^{d-1}:u\in S^{d-1}\}\to[0,\infty)$ is positive and continuous, then q can be extended to $[0,\infty)\times S^{d-1}$ such that (q1)–(q3) are satisfied. For any such extension, the right-hand side of (5.2) remains unchanged. As an example, we may choose q_1 such that $q_1(t,u)=t^{(d^2-1)/2}$ for $t=h(K,u)$ and $u\in S^{d-1}$. Then the integral in (5.2) turns into

$$\int_{\partial K}\frac{\kappa(x)^{\frac{d}{d+1}}}{\langle x,\sigma_K(x)\rangle^{d-1}}\,\mathcal{H}^{d-1}(dx)=\Omega_{d^2}(K),$$

where

$$\Omega_p(K):=\int_{\partial K}\frac{\kappa(x)^{\frac{p}{d+p}}}{\langle x,\sigma_K(x)\rangle^{\frac{(p-1)d}{d+p}}}\,\mathcal{H}^{d-1}(dx)$$

is the p-affine surface area of K (see [**26**], [**17**], [**18**], [**22**], [**44**], [**45**], [**23**], [**24**]). It has been shown that $\Omega_{d^2}(K)=\Omega_1(K^*)$; see [**18**]. Moreover, for a convex body $L\subset\mathbb{R}^d$, the equiaffine isoperimetric inequality states that

$$\Omega_1(L)\le d\alpha_d^{\frac{2}{d+1}}V(L)^{\frac{d-1}{d+1}}$$

with equality if and only if L is an ellipsoid (cf. [**27**], [**25**], [**26**], [**17**], [**5**]). Thus we get

$$\lim_{n\to\infty} n^{\frac{2}{d+1}}\,\mathbb{E}_{\mu_{q_1}}(W(K^{(n)}\cap K_1)-W(K)) \le 2dc_d\omega_d^{-\frac{d-1}{d+1}}\alpha_d^{\frac{2}{d+1}}V(K^*)^{\frac{d-1}{d+1}}$$

with equality if and only if K^* is an ellipsoid, that is, if and only if K is an ellipsoid. This can be interpreted as saying that among all convex bodies for which the volume of the polar body is fixed, ellipsoids are worst approximated asymptotically by circumscribed random polytopes (with respect to the density q_1) in the sense of the mean width.

For another application, we define

$$F(P) := f_{d-1}(P),$$

for a convex polyhedral set $P \subset \mathbb{R}^d$. It is well known that $f_0(P) = f_{d-1}(P^*)$ for a convex polytope $P \subset \mathbb{R}^d$ with $o \in \mathrm{int}(P)$. Thus, from Proposition 5.1 we get

$$\begin{aligned}
\mathbb{E}_{\mu_q}\left(f_{d-1}(K^{(n)})\right) &= \mathbb{E}_{\varrho,K^*}\left(f_{d-1}((K^*_{(n)})^*)\right)\\
&= \mathbb{E}_{\varrho,K^*}\left(\mathbf{1}\{(K^*_{(n)})^*\subset K_1\}f_{d-1}((K^*_{(n)})^*)\right)\\
&\qquad + \mathbb{E}_{\varrho,K^*}\left(\mathbf{1}\{(K^*_{(n)})^*\not\subset K_1\}f_{d-1}((K^*_{(n)})^*)\right)\\
&= \mathbb{E}_{\varrho,K^*}\left(\mathbf{1}\{(K^*_{(n)})^*\subset K_1\}f_0(K^*_{(n)})\right) + O(n\cdot\alpha^n)\\
&= \mathbb{E}_{\varrho,K^*}\left(f_0(K^*_{(n)})\right) + O(n\cdot\alpha^n),
\end{aligned}$$

where $\alpha \in (0,1)$ is a suitable constant.

The following Theorem 5.3 generalizes Theorem 2.1 in the same way as Theorem 5.2 extends Theorem 2.2.

THEOREM 5.3. *Let $K \subset \mathbb{R}^d$ be a convex body with $o \in \mathrm{int}(K)$, and let $q : [0,\infty)\times S^{d-1} \to [0,\infty)$ be a measurable function satisfying (q1)–(q3). Then*

$$\lim_{n\to\infty} n^{-\frac{d-1}{d+1}}\,\mathbb{E}_{\mu_q}(f_{d-1}(K^{(n)})) = c_d\,\omega_d{}^{-\frac{d-1}{d+1}}\int_{\partial K} q(h(K,\sigma_K(x)),\sigma_K(x))^{\frac{d-1}{d+1}}\kappa(x)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(dx).$$

The proof follows by applying Corollary 3.2 and Lemma 6.2.

6. *Polarity and an integral transformation*

In this section, we establish the required integral transformation involving the generalized Gauss curvatures of a convex body and its polar body. The main difficulty of the proof is due to the fact that we do not make any smoothness assumptions on the convex bodies that are considered.

Let $L \subset \mathbb{R}^d$ be a convex body. If the support function h_L of L is differentiable at $u \neq o$, then the gradient $\nabla h_L(u)$ of h_L at u is equal to the unique boundary point of L having u as an exterior normal vector. In particular, the gradient of h_L is a function which is homogeneous of degree zero. Note that h_L is differentiable at $\mathcal{H}^{d-1}$-almost all unit vectors. We write $D_{d-1}h_L(u)$ for the product of the principal radii of curvature of L in direction $u \in S^{d-1}$, whenever the support function h_L is twice differentiable in the generalized sense at $u \in S^{d-1}$. Note that this is the case for $\mathcal{H}^{d-1}$-almost all $u \in S^{d-1}$. The Gauss map σ_L is defined $\mathcal{H}^{d-1}$-almost everywhere on ∂L. If σ_L is differentiable in the generalized sense at $x \in \partial L$, which is the case for $\mathcal{H}^{d-1}$-almost all $x \in \partial L$, then the product of the eigenvalues of the differential is the Gauss curvature $\kappa_L(x)$. The connection to curvatures defined on the generalized normal bundle $\mathcal{N}(L)$ of L will be used in the following proof (cf. [**19**]).

LEMMA 6.1. *Let $L \subset \mathbb{R}^d$ be a convex body containing the origin in its interior. If $g : \partial L \to [0, \infty]$ is measurable, then*

$$\int_{\partial L} g(x)\kappa_L(x)^{\frac{1}{d+1}}\,\mathcal{H}^{d-1}(dx) = \int_{S^{d-1}} g(\nabla h_L(u))D_{d-1}h_L(u)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(du).$$

Proof. In the following proof, we use results and methods from [**19**], to which we refer for additional references and detailed definitions. Let $\mathcal{N}(L)$ denote the generalized normal bundle of L, and let $k_i(x,u) \in [0,\infty]$, $i = 1,\dots,d-1$, be the generalized curvatures of L, which are defined for $\mathcal{H}^{d-1}$-almost all $(x,u) \in \mathcal{N}(L)$. Expressions such as

$$\frac{k_i(x,u)^{\frac{1}{d+1}}}{\sqrt{1+k_i(x,u)^2}} \qquad \text{or} \qquad \frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}}$$

with $k_i(x,u) = \infty$ are understood as limits as $k_i(x,u) \to \infty$, and yield 0 or 1, respectively in the two given examples. As is common in measure theory, the product $0 \cdot \infty$ is defined as 0.

Our starting point is the expression

$$I := \int_{\mathcal{N}(L)} g(x) \prod_{i=1}^{d-1} \frac{k_i(x,u)^{\frac{1}{d+1}}}{\sqrt{1+k_i(x,u)^2}}\,\mathcal{H}^{d-1}(d(x,u)), \tag{6.1}$$

which will be evaluated in two different ways. A comparison of the resulting expressions yields the assertion of the lemma.

First, we rewrite I in the form

$$I = \int_{\mathcal{N}(L)} g(x) \left(\prod_{i=1}^{d-1} k_i(x,u)\right)^{-\frac{d}{d+1}} J_{d-1}\pi_2(x,u)\,\mathcal{H}^{d-1}(d(x,u)), \tag{6.2}$$

where

$$J_{d-1}\pi_2(x,u) = \prod_{i=1}^{d-1} \frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}},$$

for $\mathcal{H}^{d-1}$-almost all $(x,u) \in \mathcal{N}(L)$, is the (approximate) Jacobian of the map $\pi_2 : \mathcal{N}(L) \to S^{d-1}$, $(x,u) \mapsto u$. To check (6.2), we distinguish the following cases. If $k_i(x,u) = 0$ for some i, then the integrands on the right-hand sides of (6.1) and of (6.2) are zero, since $0 \cdot \infty = 0$ and $J_{d-1}\pi_2(x,u) = 0$. If $k_i(x,u) \neq 0$ for all i and $k_j(x,u) = \infty$ for some j, then again both integrands are zero. In all other cases the assertion is clear.

For $\mathcal{H}^{d-1}$-almost all $u \in S^{d-1}$, $\nabla h_L(u) \in \partial L$ is the unique boundary point of L which has u as an exterior unit normal vector. Then the coarea formula yields

$$I = \int_{S^{d-1}} g(\nabla h_L(u)) \left(\prod_{i=1}^{d-1} k_i(\nabla h_L(u), u)\right)^{-\frac{d}{d+1}} \mathcal{H}^{d-1}(du).$$

Using Lemma 3.4 in [**19**], we get

$$I = \int_{S^{d-1}} g(\nabla h_L(u))D_{d-1}h_L(u)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(du). \tag{6.3}$$

Now we consider also the projection $\pi_1 : \mathcal{N}(L) \to \partial L$, $(x,u) \mapsto x$, which has the (approximate) Jacobian

$$J_{d-1}\pi_1(x,u) = \prod_{i=1}^{d-1} \frac{1}{\sqrt{1+k_i(x,u)^2}},$$

for $\mathcal{H}^{d-1}$-almost all $(x,u) \in \mathcal{N}(L)$. A similar argument as before yields

$$\begin{aligned} I &= \int_{\mathcal{N}(L)} g(x) \left(\prod_{i=1}^{d-1} k_i(x,u) \right)^{\frac{1}{d+1}} J_{d-1}\pi_1(x,u)\, \mathcal{H}^{d-1}(d(x,u)) \\ &= \int_{\partial L} g(x) \left(\prod_{i=1}^{d-1} k_i(x,\sigma_L(x)) \right)^{\frac{1}{d+1}} \mathcal{H}^{d-1}(dx). \end{aligned}$$

By Lemma 3.1 in [**19**], we finally also get

$$I = \int_{\partial L} g(x)\kappa_L(x)^{\frac{1}{d+1}}\, \mathcal{H}^{d-1}(dx). \tag{6.4}$$

A comparison of equations (6.3) and (6.4) gives the required equality. □

REMARK. An alternative argument can be based on arguments similar to those used in [**17**] for the proof of the equality of two representations of the affine surface area of a convex body.

LEMMA 6.2. *Let $K \subset \mathbb{R}^d$ be a convex body with $o \in \mathrm{int}(K)$. If $f : [0,\infty) \times S^{d-1} \to [0,\infty)$ is a measurable function and $\tilde{f}(x) := f\left(\|x\|^{-1}, \|x\|^{-1}x\right)$, $x \in \partial K^*$, then*

$$\int_{\partial K^*} \tilde{f}(x)\|x\|^{-d+1}\kappa^*(x)^{\frac{1}{d+1}}\, \mathcal{H}^{d-1}(dx) = \int_{\partial K} f(h(K,\sigma_K(x)),\sigma_K(x))\kappa(x)^{\frac{d}{d+1}}\, \mathcal{H}^{d-1}(dx).$$

Proof. We apply Lemma 6.1 with $L = K^*$ and $g(x) = \tilde{f}(x)\|x\|^{-d+1}$, $x \in \partial K^*$, and thus we get

$$\begin{aligned} &\int_{\partial K^*} \tilde{f}(x)\|x\|^{-d+1}\kappa^*(x)^{\frac{1}{d+1}}\, \mathcal{H}^{d-1}(dx) \\ &\qquad = \int_{S^{d-1}} \tilde{f}(\nabla h_{K^*}(u))\|\nabla h_{K^*}(u)\|^{-d+1} D_{d-1}h_{K^*}(u)^{\frac{d}{d+1}}\, \mathcal{H}^{d-1}(du). \end{aligned}$$

Next we apply Theorem 2.2 in [**18**] (or the second part of Corollary 5.1 in [**20**]). Thus, using the fact that, for $\mathcal{H}^{d-1}$-almost all $u \in S^{d-1}$, h_{K^*} is differentiable in the generalized sense at u and $\rho(K,u)u$ is a normal boundary point of K, we have

$$D_{d-1}h_{K^*}(u)^{\frac{d}{d+1}} = \kappa(x)^{\frac{d}{d+1}} \langle u, \sigma_K(x)\rangle^{-d},$$

where $x = \rho(K,u)u \in \partial K$ and $u = \|x\|^{-1}x \in S^{d-1}$. Thus we obtain

$$\begin{aligned} &\int_{\partial K^*} \tilde{f}(x)\|x\|^{-d+1}\kappa^*(x)^{\frac{1}{d+1}}\, \mathcal{H}^{d-1}(dx) \\ &\qquad = \int_{S^{d-1}} \tilde{f}(\nabla h_{K^*}(u)) \frac{\|\nabla h_{K^*}(u)\|^{-d+1}}{\langle u, \sigma_K(\rho(K,u)u)\rangle^d}\, \kappa(\rho(K,u)u)^{\frac{d}{d+1}}\, \mathcal{H}^{d-1}(dx). \end{aligned}$$

The bijective and bilipschitz transformation $T : S^{d-1} \to \partial K$, $u \mapsto \rho(K,u)u$, has the Jacobian

$$JT(u) = \frac{\|\nabla h_{K^*}(u)\|}{h_{K^*}(u)^d}$$

for $\mathcal{H}^{d-1}$-almost all $u \in S^{d-1}$ (see the proof of Lemma 2.4 in [**18**]). Thus

$$\begin{aligned}
&\int_{\partial K^*} \tilde{f}(x)\|x\|^{-d+1}\kappa^*(x)^{\frac{1}{d+1}}\,\mathcal{H}^{d-1}(dx) \\
&= \int_{\partial K} \tilde{f}\left(\nabla h_{K^*}\left(\frac{x}{\|x\|}\right)\right)\frac{\|\nabla h_{K^*}\left(\frac{x}{\|x\|}\right)\|^{-d}}{\langle\frac{x}{\|x\|},\sigma_K(x)\rangle^d} h_{K^*}\left(\frac{x}{\|x\|}\right)^d \kappa(x)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(dx) \\
&= \int_{\partial K} \tilde{f}\left(\nabla h_{K^*}(x)\right)\frac{\|\nabla h_{K^*}(x)\|^{-d}}{\langle x,\sigma_K(x)\rangle^d} h_{K^*}(x)^d\,\kappa(x)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(dx) \\
&= \int_{\partial K} f(\|\nabla h_{K^*}(x)\|^{-1},\nabla h_{K^*}(x)/\|\nabla h_{K^*}(x)\|)\kappa(x)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(dx), \\
&= \int_{\partial K} f(h_K(\sigma_K(x)),\sigma_K(x))\kappa(x)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(dx),
\end{aligned}$$

since $h_{K^*}(x) = 1$ for $x \in \partial K$ and $x^* := \nabla h_{K^*}(x)$ satisfies $\|x^*\|^{-1} = \langle x, \sigma_K(x)\rangle$ and $x^*/\|x^*\| = \sigma_K(x)$, for $\mathcal{H}^{d-1}$-almost all $x \in \partial K$. $\square$

Acknowledgements. We are grateful for stimulating discussions with Rolf Schneider.

References

1. I. BÁRÁNY, 'Random polytopes in smooth convex bodies', *Mathematika* 39 (1992) 81–92.
2. I. BÁRÁNY, CH. BUCHTA, 'Random polytopes in a convex polytope, independence of shape, and concentration of vertices', *Math. Ann.* 297 (1993) 467–497.
3. I. BÁRÁNY, M. REITZNER, 'Random polytopes', preprint, 2008, available from `http://www.dmg.tuwien.ac.at/reitzner/reitznerselpub.html`
4. I. BÁRÁNY,, V. VU, 'Central limit theorems for Gaussian polytopes', *Math. Ann.* 35 (2007), 1593–1621.
5. K. BÖRÖCZKY JR., 'Stability of the Blaschke-Santaló and the affine isoperimetric inequality', preprint, 2008, available from `http://www.renyi.hu/~carlos/carlospub.html`
6. K.J. BÖRÖCZKY, F. FODOR, M. REITZNER, V. VÍGH, 'Mean width of inscribed random polytopes in a reasonably smooth convex body', submitted, available from `http://www.renyi.hu/~carlos/carlospub.html`
7. K.J. BÖRÖCZKY, L.M. HOFFMANN, D. HUG, 'Expectation of intrinsic volumes of random polytopes', submitted, available from `http://www.mathematik.uni-karlsruhe.de/iag4/~hug/seite/pub/`.
8. K. BÖRÖCZKY JR., M. REITZNER, 'Approximation of smooth convex bodies by random circumscribed polytopes', *Ann. Appl. Probab.* 14 (2004) 239–273.
9. K. BÖRÖCZKY JR., R. SCHNEIDER, 'The mean width of circumscribed random polytopes', submitted, available from `http://www.renyi.hu/~carlos/carlospub.html`
10. K.H. BORGWARDT, 'The Simplex Method, A Pobabilistic Analysis' (Springer, Berlin, 1987).
11. L. DÜMBGEN, G. WALTHER, 'Rates of convergence for random approximations of convex sets', *Adv. Appl. Probab.* (SGSA) 28 (1996) 384–393.
12. B. EFRON, 'The convex hull of a random set of points', *Biometrika* 52 (1965) 331–343.
13. H. FEDERER, *Geometric Measure Theory* (Springer, Berlin, 1969).
14. S. GLASAUER, P.M. GRUBER, 'Asymptotic estimates for best and stepwise approximation of convex bodies III', *Forum Math.* 9 (1997) 383–404.
15. P.M. GRUBER, 'Comparisons of best and random approximations of convex bodies by polytopes', *Rend. Circ. Mat. Palermo (2) Suppl.* 50 (1997) 189–216.
16. P.M. GRUBER, *Convex and Discrete Geometry* (Springer, Berlin, 2007).
17. D. HUG, 'Contributions to affine surface area', *Manuscripta Math.* 91 (1996) 283–301.
18. D. HUG, 'Curvature relations and affine surface area for a general convex body and it polar', *Results Math.* 29 (1996) 233–248.
19. D. HUG, 'Absolute continuity for curvature measures of convex sets I', *Math. Nachr.* 195 (1998) 139–158.
20. D. HUG, 'Absolute continuity for curvature measures of convex sets, III', *Advances Math.* 169 (2002) 92–117.
21. F.J. KALTENBACH, 'Asymptotisches Verhalten zufälliger konvexer Polyeder', doctoral thesis, Albert-Ludwigs-Universität, Freiburg i. Br., 1990.
22. K. LEICHTWEISS, *Affine geometry of convex bodies* (Johann Ambrosius Barth Verlag, Heidelberg, 1998).
23. M. LUDWIG, M. REITZNER, 'A characterization of affine surface area', *Adv. Math.* 147 (1999) 138–172.
24. M. LUDWIG, M. REITZNER, 'A classification of $SL(n)$ invariant valuations', *Ann. of Math.* (to appear).
25. E. LUTWAK, 'Selected affine isoperimetric inequalities'. In: Handbook of convex geometry, North-Holland, Amsterdam, 1993, 151–176.

26. E. LUTWAK, 'The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas', *Adv. Math.* 118 (1996) 244–294.

27. C.M. PETTY, 'Affine isoperimetric problems', Discrete geometry and convexity (New York, 1982), 113–127, Ann. New York Acad. Sci., 440, New York Acad. Sci., New York, 1985.

28. M. REITZNER, 'Random polytopes and the Efron-Stein jackknife inequality', *Ann. Prob.* 31 (2003) 2136–2166.

29. M. REITZNER, 'Central limit theorems for random polytopes', *Probab. Theory Relat. Fields* 133 (2005), 483–507.

30. A. RÉNYI, R. SULANKE, 'Über die konvexe Hülle von n zufällig gewählten Punkten', *Z. Wahrscheinlichkeitsth. verw. Geb.* 2 (1963) 75–84.

31. A. RÉNYI, R. SULANKE, 'Über die konvexe Hülle von n zufällig gewählten Punkten. II', *Z. Wahrscheinlichkeitsth. verw. Geb.* 3 (1964) 138–147.

32. A. RÉNYI, R. SULANKE, 'Zufällige konvexe Polygone in einem Ringgebiet', *Z. Wahrscheinlichkeitsth. verw. Geb.* 9 (1968) 146–157.

33. R. SCHNEIDER, Random approximation of convex sets', *J. Microsc.* 151 (1988) 211–227.

34. R. SCHNEIDER, *Convex Bodies: the Brunn-Minkowski Theory* (Cambridge University Press, Cambridge, 1993).

35. R. SCHNEIDER, 'Discrete aspects of stochastic geometry'. In: J.E. Goodman, J. O'Rourke (eds.), Handbook of Discrete and Computational Geometry, 2nd ed., CRC Press, Boca Raton 2004, pp. 255–278.

36. R. SCHNEIDER, W. WEIL, *Stochastic and Integral Geometry* (Springer, Probability and its Applications, Berlin, 2008).

37. R. SCHNEIDER, J.A. WIEACKER, 'Random polytopes in a convex body', *Z. Wahrscheinlichkeitsth. verw. Geb.* 52 (1980) 69–73.

38. C. SCHÜTT, 'Random polytopes and affine surface area', *Math. Nachr.* 170 (1994) 227–249.

39. C. SCHÜTT, E. WERNER, 'The convex floating body', *Math. Scand.* 66 (1990) 275–290.

40. J.J. SYLVESTER, 'Problem 1491', *Educational Times*, London, 1864.

41. V. VU, 'Sharp concentration of random polytopes', *Geom. Funct. Anal.* 15 (2005) 1284–1318.

42. V. VU, 'Central limit theorems for random polytopes in a smooth convex set', *Adv. Math.* 207 (2006) 221–243.

43. W. WEIL, J.A. WIEACKER, 'Stochastic geometry'. In: P.M. Gruber, J.M. Wills (eds.), Handbook of Convex Geometry, North-Holland, Amsterdam 1993, pp. 1391–1438.

44. E. WERNER, 'On L_p affine surface areas', *Indiana Univ. Math. J.* 56 (2007) 2305–2324.

45. E. WERNER, D. YE, 'New L_p affine isoperimetric inequalities', *Adv. Math.* 218 (2008) 762–780.

46. J.A. WIEACKER, 'Einige Probleme der polyedrischen Approximation' (Diplomarbeit, Albert-Ludwigs-Universität, Freiburg i. Br., 1978).

47. H. ZIEZOLD, 'Über die Eckenanzahl zufälliger konvexer Polygone', *Izv. Akad. Nauk Armjan. SSR Ser. Mat.* 5 (1970) 296–312.

Károly J. Böröczky
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
PO Box 127
H-1364 Budapest
Hungary

carlos@renyi.hu

and
Department of Geometry
Roland Eötvös University
Pázmány Péter sétány 1/C
H-1117 Budapest
Hungary

Ferenc Fodor
Department of Geometry, University of Szeged
Aradi vértanúk tere 1
H-6720 Szeged
Hungary

fodorf@math.u-szeged.hu

and
Department of Mathematics and Statistics
University of Calgary
2500 University Dr. N.W.
Calgary, Alberta
Canada
T2N 1N4

fodor@math.ucalgary.ca

Daniel Hug
Institut für Algebra und Geometrie
Universität Karlsruhe (TH), KIT
D-76133 Karlsruhe
Germany

daniel.hug@kit.edu