Chapter 2
Tensor Valuations and Their Local Versions

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Abstract The intrinsic volumes, recalled in the previous chapter, provide an array of size measurements for a convex body, one for each integer degree of homogeneity from 0 to n. For measurements and descriptions of other aspects, such as position, moments of the volume and of other size functionals, or anisotropy, tensor-valued functionals on convex bodies are useful. The classical approach leading to the intrinsic volumes, namely the Steiner formula for parallel bodies, can be extended by replacing the volume by higher moments of the volume. This leads, in a natural way, to a series of tensor-valued valuations. These so-called Minkowski tensors are introduced in the present chapter, and their properties are studied. A version of Hadwiger's theorem for tensor valuations is stated. The next natural step is a localization of the Minkowski tensors, in the form of tensor-valued measures. The essential valuation, equivariance and continuity properties of these local Minkowski tensors are collected. The main goal is then a description of the vector space of all tensor valuations on convex bodies sharing these properties. Continuity properties of local Minkowski tensors and of support measures follow from continuity properties of normal cycles of convex bodies. We establish Hölder continuity of the normal cycles of convex bodies, which provides a quantitative improvement of the aforementioned continuity property.

2.1 The Minkowski Tensors

We use the notation introduced in Chap. 1. We recall that the intrinsic volumes, certainly the most important valuations in the theory of convex bodies in Euclidean

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space, all arise from one basic valuation, the volume functional. In fact, they are generated by the Steiner formula (Chap. 1, formula (1.16)),

$$V_n(K+\rho B^n) = \sum_{j=0}^{n} \rho^{n-j} \kappa_{n-j} V_j(K), \qquad \rho \geq 0. \tag{2.1}$$

Here and in the following, $K \in \mathscr{K}^n$ denotes a convex body. The point to be kept in mind is that the evaluation of the volume of parallel bodies leads to a polynomial expansion and that the coefficients yield new valuations, which inherit some essential properties of the volume functional, but are no longer simple.

The volume functional, which we may write as

$$V_n(K) = \int_K \mathrm{d}x,$$

where dx indicates integration with respect to Lebesgue measure, has a natural vector-valued analogue, the moment vector

$$\int_K x \,\mathrm{d}x,$$

which is needed to define the centre of gravity,

$$c(K) := \frac{1}{V_n(K)} \int_K x \,\mathrm{d}x,$$

of convex bodies K with positive volume. If one wants to study moments of inertia, for example, one has to consider matrices with entries of type

$$\int_K \xi_i \xi_j \,\mathrm{d}x,$$

where $\xi_1, \dots, \xi_n$ are the coordinates of $x \in \mathbb{R}^n$ with respect to an orthonormal basis. This can be continued and leads to a series of simple valuations with values in spaces of symmetric tensors. Application to parallel bodies and polynomial expansion then reveals more general tensor-valued valuations. In the present section, we introduce these tensor valuations.

First we fix some conventions how to deal with tensors. We use the scalar product of $\mathbb{R}^n$ to identify $\mathbb{R}^n$ with its dual space. Thus, each vector $a \in \mathbb{R}^n$ is identified with the linear functional $x \mapsto a \cdot x$ from $\mathbb{R}^n$ to $\mathbb{R}$. For $r \in \mathbb{N}_0$, an *r-tensor*, or tensor of rank r, on $\mathbb{R}^n$ is defined as an r-linear mapping from $(\mathbb{R}^n)^r$ to $\mathbb{R}$. It is *symmetric* if it is invariant under permutations of its arguments. By $\mathbb{T}^r$ we denote the real vector space (with its standard topology) of symmetric r-tensors on $\mathbb{R}^n$. By definition, $\mathbb{T}^0 = \mathbb{R}$, and by the identification made above, $\mathbb{T}^1 = \mathbb{R}^n$. The *symmetric tensor product* of the symmetric tensors $a_i \in \mathbb{T}^{r_i}$, $i = 1, \dots, k$, is defined as follows. We write $s_0 = 0$, $s_i = r_1 + \cdots + r_i$ for $i = 1, \dots, k$, then

$$(a_1 \odot \cdots \odot a_k)(x_1, \ldots, x_{s_k}) := \frac{1}{s_k!} \sum_{\sigma \in \mathscr{S}(s_k)} \prod_{i=1}^{k} a_i(x_{\sigma(s_{i-1}+1)}, \ldots, x_{\sigma(s_i)})$$

for $x_1, \ldots, x_{s_k} \in \mathbb{R}^n$, where $\mathscr{S}(m)$ denotes the group of permutations of the numbers $1, \ldots, m$. Then $a_1 \odot \cdots \odot a_k \in \mathbb{T}^{r_1+\cdots+r_k}$. Thus the space of symmetric tensors (of arbitrary rank) becomes an associative, commutative graded algebra with unit. We shall always use the abbreviations $a \odot b =: ab$,

$$a_1 \odot \cdots \odot a_k =: a_1 \cdots a_k, \qquad \underbrace{a \odot \cdots \odot a}_{r} =: a^r, \qquad a^0 := 1.$$

For instance, for a vector $a \in \mathbb{R}^n$, the r-tensor a^r with $r \geq 1$ is given by

$$a^r(x_1, \ldots, x_r) = (a \cdot x_1) \cdots (a \cdot x_r), \qquad x_1, \ldots, x_r \in \mathbb{R}^n.$$

The scalar product,

$$Q(x,y) = x \cdot y, \qquad x, y \in \mathbb{R}^n,$$

is a symmetric tensor of rank two; we call Q the *metric tensor*.

Let $(e_1, \ldots, e_n)$ be an orthonormal basis of $\mathbb{R}^n$. Then the tensors $e_{i_1} \cdots e_{i_r}$ with $1 \leq i_1 \leq \cdots \leq i_r \leq n$ form a basis of $\mathbb{T}^r$. The corresponding coordinate representation of $T \in \mathbb{T}^r$ is given by

$$T = \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq n} t_{i_1 \ldots i_r} e_{i_1} \cdots e_{i_r} \tag{2.2}$$

with

$$t_{i_1 \ldots i_r} = \binom{r}{m_1 \ldots m_n} T(e_{i_1}, \ldots, e_{i_r}), \tag{2.3}$$

where m_k counts how often the number k appears among the indices $i_1, \ldots, i_r$ ($k = 1, \ldots, n$). (We remark that (2.3) should replace the formula given in [19, p. 463, line -8].)

Now we define the moment tensors, which generalize the volume. Integrals of tensor-valued functions can, of course, be defined coordinate-wise. For $r \in \mathbb{N}_0$, let

$$\Psi_r(K) := \frac{1}{r!} \int_K x^r \, \mathrm{d}x, \quad K \in \mathscr{K}^n. \tag{2.4}$$

Thus, $\Psi_r(K) \in \mathbb{T}^r$, and explicitly

$$\Psi_r(K)(y_1, \ldots, y_r) = \frac{1}{r!} \int_K (x \cdot y_1) \cdots (x \cdot y_r) \, \mathrm{d}x$$

for $y_1, \ldots, y_r \in \mathbb{R}^n$. The factor $1/r!$ in (2.4) is only for convenience. It is clear that $\Psi_r : \mathscr{K}^n \to \mathbb{T}^r$ is a simple valuation.

Immediately from (2.4) we see how Ψ_r behaves under translations. Since the binomial theorem holds for the symmetric tensor product, for $t \in \mathbb{R}^n$ we get

$$\Psi_r(K+t) = \sum_{j=0}^{r} \frac{1}{j!} \Psi_{r-j}(K) t^j. \tag{2.5}$$

Formally, this looks like an ordinary polynomial, but we have to keep in mind that here, according to our notational conventions,

$$\Psi_{r-j}(K)t^j = \Psi_{r-j}(K) \odot \underbrace{t \odot \cdots \odot t}_{j}.$$

Nevertheless, in view of (2.5) one says that Ψ_r has *polynomial behaviour* under translations.

Also the behaviour under rotations is easy to see. Let $\mathrm{O}(n)$ be the orthogonal group of $\mathbb{R}^n$. Its elements are called *rotations* of $\mathbb{R}^n$; thus, rotations in our terminology can be proper (orientation preserving) or improper. For $\vartheta \in \mathrm{O}(n)$ and for $y_1, \dots, y_r \in \mathbb{R}^n$ we have

$$\begin{aligned}
\Psi_r(\vartheta K)(y_1, \dots, y_r) &= \frac{1}{r!} \int_{\vartheta K} (x \cdot y_1) \cdots (x \cdot y_r) \, \mathrm{d}x \\
&= \frac{1}{r!} \int_K (\vartheta x \cdot y_1) \cdots (\vartheta x \cdot y_r) \, \mathrm{d}x \\
&= \frac{1}{r!} \int_K (x \cdot \vartheta^{-1} y_1) \cdots (x \cdot \vartheta^{-1} y_r) \, \mathrm{d}x \\
&= \Psi_r(K)(\vartheta^{-1} y_1, \dots, \vartheta^{-1} y_r) = (\vartheta \Psi_r(K))(y_1, \dots, y_r).
\end{aligned}$$

Thus,

$$\Psi_r(\vartheta K) = \vartheta \Psi_r(K),$$

where the usual operation of $\mathrm{O}(n)$ on $\mathbb{T}^r$ is defined by

$$(\vartheta a)(y_1, \dots, y_r) = a(\vartheta^{-1} y_1, \dots, \vartheta^{-1} y_r)$$

for $a \in \mathbb{T}^r$.

The tensor functional Ψ_r also satisfies a Steiner formula. To express it in a convenient way, we have to introduce further tensor functionals. In the following, we use the support measures Λ_k (see Chap. 1, Sec. 1.3), which are Borel measures on $\Sigma^n = \mathbb{R}^n \times \mathbb{S}^{n-1}$. The constants κ_j, ω_j were introduced in Chap. 1, (1.14).

Definition 2.1. The *Minkowski tensors* are defined by

$$\Phi_k^{r,s}(K) := \frac{1}{r!s!} \frac{\omega_{n-k}}{\omega_{n-k+s}} \int_{\Sigma^n} x^r u^s \, \Lambda_k(K, \mathrm{d}(x,u)) \tag{2.6}$$

for $k = 1, \dots, n-1$ and $r, s \in \mathbb{N}_0$. Further, we define

$$\Phi_n^{r,0}(K) := \Psi_r(K) \tag{2.7}$$

and

$$\Phi_k^{r,s} := 0 \qquad \text{if } k \notin \{0, \dots, n\} \text{ or } r \notin \mathbb{N}_0 \text{ or } s \notin \mathbb{N}_0 \text{ or } k = n,\ s \neq 0.$$

The latter definition will allow us later to extend some summations formally over all nonnegative integers.

Now we can formulate a Steiner-type formula.

Theorem 2.2. *For $r \in \mathbb{N}_0$, $K \in \mathscr{K}^n$ and $\rho \geq 0$, the formula*

$$\Psi_r(K+\rho B^n) = \sum_{k=0}^{n+r} \rho^{n+r-k} \kappa_{n+r-k} V_k^{(r)}(K) \tag{2.8}$$

holds, where

$$V_k^{(r)} = \sum_{s \in \mathbb{N}_0} \Phi_{k-r+s}^{r-s,s}. \tag{2.9}$$

For $r = 0$, formula (2.8) reduces to the ordinary Steiner formula (2.1) for the volume.

We indicate the proof of formula (2.8). For this, we need to compute an integral $\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x$ by a procedure that generalizes the transformation to polar coordinates, with the role of the unit sphere played by the boundary of a general convex body. Since such a general convex body need neither be smooth nor strictly convex, this generalized transformation formula makes use of the support measures. These satisfy themselves a Steiner formula [20, Theorem 4.2.7], of which here the following special case is relevant. We write $K_\rho := K + \rho B^n$, for $\rho \geq 0$, and define the mapping

$$\tau_\rho : \Sigma^n \to \Sigma^n, \qquad \tau_\rho(x,u) := (x+\rho u, u).$$

Then

$$2\Lambda_{n-1}(K_\rho, \cdot) = \sum_{k=0}^{n-1} \rho^{n-k-1} \omega_{n-k} \tau_\rho \Lambda_k(K, \cdot),$$

where $\tau_\rho \Lambda_k(K,\cdot)$ is the image measure (push-forward) of $\Lambda_k(K,\cdot)$ under τ_ρ. Using this, the following formula can be proved ([20, Theorem 4.2.8]).

Lemma 2.3. *Let $K \in \mathscr{K}^n$, and let $f : \mathbb{R}^n \setminus K \to \mathbb{R}$ be a nonnegative measurable function. Then*

$$\int_{\mathbb{R}^n \setminus K} f(x)\,\mathrm{d}x = \sum_{j=0}^{n-1} \omega_{n-j} \int_0^\infty t^{n-j-1} \int_{\Sigma^n} f(x+tu)\, \Lambda_j(K, \mathrm{d}(x,u))\,\mathrm{d}t. \tag{2.10}$$

To prove now formula (2.8), we first write

$$\Psi_r(K_\rho) = \Psi_r(K) + \frac{1}{r!} \int_{K_\rho \setminus K} x^r\,\mathrm{d}x.$$

To the last term we apply the transformation (2.10) coordinate-wise and obtain

$$\begin{aligned}\int_{K_\rho\setminus K} x^r\,\mathrm{d}x &= \sum_{j=0}^{n-1}\omega_{n-j}\int_0^\rho t^{n-j-1}\int_{\Sigma^n}(x+tu)^r\,\Lambda_j(K,\mathrm{d}(x,u))\,\mathrm{d}t\\ &= \sum_{j=0}^{n-1}\omega_{n-j}\int_0^\rho t^{n-j-1}\int_{\Sigma^n}\sum_{s=0}^{r}\binom{r}{s}x^{r-s}u^s t^s\,\Lambda_j(K,\mathrm{d}(x,u))\,\mathrm{d}t\\ &= \sum_{j=0}^{n-1}\sum_{s=0}^{r}\omega_{n-j}\binom{r}{s}\frac{\rho^{n-j+s}}{n-j+s}\int_{\Sigma^n}x^{r-s}u^s\,\Lambda_j(K,\mathrm{d}(x,u)).\end{aligned}$$

Introducing the index $k=j+r-s$ and using the definition (2.6), we obtain the assertion (2.8).

2.2 A Classification of Tensor Valuations

To describe our next goals, we recall Hadwiger's characterization theorem (Chap. 1, Thm. 1.23). It determines the real vector space of all mappings $\varphi:\mathscr{K}^n\to\mathbb{R}$ which are

- valuations,
- rigid motion invariant,
- continuous.

The result is that this vector space is spanned by the intrinsic volumes $V_0,\dots,V_n$. These intrinsic volume functionals are linearly independent, because they have different degrees of homogeneity and are not identically zero; hence the vector space in question has dimension $n+1$.

As the intrinsic volumes have been generalized to Minkowski tensors, it is natural to ask whether, respectively in which form, Hadwiger's characterization theorem can be extended. For tensor valuations of rank one, there is a closely analogous result.

Theorem 2.4. *The real vector space of all mappings $\psi:\mathscr{K}^n\to\mathbb{R}^n$ which are*

- *valuations,*
- *rotation equivariant, and such that $\psi(K+t)-\psi(K)$ is parallel to t,*
- *continuous,*

is spanned by the mappings

$$K\mapsto\int_K x\,C_j(K,\mathrm{d}x),\qquad j=0,\dots,n.$$

Recall from Chap. 1, Sec. 1.3, the relation between the support measures $\Lambda_j(K,\cdot)$ and the curvature measures $C_j(K,\cdot)$. The integral $\int_K x\,C_j(K,\mathrm{d}x)$ is the moment vector of the curvature measure $C_j(K,\cdot)$. Again, the vector space in question has dimension $n+1$, because the moment vectors have different degrees of homogeneity and are not identically zero. The result was proved by Hadwiger and Schneider [8]. Although it looks similar to Hadwiger's characterization theorem, its proof uses a different approach. One might wonder why the dimension of the vector space is still $n+1$. The Steiner formula for the moment vector $\int_K x\,\mathrm{d}x$ has, in fact, $n+2$ terms. However, one of these, namely $\int_{B^n} x\,\mathrm{d}x$, is identically zero.

For tensor valuations of ranks larger than one, the situation is more complicated. It remains true that each Minkowski tensor $\Phi_k^{r,s}$ defines a mapping $\Gamma : \mathscr{K}^n \to \mathbb{T}^p$, for $p = r+s$, which is a valuation and is continuous. The behaviour under isometries (combinations of rotations and translations) can be described as follows. First, we point out that in Hadwiger's theorem, 'rigid motions' are orientation preserving, whereas in the following, a 'rotation' is an element of $\mathrm{O}(n)$ and thus can be improper. The mapping Γ is *rotation covariant*, that is, if $\vartheta \in \mathrm{O}(n)$ is a rotation, then $\Gamma(\vartheta K) = \vartheta\Gamma(K)$ for all $K \in \mathscr{K}^n$. We recall that the operation of the orthogonal group appearing here is defined by

$$(\vartheta T)(y_1,\ldots,y_p) = T(\vartheta^{-1}y_1,\ldots,\vartheta^{-1}y_p) \qquad \text{for } y_1,\ldots,y_p \in \mathbb{R}^n,\, T \in \mathbb{T}^p.$$

Further, Γ has *polynomial translation behaviour*, by which we mean that

$$\Gamma(K+t) = \sum_{j=0}^{p} \frac{1}{j!}\Gamma_{p-j}(K)t^j \qquad \text{for } K \in \mathscr{K}^n,\ t \in \mathbb{R}^n,$$

with tensors $\Gamma_{p-j}(K) \in \mathbb{T}^{p-j}$, which are independent of t. (By convention, $0^0 = 1$ here.) We say that Γ is *isometry covariant* if it has both properties, rotation covariance and polynomial behaviour under translations.

One new aspect appearing for higher ranks is the following. For rank two, there is a constant mapping $\Gamma : \mathscr{K}^n \to \mathbb{T}^2$ that has all the properties listed above, namely $\Gamma(K) = Q$, the metric tensor. Since Q does not depend on K, this mapping Γ is trivially a valuation, continuous, and has polynomial behaviour under translations. Since, for $\vartheta \in \mathrm{O}(n)$,

$$Q(y_1,y_2) = y_1 \cdot y_2 = \vartheta^{-1}y_1 \cdot \vartheta^{-1}y_2 = (\vartheta Q)(y_1,y_2),$$

Γ is also rotation covariant. Since the considered properties are preserved under symmetric products, it follows that also the mappings $K \mapsto Q^m\Phi_k^{r,s}(K)$, for any $m \in \mathbb{N}_0$, share these properties with the Minkowski tensors. But this is as far as we can go, as the following characterization theorem due to Alesker [1] shows.

Theorem 2.5 (Alesker). *Let $p \in \mathbb{N}_0$. The real vector space of all mappings $\Gamma : \mathscr{K}^n \to \mathbb{T}^p$ which are*

- *valuations,*
- *isometry covariant,*
- *continuous,*

is spanned by the tensor valuations

$$Q^m\Phi_k^{r,s}, \tag{2.11}$$

where $m,r,s \in \mathbb{N}_0$ satisfy $2m+r+s=p$, where $k \in \{0,\dots,n\}$, and where $s=0$ if $k=n$.

The characterizations given in Theorem 1.23 of Chap. 1 (i.e., Hadwiger's characterization theorem) and Theorem 2.4 are special cases of this result. However, there is an essential difference: for $p \geq 2$, the spanning tensor valuations (2.11) are no longer linearly independent. They satisfy a series of linear relations, known as the *McMullen relations*. We prove these now.

The crucial relation is the identity

$$Q\Phi_n^{r-1,0} = 2\pi\Phi_{n-1}^{r,1}. \tag{2.12}$$

Explicitly, this reads

$$Q\Psi_{r-1}(K) = \frac{2}{r!}\int_{\Sigma^n} x^r u\,\Lambda_{n-1}(K,\mathrm{d}(x,u)). \tag{2.13}$$

It suffices to prove this identity for smooth convex bodies, because the general case can then be obtained by approximation. If K is smooth, we denote by $u(K,x)$ the unique outer unit normal vector of K at its boundary point x. For a smooth convex body K, the measure $2\Lambda_{n-1}(K,\cdot)$ is the image measure of the Hausdorff measure $\mathscr{H}^{n-1}$ on ∂K, the boundary of K, under the measurable mapping $x \mapsto (x,u(K,x))$ from ∂K to Σ^n. Therefore, equation (2.13) is equivalent to

$$Q\Psi_{r-1}(K) = \frac{1}{r!}\int_{\partial K} x^r u(K,x)\,\mathscr{H}^{n-1}(\mathrm{d}x). \tag{2.14}$$

To prove this, we use coordinates. We introduce an orthonormal basis $(e_1,\dots,e_n)$ of $\mathbb{R}^n$ and write $x \in \mathbb{R}^n$ in the form $x = x_1e_1+\cdots+x_ne_n$ (so $x_1,\dots,x_n$ are now Cartesian coordinates). For given $i_1,\dots,i_r,j \in \{1,\dots,n\}$, we define the vector field v by

$$v(x) := x_{i_1}\cdots x_{i_r}e_j, \qquad x \in \mathbb{R}^n.$$

To this and the convex body K we apply the divergence theorem. It says that

$$\int_K \operatorname{div} v(x)\,\mathrm{d}x = \int_{\partial K} v(x)\cdot u(K,x)\,\mathscr{H}^{n-1}(\mathrm{d}x).$$

To write this explicitly in a concise form, we use the Kronecker symbol δ and indicate by $\check{x}_m$ that x_m has to be deleted. Then we get

$$\int_K \sum_{k=1}^r \delta_{i_kj}\,x_{i_1}\cdots\check{x}_{i_k}\cdots x_{i_r}\,\mathrm{d}x = \int_{\partial K} x_{i_1}\cdots x_{i_r}(e_j\cdot u(K,x))\,\mathscr{H}^{n-1}(\mathrm{d}x).$$

Using tensor notation, this can equivalently be written as

$$\sum_{k=1}^{r} Q(e_{i_k}, e_j)\Psi_{r-1}(K)(e_{i_1}, \ldots, \check{e}_{i_k}, \ldots, e_{i_r})$$
$$= \frac{1}{(r-1)!} \int_{\partial K} x^r(e_{i_1}, \ldots, e_{i_r}) u(K,x)(e_j)\, \mathscr{H}^{n-1}(\mathrm{d}x). \tag{2.15}$$

This identity holds for arbitrary $(r+1)$-tuples $(i_1, \ldots, i_r, j)$ from $\{1, \ldots, n\}$.

To prove the identity (2.14), we have to check (only) that the $(r+1)$-tensors on either side attain the same value at any $(r+1)$-tuple $(e_{i_1}, \ldots, e_{i_{r+1}})$ of basis vectors. Now, by the definition of the symmetric tensor product, for the left side of (2.14) we have

$$(r+1)!(Q\Psi_{r-1}(K))(e_{i_1}, \ldots, e_{i_{r+1}})$$
$$= \sum_{\sigma \in \mathscr{S}(r+1)} Q(e_{i_{\sigma(1)}}, e_{i_{\sigma(2)}})\Psi_{r-1}(K)(e_{i_{\sigma(3)}}, \ldots, e_{i_{\sigma(r+1)}}). \tag{2.16}$$

For the right side of (2.14) we obtain from (2.15) that

$$\begin{aligned}
&(r+1)!\,\frac{1}{r!} \int_{\partial K} (x^r u(K,x))(e_{i_1}, \ldots, e_{i_{r+1}})\, \mathscr{H}^{n-1}(\mathrm{d}x) \\
&\quad = \frac{1}{r!} \sum_{\sigma \in \mathscr{S}(r+1)} \int_{\partial K} x^r(e_{i_{\sigma(1)}}, \ldots, e_{i_{\sigma(r)}}) u(K,x)(e_{i_{\sigma(r+1)}})\, \mathscr{H}^{n-1}(\mathrm{d}x) \\
&\quad = \frac{1}{r} \sum_{k=1}^{r} \sum_{\sigma \in \mathscr{S}(r+1)} Q(e_{i_{\sigma(k)}}, e_{i_{\sigma(r+1)}})\Psi_{r-1}(K)(e_{i_{\sigma(1)}}, \ldots, \check{e}_{i_{\sigma(k)}}, \ldots, e_{i_{\sigma(r)}}) \\
&\quad = \sum_{\sigma \in \mathscr{S}(r+1)} Q(e_{i_{\sigma(1)}}, e_{i_{\sigma(2)}})\Psi_{r-1}(K)(e_{i_{\sigma(3)}}, \ldots, e_{i_{\sigma(r+1)}}).
\end{aligned}$$

The latter agrees with (2.16). This completes the proof of (2.12).

From (2.12), further identities can be derived by applying (2.12) to the parallel bodies of a given convex body. For this, we write (2.12) in another explicit form, which is a counterpart to (2.14) for strictly convex bodies. If the convex body K is strictly convex, then to each unit vector $u \in \mathbb{S}^{n-1}$ there is a unique boundary point of K at which u is attained as outer normal vector. We denote this boundary point by $x(K,u)$. For a strictly convex body K, the measure $2\Lambda_{n-1}(K,\cdot)$ is the image measure of the area measure $S_{n-1}(K,\cdot)$ under the measurable mapping $u \mapsto (x(K,u),u)$ from $\mathbb{S}^{n-1}$ to Σ^n (for the area measure, see Sec. 1.3 of Chap. 1 or [20, Sec. 4.2]). Therefore, equation (2.13) is transformed into

$$Q\Psi_{r-1}(K) = \frac{1}{r!} \int_{\mathbb{S}^{n-1}} x(K,u)^r u\, S_{n-1}(K, \mathrm{d}u). \tag{2.17}$$

We apply this to a parallel body $K + \rho B^n$, for $\rho \geq 0$, which is also strictly convex if K is strictly convex. For the left side we get, using the Steiner formula (2.8),

$$Q\Psi_{r-1}(K + \rho B^n) = \sum_{k=0}^{n+r-1} \rho^{n+r-1-k} \kappa_{n+r-1-k} Q V_k^{(r-1)}(K). \tag{2.18}$$

To compute the right side of (2.17) for $K+\rho B^n$, we note that

$$x(K+\rho B^n,u) = x(K,u)+\rho u,$$

and hence

$$x(K+\rho B^n,u)^r = \sum_{j=0}^{r}\binom{r}{j}\rho^{r-j}x(K,u)^j u^{r-j}.$$

Further, we have to use the Steiner-type formula

$$S_{n-1}(K+\rho B^n,\cdot) = \sum_{i=0}^{n-1}\rho^{n-1-i}\binom{n-1}{i}S_i(K,\cdot)$$

(see [20, (4.36)]). Therefore, we also have

$$\begin{aligned}
&Q\Psi_{r-1}(K+\rho B^n) &(2.19)\\
&\quad= \frac{1}{r!}\sum_{i=0}^{n-1}\binom{n-1}{i}\int_{\mathbb{S}^{n-1}}\sum_{j=0}^{r}\binom{r}{j}x(K,u)^j u^{r-j+1}\,S_i(K,\mathrm{d}u)\,\rho^{n+r-1-i-j}\\
&\quad= \frac{1}{r!}\sum_{k=0}^{n+r-1}\rho^{n+r-1-k}\sum_{s=1}^{r+1}\binom{r}{s-1}\binom{n-1}{k-r-1+s} &(2.20)\\
&\qquad\times\int_{\Sigma^n}x^{r+1-s}u^s\,\Theta_{k-r-1+s}(K,\mathrm{d}(x,u)).
\end{aligned}$$

Here we have introduced new indices by $s=r+1-j$ and $k=i+j$, and instead of the measure $\Lambda_m(K,\cdot)$ we have used its re-normalization

$$\Theta_m(K,\cdot) = \frac{n\kappa_m}{\binom{n}{m}}\Lambda_m(K,\cdot).$$

Comparing the coefficients in (2.18) and (2.19), we now get

$$\begin{aligned}
&\kappa_{n+r-1-k}QV_k^{(r-1)}(K)\\
&\quad= \frac{1}{r!}\sum_{s=1}^{r+1}\binom{r}{s-1}\binom{n-1}{k-r-1+s}\int_{\Sigma^n}x^{r+1-s}u^s\,\Theta_{k-r-1+s}(K,\mathrm{d}(x,u)).
\end{aligned}$$

With the help of the identity $2\pi\kappa_m = \omega_{m+2}$, this can be simplified. Replacing $r+1$ by r, we obtain

$$QV_k^{(r-2)}(K) = 2\pi\sum_{s\in\mathbb{N}_0}s\Phi_{k-r+s}^{r-s,s}(K). \tag{2.21}$$

So far, this identity has been proved for strictly convex bodies K. By approximation, this result can be extended to general convex bodies.

Now, multiplying (2.9) (with r replaced by $r-2$) by Q and comparing with (2.21), we immediately get the McMullen relations. McMullen [15] proved these relations in a different way, namely first for polytopes.

Theorem 2.6 (McMullen). *For $r \in \mathbb{N}$ with $r \geq 2$ and $k \in \{0,\dots,n+r-2\}$,*

$$Q \sum_{s \in \mathbb{N}_0} \Phi_{k-r+s}^{r-s,s-2} = 2\pi \sum_{s \in \mathbb{N}_0} s \Phi_{k-r+s}^{r-s,s}. \tag{2.22}$$

For $r = 1$, relation (2.22) also holds, but only expresses the well-known fact that

$$\int_{\mathbb{S}^{n-1}} u\, S_j(K, \mathrm{d}u) = 0$$

for $j = 0,\dots,n-1$. For rank two, the McMullen relations are given by

$$Q\Phi_k^{0,0} = 2\pi\Phi_{k-1}^{1,1} + 4\pi\Phi_k^{0,2}, \quad k = 0,\dots,n.$$

We recall that

$$\begin{aligned}
&\Phi_k^{0,0}(K) = V_k, \\
&\Phi_{k-1}^{1,1}(K) = a_k \int_{\Sigma^n} xu\, \Lambda_{k-1}(K, \mathrm{d}(x,u)) \qquad \text{for } k \geq 1,\ \Phi_{-1}^{1,1}(K) = 0, \\
&\Phi_k^{0,2}(K) = b_k \int_{\Sigma^n} u^2\, \Lambda_k(K, \mathrm{d}(x,u)) \qquad \text{for } k \leq n-1,\ \Phi_n^{0,2}(K) = 0,
\end{aligned}$$

with positive constants a_k, b_k.

Now the question arises whether the McMullen relations are essentially the only linear dependences between the basic tensor valuations $Q^m\Phi_k^{r,s}$. This is, in fact, true. The following was proved by Hug, Schneider and Schuster [13].

Theorem 2.7. *Any nontrivial linear relation between basic tensor valuations $Q^m\Phi_k^{r,s}$ can be obtained by multiplying suitable McMullen relations by powers of Q and by taking linear combinations of relations obtained in this way.*

This result opened the way to determine bases and dimensions of the vector spaces in question. Let $T_{p,k}$ denote the real vector space of all mappings $\mathscr{K}^n \to \mathbb{T}^p$ that are continuous, isometry covariant valuations and homogeneous of degree k. Theorem 3.1 of [13] gives an explicit formula for the dimension of $T_{p,k}$. As an example for explicit bases, we present here the case of rank two:

- $T_{2,0}$: a basis is $\{Q\Phi_0^{0,0}\}$.
- $T_{2,1}$: a basis is $\{\Phi_1^{0,2}, Q\Phi_1^{0,0}\}$.
- $T_{2,k}$ for $k = 2,\dots,n-1$: a basis is $\{\Phi_k^{0,2}, \Phi_{k-2}^{2,0}, Q\Phi_k^{0,0}\}$.
- $T_{2,n}$: a basis is $\{\Phi_{n-2}^{2,0}, Q\Phi_n^{0,0}\}$.
- $T_{2,k}$ for $k = n+1, n+2$: a basis is $\{\Phi_{k-2}^{2,0}\}$.

Thus, the vector space of continuous, isometry covariant tensor valuations of rank two has dimension $3n+1$.

2.3 Local Tensor Valuations

In the same way as the intrinsic volumes have local versions, the support measures, so the Minkowski tensors have natural measure-valued extensions. We abbreviate now the normalizing factor appearing in (2.6) by

$$c_{n,k}^{r,s} := \frac{1}{r!s!} \frac{\omega_{n-k}}{\omega_{n-k+s}}$$

and define the *local Minkowski tensors* by

$$\phi_k^{r,s}(K,\eta) := c_{n,k}^{r,s} \int_\eta x^r u^s \, \Lambda_k(K, \mathrm{d}(x,u)) \tag{2.23}$$

for $\eta \in \mathscr{B}(\Sigma^n)$, the σ-algebra of Borel sets in Σ^n, and for $r,s \in \mathbb{N}_0$, $k \in \{0,\dots,n-1\}$. These local tensor valuations can also be introduced in a way that generalizes the introduction of the support measures by means of a local Steiner formula (see [20, Thm. 4.2.1]). For this, we define, for $K \in \mathscr{K}^n$ and $\eta \in \mathscr{B}(\Sigma^n)$, a tensor in $\mathbb{T}^{r+s}$ by

$$\mathscr{V}_\rho^{r,s}(K,\eta) := \int_{K_\rho \setminus K} \mathbf{1}_\eta \left(p_K(x), u_K(x)\right) p_K(x)^r \left(x - p_K(x)\right)^s \mathrm{d}x \tag{2.24}$$

for $\rho \geq 0$ and $r,s \in \mathbb{N}_0$. Here $\mathbf{1}_\eta$ is the characteristic function of the set η and $p_K(x)$ denotes the point in K nearest to x; the vector $u_K(x) := (x - p_K(x))/\|x - p(K,x)\|$ points from $p_K(x)$ to x, for $x \notin K$. (Variants of the tensor (2.24) have been introduced in [16] and [11], aiming at applications.) Noting that for (x,u) in the support of the measure $\Lambda_j(K,\cdot)$ and $t > 0$ the relations $p_K(x+tu) = x$ and $u_K(x+tu) = u$ hold, we obtain from Lemma 2.3 that

$$\mathscr{V}_\rho^{r,s}(K,\eta) = r!s! \sum_{j=0}^{n-1} \rho^{n-j+s} \kappa_{n-j+s} \phi_j^{r,s}(K,\eta). \tag{2.25}$$

Equation (2.23) defines a mapping $\phi_k^{r,s}$ from $\mathscr{K}^n \times \mathscr{B}(\Sigma^n)$ into $\mathbb{T}^{r+s}$. We want to list the properties of this mapping and collect, therefore, the most important properties which a general mapping $\Gamma : \mathscr{K}^n \times \mathscr{B}(\Sigma^n) \to \mathbb{T}^p$ may have. For $\eta \in \mathscr{B}(\Sigma^n)$, $t \in \mathbb{R}^n$ and $\vartheta \in \mathrm{O}(n)$, we write $\eta + t := \{(x+t,u) : (x,u) \in \eta\}$ and $\vartheta\eta := \{(\vartheta x, \vartheta u) : (x,u) \in \eta\}$. Moreover, recall from Chap. 1, Sec. 1.3, that $\mathbf{nc}(K) = \{(p_K(x), u_K(x)) : x \in \mathbb{R}^n \setminus K\}$ denotes the normal bundle of K. The following properties will play an important role.

- Γ has *polynomial translation behaviour of degree q*, where $0 \leq q \leq p$, if

$$\Gamma(K+t, \eta+t) = \sum_{j=0}^{q} \frac{1}{j!} \Gamma_{p-j}(K,\eta) t^j \tag{2.26}$$

with tensors $\Gamma_{p-j}(K,\eta) \in \mathbb{T}^{p-j}$, for all $K \in \mathscr{K}^n$, $\eta \in \mathscr{B}(\Sigma^n)$ and $t \in \mathbb{R}^n$ (the factor $1/j!$ is convenient); here $\Gamma_p = \Gamma$. In particular, Γ is called *translation invariant* if it is translation covariant of degree zero.

- Γ is *rotation covariant* if $\Gamma(\vartheta K, \vartheta\eta) = \vartheta\Gamma(K,\eta)$ for all $K \in \mathscr{K}^n$, $\eta \in \mathscr{B}(\Sigma^n)$ and $\vartheta \in \mathrm{O}(n)$.
- Γ is *isometry covariant* (of degree q) if it has polynomial translation behaviour of some degree $q \leq p$ (and hence of degree p) and is rotation covariant.
- Γ is *locally defined* if for $\eta \in \mathscr{B}(\Sigma^n)$ and $K, K' \in \mathscr{K}^n$ with $\eta \cap \mathbf{nc}(K) = \eta \cap \mathbf{nc}(K')$ the equality $\Gamma(K,\eta) = \Gamma(K',\eta)$ holds.
- If $\Gamma(K,\cdot)$ is a $\mathbb{T}^p$-valued measure for each $K \in \mathscr{K}^n$, then Γ is *weakly continuous* if for each sequence $(K_i)_{i\in\mathbb{N}}$ of convex bodies in $\mathscr{K}^n$ converging to a convex body K the relation
$$\lim_{i\to\infty} \int_{\Sigma^n} f \, \mathrm{d}\Gamma(K_i,\cdot) = \int_{\Sigma^n} f \, \mathrm{d}\Gamma(K,\cdot)$$
holds for all continuous functions $f : \Sigma^n \to \mathbb{R}$.

In the previous definitions, the set $\mathscr{K}^n$ may be replaced by $\mathscr{P}^n$.

Returning to the local Minkowski tensors, we note that from the properties of the support measures, the following can be deduced for each $\Gamma = \phi_k^{r,s}$.

- For each $K \in \mathscr{K}^n$, $\Gamma(K,\cdot)$ is a $\mathbb{T}^{r+s}$-valued measure.
- Γ is weakly continuous.
- For each $\eta \in \mathscr{B}(\Sigma^n)$, $\Gamma(\cdot,\eta)$ is measurable.
- For each $\eta \in \mathscr{B}(\Sigma^n)$, $\Gamma(\cdot,\eta)$ is a valuation.
- The mapping Γ is isometry covariant.
- The mapping Γ is locally defined.

It will be the main goal of the rest of this chapter to determine all mappings with of these properties. In fact, it will turn out that some properties are consequences of the others.

2.4 A Characterization Result for Local Tensor Valuations on Polytopes

In a first step to achieve the goal just formulated, we consider local tensor valuations on the space $\mathscr{P}^n$ of polytopes.

Let $P \in \mathscr{P}^n$ be a polytope. By $\mathscr{F}_k(P)$ we denote the set of k-dimensional faces of P, for $k \in \{0,\dots,n\}$. For $F \in \mathscr{F}_k(P)$, the set $\nu(P,F) = N(P,F) \cap \mathbb{S}^{n-1}$ is the set of outer unit normal vectors of P at its face F (see [20], Sec. 2.4, for the normal cone $N(P,F)$). From a representation of the support measures for polytopes (see [20], (4.3)), one can deduce that the local Minkowski tensors of a polytope P have the explicit representation

$$\phi_k^{r,s}(P,\eta) = C_{n,k}^{r,s} \sum_{F\in\mathscr{F}_k(P)} \int_F \int_{\nu(P,F)} \mathbf{1}_\eta(x,u) x^r u^s \, \mathscr{H}^{n-k-1}(\mathrm{d}u) \, \mathscr{H}^k(\mathrm{d}x), \qquad (2.27)$$

for $k \in \{0,\dots,n-1\}$ and $r,s \in \mathbb{N}_0$, where

$$C_{n,k}^{r,s} := (r!s!\omega_{n-k+s})^{-1}. \tag{2.28}$$

We point out that the integrations in (2.27) are only with respect to Hausdorff measures. The structure of (2.27) should be well understood, since it plays an important role in the following.

If one studies valuations on polytopes, it is always advisable to see how far one gets without the assumption of continuity. Theorem 1.31 from Chap. 1, for example, does not need any continuity assumption. However, without this assumption, there are mappings on $\mathscr{P}^n$ which share the preceding properties with the local Minkowski tensors, but are far more general. Hence, a possible classification theorem has to take these into account.

To define these generalizations, we associate with each face F of a polytope the linear subspace that is a translate of the affine hull of F. We denote this subspace by $L(F)$ and call it the *direction space* of F. For a linear subspace L of $\mathbb{R}^n$, we denote by $\pi_L : \mathbb{R}^n \to L$ the orthogonal projection. Then we define $Q_L \in \mathbb{T}^2$ by

$$Q_L(a,b) := \pi_L a \cdot \pi_L b \qquad \text{for } a,b \in \mathbb{R}^n.$$

We note that $Q_{\vartheta L} = \vartheta Q_L$ for $\vartheta \in \mathrm{O}(n)$.

Now we define the *generalized local Minkowski tensors* by extending (2.27) in the following way:

$$\begin{aligned}\phi_k^{r,s,j}(P,\eta) \\ := C_{n,k}^{r,s} \sum_{F \in \mathscr{F}_k(P)} Q_{L(F)}^j \int_F \int_{\nu(P,F)} \mathbf{1}_\eta(x,u) x^r u^s \,\mathscr{H}^{n-k-1}(\mathrm{d}u)\,\mathscr{H}^k(\mathrm{d}x),\end{aligned} \tag{2.29}$$

for $r,s,j,k \in \mathbb{N}_0$ with $1 \le k \le n-1$. This definition is supplemented by

$$\phi_0^{r,s,0} := \phi_0^{r,s},$$

but $\phi_0^{r,s,j}$ remains undefined for $j \ge 1$. Each mapping $\Gamma = \phi_k^{r,s,j}$ has the following properties. It is isometry covariant and locally defined. For each $P \in \mathscr{P}^n$, $\Gamma(P,\cdot)$ is a $\mathbb{T}^p$-valued measure, with $p = 2j+r+s$. For each $\eta \in \mathscr{B}(\Sigma^n)$, $\Gamma(\cdot,\eta)$ is a valuation. The first of these properties are easy to see; the proof of the last one uses Theorem 1.7 of Chap. 1; we refer to [12], Theorem 3.3, for the details.

Now we can state a characterization theorem. It is motivated by Theorem 2.5 of this chapter and Theorem 1.31 of Chap. 1.

Theorem 2.8. *For $p \in \mathbb{N}_0$, let $T_p(\mathscr{P}^n)$ denote the real vector space of all mappings*

$$\Gamma : \mathscr{P}^n \times \mathscr{B}(\Sigma^n) \to \mathbb{T}^p$$

with the following properties.

(a) *$\Gamma(P,\cdot)$ is a $\mathbb{T}^p$-valued measure, for each $P \in \mathscr{P}^n$;*

(b) Γ *is isometry covariant;*
(c) Γ *is locally defined.*

Then a basis of $T_p(\mathscr{P}^n)$ *is given by the mappings*

$$Q^m \phi_k^{r,s,j},$$

where $m,r,s,j \in \mathbb{N}_0$ *satisfy* $2m+2j+r+s=p$, *where* $k \in \{0,\dots,n-1\}$, *and where* $j=0$ *if* $k \in \{0,n-1\}$.

That only $j=0$ appears if $k=n-1$, is due to the easily proved identity

$$\phi_{n-1}^{r,s,j} = \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{(s+2i)!\omega_{1+s+2i}}{s!\omega_{1+s}} Q^{j-i} \phi_{n-1}^{r,s+2i}. \tag{2.30}$$

Theorem 2.8 is a stronger version of a theorem proved in [19]. Some modifications, including the linear independence result, were proved in [12]. We state this linear independence as a separate theorem.

Theorem 2.9. *Let* $p \in \mathbb{N}_0$. *On* $\mathscr{P}^n$, *the generalized local Minkowski tensors* $Q^m\phi_k^{r,s,j}$ *with*

$$m,r,s,j \in \mathbb{N}_0,\ 2m+2j+r+s=p,\ k \in \{0,\dots,n-1\},$$
$$\textit{and } j=0 \textit{ if } k \in \{0,n-1\},$$

are linearly independent.

The proof starts with a general linear relation and takes advantage of the fact that it involves general Borel sets. This generality, together with homogeneity considerations, can be used to simplify the relation. The simplified relation is then applied to a tuple $(x,\dots,x)$ of vectors $x = x_1e_1+\cdots+x_ne_n$, and from the fact that the resulting polynomial in $x_1,\dots,x_n$ is zero, one can deduce that all coefficients must be zero.

We shall now describe the main steps and ideas of the proof of Theorem 2.8 (the details are found in [12] and [19]). For this, we suppose that

$$\Gamma : \mathscr{P}^n \times \mathscr{B}(\Sigma^n) \to \mathbb{T}^p$$

is a mapping which has the following properties.

- For each $K \in \mathscr{P}^n$, $\Gamma(K,\cdot)$ is a $\mathbb{T}^p$-valued measure.
- Γ is isometry covariant.
- Γ is locally defined.

That Γ is isometry covariant, includes that it has polynomial translation behaviour of some degree q. Thus, there are mappings $\Gamma_{p-j} : \mathscr{P}^n \times \mathscr{B}(\Sigma^n) \to \mathbb{T}^{p-j}$, $j=0,\dots,q$, (possibly zero for some j and with $\Gamma_p = \Gamma$) such that

$$\Gamma(K+t,\eta+t) = \sum_{j=0}^{q} \frac{1}{j!} \Gamma_{p-j}(K,\eta)t^j$$

for all $K \in \mathscr{P}^n$, $\eta \in \mathscr{B}(\Sigma^n)$ and $t \in \mathbb{R}^n$. This implies similar behaviour of the coefficient tensors, namely

$$\Gamma_{p-j}(K+t, \eta+t) = \sum_{r=0}^{q-j} \frac{1}{r!} \Gamma_{p-j-r}(K, \eta) t^r \tag{2.31}$$

for $j = 0, \dots, q$ and all $K \in \mathscr{P}^n$, $\eta \in \mathscr{B}(\Sigma^n)$ and $t \in \mathbb{R}^n$, in particular (case $j = q$),

$$\Gamma_{p-q}(K+t, \eta+t) = \Gamma_{p-q}(K, \eta).$$

Properties of Γ_{p-j} can be derived from those of Γ, by means of the following relation. There are constants a_{jm} ($j = 0, \dots, q$, $m = 1, \dots, q+1$), depending only on q, j, m, such that

$$\Gamma_{p-j}(K, \eta) t^j = \sum_{m=1}^{q+1} a_{jm} \Gamma(K+mt, \eta+mt) \tag{2.32}$$

for all $K \in \mathscr{P}^n$, $\eta \in \mathscr{B}(\Sigma^n)$ and $t \in \mathbb{R}^n$. In particular, we can deduce that $\Gamma_{p-j}(K, \cdot)$ is a $\mathbb{T}^{p-j}$-valued measure and that

$$\Gamma_{p-j}(\vartheta K, \vartheta \eta) = \vartheta \Gamma_{p-j}(K, \eta) \tag{2.33}$$

for $\vartheta \in \mathrm{O}(n)$. Together with (2.31) this shows that also Γ_{p-j} is isometry covariant.

Lemma 2.10. *For each $K \in \mathscr{P}^n$, the measure $\Gamma(K, \cdot)$ is concentrated on* $\mathbf{nc}(K)$.

The proof uses that Γ is locally defined and has polynomial translation behaviour. Further it uses that the only translation invariant finite signed measure on the bounded Borel sets of $\mathbb{R}^n$ is Lebesgue measure, up to a constant factor.

The essential step to prove Theorem 2.8 is the translation invariant case, that is, the following result.

Theorem 2.11. *Let $p \in \mathbb{N}_0$. Let $\Gamma : \mathscr{P}^n \times \mathscr{B}(\Sigma^n) \to \mathbb{T}^p$ be a mapping with the following properties.*

(a) *$\Gamma(P, \cdot)$ is a $\mathbb{T}^p$-valued measure, for each $P \in \mathscr{P}^n$;*
(b) *Γ is translation invariant and rotation covariant;*
(c) *Γ is locally defined.*

Then Γ is a linear combination, with constant coefficients, of the mappings

$$Q^m \phi_k^{0,s,j},$$

where $m, s, j \in \mathbb{N}_0$ satisfy $2m + 2j + s = p$, where $k \in \{0, \dots, n-1\}$, and where $j = 0$ if $k \in \{0, n-1\}$.

If this has been proved, then one can use the properties of the coefficient tensors Γ_{p-j} mentioned above, to give for Theorem 2.8 an inductive proof, which step by step reduces the degree of the polynomial translation behaviour of Γ.

Now we indicate some ideas in the proof of Theorem 2.11. To show, as we have to do, an equality for measures on $\mathscr{B}(\Sigma^n)$, it is sufficient to prove equality on product sets $\beta \times \omega$ with $\beta \in \mathscr{B}(\mathbb{R}^n)$ and $\omega \in \mathscr{B}(\mathbb{S}^{n-1})$. Let $P \in \mathscr{P}^n$. By Lemma 2.10, $\Gamma(P,\cdot)$ is concentrated on $\mathbf{nc}(P)$. The polytope P is the disjoint union of the relative interiors of its facets. Therefore,

$$\Gamma(P,\beta \times \omega) = \sum_{k=0}^{n-1} \sum_{F \in \mathscr{F}_k(P)} \Gamma\big(P,(\beta \cap \operatorname{relint} F) \times (\omega \cap \nu(P,F))\big). \tag{2.34}$$

Consequently, it is sufficient to determine $\Gamma(P,\beta \times \omega)$ for the case where $\beta \subset \operatorname{relint} F$ and $\omega \subset \nu(P,F)$, for some face $F \in \mathscr{F}_k(P)$.

Therefore, we may restrict ourselves to the following situation. We are given a number $k \in \{0,\dots,n-1\}$, a k-dimensional linear subspace $L \subset \mathbb{R}^n$, a bounded Borel set $\beta \subset L$, a Borel set $\omega \subset \mathbb{S}^{n-1} \cap L^\perp$, a k-dimensional polytope $F \subset L$ with $\beta \subset \operatorname{relint} F$. It suffices to determine $\Gamma(F,\beta \times \omega)$ in this case.

First, we fix ω and use the standard characterization of Lebesgue measure in L to show that

$$\Gamma(F,\beta \times \omega) = a(L,\omega)\mathscr{H}^k(\beta),$$

where the constant $a(L,\omega)$ is a tensor in $\mathbb{T}^p$ that depends on the subspace L and the Borel set ω. The main task is to determine this tensor function. It has an important covariance property, namely

$$a(\vartheta L, \vartheta\omega) = \vartheta a(L,\omega) \qquad \text{for } \vartheta \in \mathrm{O}(n)$$

and

$$\vartheta a(L,\omega) = a(L,\omega) \qquad \text{if } \vartheta \text{ fixes } L^\perp \text{ pointwise.}$$

From this, it is deduced in [19] that

$$a(L,\omega) = \sum_{j=0}^{\lfloor p/2 \rfloor} Q_L^j \sum_{i=0}^{\lfloor p/2 \rfloor} c_{pkij} Q_{L^\perp}^i \int_\omega u^{p-2j-2i} \mathscr{H}^{n-k-1}(\mathrm{d}u) \tag{2.35}$$

with real constants c_{pkij}. Once this has been proved, things can be put together to finish the proof of Theorem 2.11.

The only hints we can give here to the proof of (2.35) is the formulation of two lemmas. The first exhibits the crucial point where the tensors Q_L enter the scene.

Lemma 2.12. *Let $L \subset \mathbb{R}^n$ be a linear subspace. Let $r \in \mathbb{N}_0$, let $T \in \mathbb{T}^r$ be a tensor satisfying $\vartheta T = T$ for each $\vartheta \in \mathrm{O}(n)$ that fixes $L^\perp$ pointwise. Then*

$$T = \sum_{j=0}^{\lfloor r/2 \rfloor} Q_L^j \pi_{L^\perp}^* T^{(r-2j)}$$

with tensors $T^{(r-2j)} \in \mathbb{T}^{r-2j}(L^\perp)$, $j = 0,\dots,\lfloor r/2 \rfloor$.

Here $\mathbb{T}^p(L^\perp)$ denotes the space of p-tensors on $L^\perp$, and for $T \in \mathbb{T}^p(L^\perp)$ we have used the notation

$$(\pi^*_{L^\perp} T)(x_1, \dots, x_p) := T(\pi_{L^\perp} x_1, \dots, \pi_{L^\perp} x_p) \quad \text{for } x_1, \dots, x_p \in \mathbb{R}^n.$$

The proof of Lemma 2.12 is based on the fact that the algebra of symmetric tensors on $\mathbb{R}^n$ is isomorphic to the polynomial algebra on $\mathbb{R}^n$, and it uses some manipulations with polynomials.

The second crucial lemma deals with rotation covariant tensor measures on the sphere.

Lemma 2.13. *Let $r \in \mathbb{N}_0$, and let $\mu : \mathscr{B}(\mathbb{S}^{n-1}) \to \mathbb{T}^r$ be a $\mathbb{T}^r$-valued measure satisfying*

$$\mu(\vartheta\omega) = (\vartheta\mu)(\omega) \qquad \textit{for all } \omega \in \mathscr{B}(\mathbb{S}^{n-1}) \textit{ and all } \vartheta \in \mathrm{O}(n).$$

Then

$$\mu(\omega) = \sum_{j=0}^{\lfloor r/2 \rfloor} a_j Q^j \int_\omega u^{r-2j} \, \mathscr{H}^{n-1}(\mathrm{d}u), \qquad \omega \in \mathscr{B}(\mathbb{S}^{n-1}),$$

with real constants a_j, $j = 0, \dots, \lfloor r/2 \rfloor$.

A first step of the proof uses that the total variation measure of μ is rotation invariant and hence a constant multiple of spherical Lebesgue measure. Then the Radon–Nikodym theorem, applied coordinate-wise, yields a representation

$$\mu(\omega) = \int_\omega f \, \mathrm{d}\mathscr{H}^{n-1}, \qquad \omega \in \mathscr{B}(\mathbb{S}^{n-1}),$$

with an almost everywhere defined measurable mapping $f : \mathbb{S}^{n-1} \to \mathbb{T}^r$. A special case of Lemma 2.12 together with the covariance property and Lebesgue's differentiation theorem can then be used to determine the function f.

2.5 The Characterization Result on General Convex Bodies

If we want to extend Theorem 2.8 from polytopes to general convex bodies, we certainly need some continuity assumption. This raises the question whether $\phi_k^{r,s,j}$ has a weakly continuous extension from polytopes to general convex bodies. To make this question more precise, let

$$\Gamma : \mathscr{K}^n \times \mathscr{B}(\Sigma^n) \to \mathbb{T}^p \tag{2.36}$$

be a mapping and consider the following properties, which it may or may not have:

(A) $\Gamma(K, \cdot)$ is a $\mathbb{T}^p$-valued measure, for each $K \in \mathscr{K}^n$;
(B) Γ is isometry covariant;

(C) Γ is locally defined;
(D) Γ is weakly continuous.

Question. For given $k \in \{0,\dots,n-1\}$ and $r,s,j \in \mathbb{N}_0$, is there a mapping Γ as in (2.36) having properties (A) – (D) and satisfying $\Gamma(P,\cdot) = \phi_k^{r,s,j}(P,\cdot)$ for $P \in \mathscr{P}^n$?

This is trivially true for $k=0$, since $\phi_0^{r,s,0} = \phi_0^{r,s}$ by definition, and $\phi_0^{r,s,j}$ is not defined for $j \geq 1$. It is also true for $k = n-1$, since we may define

$$\phi_{n-1}^{r,s,j}(K) := \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{(s+2i)!\,\omega_{1+s+2i}}{s!\,\omega_{1+s}} Q^{j-i} \phi_{n-1}^{r,s+2i}(K)$$

for $K \in \mathscr{K}^n$; by (2.30), this is consistent with the case of polytopes. The weak continuity of $\phi_{n-1}^{r,s+2i}$ follows from (2.23) and the weak continuity of the support measures. Further, the answer is affirmative if $j=0$, since $\phi_k^{r,s,0}(P,\cdot) = \phi_k^{r,s}(P,\cdot)$ for $P \in \mathscr{P}^n$, and we can define $\phi_k^{r,s,0}(K,\cdot) = \phi_k^{r,s}(K,\cdot)$ for $K \in \mathscr{K}^n$. It remains to consider the cases of $\phi_k^{r,s,j}$ where $1 \leq k \leq n-2$ and $j \geq 1$.

Proposition 2.14. *For $k \in \{1,\dots,n-2\}$ and $r,s \in \mathbb{N}_0$, the answer to the question above is affirmative for $j=1$.*

Postponing the proof of this proposition to Section 2.6, we can now state the following characterization theorem. It includes the fact that the statement of Proposition 2.14 does not extend to $j > 1$.

Theorem 2.15. *For $p \in \mathbb{N}_0$, let $T_p(\mathscr{K}^n)$ denote the real vector space of all mappings $\Gamma : \mathscr{K}^n \times \mathscr{B}(\Sigma^n) \to \mathbb{T}^p$ with properties* (A)–(D).

A basis of $T_p(\mathscr{K}^n)$ is given by the mappings

$$Q^m \phi_k^{r,s,j}, \qquad k \in \{0,\dots,n-1\},\ m,r,s \in \mathbb{N}_0,\ j \in \{0,1\},$$

where $2m+2j+r+s = p$ and $j=0$ if $k \in \{0,n-1\}$.

As in the case of polytopes, where Theorem 2.8 follows from Theorem 2.11, it suffices to consider the translation invariant case. By an inductive argument, which was already used by Alesker [1] in his proof of Theorem 2.5, Theorem 2.15 can be deduced from the following result. We also observe that linear independence can be deduced from Theorem 2.9.

Theorem 2.16. *Let $p \in \mathbb{N}_0$. Let $\Gamma : \mathscr{K}^n \times \mathscr{B}(\Sigma^n) \to \mathbb{T}^p$ be a mapping with the properties* (A), (C), (D) *and*

(B') *Γ is translation invariant and rotation covariant.*

Then Γ is a linear combination, with constant coefficients, of the mappings

$$Q^m \phi_k^{0,s,j}, \qquad k \in \{0,\dots,n-1\},\ m,s \in \mathbb{N}_0,\ j \in \{0,1\},$$

where $2m+2j+s = p$ and $j=0$ if $k \in \{0,n-1\}$.

For the proof, some further simplifications are possible. If Γ satisfies the assumptions of Theorem 2.16, then it is not difficult to see (cf. [12], Lemma 3.5) that $\Gamma = \sum_{k=0}^{n-1} \Gamma_k$, where Γ_k is a mapping with the same properties which is, moreover, homogeneous of degree k. Therefore, to prove Theorem 2.16, we can and will assume in addition that Γ is homogeneous of some degree $k \in \{0, \dots, n-1\}$. If $k \in \{0, n-1\}$, then Theorem 2.11 shows that the restriction of Γ to $\mathscr{P}^n$ is a linear combination of mappings $Q^m \phi_k^{0,s}$, and by weak continuity this holds also for Γ on $\mathscr{K}^n$. Hence, we can assume now that Γ is homogeneous of some degree $k \in \{1, \dots, n-2\}$ (and, therefore, $n \geq 3$). Under these assumptions, Theorem 2.11 implies that there are constants c_{mjs} (only finitely many of them different from zero) such that

$$\Gamma(P, \cdot) = \sum_{\substack{m,j,s \geq 0 \\ 2m+2j+s=p}} c_{mjs} Q^m \phi_k^{0,s,j}(P, \cdot) \qquad \text{for } P \in \mathscr{P}^n.$$

Since Γ and $\phi_k^{0,s,0}$, and by the postponed Proposition 2.14 also $\phi_k^{0,s,1}$, are weakly continuous, the mapping Γ' defined by

$$\Gamma' := \Gamma - \sum_{\substack{m,j,s \geq 0,\, j \leq 1 \\ 2m+2j+s=p}} c_{mjs} Q^m \phi_k^{0,s,j} \tag{2.37}$$

has the properties (A), (B'), (C), (D), and for $P \in \mathscr{P}^n$ it satisfies

$$\Gamma'(P, \cdot) = \sum_{\substack{m,s \geq 0,\, j \geq 2 \\ 2m+2j+s=p}} c_{mjs} Q^m \phi_k^{0,s,j}(P, \cdot).$$

Theorem 2.16 and thus Theorem 2.15 is proved if we show that Γ' is identically zero. We sketch the main ideas leading to this result and refer to [12] for the details.

The strategy of the proof is indicated by the following lemma. We write

$$\Gamma'(K, f) := \int_{\Sigma^n} f(u) \Gamma'(K, \mathrm{d}(x, u))$$

for $K \in \mathscr{K}^n$ and continuous real functions f on the unit sphere $\mathbb{S}^{n-1}$.

Lemma 2.17. *If the function Γ' defined by* (2.37) *is not identically zero, then there exist a convex body $K \in \mathscr{K}^n$, a continuous function f on $\mathbb{S}^{n-1}$, a p-tuple E of vectors from $\mathbb{R}^n$, and a rotation $\vartheta \in \mathrm{O}(n)$ such that K and f are invariant under ϑ, but $\Gamma'(K, f)(\vartheta E) \neq \Gamma'(K, f)(E)$.*

If this is proved, then it follows from the invariance of K and f under ϑ and from the rotation covariance of Γ' that

$$\Gamma'(K, f)(\vartheta E) = \Gamma'(\vartheta K, \vartheta f)(\vartheta E) = \Gamma'(K, f)(E),$$

which is a contradiction. The conclusion is that $\Gamma' \equiv 0$, which proves the theorem.

The (lengthy) proof of Lemma 2.17 constructs a sequence $(P_i)_{i\in\mathbb{N}}$ of polytopes converging to a convex body K, such that K has a symmetry ϑ (a rotation mapping K into itself) with the following property. For each i, the rotation ϑ is not a symmetry of P_i, and this fact can be strengthened as follows. If Γ' is not identically zero, then there are a continuous function f on $\mathbb{S}^{n-1}$, invariant under ϑ, and a p-tuple E of vectors from $\mathbb{R}^n$, such that

$$|\Gamma'(P_i,f)(\vartheta E)-\Gamma'(P_i,f)(E)|\geq c>0. \tag{2.38}$$

The function f, the p-tuple E and the constant c are independent of i. By the weak continuity of Γ', it then follows that $|\Gamma'(K,f)(\vartheta E)-\Gamma'(K,f)(E)|\geq c>0$.

The polytopes P_i are constructed as follows (we describe the construction for $n\geq 4$; a modification is necessary for $n=3$). Let $(e_1,\dots,e_n)$ be the standard orthonormal basis of $\mathbb{R}^n$, and identify $\mathrm{lin}\{e_1,\dots,e_{n-1}\}$ with $\mathbb{R}^{n-1}$. In $\mathbb{R}^{n-1}$, we consider the lattice

$$\mathbb{Z}^{n-1}:=\{m_1e_1+\cdots+m_{n-1}e_{n-1}:m_1,\dots,m_{n-1}\in\mathbb{Z}\}.$$

Its points are the vertices of a tessellation of $\mathbb{R}^{n-1}$ into $(n-1)$-cubes. We lift the homothets of this lattice to a paraboloid of revolution. For this, we define the lifting map $\ell:\mathbb{R}^{n-1}\to\mathbb{R}^n$ by $\ell(x):=x+\|x\|^2e_n$ for $x\in\mathbb{R}^{n-1}$. For $t>0$ we define the polyhedral set

$$R_t:=\mathrm{conv}\,\ell(2t\mathbb{Z}^{n-1}).$$

It is well known and easy to see that under orthogonal projection to $\mathbb{R}^{n-1}$, the facets of R_t project into the cubes of the tessellation induced by $2t\mathbb{Z}^{n-1}$. With $H_h^-:=\{y\in\mathbb{R}^n:y\cdot e_n\leq h\}$ for suitable $h>0$, we define

$$P_i:=R_{1/i}\cap H_h^- \quad\text{and}\quad K:=\mathrm{epi}\,\ell\cap H_h^-.$$

Then P_i is a convex polytope, and $P_i\to K$ for $i\to\infty$.

The details of the estimates leading to (2.38) (if $h>0$ is sufficiently small) are found in [12].

We point out, however, that the last argument of the proof given there (which concerns the case $n=3$) needs a correction, and we replace the reasoning on page 1561 of [12] by the following.

Let

$$F(\lambda):=\sum_{j=2}^{d}c_j\sum_{r=0}^{d-1}\left(\lambda\cos r\beta_d+\sqrt{1-\lambda^2}\sin r\beta_d\right)^{2j},\quad \lambda\in[0,1],$$

where $d\in\mathbb{N}$, $d\geq 2$, $c_j\in\mathbb{R}$, $c_d\neq 0$, $\beta_d=\pi/d$. We have to show that F is not constant. First we note that $F(\lambda)=P(\lambda)+\sqrt{1-\lambda^2}\,Q(\lambda)$ with polynomials P and Q, in particular,

$$P(\lambda)=\sum_{j=2}^{d}c_j\sum_{r=0}^{d-1}\sum_{\ell=0}^{j}\binom{2j}{2\ell}\lambda^{2\ell}(\cos r\beta_d)^{2\ell}(1-\lambda^2)^{j-\ell}(\sin r\beta_d)^{2j-2\ell}.$$

Suppose, to the contrary, that $F(\lambda) = c$ for $\lambda \in [0,1]$, with a constant c. Then

$$(P(\lambda) - c)^2 = (1-\lambda)(1+\lambda)Q(\lambda)^2 \tag{2.39}$$

for $\lambda \in [0,1]$ and hence for all $\lambda \in \mathbb{R}$. If $P - c$ and Q are not identically zero, then the multiplicity of 1 as a root of either $(P-c)^2$ or Q^2 is even, but according to (2.39), for $(P-c)^2$ it is odd. This contradiction shows, in particular, that P is constant. However, we have

$$\begin{aligned}\lim_{\lambda\to\infty} \lambda^{-2d} P(\lambda) &= c_d \sum_{r=0}^{d-1} \sum_{\ell=0}^{d} \binom{2d}{2\ell} (-1)^{d-\ell} (\cos r\beta_d)^{2\ell} (\sin r\beta_d)^{2d-2\ell} \\ &= c_d \sum_{r=0}^{d-1} \mathrm{Re}\,(\cos r\beta_d + \mathrm{i} \sin r\beta_d)^{2d} \\ &= c_d \mathrm{Re} \sum_{r=0}^{d-1} \exp\left(r\frac{\pi}{d}\mathrm{i}\cdot 2d\right) = dc_d \neq 0,\end{aligned}$$

a contradiction

2.6 A Weakly Continuous Extension

The main purpose of this section is to sketch the proof of Proposition 2.14, which was formulated in the previous section. Moreover, for an arbitrary convex body K we shall give an explicit description of $\phi_k^{r,s,1}(K,\cdot)$ as an integral over the normal bundle of K involving generalized curvatures and principal directions of curvature, which is then specialized for smooth convex bodies.

It is well known that the map $K \mapsto \Lambda_j(K,\cdot)$ is weakly continuous on $\mathscr{K}^n$. This follows most easily from the weak continuity of the local parallel volume map $K \mapsto \mathscr{H}^n(M_\rho(K,\cdot))$, for all $\rho > 0$. As an immediate consequence we obtain that $K \mapsto \phi_k^{r,s,0}(K,\cdot)$ is weakly continuous. In order to show that $P \mapsto \phi_k^{r,s,1}(P,\cdot)$ has a weakly continuous extension from polytopes to general convex bodies, we shall proceed in a different way. The starting point is a description of the support measure $\Lambda_k(K,\cdot)$ of K by means of a current, the normal cycle T_K of K, evaluated at suitably chosen differential forms φ_k (the Lipschitz–Killing curvature forms), as first explained in [21]. From the continuity of the map $K \mapsto T_K$ (in a suitable topology), it follows again that the support measures are weakly continuous. The main task then is to find suitable tensor-valued differential forms $\varphi_k^{r,s}$ such that T_P evaluated at $\varphi_k^{r,s}$ yields $\phi_k^{r,s,1}(P,\cdot)$, for an arbitrary polytope P.

We start with some basic terminology and facts of multilinear algebra and geometric measure theory (see [6]), which will also be useful in the final section. Let V be a finite-dimensional real vector space. Then $\bigwedge_m V$, for $m \in \mathbb{N}_0$, denotes the vector space of *m-vectors* of V, and $\bigwedge^m V$ is the vector space of all m-linear alternating maps from V^m to $\mathbb{R}$, whose elements are called *m-covectors*. The map $\bigwedge^m V \to \mathrm{Hom}(\bigwedge_m V, \mathbb{R})$, which assigns to $f \in \bigwedge^m V$ the homomorphism $v_1 \wedge \cdots \wedge v_m \mapsto f(v_1, \ldots, v_m)$, allows us to identify $\bigwedge^m V$ and $\mathrm{Hom}(\bigwedge_m V, \mathbb{R})$. By this identification, the dual pairing of elements $a \in \bigwedge_m V$ and $\varphi \in \bigwedge^m V$ can be defined by $\langle a, \varphi \rangle := \varphi(a)$. If V' is another finite-dimensional vector space and $f : V \to V'$ is a linear map, then a linear map $\bigwedge_m f : \bigwedge_m V \to \bigwedge_m V'$ is determined by $(\bigwedge_m f)(v_1 \wedge \cdots \wedge v_m) = f(v_1) \wedge \cdots \wedge f(v_m)$, for all $v_1, \ldots, v_m \in V$.

To introduce the normal cycle T_K of $K \in \mathscr{K}^n$, we remark that the normal bundle $\mathbf{nc}(K) \subset \mathbb{R}^{2n}$ of K is an $(n-1)$-rectifiable set. In fact, the map $F : \partial K_1 \to \mathbb{R}^n \times \mathbb{S}^{n-1}$ given by $F(x) := (p_K(x), u_K(x))$ is bi-Lipschitz, and hence the image $\mathbf{nc}(K)$ is an $(n-1)$-rectifiable subset of $\mathbb{R}^{2n}$. Therefore, for $\mathscr{H}^{n-1}$-almost all $(x,u) \in \mathbf{nc}(K)$, the set of $(\mathscr{H}^{n-1} \llcorner \mathbf{nc}(K), n-1)$ approximate tangent vectors at (x,u) is an $(n-1)$-dimensional linear subspace of $\mathbb{R}^{2n}$, which is denoted by $\mathrm{Tan}^{n-1}(\mathscr{H}^{n-1} \llcorner \mathbf{nc}(K), (x,u))$. This *approximate tangent space* is spanned by an orthonormal basis $(a_1(x,u), \ldots, a_{n-1}(x,u))$, where

$$a_i(x,u) := \left(\frac{1}{\sqrt{1+k_i(x,u)^2}}\, b_i(x,u), \frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}}\, b_i(x,u) \right)$$

and where $(b_1(x,u), \ldots, b_{n-1}(x,u))$ is a suitable orthonormal basis of $u^\perp$, which is chosen so that $(b_1(x,u), \ldots, b_{n-1}(x,u), u)$ has the same orientation as the standard basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$. Here, $k_i(x,u) \in [0,\infty]$ for $i = 1, \ldots, n-1$ with the usual convention

$$\frac{1}{\sqrt{1+k_i(x,u)^2}} = 0 \quad \text{and} \quad \frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}} = 1 \qquad \text{if } k_i(x,u) = \infty.$$

The dependence of a_i, b_i, k_i on K is not made explicit by our notation. We remark that b_i, k_i, $i = 1, \ldots, n-1$, are essentially uniquely determined (see [18, Prop. 3, Lemma 2]). The numbers $k_i(x,u)$ can be interpreted as generalized curvatures with corresponding generalized principal directions of curvature $b_i(x,u)$. Moreover, we can assume that $b_i(x+\varepsilon u, u) = b_i(x,u)$, independent of $\varepsilon > 0$, where $(x,u) \in \mathbf{nc}(K)$ and $(x+\varepsilon u, u) \in \mathbf{nc}(K_\varepsilon)$. For $\mathscr{H}^{n-1}$-almost all $(x,u) \in \mathbf{nc}(K)$,

$$a_K(x,u) := a_1(x,u) \wedge \cdots \wedge a_{n-1}(x,u)$$

is an $(n-1)$-vector, which fixes an orientation of the approximate tangent space $\mathrm{Tan}^{n-1}(\mathscr{H}^{n-1} \llcorner \mathbf{nc}(K), (x,u))$. Then

$$T_K := \left(\mathscr{H}^{n-1} \llcorner \mathbf{nc}(K)\right) \wedge a_K$$

defines an $(n-1)$-dimensional current in $\mathbb{R}^{2n}$, the *normal cycle* of K. Explicitly, we have

$$T_K(\varphi) = \int_{\mathbf{nc}(K)} \langle a_K(x,u), \varphi(x,u) \rangle\, \mathscr{H}^{n-1}(d(x,u)),$$

for all $\mathscr{H}^{n-1} \llcorner \mathbf{nc}(K)$-integrable functions $\varphi : \mathbb{R}^{2n} \to \bigwedge^{n-1} \mathbb{R}^{2n}$. Note that T_K is a rectifiable current, which has compact support, and thus T_K can be defined for a larger class of functions than just for the class of smooth differential forms.

In order to define the Lipschitz–Killing forms φ_k, $k \in \{0,\dots,n-1\}$, let $\Pi_1 : \mathbb{R}^{2n} \to \mathbb{R}^n$, $(x,u) \mapsto x$, and $\Pi_2 : \mathbb{R}^{2n} \to \mathbb{R}^n$, $(x,u) \mapsto u$. Let Ω_n be the volume form on $\mathbb{R}^n$ with the orientation chosen so that $\Omega_n(e_1,\dots,e_n) = \langle e_1 \wedge \cdots \wedge e_n, \Omega_n \rangle = 1$. Then differential forms $\varphi_k : \mathbb{R}^{2n} \to \bigwedge^{n-1} \mathbb{R}^{2n}$, $k \in \{0,\dots,n-1\}$, of degree $n-1$ on $\mathbb{R}^{2n}$ are defined by

$$\begin{aligned}\varphi_k(x,u)(\xi_1,\dots,\xi_{n-1}) &:= \frac{1}{k!(n-1-k)!\,\omega_{n-k}} \\ &\quad \times \sum_{\sigma \in \mathscr{S}(n-1)} \operatorname{sgn}(\sigma) \Big\langle \bigwedge_{i=1}^{k} \Pi_1 \xi_{\sigma(i)} \wedge \bigwedge_{i=k+1}^{n-1} \Pi_2 \xi_{\sigma(i)} \wedge u, \Omega_n \Big\rangle,\end{aligned}$$

where $(x,u) \in \mathbb{R}^{2n}$, $\xi_1,\dots,\xi_{n-1} \in \mathbb{R}^{2n}$, and $\mathscr{S}(n-1)$ denotes the set of all permutations of $\{1,\dots,n-1\}$. Then, writing

$$\mathbb{K}(x,u) := \prod_{i=1}^{n-1} \sqrt{1 + k_i(x,u)^2},$$

we have

$$\langle a_K(x,u), \varphi_k(x,u) \rangle = \frac{1}{\omega_{n-k}} \sum_{|I|=n-1-k} \frac{\prod_{i \in I} k_i(x,u)}{\mathbb{K}(x,u)}$$

for $\mathscr{H}^{n-1}$-almost all $(x,u) \in \mathbf{nc}(K)$. The summation extends over all subsets I of $\{1,\dots,n-1\}$ of cardinality $n-1-k$, where a product over an empty set is defined as 1. Then, for $\eta \in \mathscr{B}(\Sigma^n)$,

$$T_K(\mathbf{1}_\eta \varphi_k) = \Lambda_k(K,\eta),$$

which provides a representation of the kth support measure of K in terms of the normal cycle of K, evaluated at the kth Lipschitz–Killing form φ_k.

The construction of suitable tensor-valued differential forms $\varphi_k^{r,s}$ is slightly more involved. For the explicit definition, we refer to [12], Section 4. We simply remark that the map

$$\varphi_k^{r,s} : \mathbb{R}^{2n} \to \bigwedge^{n-1}(\mathbb{R}^{2n}, \mathbb{T}^{r+s+2}), \qquad (x,u) \mapsto \varphi_k^{r,s}(x,u),$$

is a differential form of degree $n-1$ on $\mathbb{R}^{2n}$ with coefficients in $\mathbb{T}^{r+s+2}$ (see [6, p. 351]). In particular, $\langle a, \varphi_k^{r,s}(x,u) \rangle \in \mathbb{T}^{r+s+2}$, for all $(x,u) \in \mathbb{R}^{2n}$ and $a \in \bigwedge_{n-1} \mathbb{R}^{2n}$, where we identify $\bigwedge^{n-1}(\mathbb{R}^{2n}, W)$ and $\mathrm{Hom}(\bigwedge_{n-1} \mathbb{R}^{2n}, W)$, for an arbitrary vector space W. A straightforward calculation shows that

$$\langle \vartheta a, \varphi_k^{r,s}(\vartheta x, \vartheta u) \rangle = \vartheta \langle a, \varphi_k^{r,s}(x,u) \rangle,$$

for all $\vartheta \in \mathrm{O}(n)$, where in each case the natural operation of the rotation group is used (in particular, $\vartheta\xi := (\vartheta p, \vartheta q)$ for $\xi = (p,q) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$). As a result of the construction and by some calculations, which use that for a polytope P we have $k_i(x,u) = 0$ if and only if $b_i(x,u) \in L(F)$ and $k_i(x,u) = \infty$ otherwise, we obtain

$$T_P(\mathbf{1}_\eta \varphi_k^{r,s}) = \phi_k^{r,s,1}(P,\eta)$$

for all $P \in \mathscr{P}^n$ and $\eta \in \mathscr{B}(\Sigma^n)$.

It is known that

- T_K is a cycle for $K \in \mathscr{K}^n$ (see [17, Prop. 2.6]);
- the map $K \mapsto T_K$ is a valuation on $\mathscr{K}^n$ (see [17, Thm. 2.2]);
- $T_{K_i} \to T_K$ in the dual flat seminorm for currents, if $K_i, K \in \mathscr{K}^n$, $i \in \mathbb{N}$, and $K_i \to K$ in the Hausdorff metric, as $i \to \infty$ (see [17, Thm. 3.1], and for the dual flat seminorm, [6, Sec. 4.1.12, p. 367]).

In the next section, we prove a strengthened form of the continuity assertion stated in the third point, namely local Hölder continuity of the normal cycles of convex bodies with respect to the Hausdorff metric and the dual flat seminorm.

The third point above implies that if $f : \mathbb{R}^{2n} \to \mathbb{R}$ is of class C^∞, then the map

$$\mathscr{K}^n \to \mathbb{R}, \qquad K \mapsto T_K(f\varphi_k^{r,s}),$$

is continuous. But then the same is true if f is merely continuous. Hence, $(K,\eta) \mapsto T_K\left(\mathbf{1}_\eta \varphi_k^{r,s}\right)$ is the weakly continuous extension of the map $(P,\eta) \mapsto \phi_k^{r,s,1}(P,\eta)$ from polytopes P to general convex bodies. Moreover, we have the following result (with (A) – (D) as formulated at the beginning of Sec. 2.5).

Theorem 2.18. *The map $\mathscr{K}^n \times \mathscr{B}(\Sigma^n) \to \mathbb{T}^{r+s+2}$, $(K,\eta) \mapsto T_K\left(\mathbf{1}_\eta \varphi_k^{r,s}\right)$, satisfies the properties* (A) – (D).

The next corollary then is an immediate consequence.

Corollary 2.19. *Let $r,s \in \mathbb{N}_0$ and $k \in \{1,\dots,n-2\}$. Then, for each $\eta \in \mathscr{B}(\Sigma^n)$, the map $K \mapsto \phi_k^{r,s,1}(K,\eta)$ is a valuation and Borel measurable on $\mathscr{K}^n$.*

Since the global functionals $\phi_k^{r,s,1}(P,\Sigma^n)$ are continuous, Alesker's characterization theorem must yield a representation for them. Such a representation was explicitly known before. In fact, for $r = 0$ it follows from another relation by McMullen (see [15, p. 269] and [14, Lemma 3.3]) that

$$\phi_k^{0,s,1}(P,\Sigma^n) = Q\Phi_k^{0,s}(P) - 2\pi(s+2)\,\Phi_k^{0,s+2}(P).$$

The general case is covered by [14, p. 505].

It is instructive to express the new local tensor valuations $\phi_k^{r,s,1}(K,\cdot)$ for a general convex body K in terms of the generalized curvatures $k_i(x,u)$ and the corresponding principal directions of curvature $b_i(x,u)$, $i = 1,\dots,n-1$. A short calculation shows that

$$\phi_k^{r,s,1}(K,\eta) \tag{2.40}$$
$$= C_{n,k}^{r,s} \int_{\eta \cap \mathbf{nc}(K)} x^r u^s \sum_{i=1}^{n-1} b_i(x,u)^2 \sum_{\substack{|I|=n-1-k \\ i \notin I}} \frac{\prod_{j \in I} k_j(x,u)}{\mathbb{K}(x,u)} \, \mathscr{H}^{n-1}(\mathrm{d}(x,u)).$$

If $k = 1$, then

$$\phi_1^{r,s,1}(K,\eta) = C_{n,1}^{r,s} \int_{\eta \cap \mathbf{nc}(K)} x^r u^s \sum_{i=1}^{n-1} b_i(x,u)^2 \frac{\prod_{j: j \neq i} k_j(x,u)}{\mathbb{K}(x,u)} \, \mathscr{H}^{n-1}(\mathrm{d}(x,u)),$$

and for $k = n-2$, we have

$$\phi_{n-2}^{r,s,1}(K,\eta) = C_{n,n-2}^{r,s} \int_{\eta \cap \mathbf{nc}(K)} x^r u^s \sum_{i=1}^{n-1} b_i(x,u)^2 \sum_{j: j \neq i} \frac{k_j(x,u)}{\mathbb{K}(x,u)} \, \mathscr{H}^{n-1}(\mathrm{d}(x,u)).$$

For $n = 3$, these two special cases coincide and we get

$$\phi_1^{r,s,1}(K,\eta) = C_{3,1}^{r,s} \int_{\eta \cap \mathbf{nc}(K)} x^r u^s \frac{k_1(x,u) b_2(x,u)^2 + k_2(x,u) b_1(x,u)^2}{\mathbb{K}(x,u)} \, \mathscr{H}^2(\mathrm{d}(x,u)).$$

For a convex body of class C^2, we write u_x for the unique exterior unit normal of K at the boundary point $x \in \partial K$ of K. An application of the coarea formula then yields

$$\phi_k^{r,s,1}(K,\eta) = C_{n,k}^{r,s} \int_{\partial K} \mathbf{1}_\eta(x,u_x) x^r u_x^s \sum_{i=1}^{n-1} b_i(x)^2 \sum_{\substack{|I|=n-1-k \\ i \notin I}} \prod_{j \in I} k_j(x) \, \mathscr{H}^{n-1}(\mathrm{d}x),$$

where the $k_j(x)$ are the principal curvatures and the unit vectors $b_j(x)$ give the principal directions of curvature of K at $x \in \partial K$. In particular, for a convex body K in $\mathbb{R}^3$ with a C^2 boundary we get

$$\phi_1^{r,s,1}(K,\eta) = C_{3,1}^{r,s} \int_{\partial K} \mathbf{1}_\eta(x,u_x) x^r u_x^s \big(k_1(x) b_2(x)^2 + k_2(x) b_1(x)^2\big) \, \mathscr{H}^2(\mathrm{d}x).$$

2.7 Hölder Continuity of Normal Cycles of Convex Bodies

The normal cycle T_K of a convex body K in $\mathbb{R}^n$ has a useful continuity property, which we have used in the previous section. If K_i, $i \in \mathbb{N}$, and K are convex bodies in $\mathbb{R}^n$ and $K_i \to K$ in the Hausdorff metric, as $i \to \infty$, then $T_{K_i} \to T_K$ in the dual flat seminorm for currents (cf. [6, Sec. 4.1.12, p. 367]). This was stated without proof in [22, p. 251] and was proved in [17, Thm. 3.1]; see also [7, Thm. 3.1]. The continuity property has been used in the theory of valuations on manifolds (see, for instance, [2]). It is also a crucial ingredient in [12], in the course of the proof of a classification theorem for local tensor valuations on the space of convex bodies, as we have seen in the previous section.

The purpose of this section is to obtain a quantitative improvement of the preceding continuity result, in the form of a Hölder estimate. As usual we equip $\mathscr{K}^n$ with the Hausdorff metric d_H. We denote by $\mathscr{E}^{n-1}(\mathbb{R}^{2n}) = \mathscr{E}(\mathbb{R}^{2n}, \bigwedge^{n-1}\mathbb{R}^{2n})$ the vector space of all differential forms of degree $n-1$ on $\mathbb{R}^{2n}$ with real coefficients of class C^∞.

Theorem 2.20. *Let $K,L \in \mathscr{K}^n$, and let $M \subset \mathbb{R}^{2n}$ be a compact convex set containing $K_1 \times \mathbb{S}^{n-1}$ and $L_1 \times \mathbb{S}^{n-1}$. Then, for each $\varphi \in \mathscr{E}^{n-1}(\mathbb{R}^{2n})$,*

$$|T_K(\varphi) - T_L(\varphi)| \le C(M,\varphi)\, d_H(K,L)^{\frac{1}{2n+1}},$$

where $C(M,\varphi)$ is a constant which depends (for given dimension) on M and on the Lipschitz constant and the sup-norm of φ on M.

According to the definition of the dual flat seminorm, this result can be interpreted as local Hölder continuity of the normal cycles of convex bodies with respect to the Hausdorff metric and the dual flat seminorm. A similar, but essentially different quantitative result is obtained in [4, Thm. 2]. It refers to more general sets and is, therefore, less explicit. On the other hand, its restriction to convex bodies does not yield the present result, since at least one of the sets in [4] has to be bounded by a submanifold of class C^2. We have not been able to decide whether the stability exponent $1/(2n+1)$ in Theorem 2.20 can be improved.

It remains to prove Theorem 2.20. We continue to use the same notation as in Federer's [6] book, in order to facilitate the comparison. For the scalar product of vectors $x,y \in \mathbb{R}^n$, however, we continue to write $x \cdot y$; the induced norm is denoted by $|\cdot|$. The same notation is used also for other Euclidean spaces which will come up in the following. We identify $\mathbb{R}^n$ and its dual space via the given scalar product.

Given an inner product space $(V,\cdot)$ with norm $|\cdot|$ we obtain an inner product on $\bigwedge_m V$. For $\xi,\eta \in \bigwedge_m V$ with $\xi = v_1 \wedge \cdots \wedge v_m$ and $\eta = w_1 \wedge \cdots \wedge w_m$, where $v_i, w_j \in V$, we define $\xi \cdot \eta = \det(\langle v_i, w_j\rangle_{i,j=1}^m)$. This is independent of the particular representation of ξ,η. For general $\xi,\eta \in \bigwedge_m V$ the inner product is defined by linear extension, and then we put $|\xi| := \sqrt{\xi \cdot \xi}$ for $\xi \in \bigwedge_m V$. If $(b_1,\dots,b_n)$ is an orthonormal basis of V, then the m-vectors $b_{i_1} \wedge \cdots \wedge b_{i_m}$ with $1 \le i_1 < \cdots < i_m \le n$ form an orthonormal basis of $\bigwedge_m V$. Moreover, if $\xi \in \bigwedge_p V$ or $\eta \in \bigwedge_q V$ is simple, then

$$|\xi \wedge \eta| \le |\xi|\,|\eta|. \tag{2.41}$$

Let $(b_1,\dots,b_n)$ be an orthonormal basis of V, and let $(b_1^*,\dots,b_n^*)$ be the dual basis in $V^* = \bigwedge^1 V$. We endow $\bigwedge^m V$ (which is identified with $\bigwedge_m V^*$) with the inner product for which the vectors $b_{i_1}^* \wedge \cdots \wedge b_{i_m}^*$, for $1 \le i_1 < \cdots < i_m \le n$, are an orthonormal basis. Then

$$|\langle \xi, \Phi\rangle| \le |\xi|\,|\Phi| \tag{2.42}$$

for $\xi \in \bigwedge_m V$ and $\Phi \in \bigwedge^m V$. The preceding facts are essentially taken from [6, Section 1.7].

Finally, if V is an n-dimensional inner product space, then *comass* and *mass* are defined as in [6, Section 1.8]. In particular, for $\Phi \in \bigwedge^m V$ the comass $\|\Phi\|$ of Φ satisfies $\|\Phi\| = |\Phi|$ if Φ is simple. Moreover, for $\xi \in \bigwedge_m V$ the mass $\|\xi\|$ of ξ satisfies $\|\xi\| = |\xi|$ if ξ is simple.

The proof of Theorem 2.20 will be preceded by a sequence of lemmas. In order to obtain an upper bound for $|T_K - T_L|$, we first establish an upper bound for $|T_{A_\varepsilon} - T_A|$, for $A \in \{K, L\}$ and $\varepsilon \in [0,1]$, which is done in Lemma 2.21. Then we derive an upper bound for $|T_{K_\varepsilon} - T_{L_\varepsilon}|$ under the assumption that the Hausdorff distance of K and L is sufficiently small. This bound is provided in Lemma 2.26, which in turn is based on four preparatory lemmas.

Lemma 2.21. *Let $K \in \mathscr{K}^n$ and $\varepsilon \in [0,1]$. Let $\varphi \in \mathscr{E}^{n-1}(\mathbb{R}^{2n})$. Then*

$$|T_{K_\varepsilon}(\varphi) - T_K(\varphi)| \le C(K,\varphi)\,\varepsilon,$$

where $C(K,\varphi)$ is a real constant, which depends on the maximum and the Lipschitz constant of φ on $K_1 \times \mathbb{S}^{n-1}$ and on $\mathscr{H}^{n-1}(\partial K_1)$.

Proof. We consider the bi-Lipschitz map

$$F_\varepsilon : \mathbf{nc}(K) \to \mathbf{nc}(K_\varepsilon), \qquad (x,u) \mapsto (x+\varepsilon u, u).$$

The extension of F_ε to all $(x,u) \in \mathbb{R}^{2n}$ by $F_\varepsilon(x,u) := (x+\varepsilon u, u)$ is differentiable for all $(x,u) \in \mathbb{R}^{2n}$. By [6, Theorem 3.2.22 (1)], for $\mathscr{H}^{n-1}$-almost all $(x,u) \in \mathbf{nc}(K)$ the approximate Jacobian of F_ε satisfies

$$\operatorname{ap} J_{n-1} F_\varepsilon(x,u) = \|\textstyle\bigwedge_{n-1} \operatorname{ap} DF_\varepsilon(x,u) a_K(x,u)\| > 0, \tag{2.43}$$

and the simple orienting $(n-1)$-vectors $a_K(x,u)$ and $a_{K_\varepsilon}(x+\varepsilon u, u)$ are related by

$$a_{K_\varepsilon}(x+\varepsilon u, u) = \frac{\bigwedge_{n-1} \operatorname{ap} DF_\varepsilon(x,u) a_K(x,u)}{\|\bigwedge_{n-1} \operatorname{ap} DF_\varepsilon(x,u) a_K(x,u)\|}. \tag{2.44}$$

The orientations coincide, since

$$\langle \textstyle\bigwedge_{n-1}(\Pi_1 + \rho\,\Pi_2) a_K(x,u) \wedge u, \Omega_n \rangle > 0$$

for all $\rho > 0$. Thus, first using the coarea theorem [6, Theorem 3.2.22] and then (2.43) and (2.44), we get

$$\begin{aligned}
T_{K_\varepsilon}(\varphi) &= \int_{\mathbf{nc}(K_\varepsilon)} \langle a_{K_\varepsilon}, \varphi \rangle \, \mathrm{d}\mathscr{H}^{n-1} \\
&= \int_{\mathbf{nc}(K)} \langle a_{K_\varepsilon} \circ F_\varepsilon(x,u), \varphi \circ F_\varepsilon(x,u) \rangle \operatorname{ap} J_{n-1} F_\varepsilon(x,u) \, \mathscr{H}^{n-1}(\mathrm{d}(x,u)) \\
&= \int_{\mathbf{nc}(K)} \langle \textstyle\bigwedge_{n-1} \operatorname{ap} DF_\varepsilon(x,u) a_K(x,u), \varphi \circ F_\varepsilon(x,u) \rangle \, \mathscr{H}^{n-1}(\mathrm{d}(x,u)).
\end{aligned}$$

By the triangle inequality, we obtain

$$\begin{aligned}
&|T_{K_\varepsilon}(\varphi) - T_K(\varphi)| \\
&\quad \le \int_{\mathbf{nc}(K)} \Big\{ \big|\big\langle \big(\textstyle\bigwedge_{n-1} \mathrm{ap}\, DF_\varepsilon(x,u) - \textstyle\bigwedge_{n-1} \mathrm{id}\big) a_K(x,u), \varphi \circ F_\varepsilon(x,u) \big\rangle\big| \\
&\qquad + |\langle a_K(x,u), \varphi(x+\varepsilon u, u) - \varphi(x,u)\rangle| \Big\} \mathscr{H}^{n-1}(\mathrm{d}(x,u)).
\end{aligned}$$

We have

$$\begin{aligned}
&|\langle (\textstyle\bigwedge_{n-1} \mathrm{ap}\, DF_\varepsilon(x,u) - \textstyle\bigwedge_{n-1} \mathrm{id})\, a_K(x,u), \varphi \circ F_\varepsilon(x,u)\rangle| \\
&\qquad \le |\varphi(x+\varepsilon u, u)|\, |(\textstyle\bigwedge_{n-1} \mathrm{ap}\, DF_\varepsilon(x,u) - \textstyle\bigwedge_{n-1} \mathrm{id})\, a_K(x,u)|,
\end{aligned}$$

where we used (2.42). Now $a_K(x,u)$ is of the form $\bigwedge_{i=1}^{n-1}(v_i,w_i)$ with suitable $(v_i,w_i) \in \mathbb{R}^{2n}$ and $|v_i|^2 + |w_i|^2 = 1$. Moreover, we have $DF_\varepsilon(x,u)(v,w) = (v+\varepsilon w, w)$, for all $(v,w) \in \mathbb{R}^{2n}$. Writing $z_i^0 := v_i$, $z_i^1 := w_i$, we have

$$\begin{aligned}
&\big|(\textstyle\bigwedge_{n-1} \mathrm{ap}\, DF_\varepsilon(x,u) - \textstyle\bigwedge_{n-1} \mathrm{id})\, a_K(x,u)\big| \\
&\quad = \Big| \bigwedge_{i=1}^{n-1} (v_i + \varepsilon w_i, w_i) - \bigwedge_{i=1}^{n-1} (v_i, w_i) \Big| \\
&\quad = \Big| \sum_{\alpha_i \in \{0,1\}} \varepsilon^{\sum \alpha_i} \bigwedge_{i=1}^{n-1} (z_i^{\alpha_i}, w_i) - \bigwedge_{i=1}^{n-1} (z_i^0, w_i) \Big| \\
&\quad \le \varepsilon \sum_{\alpha_i \in \{0,1\}, \sum \alpha_i \ge 1} \Big| \bigwedge_{i=1}^{n-1} (z_i^{\alpha_i}, w_i) \Big| \\
&\quad \le c(n)\varepsilon,
\end{aligned}$$

where we used (2.41) and the fact that $|(v_i,w_i)| = 1$ and $|(w_i,w_i)| \le 2$. We deduce that

$$|\varphi(x+\varepsilon u, u)|\big|(\textstyle\bigwedge_{n-1} \mathrm{ap}\, DF_\varepsilon(x,u) - \textstyle\bigwedge_{n-1} \mathrm{id}) a_K(x,u)\big| \le C_1(K,\varphi)\varepsilon.$$

Furthermore, again by (2.42) we get

$$|\langle a_K(x,u), \varphi(x+\varepsilon u, u) - \varphi(x,u)\rangle| \le |\varphi(x+\varepsilon u, u) - \varphi(x,u)| \le C_2(K,\varphi)\,\varepsilon.$$

Thus we conclude that

$$|T_{K_\varepsilon}(\varphi) - T_K(\varphi)| \le C_3(K,\varphi)\,\varepsilon\, \mathscr{H}^{n-1}(\mathbf{nc}(K)).$$

Since $F : \partial K_1 \to \mathbf{nc}(K)$, $z \mapsto (p(K,z), z - p(K,z))$, is Lipschitz with Lipschitz constant bounded from above by 3, the assertion follows. $\square$

A convex body $K \in \mathscr{K}^n$ is said to be ε-smooth (for some $\varepsilon > 0$), if $K = K' + \varepsilon B^n$ for some $K' \in \mathscr{K}^n$. For a nonempty set $A \subset \mathbb{R}^n$, we define the distance from A

to $x \in \mathbb{R}^n$ by $d(A,x) := \inf\{|a-x| : a \in A\}$. The signed distance is defined by $d^*(A,x) := d(A,x) - d(\mathbb{R}^n \setminus A, x)$, $x \in \mathbb{R}^n$, if $A, \mathbb{R}^n \setminus A \neq \emptyset$. If K is ε-smooth, then ∂K has positive reach. More precisely, if $x \in \mathbb{R}^n$ satisfies $d(\partial K, x) < \varepsilon$, then there is a unique point $p(\partial K, x) \in \partial K$ such that $d(\partial K, x) = |p(\partial K, x) - x|$.

Lemma 2.22. *Let $\varepsilon \in (0,1)$ and $\delta \in (0, \varepsilon/2)$. Let $K, L \in \mathscr{K}^n$ be ε-smooth and assume that $d_H(K,L) \leq \delta$. Then*

$$p : \partial K \to \partial L, \qquad x \mapsto p(\partial L, x),$$

is well-defined, bijective, bi-Lipschitz with $\mathrm{Lip}(p) \leq \varepsilon/(\varepsilon-\delta)$*, and $|p(x) - x| \leq \delta$ for all $x \in \partial K$.*

Proof. Since $d_H(K,L) \leq \delta$, we have $K \subset L + \delta B^n$, $L \subset K + \delta B^n$, and a separation argument yields that

$$\{x \in L : d(\partial L, x) \geq \delta\} \subset K. \tag{2.45}$$

This shows that $\partial K \subset \{z \in \mathbb{R}^n : d(\partial L, z) \leq \delta\}$ and therefore the map p is well-defined on ∂K and $|p(x) - x| \leq \delta$ for all $x \in \partial K$. By [5, Theorem 4.8 (8)] it follows that $\mathrm{Lip}(p) \leq \varepsilon/(\varepsilon - \delta)$. Since L is ε-smooth, for $y \in \partial L$ there is a unique exterior unit normal of L at y, which we denote by $u =: u_L(y)$. Put $y_0 := y - \varepsilon u$ and note that $y_0 + (\varepsilon - \delta)B^n \subset K \cap L$ by (2.45). Then $x \in \partial K$ is uniquely determined by the condition $\{x\} = (y_0 + [0,\infty)u) \cap \partial K$ and satisfies $p(x) = y$. This shows that p is surjective.

Now let $x_1, x_2 \in \partial K$ satisfy $p(x_1) = p(x_2) =: p_0 \in \partial L$. Since there is a ball B of radius ε with $p_0 \in B \subset L$, the points $x_1, x_2 \in \partial K$ are on the line through p_0 and the center of B. By (2.45), they cannot be on different sides of p_0, hence $x_1 = x_2$. This shows that the map p is also injective. If $d^*(\partial K, \cdot) : \mathbb{R}^n \to \partial K$ denotes the signed distance function of ∂K, then $q : \partial L \to \partial K$, $z \mapsto z - d^*(\partial K, z) u_L(z)$, is the inverse of p. Since the signed distance function is Lipschitz, Lemma 2.23 below shows that q is Lipschitz as well. ☐

The following lemma provides a simple argument for the fact that the spherical image map of an ε-smooth convex body is Lipschitz with Lipschitz constant at most $1/\varepsilon$.

Lemma 2.23. *Let $K \in \mathscr{K}^n$ be ε-smooth, $\varepsilon > 0$. Then the spherical image map u_K is Lipschitz with Lipschitz constant $1/\varepsilon$.*

Proof. Let $x, y \in \partial K$, and define $u := u_K(x)$, $v := u_K(y)$. Then $x - \varepsilon u + \varepsilon v \in x - \varepsilon u + \varepsilon B^n \subset K$, and hence $(x - \varepsilon u + \varepsilon v - y) \cdot v \leq 0$. This yields

$$\varepsilon(v - u) \cdot v \leq (y - x) \cdot v. \tag{2.46}$$

By symmetry, we also have $\varepsilon(u - v) \cdot u \leq (x - y) \cdot u$, and therefore

$$\varepsilon(v - u) \cdot (-u) \leq (y - x) \cdot (-u). \tag{2.47}$$

Addition of (2.46) and (2.47) yields

$$\varepsilon|v-u|^2 \leq (y-x)\cdot(v-u) \leq |y-x||v-u|,$$

which implies the assertion. □

Lemma 2.24. *Let $\varepsilon \in (0,1)$ and $\delta \in (0,\varepsilon/2)$. Let $K,L \in \mathscr{K}^n$ be ε-smooth and assume that $d_H(K,L) \leq \delta$. Put $p(x) := p(\partial L, x)$ for $x \in \partial K$. Then*

$$G : \mathbf{nc}(K) \to \mathbf{nc}(L), \qquad (x,u) \mapsto (p(x), u_L(p(x))),$$

is bijective, bi-Lipschitz with $\mathrm{Lip}(G) \leq 2/(\varepsilon-\delta) \leq 4/\varepsilon$*, and*

$$|G(x,u)-(x,u)| \leq \delta + 2\sqrt{\delta/\varepsilon}$$

for all $(x,u) \in \mathbf{nc}(K)$.

Proof. It follows from Lemma 2.22 that G is bijective. Then, for $(x,u),(y,v) \in \mathbf{nc}(K)$ we get

$$\begin{aligned}
|G(x,u)-G(y,v)| &\leq |p(x)-p(y)| + |u_L(p(x)) - u_L(p(y))| \\
&\leq \frac{\varepsilon}{\varepsilon-\delta}|x-y| + \frac{1}{\varepsilon}\frac{\varepsilon}{\varepsilon-\delta}|x-y| \\
&\leq \frac{\varepsilon+1}{\varepsilon-\delta}|x-y| \\
&\leq \frac{2}{\varepsilon-\delta}|(x,u)-(y,v)|,
\end{aligned}$$

where we have used again Lemma 2.22 and Lemma 2.23. Let $x \in \partial K$ and $z := p(x) \in \partial L$. We want to bound $u_L(z)\cdot u_K(x)$ from below. If $x \notin L$, then

$$\mathrm{conv}\big(\{x\} \cup (z - \varepsilon u_L(z) + (\varepsilon-\delta)B^n)\big) \subset K,$$

and therefore

$$u_L(z)\cdot u_K(x) \geq \frac{\varepsilon-\delta}{\varepsilon+\delta} \geq 1 - \frac{2\delta}{\varepsilon}.$$

If $x \in L$, then in a similar way we obtain

$$u_L(z)\cdot u_K(x) \geq \frac{\varepsilon-\delta}{\varepsilon} \geq 1 - \frac{\delta}{\varepsilon},$$

hence

$$u_L(z)\cdot u_K(x) \geq 1 - \frac{2\delta}{\varepsilon} \tag{2.48}$$

holds for all $x \in \partial K$. Thus

$$|u_L(z) - u_K(x)| \leq 2\sqrt{\delta/\varepsilon},$$

which finally implies that, for all $(x,u) \in \mathbf{nc}(K)$,

$$\begin{aligned}|G(x,u)-(x,u)| &\le |p(x)-x|+|u_L(p(x))-u_K(x)|\\ &\le \delta+2\sqrt{\delta/\varepsilon}.\end{aligned}$$

Since $G^{-1}:\mathbf{nc}(L)\to\mathbf{nc}(K)$ is given by $G^{-1}(z,u)=(q(z),u_K(q(z)))$ (with q as defined in the proof of Lemma 2.22), it follows that also G^{-1} is Lipschitz. □

Next we show that, under the assumptions of Lemma 2.25, $\bigwedge_{n-1}DG(x,u)$ is an orientation preserving map from the approximate tangent space of $\mathbf{nc}(K)$ to the approximate tangent space of $\mathbf{nc}(L)$. It seems that a corresponding fact is not provided in the proofs of related assertions in the literature.

Lemma 2.25. *Let $\varepsilon\in(0,1)$ and $\delta\in(0,\varepsilon/(4n))$. Let $K,L\in\mathscr{K}^n$ be ε-smooth and assume that $d_H(K,L)\le\delta$. Then, for $\mathscr{H}^{n-1}$-almost all $(x,u)\in\mathbf{nc}(K)$, the $(n-1)$-vector $\bigwedge_{n-1}DG(x,u)a_K(x,u)\in\mathrm{Tan}^{n-1}(\mathscr{H}^{n-1}\llcorner\mathbf{nc}(L),G(x,u))$ has the same orientation as $a_L(G(x,u))$.*

Proof. Let $x\in\partial K$, $u:=u_K(x)$, and $\bar{x}:=p(x)$, hence $d(\partial L,x)=|x-\bar{x}|$. The orientation of $\mathrm{Tan}^{n-1}(\partial K,x)$ is determined by an arbitrary orthonormal basis $(b_1(x),\dots,b_{n-1}(x))$ of $u^\perp$ with $\Omega_n(b_1(x),\dots,b_{n-1}(x),u)=1$. Similarly, any orthonormal basis $(\bar{b}_1(\bar{x}),\dots,\bar{b}_{n-1}(\bar{x}),\bar{u})$ with $\bar{u}:=u_L(p(x))$ determines the orientation of the space $\mathrm{Tan}^{n-1}(\partial L,p(x))$. Since G is bi-Lipschitz, we can assume that $(x,u)\in\mathbf{nc}(K)$ is such that all differentials exist that are encountered in the proof. Moreover, we can also assume that $\bigwedge_{n-1}DG(x,u)a_K(x,u)$ spans $\mathrm{Tan}^{n-1}(\mathscr{H}^{n-1}\llcorner\mathbf{nc}(L),G(x,u))$, where we write again G for a Lipschitz extension of the given map G to $\mathbb{R}^{2n}$. In the following, we put $b_i:=b_i(x)$ and $\bar{b}_i:=\bar{b}_i(\bar{x})$ for $i=1,\dots,n-1$.

By our previous discussion, the differentials of the maps $\mathbf{nc}(K)\to\partial K$, $(x,u)\mapsto x$, and $\partial L\to\mathbf{nc}(L)$, $z\mapsto(z,u_L(z))$, are orientation preserving. Hence, it remains to be shown that the differential of $p:\partial K\to\partial L$, $x\mapsto p(x)$, is orientation preserving, that is,

$$\Delta:=\Omega_n(Dp(x)(b_1),\dots,Dp(x)(b_{n-1}),\bar{u})>0.$$

First, we assume that $x\neq\bar{x}$, that is, $x\notin\partial L$. Since $Dp(x)(\bar{u})=0$, we get

$$Dp(x)(b_i)=\sum_{j=1}^{n-1}b_i\cdot\bar{b}_j\,Dp(x)(\bar{b}_j),$$

and thus

$$\Delta=\det(B)\,\Omega_n(Dp(x)(\bar{b}_1),\dots,Dp(x)(\bar{b}_{n-1}),\bar{u}),$$

where $B=(B_{ij})$ with $B_{ij}:=b_i\cdot\bar{b}_j$ for $i,j\in\{1,\dots,n-1\}$. We choose $\bar{b}_1,\dots,\bar{b}_{n-1}$ as principal directions of curvature of ∂L at $\bar{x}=p(x)$. Then $Dp(x)(\bar{b}_i)=\tau_i\bar{b}_i$ with

$$\tau_i:=1-d(\partial L,x)k_i\left(\partial L,\bar{x},\frac{x-\bar{x}}{|x-\bar{x}|}\right)>0,$$

for $i=1,\dots,n-1$. Here we use that L is ε-smooth, hence ∂L has positive reach, $d(\partial L,x)<\varepsilon$ and

$$\left|k_i\left(\partial L, \bar{x}, \frac{x-\bar{x}}{|x-\bar{x}|}\right)\right| \leq 1/\varepsilon.$$

Hence it follows that $\Delta > 0$ if we can show that $\det(B) > 0$. Let $\tilde{B} = (\tilde{B}_{ij})$ be defined by $\tilde{B}_{ij} := B_{ij}$, $\tilde{B}_{in} := b_i \cdot \bar{u}$, $\tilde{B}_{nj} := u \cdot \bar{b}_j$, and $\tilde{B}_{nn} := u \cdot \bar{u}$, for $i, j \in \{1, \ldots, n-1\}$. Then

$$\begin{aligned}1 &= \Omega_n(b_1, \ldots, b_{n-1}, u)\, \Omega_n(\bar{b}_1, \ldots, \bar{b}_{n-1}, \bar{u}) = \det(\tilde{B})\\ &\leq u \cdot \bar{u} \det(B) + \sum_{i=1}^{n-1} |b_i \cdot \bar{u}| \cdot 1\\ &\leq u \cdot \bar{u} \det(B) + \sqrt{n-1}\sqrt{1-(u \cdot \bar{u})^2}.\end{aligned}$$

From (2.48) and our assumptions, we get $u \cdot \bar{u} \geq 1 - (2\delta)/\varepsilon \geq 1 - 1/(2n)$, and therefore

$$\sqrt{1-(u \cdot \bar{u})^2} \leq \sqrt{1/n}.$$

Thus

$$1 < u \cdot \bar{u} \det(B) + 1,$$

which implies that $\det(B) > 0$.

Finally, we have to consider the case where $x \in \partial L$. For $\mathscr{H}^{n-1}$-almost all $x \in \partial K \cap \partial L$, we have $\operatorname{Tan}^{n-1}(\mathscr{H}^{n-1} \llcorner (\partial K \cap \partial L), x) = u^{\perp}$ and $Dp(x) = \mathrm{id}_{u^{\perp}}$, since $p(z) = z$ for all $z \in \partial K \cap \partial L$. Hence, $\Delta = \Omega_n(b_1, \ldots, b_{n-1}, \bar{u}) = u \cdot \bar{u} > 0$. $\square$

Lemma 2.26. *Let $\varepsilon \in (0,1)$ and $\delta \in (0, \varepsilon/(4n))$. Let $K, L \in \mathscr{K}^n$ be ε-smooth and assume that $d_H(K,L) \leq \delta$. Let $M \subset \mathbb{R}^{2n}$ be a compact convex set containing $K_{1-\varepsilon} \times \mathbb{S}^{n-1}$ and $L_{1-\varepsilon} \times \mathbb{S}^{n-1}$ in its interior. Then*

$$|T_K(\varphi) - T_L(\varphi)| \leq C(M, \varphi)(4/\varepsilon)^{n-1}(\delta + 2\sqrt{\delta/\varepsilon})$$

for $\varphi \in \mathscr{E}^{n-1}(\mathbb{R}^{2n})$, where $C(M, \varphi)$ is a constant which depends on the sup-norm and the Lipschitz constant of φ on M, and on $\mathscr{H}^{n-1}(\partial K_1)$.

Proof. Let G be as in Lemma 2.24 (or a Lipschitz extension to the whole space with the same Lipschitz constant). Then [6, Theorem 4.1.30] implies that

$$T_L = G_\sharp T_K,$$

since $\bigwedge_{n-1} DG$ preserves the orientation of the orienting $(n-1)$-vectors, by Lemma 2.25. (In [17] a corresponding fact is stated without further comment.) Recall the definitions of the dual flat metric $\mathbf{F}_M$ from [6, 4.1.12] and of the mass $\mathbf{M}$ (of a current) from [6, p. 358]. Using [6, 4.1.14], $\partial T_K = 0$, the fact that T_K has compact support contained in the interior of M and Lemma 2.24, we get

$$\begin{aligned}\mathbf{F}_M(T_L - T_K) &= \mathbf{F}_M(G_\sharp T_K - T_K)\\ &\leq \mathbf{M}(T_K) \cdot \|G - \mathrm{id}\|_{\mathbf{nc}(K),\infty} \cdot (4/\varepsilon)^{n-1}\\ &\leq \mathscr{H}^{n-1}(\partial K_1)(4/\varepsilon)^{n-1}(\delta + 2\sqrt{\delta/\varepsilon}),\end{aligned}$$

where $\|G-\mathrm{id}\|_{\mathbf{nc}(K),\infty} := \sup\{|G(x,u)-(x,u)| : (x,u)\in \mathbf{nc}(K)\}$. The assertion now follows from the definition of $\mathbf{F}_M$, since $\|d\varphi\|$ can be bounded in terms of the sup-norm and the Lipschitz constant of φ on M. □

Now we are in a position to complete the proof of Theorem 2.20.

Proof of Theorem 2.20. Let $\varphi\in\mathscr{E}^{n-1}(\mathbb{R}^{2n})$. Let $\delta := d_H(K,L)>0$ and $\varepsilon := \delta^{\frac{1}{2n+1}}$. Assume that $\delta<(4n)^{-\frac{2n+1}{2n}}$. Then $\delta<\varepsilon/(4n)$. Lemma 2.21 implies that

$$\begin{aligned}|T_K(\varphi)-T_{K_\varepsilon}(\varphi)| &\le C(M,\varphi)\,\varepsilon,\\ |T_L(\varphi)-T_{L_\varepsilon}(\varphi)| &\le C(M,\varphi)\,\varepsilon.\end{aligned}$$

Since K_ε and L_ε are ε-smooth, $d_H(K_\varepsilon,L_\varepsilon)=\delta$, $(K_\varepsilon)_{1-\varepsilon}=K_1$ and $(L_\varepsilon)_{1-\varepsilon}=L_1$, Lemma 2.26 shows that

$$|T_{K_\varepsilon}(\varphi)-T_{L_\varepsilon}(\varphi)| \le C(M,\varphi)(4/\varepsilon)^{n-1}(\delta+2\sqrt{\delta/\varepsilon}).$$

The triangle inequality then yields

$$\begin{aligned}|T_K(\varphi)-T_L(\varphi)| &\le C_4(M,\varphi)(\varepsilon+\varepsilon^{1-n}\delta+\varepsilon^{1-n}\sqrt{\delta/\varepsilon})\\ &\le C_5(M,\varphi)\,\delta^{\frac{1}{2n+1}}.\end{aligned}$$

If $\delta\ge(4n)^{-\frac{2n+1}{2n}}$, we simply adjust the constant. □

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