Integral Geometry and Algebraic Structures for Tensor Valuations

Andreas Bernig and Daniel Hug

Abstract In this survey, we consider various integral geometric formulas for tensor-valued valuations that have been obtained by different methods. Furthermore we explain in an informal way recently introduced algebraic structures on the space of translation invariant, smooth tensor valuations, including convolution, product, Poincaré duality and Alesker-Fourier transform, and their relation to kinematic formulas for tensor valuations. In particular, we describe how the algebraic viewpoint leads to new intersectional kinematic formulas and substantially simplified Crofton formulas for translation invariant tensor valuations. We also highlight the connection to general integral geometric formulas for area measures.

1 Introduction

An important part of integral geometry is devoted to the investigation of integrals (mean values) of the form

$$\int_{\mathrm{G}} \varphi(K \cap gL)\, \mu(\mathrm{d}g),$$

where $K, L \subset \mathbb{R}^n$ are sets from a suitable intersection stable class of sets, G is a group acting on $\mathbb{R}^n$ and thus on its subsets, μ is a Haar measure on G, and φ is a functional with values in some vector space W. Common choices for W are the reals or the space of signed Radon measures. Instead of the intersection, Minkowski addition is another natural choice for a set operation which has been studied. The principle aim then is to express such integrals by means of basic geometric functionals of K

Andreas Bernig
Institut für Mathematik, Goethe-Universität Frankfurt, Robert-Mayer-Str. 10, 60054 Frankfurt, Germany, e-mail: bernig@math.uni-frankfurt.de,
Daniel Hug
Karlsruhe Institute of Technology (KIT), Department of Mathematics, D-76128 Karlsruhe, Germany, e-mail: daniel.hug@kit.edu

and L. Depending on the specific framework, such as the class of sets or the type of functional under consideration, different methods have been developed to establish integral geometric formulas, ranging from classical convexity, differential geometry, geometric measure theory to the theory of valuations. The interplay between the theory of valuations and integral geometry, although a classical topic in convexity, has been expanded and deepened considerably in recent years. In the present survey, we explore the integral geometry of tensor-valued functionals. This study suggests and requires generalizations in the theory of valuations which are of independent interest.

Therefore, we describe how some algebraic operations known for smooth translation invariant scalar-valued valuations (product, convolution, Alesker-Fourier transform) can be extended to smooth translation invariant tensor-valued valuations. Although these extensions are straightforward to define, they encode various integral geometric formulas for tensor valuations, like Crofton-type formulas, rotation sum formulas (also called additive kinematic formulas) and intersectional kinematic formulas. Even in the easiest case of translation invariant and $\mathrm{O}(n)$-covariant tensor valuations, explicit formulas are hard to obtain by classical methods. With the present algebraic approach, we are able to simplify the constants in Crofton-type formulas for tensor valuations, and to formulate a new type of intersectional kinematic formulas for tensor valuations. For the latter we show how such formulas can be explicitly calculated in the $\mathrm{O}(n)$-covariant case. As an important byproduct, we compute the Alesker-Fourier transform on a certain class of smooth valuations, called spherical valuations. This result is of independent interest and is the technical heart of the computation of the product of tensor valuations.

2 Tensor Valuations

The present chapter is based on the general introduction to valuations in Chap. 1 and on the description and structural analysis of tensor valuations contained in Chap. 2. The algebraic framework for the investigation of scalar valuations, which has already proved to be very useful in integral geometry, is outlined in Chap. 4. In these chapters relevant background information is provided, including references to previous work, motivation and hints to applications. The latter are also discussed in other parts of this volume, especially in Chaps. 10, 12 and 13.

Let us fix our notation and recall some basic structural facts. We will write V for a finite-dimensional real vector space. Sometimes we fix a Euclidean structure on V, which allows us to identify V with Euclidean space $\mathbb{R}^n$. The space of compact convex sets (including the empty set) is denoted by $\mathscr{K}(V)$ (or $\mathscr{K}^n$ if $V=\mathbb{R}^n$). The vector space of translation invariant, continuous scalar valuations is denoted by $\mathrm{Val}(V)$ (or simply by Val if the vector space V is clear from the context). The smooth valuations in $\mathrm{Val}(V)$ constitute an important subspace for which we write $\mathrm{Val}^\infty(V)$; see Definition 4.5 and Remark **??**, Definition 5.5 and Proposition 5.8, and Section 6.3. There is a natural decomposition of $\mathrm{Val}(V)$ (and then also of $\mathrm{Val}^\infty(V)$)

into subspaces of different parity and different degrees of homogeneity, hence

$$\mathrm{Val}(V) = \bigoplus_{\substack{m=0\\ \varepsilon=\pm}}^{n} \mathrm{Val}_m^{\varepsilon}(V),$$

if V has dimension n, and similarly

$$\mathrm{Val}^{\infty}(V) = \bigoplus_{\substack{m=0\\ \varepsilon=\pm}}^{n} \mathrm{Val}_m^{\varepsilon,\infty}(V);$$

see Sects. 1.4, 4.1 and Theorem 5.1.

Our main focus will be on valuations with values in the space of symmetric tensors of a given rank $p \in \mathbb{N}_0$, for which we write $\mathrm{Sym}^p \mathbb{R}^n$ or simply Sym^p if the underlying vector space is clear from the context (resp., $\mathrm{Sym}^p V$ in case of a general vector space V). Here we deviate from the notation $\mathbb{T}^p$ used in Chaps. 1 and 2. The spaces of symmetric tensors of different ranks can be combined to form a graded algebra in the usual way. By a tensor valuation we mean a valuation on $\mathscr{K}(V)$ with values in the vector space of tensors of a fixed rank, say $\mathrm{Sym}^p(V)$. For the space of translation invariant, continuous tensor valuations with values in $\mathrm{Sym}^p(V)$ we write $\mathrm{TVal}^p(V)$; cf. the notation in Chapters 4, 6 and Definition 5.38. This vector space can be identified with $\mathrm{Val}(V) \otimes \mathrm{Sym}^p(V)$ (or $\mathrm{Val} \otimes \mathrm{Sym}^p$, for short). If we restrict to smooth tensor valuations, we add the superscript ∞, that is $\mathrm{TVal}^{p,\infty}(V)$. It is clear that McMullen's decomposition extends to tensor valuations, hence

$$\mathrm{TVal}^p(V) = \bigoplus_{\substack{m=0\\ \varepsilon=\pm}}^{n} \mathrm{TVal}_m^{p,\varepsilon}(V),$$

if $\dim(V) = n$. The corresponding decomposition is also available for smooth tensor-valued valuations or valuations covariant (or invariant) with respect to a compact subgroup G of the orthogonal group which acts transitively on the unit sphere. The vector spaces of tensor valuations satisfying an additional covariance condition with respect to such a group G is finite-dimensional and consists of smooth valuations only (cf. Example 4.6 and Theorem 5.15). In the following, we will only consider rotation covariant valuations (see Chapter 2).

2.1 Examples of Tensor Valuations

In the following, we mainly consider translation invariant tensor valuations. However, we start with recalling general Minkowski tensors, which are translation covariant but not necessarily translation invariant. For Minkowski tensors, and hence for all isometry covariant continuous tensor valuations, we first state a general Crofton formula. The major part of this contribution is then devoted to translation invariant, rotation covariant, continuous tensor valuations. In this framework, we explain how algebraic structures can be introduced and how they are related to Crofton formulas as well as to additive and intersectional kinematic formulas. Crofton formulas for tensor-valued curvature measures are the subject of Chap. **??**.

For $k \in \{0, \dots, n-1\}$ and $K \in \mathscr{K}^n$, let $\Lambda_0(K, \cdot), \dots, \Lambda_{n-1}(K, \cdot)$ denote the support measures associated with K (see Sect. 1.3). They are Borel measures on $\Sigma^n := \mathbb{R}^n \times \mathbb{S}^{n-1}$ which are concentrated on the normal bundle $\mathbf{nc}\,K$ of K. Let κ_n denote the volume of the unit ball and $\omega_n = n\kappa_n$ the volume of its boundary, the unit sphere. Using the support measures, we recall from Sects. 1.3 or 2.1 that the Minkowski tensors are defined by

$$\Phi_k^{r,s}(K) := \frac{1}{r!s!}\frac{\omega_{n-k}}{\omega_{n-k+s}}\int_{\Sigma^n} x^r u^s \,\Lambda_k(K, \mathrm{d}(x,u)),$$

for $k \in \{0, \dots, n-1\}$ and $r, s \in \mathbb{N}_0$, and

$$\Phi_n^{r,0}(K) := \frac{1}{r!}\int_K x^r \,\mathrm{d}x.$$

In addition, we define $\Phi_k^{r,s} := 0$ for all other choices of indices. Clearly, the tensor valuations $\Phi_k^{0,s}$ and $\Phi_n^{0,0}$, which are obtained by choosing $r = 0$, are translation invariant. However, these are not the only translation invariant examples, since e.g. $\Phi_{k-1}^{1,1}$, for $k \in \{1, \dots, n\}$, also satisfies $\Phi_{k-1}^{1,1}(K+t) = \Phi_{k-1}^{1,1}(K)$ for all $K \in \mathscr{K}^n$ and $t \in \mathbb{R}^n$.

Further examples of continuous, isometry covariant tensor valuations are obtained by multiplying the Minkowski tensors with powers of the metric tensor Q and by taking linear combinations. As shown by Alesker [1, 2], no other examples exist (see also Theorem 2.5). In the following, we write

$$\begin{aligned}\Phi_k^s(K) :&= \Phi_k^{0,s}(K) = \frac{1}{s!}\frac{\omega_{n-k}}{\omega_{n-k+s}}\int_{\Sigma^n} u^s \,\Lambda_k(K, \mathrm{d}(x,u)) \\ &= \binom{n-1}{k}\frac{1}{\omega_{n-k+s}s!}\int_{\mathbb{S}^{n-1}} u^s \,S_k(K, \mathrm{d}u),\end{aligned}$$

for $k \in \{0, \dots, n-1\}$, where we used the kth area measure $S_k(K, \cdot)$ of K, a Borel measure on $\mathbb{S}^{n-1}$ defined by

$$S_k(K, \cdot) := \frac{n\kappa_{n-k}}{\binom{n}{k}}\Lambda_k(K, \mathbb{R}^n \times \cdot).$$

In addition, we define $\Phi_n^0 := V_n$ and $\Phi_n^s := 0$ for $s > 0$. The normalization is such that $\Phi_k^0 = V_k$, for $k \in \{0, \dots, n\}$, where V_k is the kth intrinsic volume. Clearly, the tensor valuations $Q^i\Phi_k^s$, for $k \in \{0, \dots, n\}$ and $i, s \in \mathbb{N}_0$, are continuous, translation invariant, $\mathrm{O}(n)$-covariant, homogeneous of degree k and have tensor rank $2i + s$. We have $\Phi_n^s \equiv 0$ for $s \neq 0$, and $\Phi_0^s(K)$ is independent of K. Hence, we usually exclude these trivial cases. Apart from these, Alesker showed that for each fixed $k \in \{1, \ldots, n-1\}$ the valuations

$$Q^i \Phi_k^s, \quad i, s \in \mathbb{N}_0, 2i + s = p, s \neq 1,$$

form a basis of the vector space of all continuous, translation invariant, $\mathrm{O}(n)$-covariant tensor valuations of rank p which are homogeneous of degree k. The fact that these valuations span the corresponding vector space is implied by [1, Prop. 4.9] (and [2]), the proof is based in particular on basic representation theory. A result of Weil [17, Thm. 3.5] states that differences of area measure of order k, for any fixed $k \in \{1, \ldots, d-1\}$, are dense in the vector space of differences of finite, centered Borel measures on the unit sphere. From this the asserted linear independence of the tensor valuations can be inferred. We also refer to Sect. 6.5 where the present case is discussed as an example of a very general representation theoretic theorem.

The situation for general tensor valuations (which are not necessarily translation invariant) is more complicated. As explained in Chap. 2, the valuations $Q^i \Phi_k^{r,s}$ span the corresponding vector space, but there exist linear dependences between these functionals. Although all linear relations are known and the dimension of the corresponding vector space (for fixed rank and degree of homogeneity) has been determined, the situation here is not perfectly understood.

In the following, it will often (but not always) be sufficient to neglect the metric tensor powers Q^i and just consider the tensor valuations Φ_k^s, since the metric tensor commutes with the algebraic operations to be considered.

2.2 Integral Geometric Formulas

Let $\mathrm{A}(n,k)$, for $k \in \{0, \ldots, n\}$, denote the affine Grassmannian of k-flats in $\mathbb{R}^n$, and let μ_k denote the motion invariant measure on $\mathrm{A}(n,k)$ normalized as in [13, 14]. The Crofton formulas to be discussed below relate the integral mean

$$\int_{\mathrm{A}(n,k)} \Phi_j^{r,s}(K \cap E)\, \mu_k(\mathrm{d}E)$$

of the tensor valuation $\Phi_j^{r,s}(K \cap E)$ of the intersection of K with flats $E \in \mathrm{A}(n,k)$ to tensor valuations of K. Guessing from the scalar case, one would expect that only tensor valuations of the form $Q^i \Phi_{n-k+j}^{r',s'}(K)$ are required. It turns out, however, that for general r the situation is more involved.

The following Crofton formulas for Minkowski tensors have been established in [7]. Since $\Phi_j^{r,s}(K \cap E) = 0$ if $k < j$, we only have to consider the cases where $k \geq j$.

We start with the basic case $k = j$, in which the Crofton formula has a particularly simple form.

Theorem 2.1. *For $K \in \mathscr{K}^n$, $r, s \in \mathbb{N}_0$ and $0 \leq k \leq n-1$,*

$$\int_{\mathrm{A}(n,k)} \Phi_k^{r,s}(K\cap E)\,\mu_k(\mathrm{d}E) = \begin{cases} \tilde{\alpha}_{n,k,s}\, Q^{\frac{s}{2}}\,\Phi_n^{r,0}(K), & \text{if } s \text{ is even,}\\ 0, & \text{if } s \text{ is odd,}\end{cases}$$

where

$$\tilde{\alpha}_{n,k,s} := \frac{1}{(4\pi)^{\frac{s}{2}}\left(\frac{s}{2}\right)!}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-k+s}{2}\right)}{\Gamma\left(\frac{n+s}{2}\right)\Gamma\left(\frac{n-k}{2}\right)}.$$

This result essentially follows from Fubini's theorem, combined with a relation due to McMullen, which connects the Minkowski tensors of $K\cap E$ and the Minkowski tensors of $K\cap E$, defined with respect to the flat E as the ambient space (see (4) for a precise statement).

The main case $j<k$ is considered in the next theorem.

Theorem 2.2. *Let $K\in\mathscr{K}^n$ and $k,j,r,s\in\mathbb{N}_0$ with $0\le j<k\le n-1$. Then*

$$\begin{aligned}\int_{\mathrm{A}(n,k)} &\Phi_j^{r,s}(K\cap E)\,\mu_k(\mathrm{d}E)\\ &= \sum_{z=0}^{\lfloor\frac{s}{2}\rfloor}\chi^{(1)}_{n,j,k,s,z}Q^z\Phi^{r,s-2z}_{n+j-k}(K) + \sum_{z=0}^{\lfloor\frac{s}{2}\rfloor-1}\chi^{(2)}_{n,j,k,s,z}Q^z\\ &\qquad\times\sum_{l=0}^{s-2z-1}\left(2\pi l\Phi^{r+s-2z-l,l}_{n+j-k-s+2z+l}(K) - Q\Phi^{r+s-2z-l,l-2}_{n+j-k-s+2z+l}(K)\right),\end{aligned} \tag{1}$$

where $\chi^{(1)}_{n,j,k,s,z}$ and $\chi^{(2)}_{n,j,k,s,z}$ are explicitly known constants.

The constants $\chi^{(1)}_{n,j,k,s,z}$ and $\chi^{(2)}_{n,j,k,s,z}$ only depend on the indicated lower indices. It is remarkable that they are independent of r. Moreover, the right-hand side of this Crofton formula also involves other tensor valuations than $\Phi^{r',s'}_{n-k+j}(K)$. For instance, in the special case where $n=3$, $k=2$, $j=0$, $r=1$ and $s=2$, Theorem 2.2 yields that

$$\int_{A(3,2)}\Phi_0^{1,2}(K\cap E)\,\mu_2(\mathrm{d}E) = \frac{1}{3}\Phi_1^{1,2}(K) + \frac{1}{24\pi}Q\Phi_1^{1,0}(K) + \frac{1}{6}\Phi_0^{2,1}(K).$$

It can be shown that it is not possible to write $\Phi_0^{2,1}$ as a linear combination of $\Phi_1^{1,2}$ and $Q\Phi_1^{1,0}$, which are the only other Minkowski tensors of rank 3 and homogeneity degree 2.

The explicit expressions obtained for the constants $\chi^{(1)}_{n,j,k,s,z}$ and $\chi^{(2)}_{n,j,k,s,z}$ in [7] require a multiple (five-fold) summation over products and ratios of binomial coefficients and Gamma functions. Some progress which can be made in simplifying this representation is described in Chap. **??**.

Since the tensor valuations on the right-hand side of the Crofton formula (1) are not linearly independent, the specific representation is not unique. Using the linear relation due to McMullen, the result can also be expressed in the form

$$\begin{aligned}\int_{\mathrm{A}(n,k)} &\Phi_j^{r,s}(K\cap E)\,\mu_k(\mathrm{d}E)\\ &=\sum_{z=0}^{\lfloor\frac{s}{2}\rfloor}\chi^{(1)}_{n,j,k,s,z}Q^z\Phi^{r,s-2z}_{n+j-k}(K)+\sum_{z=0}^{\lfloor\frac{s}{2}\rfloor-1}\chi^{(2)}_{n,j,k,s,z}Q^z\\ &\qquad\times\sum_{l\geq s-2z}\Big(Q\Phi^{r+s-2z-l,l-2}_{n+j-k-s+2z+l}(K)-2\pi l\,\Phi^{r+s-2z-l,l}_{n+j-k-s+2z+l}(K)\Big)\end{aligned}\tag{2}$$

with the same constants as before. From (2) we now deduce the Crofton formula for the translation invariant tensor valuations Φ_j^s. For $r=0$, the sum $\sum_{l\geq s-2z}$ on the right-hand side of (2) is non-zero only if $l=s-2z$. Therefore, after some index shift (and discussion of the 'boundary cases' $z=0$ and $z=\lfloor\frac{s}{2}\rfloor$), we obtain

$$\int_{\mathrm{A}(n,k)}\Phi_j^s(K\cap E)\,\mu_k^n(\mathrm{d}E)=\sum_{z=0}^{\lfloor\frac{s}{2}\rfloor}\chi^{(*)}_{n,j,k,s,z}Q^z\Phi^{s-2z}_{n+j-k}(K)\tag{3}$$

for $j<k$, where

$$\chi^{(*)}_{n,j,k,s,z}=\chi^{(1)}_{n,j,k,s,z}+\chi^{(2)}_{n,j,k,s,z-1}-2\pi(s-2z)\chi^{(2)}_{n,j,k,s,z}.$$

Since the right-hand side of (3) is uniquely determined by the left-hand side and the tensor valuations on the right-hand side are linearly independent, the constant $\chi^{(*)}_{n,j,k,s,z}$ is uniquely determined. Using the expression which is obtained for $\chi^{(*)}_{n,j,k,s,z}$ from the constants $\chi^{(1)}_{n,j,k,s,z}$ and $\chi^{(2)}_{n,j,k,s,z}$ provided in [7], it seems to be a formidable task to get a reasonably simple expression for this constant. If $j=k$, then Theorem 2.1 shows that (3) remains true if we define $\chi_{n,k,k,s,\lfloor\frac{s}{2}\rfloor}:=\tilde{\alpha}_{n,k,s}$ if s is even, and as zero in all other cases. As we will see, the approach of algebraic integral geometry to (3) will reveal that $\chi^{(*)}_{n,j,k,s,z}$ has indeed a surprisingly simple expression.

To compare the algebraic approach with the one used in [7], and extended to tensorial curvature measures in Chap. **??**, we point out that the result of Theorem 2.2 is complemented by and in fact is based on an intrinsic Crofton formula, where the tensor valuation $\Phi_j^{r,s}(K\cap E)$ is replaced by $\Phi_{j,E}^{r,s}(K\cap E)$. The latter is the tensor valuation of the intersection $K\cap E$, determined with respect to E as the ambient space but considered as a tensor in $\mathbb{R}^n$(see Section **??** or [7] for an explicit definition). The two tensors are connected by the relation

$$\Phi_j^{r,s}(K\cap E)=\sum_{m\geq 0}\frac{Q(E^\perp)^m}{(4\pi)^m m!}\Phi_{j,E}^{r,s-2m}(K\cap E),\tag{4}$$

due to McMullen [11, Theorem 5.1] (see also [7]), where $Q(E^\perp)$ is the metric tensor of the linear subspace orthogonal to the direction space of E but again considered as a tensor in $\mathbb{R}^n$, that is, $Q(E^\perp)=e_{k+1}^2+\ldots+e_n^2$, where $e_{k+1},\ldots,e_n$ is an orthonormal basis of $E^\perp$. Note that for $s=0$ we get $\Phi_j^{r,0}(K\cap E)=\Phi_{j,E}^{r,0}(K\cap E)$, since the intrinsic volumes and the suitably normalized curvature measures are independent of the ambient space.

The intrinsic Crofton formula for

$$\int_{\mathrm{A}(n,k)} \Phi_{j,E}^{r,s}(K \cap E)\, \mu_k(\mathrm{d}E)$$

has the same structure as the extrinsic Crofton formula stated in Theorem 2.2, but the constants are different. Apart from reducing the number of summations required for determining the constants, progress in understanding the structure of these (intrinsic and extrinsic) integral geometric formulas can be made by localizing the Minkowski tensors. This is the topic of Chapter **??**.

Crofton and intersectional kinematic formulas for Minkowski tensors $\Phi_j^{r,s}$ with $s = 0$ are special cases of corresponding (more general) integral geometric formulas for curvature measures. For example, we have

$$\int_{\mathrm{A}(n,k)} \Phi_j^{r,0}(K \cap E)\, \mu_k(\mathrm{d}E) = a_{njk}\, \Phi_{n+j-k}^{r,0}(K) \tag{5}$$

and

$$\int_{\mathrm{G}_n} \Phi_j^{r,0}(K \cap gM)\, \mu(\mathrm{d}g) = \sum_{k=j}^{n} a_{njk}\, \Phi_{n+j-k}^{r,0}(K) V_k(M), \tag{6}$$

where G_n is the Euclidean motion group, μ is the suitably normalized Haar measure and the (simple) constants a_{njk} are known explicitly. Therefore, we focus on the case $s \neq 0$ (and $r = 0$) in the following.

A close connection between Crofton formulas and intersectional kinematic formulas follows from Hadwiger's general integral geometric theorem (see [14, Theorem 5.1.2]). It states that for any continuous valuation φ on the space of convex bodies and for all $K, M \in \mathscr{K}^n$, we have

$$\int_{\mathrm{G}_n} \varphi(K \cap gM)\, \mu(\mathrm{d}g) = \sum_{k=0}^{n} \int_{\mathrm{A}(n,k)} \varphi(K \cap E)\, \mu_k(\mathrm{d}E)\, V_k(M). \tag{7}$$

Hence, if a Crofton formula for the functional φ is available, then an intersectional kinematic formula is an immediate consequence. This statement includes also tensor-valued functionals, since (7) can be applied coordinate-wise. In particular, this shows that (6) can be obtained from (5). In the same way, Theorem 2.2 and the special case shown in (3) imply kinematic formulas for intersections of convex bodies, one fixed the other moving. Thus, for instance, we obtain

$$\int_{\mathrm{G}_n} \Phi_j^{s}(K \cap gM)\, \mu(\mathrm{d}g) = \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{k=j}^{n} \chi_{n,j,k,s,z}^{(*)}\, Q^z \Phi_{n+j-k}^{s-2z}(K) V_k(M). \tag{8}$$

These results are related to and in fact inspired general integral geometric formulas for area measures (see [10]). The starting point is a local version of Hadwiger's general integral geometric theorem for measure-valued valuations. To state it, let

$\mathscr{M}^+(\mathbb{S}^{n-1})$ be the cone of non-negative measures in the vector space $\mathscr{M}(\mathbb{S}^{n-1})$ of finite Borel measures on the unit sphere.

Theorem 2.3. *Let $\varphi : \mathscr{K}^n \to \mathscr{M}^+(\mathbb{S}^{n-1})$ be a continuous and additive mapping with $\varphi(\emptyset, \cdot) = 0$ (the zero measure). Then, for $K, M \in \mathscr{K}^n$ and Borel sets $A \subset \mathbb{S}^{n-1}$,*

$$\int_{\mathrm{G}_n} \varphi(K \cap gM, A)\,\mu(\mathrm{d}g) = \sum_{k=0}^{n} [T_{n,k}\varphi(K,\cdot)](A) V_k(M), \tag{9}$$

with (the Crofton operator) $T_{n,k} : \mathscr{M}^+(\mathbb{S}^{n-1}) \to \mathscr{M}^+(\mathbb{S}^{n-1})$ given by

$$T_{n,k}\varphi(K,\cdot) := \int_{\mathrm{A}(n,k)} \varphi(K \cap E, \cdot)\,\mu_k(\mathrm{d}E), \quad k = 0, \dots, n.$$

We want to apply this result to area measures of convex bodies, hence we need a Crofton formula for area measures. The statement of such a Crofton formula is based on Fourier operators I_p, for $p \in \{-1, 0, 1, \dots, n\}$, which act on C^∞ functions on $\mathbb{S}^{n-1}$. For $f \in C^\infty(\mathbb{S}^{n-1})$, let f_p be the extension of f to $\mathbb{R}^n \setminus \{0\}$ which is homogeneous of degree $-n+p$, and let $\hat{f}_p$ be the distributional Fourier transform of f_p. For $0 < p < n$, the restriction $I_p(f)$ of $\hat{f}_p$ to the unit sphere is again a smooth function. Let $\mathscr{H}^n_s$ denote the space of spherical harmonics of degree s. Recall that a spherical harmonic of degree s is the restriction to the unit sphere of a homogeneous polynomial p of degree s on $\mathbb{R}^n$ which satisfies $\Delta p = 0$ (and hence is called harmonic), where Δ is the Laplace operator. We refer to [13] for more information on spherical harmonics. Since I_p intertwines the group action of $\mathrm{SO}(n)$, we have $I_p(f_s) = \lambda_s(n,p)\, f_s$ for $f_s \in \mathscr{H}^n_s$ and some $\lambda_s(n,p) \in \mathbb{C}$. It is known that

$$\lambda_s(n,p) = \pi^{\frac{n}{2}} 2^p\, \mathbf{i}^s \frac{\Gamma\left(\frac{s+p}{2}\right)}{\Gamma\left(\frac{s+n-p}{2}\right)}.$$

Note that $\lambda_s(n,p)$ is purely imaginary if s is odd, and real if s is even. See [10] for a summary of the main properties of this Fourier operator and [9, 8] for a detailed exposition.

Using the connection to mean section bodies (see [8]) and the Fourier operators I_p, the following Crofton formula for area measures has been established in [10, Theorem 3.1].

Theorem 2.4. *Let $1 \le j < k \le n$ and $K \in \mathscr{K}^n$. Then*

$$\int_{\mathrm{A}(n,k)} S_j(K \cap E, \cdot)\,\mu_k(\mathrm{d}E) = a(n,j,k) I_j I_{k-j} S_{n+j-k}(-K, \cdot) \tag{10}$$

with

$$a(n,j,k) := \frac{j}{2^n \pi^{(n+k)/2}(n+j-k)} \frac{\Gamma(\frac{k+1}{2})\Gamma(n-j)}{\Gamma(\frac{n+1}{2})\Gamma(k-j)}.$$

Let I^* be the reflection operator $(I^* f)(u) = f(-u)$, $u \in \mathbb{S}^{n-1}$, for a function f on the unit sphere. The operator $T_{n,j,k} := a(n,j,k) I_j I_{k-j} I^*$, for $1 \le j < k \le n$, and the identity operator $T_{n,j,n}$ act as linear operators on $\mathscr{M}(\mathbb{S}^{n-1})$ and can be used to express (10) in the form

$$\int_{\mathrm{A}(n,k)} S_j(K \cap E, \cdot)\, \mu_k(\mathrm{d}E) = T_{n,j,k} S_{n+j-k}(K, \cdot). \tag{11}$$

This is also true for $k = j < n$ if we define

$$T_{n,j,j} S_n(K, \cdot) := \binom{n-1}{j}^{-1} \frac{\omega_{n-j}}{\omega_n} V_n(K) \sigma,$$

where σ is spherical Lebesgue measure. Combining equations (9) and (11), we obtain a kinematic formula for area measures. Using again the operator $T_{n,j,k}$, it can be stated in the form

$$\int_{\mathrm{G}_n} S_j(K \cap gM, A)\, \mu(\mathrm{d}g) = \sum_{k=j}^{n} [T_{n,j,k} S_{n+j-k}(K, \cdot)](A) V_k(M),$$

for $j = 1, \ldots, n-1$. Since the Fourier operators act as multiplier operators on spherical harmonics, it follows that Theorem 2.4 can be rewritten in the form

$$\begin{aligned} &\int_{\mathrm{A}(n,k)} \int_{\mathbb{S}^{n-1}} f_s(u)\, S_j(K \cap E, \mathrm{d}u)\, \mu_k(\mathrm{d}E) \\ &\qquad = a_s(n,j,k) \int_{\mathbb{S}^{n-1}} f_s(u)\, S_{n+j-k}(K, \mathrm{d}u), \end{aligned} \tag{12}$$

where $f_s \in \mathscr{H}_s^n$ and $a_s(s,j,k) := a(n,j,k) b_s(n,j,k)$ with

$$b_s(n,j,k) := 2^k \pi^n \frac{\Gamma\left(\frac{s+j}{2}\right) \Gamma\left(\frac{s+k-j}{2}\right)}{\Gamma\left(\frac{s+n-j}{2}\right) \Gamma\left(\frac{s+n-k+j}{2}\right)}.$$

In addition to Crofton and intersectional kinematic formulas, there is another classical type of integral geometric formula. Since they involve rotations and Minkowski sums of convex bodies, it is justified to call them rotation sum formulas. Let $\mathrm{SO}(n)$ denote the group of rotations and let ν denote the Haar probability measure on this group. A general form of such a formula can again be stated for area measures. Let $K, M \in \mathscr{K}^n$ be convex bodies and let $\alpha, \beta \subset \mathbb{S}^{n-1}$ be Borel sets. Then [13, Theorem 4.4.6] can be written in the form

$$\begin{aligned} &\int_{\mathrm{SO}(n)} \int_{\mathbb{S}^{n-1}} \mathbf{1}_\alpha(u) \mathbf{1}_\beta(\rho^{-1} u)\, S_j(K + \rho M, \mathrm{d}u)\, \nu(\mathrm{d}\rho) \\ &\qquad = \frac{1}{\omega_n} \sum_{k=0}^{j} \binom{j}{k} S_k(K, \alpha) S_{j-k}(M, \beta). \end{aligned} \tag{13}$$

More generally, by the inversion invariance of the Haar measure ν, by basic measure theoretic extension arguments, and by an application of (13) to the coordinate functions of an arbitrary continuous function $f : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathrm{Sym}^{s_1} \otimes \mathrm{Sym}^{s_2}$, for given $s_1, s_2 \in \mathbb{N}_0$, we obtain

$$\int_{\mathrm{SO}(n)} \int_{\mathbb{S}^{n-1}} f(u, \rho u)\, S_j(K + \rho^{-1}M, \mathrm{d}u)\, \nu(\mathrm{d}\rho) = \frac{1}{\omega_n} \sum_{k=0}^{j} \binom{j}{k} \int_{(\mathbb{S}^{n-1})^2} f(u,v)\, \big(S_k(K,\cdot) \times S_{j-k}(M,\cdot)\big)\, (\mathrm{d}(u,v)).$$

To simplify constants, we define

$$\phi_k^s(K) := \int_{\mathbb{S}^{n-1}} u^s\, S_k(K, \mathrm{d}u). \tag{14}$$

Choosing $f(u,v) = u^{s_1} \otimes v^{s_2}$, we thus get

$$\begin{aligned} &\int_{\mathrm{SO}(n)} (\mathrm{id}^{\otimes s_1} \otimes \rho^{\otimes s_2}) \phi_j^{s_1+s_2}(K + \rho^{-1}M)\, \nu(\mathrm{d}\rho) \\ &\quad = \int_{\mathrm{SO}(n)} \int_{\mathbb{S}^{n-1}} u^{s_1} \otimes (\rho u)^{s_2}\, S_j(K + \rho^{-1}M, \mathrm{d}u)\, \nu(\mathrm{d}\rho) \\ &\quad = \frac{1}{\omega_n} \sum_{k=0}^{j} \binom{j}{k} \int_{(\mathbb{S}^{n-1})^2} u^{s_1} \otimes v^{s_2}\, \big(S_k(K,\cdot) \times S_{j-k}(M,\cdot)\big)\, (\mathrm{d}(u,v)), \end{aligned} \tag{15}$$

and hence

$$\int_{\mathrm{SO}(n)} (\mathrm{id}^{\otimes s_1} \otimes \rho^{\otimes s_2}) \phi_j^{s_1+s_2}(K + \rho^{-1}M)\, \nu(\mathrm{d}\rho) = \frac{1}{\omega_n} \sum_{k=0}^{j} \binom{j}{k} \phi_k^{s_1}(K) \otimes \phi_{j-k}^{s_2}(M).$$

Up to the different normalization, this is the additive kinematic formula for tensor valuations stated in [6, Theorem 5]. In particular,

$$\int_{\mathrm{SO}(n)} \phi_j^s(K + \rho M)\, \nu(\mathrm{d}\rho) = \frac{1}{\omega_n} \sum_{k=0}^{j} \binom{j}{k} \phi_k^s(K) S_{j-k}(M),$$

where $S_i(M) := S_i(M, \mathbb{S}^{n-1}) = n\kappa_{n-i} \binom{n}{i}^{-1} V_i(M)$.

In the following section, we develop basic algebraic structures for tensor valuations and provide applications to integral geometry. From this approach, we will obtain a Crofton formula for the tensor valuations Φ_k^s, but also for another set of tensor valuations, denoted by Ψ_k^s, for which the Crofton formula has 'diagonal form'. Moreover, we will study more general intersectional kinematic formulas than the one considered in (8) and describe the connection between intersectional and additive kinematic formulas. In the course of our analysis, we determine Alesker's Fourier operator for spherical valuations, that is, valuations obtained by integration of a spherical harmonic (or, more generally, any smooth spherical function) against an area measure.

3 Algebraic Structures on Tensor Valuations

Recall that $\mathrm{Val} = \mathrm{Val}(\mathbb{R}^n)$ denotes the Banach space of translation invariant continuous valuations on $V = \mathbb{R}^n$, and $\mathrm{Val}^\infty = \mathrm{Val}^\infty(\mathbb{R}^n)$ is the dense subspace of smooth valuations; see Chapters 4 and 5 for more information. In this section, we first discuss the extension of basic operations and transformations from scalar valuations to tensor-valued valuations. The scalar case is described in Chap. 4.

In the following, we usually work in Euclidean space $\mathbb{R}^n$ with the Lebesgue measure and the volume functional V_n on convex bodies. Since some of the results are also stated in invariant terms, we write vol for a volume measure on V, that is, a choice of a translation invariant locally finite Haar measure on an n-dimensional vector space V. Of course, in case $V = \mathbb{R}^n$ we always use V_n as a specific choice of the restriction of a volume measure vol to $\mathscr{K}^n$ (the corresponding choice is made for $V = \mathbb{R}^n \times \mathbb{R}^n$).

3.1 Product

Existence and uniqueness of the product of smooth valuations is provided by the following result; see also Sect. 4.2 for the more general construction of an exterior product between smooth scalar-valued valuations on possibly different vector spaces.

Proposition 2.5. *Let $\phi_1, \phi_2 \in \mathrm{Val}^\infty$ be smooth valuations on $\mathbb{R}^n$ given by*

$$\phi_i(K) = \mathrm{vol}(K + A_i), \quad K \in \mathscr{K}^n,$$

where $A_1, A_2 \in \mathscr{K}^n$ are smooth convex bodies with positive Gauss curvature at every boundary point. Let $\Delta : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ be the diagonal embedding. Then

$$\phi_1 \cdot \phi_2(K) := \mathrm{vol}(\Delta K + A_1 \times A_2), \quad K \in \mathscr{K}^n,$$

extends by continuity and bilinearity to a product on Val^∞.

The product is compatible with the degree of a valuation (i.e., if ϕ_i has degree k_i, then $\phi_1 \cdot \phi_2$ has degree $k_1 + k_2$ if $k_1 + k_2 \leq n$), and more generally with the action of the group $\mathrm{GL}(n)$.

We can extend the product component-wise from smooth scalar-valued valuations to smooth tensor-valued valuations. To see this, let V be a finite-dimensional vector space, $V = \mathbb{R}^n$ say, and $s_1, s_2 \in \mathbb{N}_0$. Let $\Phi_i \in \mathrm{TVal}^{s_i,\infty}(V)$ for $i = 1, 2$. Let $w_1, \ldots, w_m$ be a basis of $\mathrm{Sym}^{s_1} V$, and let $u_1, \ldots, u_l$ be a basis of $\mathrm{Sym}^{s_2} V$. Then there are $\phi_i, \psi_j \in \mathrm{Val}^\infty(V)$, $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, l\}$, such that

$$\Phi_1(K) = \sum_{i=1}^m \phi_i(K) w_i \quad \text{and} \quad \Phi_2(K) = \sum_{j=1}^l \psi_j(K) u_j$$

for $K \in \mathscr{K}(V)$. Now we would like to define (omitting the obvious ranges of the indices)

$$(\Phi_1 \cdot \Phi_2)\,(K) := \sum_{i,j} (\phi_i \cdot \psi_j)(K) w_i u_j.$$

The dot on the right-hand side is the product of the smooth valuations ϕ_i, ψ_j, and $w_i u_j \in \mathrm{Sym}^{s_1+s_2} V$ denotes the symmetric tensor product of the symmetric tensors $w_i \in \mathrm{Sym}^{s_1} V$ and $u_j \in \mathrm{Sym}^{s_2} V$.

Let us verify that this definition is independent of the chosen bases. For this, let $w'_i = \sum_j c_{ij} w_j$ with some invertible matrix (c_{ij}), and let $u'_i = \sum_j e_{ij} u_j$ with an invertible matrix (e_{ij}).

If

$$\Phi_1(K) = \sum_i \phi'_i(K) w'_i = \sum_i \phi_i(K) w_i,$$

then a comparison of coefficients yields that $\phi'_i = \sum_j c^{ji} \phi_j$, where (c^{ji}) denotes the matrix inverse. Similarly, from

$$\Phi_2(K) = \sum_i \psi'_j(K) u'_i = \sum_i \psi_i(K) u_i,$$

we conclude that $\psi'_i = \sum_j e^{ji} \psi_j$, where (e^{ji}) denotes the matrix inverse. Therefore, we have

$$\begin{aligned}
\sum_{i,j} (\phi'_i \cdot \psi'_j) w'_i u'_j &= \sum_{i,j,b_1,b_2} \left(\sum_{a_1,b_1} c^{a_1 i} \phi_{a_1} \cdot e^{b_1 j} \psi_{b_1} \right) \sum_{a_2,b_2} c_{ia_2} w_{a_2} e_{jb_2} u_{b_2} \\
&= \sum_{a_1,a_2,b_1,b_2} \underbrace{\left(\sum_{i,j} c^{a_1 i} c_{ia_2} e^{b_1 j} e_{jb_2} \right)}_{=\delta^{a_1}_{a_2} \delta^{b_1}_{b_2}} (\phi_{a_1} \cdot \psi_{b_1}) w_{a_2} \cdot u_{b_2} \\
&= \sum_{a,b} (\phi_a \cdot \psi_a) w_a \cdot u_b,
\end{aligned}$$

which proves the asserted independence of the representation.

Thus, recalling that $\mathrm{TVal}^s_m(V)$ denotes the vector space of translation invariant continuous valuations on $\mathscr{K}(V)$ which are homogeneous of degree m and take values in the vector space $\mathrm{Sym}^s V$ of symmetric tensors of rank s over V, and that $\mathrm{TVal}^{s,\infty}_m(V)$ is the subspace consisting of the smooth elements of this vector space, we have

$$\Phi_1 \cdot \Phi_2 \in \mathrm{TVal}^{s_1+s_2,\infty}_{k+l}(V), \quad k+l \le n,$$

for $\Phi_1 \in \mathrm{TVal}^{s_1,\infty}_k(V)$, $\Phi_2 \in \mathrm{TVal}^{s_2,\infty}_l(V)$ and $k,l \in \{0,\dots,n\}$.

A similar description and similar arguments can be given for the operations considered in the following subsections.

3.2 Convolution

Similarly as for the product of valuations, an explicit definition of the convolution of two valuations is given only for a suitable subclass of valuations (cf. Sect. 4.3).

Proposition 2.6. *Let $\phi_1, \phi_2 \in \mathrm{Val}^\infty$ be smooth valuations on $\mathbb{R}^n$ given by*

$$\phi_i(K) = \mathrm{vol}(K + A_i),$$

where A_1, A_2 are smooth convex bodies with positive Gauss curvature at every boundary point. Then

$$\phi_1 * \phi_2(K) := \mathrm{vol}(K + A_1 + A_2),$$

extends by continuity and bilinearity to a product (which is called convolution) on Val^∞.

Written in invariant terms, the convolution is a bilinear map

$$\mathrm{Val}^\infty(V) \otimes \mathrm{Dens}(V^*) \times \mathrm{Val}^\infty(V) \otimes \mathrm{Dens}(V^*) \to \mathrm{Val}^\infty(V) \otimes \mathrm{Dens}(V^*),$$

where $\mathrm{Dens}(V^*)$ is the one-dimensional space of translation invariant, locally finite complex-valued Haar measures (Lebesgue measures, see Sect. 4.3) on the dual space V^*. It is compatible with the action of the group $\mathrm{GL}(n)$ and with the codegree of a valuation (i.e., if ϕ_i has degree k_i, then $\phi_1 * \phi_2$ has degree $k_1 + k_2 - n$ if $k_1 + k_2 \geq n$).

The convolution can be extended component-wise to a convolution on the space of translation invariant smooth tensor valuations. Hence we have

$$\Phi_1 * \Phi_2 \in \mathrm{TVal}_{k+l-n}^{s_1+s_2,\infty}(V), \quad k + l \geq n,$$

for $\Phi_1 \in \mathrm{TVal}_k^{s_1,\infty}(V)$, $\Phi_2 \in \mathrm{TVal}_l^{s_2,\infty}(V)$ and $k, l \in \{0, \ldots, n\}$. This is analogous to the definition and computation in the previous subsection.

3.3 Alesker-Fourier Transform

Alesker introduced an operation on smooth valuations, now called Alesker-Fourier transform (cf. Sect. 4.4). It is a map $\mathbb{F} : \mathrm{Val}^\infty(\mathbb{R}^n) \to \mathrm{Val}^\infty(\mathbb{R}^n)$ which reverses the degree of homogeneity, that is,

$$\mathbb{F} : \mathrm{Val}_k^\infty(\mathbb{R}^n) \to \mathrm{Val}_{n-k}^\infty(\mathbb{R}^n),$$

and which transforms product into convolution of smooth valuations, more precisely, we have

$$\mathbb{F}(\phi_1 \cdot \phi_2) = \mathbb{F}(\phi_1) * \mathbb{F}(\phi_2). \tag{16}$$

On valuations which are smooth and even, the Alesker-Fourier transform can easily be described in terms of Klain functions as follows. Let $\phi \in \mathrm{Val}_k^{\infty,+}(\mathbb{R}^n)$ (the space of smooth and even valuations which are homogeneous of degree k). Then the restriction of ϕ to a k-dimensional subspace E is a multiple $\mathrm{Kl}_\phi(E)$ of the volume, and the resulting function (Klain function) Kl_ϕ determines ϕ. Then

$$\mathrm{Kl}_{\mathbb{F}\phi}(E) = \mathrm{Kl}_\phi(E^\perp)$$

for every $(n-k)$-dimensional subspace E.

As an example (and consequence of the relation to Klain functions), the intrinsic volumes satisfy

$$\mathbb{F}(V_k) = V_{n-k}, \tag{17}$$

where $V_0, \dots, V_n$ denote the intrinsic volumes on $\mathscr{K}^n$.

The description in the odd case is more involved and it is preferable to describe it in invariant terms (i.e., without referring to a Euclidean structure).

Let V be an n-dimensional real vector space. Then

$$\mathbb{F} : \mathrm{Val}_k^\infty(V) \to \mathrm{Val}_{n-k}^\infty(V) \otimes \mathrm{Dens}(V^*),$$

where Dens denotes the one-dimensional space of densities (Lebesgue measures). This map commutes with the action of $\mathrm{GL}(V)$ on both sides. Applying it twice (and using the identification $\mathrm{Dens}(V^*) \otimes \mathrm{Dens}(V) \cong \mathbb{C}$), it satisfies the Plancherel type formula

$$(\mathbb{F}^2\phi)(K) = \phi(-K).$$

Working again on Euclidean space $V = \mathbb{R}^n$, we can extend the Alesker-Fourier transform component-wise to a map $\mathbb{F} : \mathrm{TVal}^{s,\infty} \to \mathrm{TVal}^{s,\infty}$ such that

$$\mathbb{F} : \mathrm{TVal}_k^{s,\infty} \to \mathrm{TVal}_{n-k}^{s,\infty}.$$

It is not an easy task to determine the Fourier transform of valuations other than the intrinsic volumes.

3.4 Example: Intrinsic Volumes

As an example, let us compute the Alesker product of intrinsic volumes $V_0, \dots, V_n$ in $\mathbb{R}^n$. We complement the definition of the intrinsic volumes by $V_l := 0$ for $l < 0$. Let $\mathrm{vol} = V_n$ denote the volume measure on $\mathscr{K}^n$.

Recall Steiner's formula (1.16) which states that

$$\mathrm{vol}(K + rB) = \sum_{i=0}^{n} V_{n-i}(K)\kappa_i r^i, \qquad r \geq 0.$$

Now we fix $r \geq 0$ and $s \geq 0$ and define the smooth valuations $\phi_1(K) := \mathrm{vol}(K + rB)$ and $\phi_2(K) := \mathrm{vol}(K + sB)$. Then

$$\phi_1 * \phi_2(K) = \mathrm{vol}(K + rB + sB) = \mathrm{vol}(K + (r+s)B) = \sum_{k=0}^{n} V_{n-k}(K)\kappa_k (r+s)^k,$$

hence

$$\phi_1 * \phi_2 = \sum_{i,j=0}^{n} V_{n-i-j}\kappa_{i+j}\binom{i+j}{i} r^i s^j.$$

On the other hand, since $\phi_1 = \sum_{i=0}^{n} V_{n-i}\kappa_i r^i$ and $\phi_2 = \sum_{i=0}^{n} V_{n-i}\kappa_i s^i$, we obtain

$$\phi_1 * \phi_2 = \sum_{i,j=0}^{n} V_{n-i} * V_{n-j}\kappa_i\kappa_j r^i s^j.$$

Now we compare the coefficient of $r^i s^j$ in these equations and get

$$V_{n-i-j}\kappa_{i+j}\binom{i+j}{i} = V_{n-i} * V_{n-j}\kappa_i\kappa_j.$$

Writing i instead of $n-i$ and j instead of $n-j$, we obtain

$$V_i * V_j = \begin{bmatrix} 2n-i-j \\ n-i \end{bmatrix} V_{i+j-n}, \tag{18}$$

where we used the flag coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} := \binom{n}{k}\frac{\kappa_n}{\kappa_k\kappa_{n-k}}, \qquad k \in \{0, \dots, n\}.$$

Taking Alesker-Fourier transform on both sides yields

$$V_i \cdot V_j = \begin{bmatrix} i+j \\ i \end{bmatrix} V_{i+j}. \tag{19}$$

The computation of convolution and product of tensor valuations follows the same scheme: first one computes the convolution of tensor valuations, which can be considered easier. Then one applies the Alesker-Fourier transform to obtain the product. However, in the tensor-valued case it is much harder to write down the Alesker-Fourier transform in an explicit way. This step is the technical heart of our approach.

3.5 Poincaré Duality

The product of smooth translation invariant valuations as well as the convolution both satisfy a version of Poincaré duality, which moreover are identical up to a sign.

To state this more precisely, recall that the vector spaces $\mathrm{Val}_k = \mathrm{Val}_k(\mathbb{R}^n)$, $k \in \{0,n\}$, are one-dimensional and spanned by the Euler-characteristic $\chi = V_0$ and the volume functional $V_n = \mathrm{vol}$, that is, $\mathrm{Val}_0 \cong \mathbb{R} \cdot \chi$ and $\mathrm{Val}_n \cong \mathbb{R} \cdot \mathrm{vol}$. We denote by $\phi_0, \phi_n \in \mathbb{R}$ the component of $\phi \in \mathrm{Val}$ of degree 0 and n, respectively.

Proposition 2.7. *The pairings*

$$\mathrm{Val}_k^\infty \times \mathrm{Val}_{n-k}^\infty \to \mathbb{R}, \quad (\phi_1, \phi_2) \mapsto (\phi_1 \cdot \phi_2)_n,$$

and

$$\mathrm{Val}_k^\infty \times \mathrm{Val}_{n-k}^\infty \to \mathbb{R}, \quad (\phi_1, \phi_2) \mapsto (\phi_1 * \phi_2)_0,$$

are perfect, that is, the induced maps

$$\mathrm{pd}_m, \mathrm{pd}_c : \mathrm{Val}_k^\infty \to \mathrm{Val}_{n-k}^{\infty,*}$$

are injective with dense image. Moreover,

$$\mathrm{pd}_c = \begin{cases} \mathrm{pd}_m & \text{on } \mathrm{Val}_k^+ . \\ -\mathrm{pd}_m & \text{on } \mathrm{Val}_k^- . \end{cases}$$

To illustrate this proposition and to highlight the difference between the two pairings, let us compute them on an easy example. Let $\phi_i(K) := \mathrm{vol}(K + A_i)$, where A_i, $i \in \{1,2\}$, are smooth convex bodies with positive Gauss curvature. Then $\phi_1 * \phi_2(K) = \mathrm{vol}(K + A_1 + A_2)$, and hence $(\phi_1 * \phi_2)_0 = \mathrm{vol}(A_1 + A_2)$.

On the other hand, $\phi_1 \cdot \phi_2(K) = \mathrm{vol}_{2n}(\Delta K + A_1 \times A_2)$. Using Fubini's theorem, one rewrites this as

$$\phi_1 \cdot \phi_2(K) = \int_{\mathbb{R}^n} \phi_2((x - A_1) \cap K)\, \mathrm{d}x.$$

Taking for K a large ball reveals that $(\phi_1 \cdot \phi_2)_n = \phi_2(-A_1) = \mathrm{vol}(A_2 - A_1)$. If $A_1 = -A_1$, then ϕ_1 is even and both pairings agree indeed.

The extension of Poincaré duality to tensor-valued valuations is postponed to Sect. 4.1 where it is required for the description of the relation between additive and intersectional kinematic formulas for tensor valuations.

3.6 Explicit Computations in the $\mathrm{O}(n)$-Equivariant Case

In this subsection, we outline the explicit computation of product, convolution and Alesker-Fourier transform in the $\mathrm{O}(n)$-equivariant case. Depending on the situation, we will either use the basis consisting of the tensor valuations $Q^i \Phi_k^{s-2i}$ or the basis consisting of the tensor valuations $Q^i \Psi_k^{s-2i}$. The latter are defined in the following proposition.

Proposition 2.8. *The following statements hold.*

(i) *For $0 \leq k < n$ and $s \neq 1$, define*

$$\Psi_k^s := \Phi_k^s + \sum_{j=1}^{\lfloor \frac{s}{2} \rfloor} \frac{(-1)^j \Gamma(\frac{n-k+s}{2}) \Gamma(\frac{n}{2}+s-1-j)}{(4\pi)^j j! \Gamma(\frac{n-k+s}{2}-j) \Gamma(\frac{n}{2}+s-1)} Q^j \Phi_k^{s-2j}$$

and let $\Psi_n^0 := \Phi_n^0$. Then Ψ_k^s is the trace free part of Φ_k^s. In particular, $\Psi_k^s \equiv \Phi_k^s$ mod Q.

(ii) *For $0 \leq k < n$ and $s \neq 1$, Φ_k^s can be written in terms of $\Psi_k^{s'}$ as*

$$\Phi_k^s = \Psi_k^s + \sum_{j=1}^{\lfloor \frac{s}{2} \rfloor} \frac{\Gamma\left(\frac{n-k+s}{2}\right) \Gamma(\frac{n}{2}+s-2j)}{(4\pi)^j j! \Gamma(\frac{n-k+s}{2}-j) \Gamma(\frac{n}{2}+s-j)} Q^j \Psi_k^{s-2j}.$$

The inversion which is needed to derive (ii) from (i) can be accomplished with the help of Zeilberger's algorithm.

The first and easier step in the explicit calculations of algebraic structures for tensor valuations is to compute the convolution of two tensor valuations. Since Φ_k^s is smooth (i.e., each component is a smooth valuation), we may write

$$\Phi_k^s(K) = \int_{\mathrm{nc}(K)} \omega_{k,s},$$

where $\omega_{k,s}$ is a smooth $(n-1)$-form on the sphere bundle $\mathbb{R}^n \times S^{n-1}$ with values in $\operatorname{Sym}^s \mathbb{R}^n$. Next, for valuations represented by differential forms, there is an easy formula for the convolution, which involves only some linear and bilinear operations (a kind of Hodge star and a wedge product). The resulting formula states that, for $k, l \leq n$ with $k + l \geq n$ and $s_1, s_2 \neq 1$, we have

$$\Phi_k^{s_1} * \Phi_l^{s_2} = \frac{\omega_{s_1+s_2+2n-k-l}}{\omega_{s_1+n-k}\omega_{s_2+n-l}} \frac{(n-k)(n-l)}{2n-k-l} \cdot \\ \cdot \binom{2n-k-l}{n-k} \binom{s_1+s_2}{s_1} \frac{(s_1-1)(s_2-1)}{1-s_1-s_2} \Phi_{k+l-n}^{s_1+s_2},$$

or, using the normalization (14) which is more convenient for this purpose,

$$\phi_k^{s_1} * \phi_l^{s_2} = n \frac{\binom{k+l}{n}}{\binom{k+l}{k}} \frac{(s_1-1)(s_2-1)}{1-s_1-s_2} \phi_{k+l-n}^{s_1+s_2}.$$

The computation of the Alesker-Fourier transform of tensor valuations is the main step and will be explained in the next subsection. For $0 \leq k \leq n$ and $s \neq 1$, the result is

$$\mathbb{F}(\Psi_k^s) = \mathbf{i}^s\, \Psi_{n-k}^s,$$
$$\mathbb{F}(\Phi_k^s) = \mathbf{i}^s \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \frac{(-1)^j}{(4\pi)^j j!} Q^j \Phi_{n-k}^{s-2j}.$$

Finally, the product of two tensor valuations can be computed once the convolution and the Alesker-Fourier transform are known, see (16). The result is a bit more involved than the formulas for convolution and Alesker-Fourier transform. The reason is that the formula for the convolution is best described in terms of the tensor valuations Φ_k^s, while the description of the Alesker-Fourier transform has a simpler expression for the Ψ_k^s.

After some algebraic manipulations (which make use of Zeilberger's algorithm), we arrive at

$$\begin{aligned}\Phi_k^{s_1} \cdot \Phi_l^{s_2} = \frac{kl}{k+l}\binom{k+l}{k} \sum_{\substack{a=0 \\ 2a \neq s_1+s_2-1}}^{\lfloor \frac{s_1+s_2}{2} \rfloor} \frac{1}{(4\pi)^a a!} \Bigg(\sum_{m=0}^{a} \sum_{i=\max\{0, m-\lfloor \frac{s_2}{2} \rfloor\}}^{\min\{m, \lfloor \frac{s_1}{2} \rfloor\}} \\ (-1)^{a-m} \binom{a}{m}\binom{m}{i} \frac{\omega_{s_1+s_2-2m+k+l}}{\omega_{s_1-2i+k}\,\omega_{s_2-2m+2i+l}} \binom{s_1+s_2-2m}{s_1-2i}. \\ \cdot \frac{(s_1-2i-1)(s_2-2m+2i-1)}{1-s_1-s_2+2m} \Bigg) Q^a \Phi_{k+l}^{s_1+s_2-2a}. \qquad (20)\end{aligned}$$

Here $0 \leq k, l$ with $k+l \leq n$ and $s_1, s_2 \neq 1$. It seems that there is no simple closed expression for the inner sum.

3.7 Tensor Valuations Versus Scalar-Valued Valuations

The interplay between tensor valuations and scalar-valued valuations will be essential in the computation of the Alesker-Fourier transform. We therefore explain this now is some more detail.

We first need some facts from representation theory. It is well-known that equivalence classes of complex irreducible (finite-dimensional) representations of $\mathrm{SO}(n)$ are indexed by their highest weights. The possible highest weights are tuples $(\lambda_1, \lambda_2, \ldots, \lambda_{\lfloor \frac{n}{2} \rfloor})$ of integers such that

1. $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\lfloor \frac{n}{2} \rfloor} \geq 0$ if n is odd,
2. $\lambda_1 \geq \lambda_2 \geq \ldots \geq |\lambda_{\frac{n}{2}}| \geq 0$ if n is even.

Given $\lambda = (\lambda_1, \ldots, \lambda_{\lfloor \frac{n}{2} \rfloor})$ satisfying this condition, we will denote the corresponding equivalence class of representations by Γ_λ.

The decomposition of the $\mathrm{SO}(n)$-module Val_k has recently been obtained in [3].

Theorem 2.9 ([3]). *There is an isomorphism of* $\mathrm{SO}(n)$*-modules*

$$\mathrm{Val}_k \cong \bigoplus_\lambda \Gamma_\lambda,$$

where λ ranges over all highest weights such that $|\lambda_2| \leq 2$, $|\lambda_i| \neq 1$ for all i and $\lambda_i = 0$ for $i > \min\{k, n-k\}$. In particular, these decompositions are multiplicity-free.

Let Γ be an irreducible representation of $\mathrm{SO}(n)$ and Γ^* its dual. The space of k-homogeneous $\mathrm{SO}(n)$-equivariant Γ-valued valuations (i.e., maps $\Phi : \mathscr{K} \to \Gamma$ such that $\Phi(gK) = g\Phi(K)$ for all $g \in \mathrm{SO}(n)$) is $(\mathrm{Val}_k \otimes \Gamma)^{\mathrm{SO}(n)} = \mathrm{Hom}_{\mathrm{SO}(n)}(\Gamma^*, \mathrm{Val}_k)$. By Theorem 2.9, Γ^* appears in the decomposition of Val_k precisely if Γ appears, and in this case the multiplicity is 1. By Schur's lemma it follows that $\dim(\mathrm{Val}_k \otimes \Gamma)^{\mathrm{SO}(n)} = 1$ in this case.

Let us construct the (unique up to scale) equivariant Γ-valued valuation explicitly. Denote by $\mathrm{Val}_k(\Gamma)$ the Γ-isotypical summand, which is isomorphic to Γ since Val_k is multiplicity free.

Let $\phi_1, \ldots, \phi_m$ be a basis of $\mathrm{Val}_k(\Gamma)$. These elements play two different roles: first we can look at them as valuations, i.e., elements of Val_k. Second, we may think of $\phi_1, \ldots, \phi_m$ as basis of the irreducible representation Γ. The action of $\mathrm{SO}(n)$ on this basis is given by

$$g\phi_i = \sum_j c_i^j(g)\phi_j,$$

where $(c_i^j(g))_{i,j}$ is a matrix depending on g. The map $g \mapsto (c_i^j(g))_{i,j}$ is a homomorphism of Lie groups $\mathrm{SO}(n) \to \mathrm{GL}(m)$.

Let $\phi_1^*, \ldots, \phi_m^*$ be the dual basis of Γ^*. Then

$$g\phi_i^* = \sum_j (c_i^j(g))^{-t}\phi_j = \sum_j c_j^i(g^{-1})\phi_j,$$

Using the double role played by the ϕ_i mentioned above, we set

$$\Phi(K) := \sum_i \phi_i(K)\phi_i^* \in \Gamma^*. \tag{21}$$

We claim that Φ is an $\mathrm{O}(n)$-equivariant valuation with values in Γ^*. Indeed, we compute

$$\begin{aligned}
\Phi(gK) &= \sum_i \phi_i(gK)\phi_i^* \\
&= \sum_i (g^{-1}\phi_i)(K)\phi_i^* \\
&= \sum_{i,j} c_i^j(g^{-1})\phi_j(K)\phi_i^* \\
&= \sum_j \phi_j(K) \sum_i c_i^j(g^{-1})\phi_i^* \\
&= \sum_j \phi_j(K) g\phi_j^* \\
&= g(\Phi(K)).
\end{aligned}$$

Conversely, we now start with an equivariant Γ^*-valued continuous translation invariant valuation Φ of degree k. Let $w_1,\dots,w_m$ be a basis of Γ^*. Then we may look at the components of Φ, i.e., we decompose

$$\Phi(K) = \sum_i \phi_i(K) w_i$$

with $\phi_i \in \mathrm{Val}_k$. Let the action of $\mathrm{SO}(n)$ on Γ^* be given by

$$g w_i = \sum_j a_i^j(g) w_j.$$

We have

$$\begin{aligned}
\Phi(gK) &= \sum_i \phi_i(gK) w_i = \sum_i (g^{-1}\phi_i)(K) w_i, \\
g(\Phi(K)) &= \sum_j \phi_j(K) g w_j = \sum_{i,j} \phi_j(K) a_j^i(g) w_i.
\end{aligned}$$

Comparing coefficients yields $g^{-1}\phi_i = \sum_j a_j^i(g)\phi_j$, or

$$g\phi_i = \sum_j a_j^i(g^{-1})\phi_j.$$

This shows that the subspace of Val_k spanned by $\phi_1,\dots,\phi_m$ is isomorphic to Γ.

In summary, we have shown the following fact.

Each $\mathrm{SO}(n)$*-irreducible representation* Γ *appearing in the decomposition of* Val_k *corresponds to the (unique up to scale)* Γ^**-valued continuous translation invariant valuation* Φ *from* (21)*. Conversely, the coefficients of a* Γ^**-valued continuous translation invariant valuation span a subspace of* Val_k *isomorphic to* Γ*.*

Let us now discuss the special case of symmetric tensor valuations. The $\mathrm{SO}(n)$-representation space Sym^s is (in general) not irreducible. Indeed, the trace map $\mathrm{tr}: \mathrm{Sym}^s \to \mathrm{Sym}^{s-2}$ commutes with $\mathrm{SO}(n)$, hence its kernel is an invariant subspace.

This subspace turns out to be the irreducible representation $\Gamma_{(s,0,\dots,0)}$ and can be identified with the space $\mathscr{H}_s^n$ of spherical harmonics of degree s.

Since the trace map is onto, we get the following decomposition.

$$\mathrm{Sym}^s \cong \bigoplus_j \mathscr{H}^n_{s-2j}.$$

Instead of studying Sym^s-valued valuations, we can therefore study $\mathscr{H}_s^n$-valued valuations. For $s \neq 1$ and $1 \leq k \leq n-1$, the representation $\mathscr{H}_s^n$ appears in Val_k with multiplicity 1. Since $\mathscr{H}_s^n$ is self-dual, the construction sketched above yields in the special case $\Gamma := \mathscr{H}_s^n$ a unique (up to scale) $\mathscr{H}_s^n$-valued equivariant continuous translation invariant valuation homogeneous of degree k, which we denoted by Ψ_k^s.

3.8 The Alesker-Fourier Transform

As we have seen in the previous subsection, the study of (symmetric) tensor valuations and the study of the $\mathscr{H}^s$-isotypical summand of Val_k are equivalent. For the computation of the Alesker-Fourier transform, it is easier to work with scalar-valued valuations. Let us first define a particular class of valuations, called spherical valuations.

Let f be a smooth function on S^{n-1}. For $k \in \{0,\dots,n-1\}$, we define a valuation $\mu_{k,f} \in \mathrm{Val}_k(\mathbb{R}^n)$ by

$$\mu_{k,f}(K) := \binom{n-1}{k} \frac{1}{\omega_{n-k}} \int_{S^{n-1}} f(y)\, S_k(K, \mathrm{d}y).$$

Such valuations are called spherical (see also the recent preprint [15]). Here the normalization is chosen such that for $f \equiv 1$ we have $\mu_{k,f} = V_k$, $k \in \{0,\dots,n-1\}$. By Subsection 3.7, the components of an $\mathrm{SO}(n)$-equivariant tensor valuation are spherical. Since the Alesker-Fourier transform of such a tensor valuation is defined component-wise, it suffices to compute the Alesker-Fourier transform of spherical valuations.

In this subsection, we sketch this (rather involved) computation. The first and easy observation is that, by Schur's lemma, there exist constants $c_{n,k,s} \in \mathbb{C}$ which only depend on n,k,s such that

$$\mathbb{F}(\mu_{k,f}) = c_{n,k,s}\mu_{n-k,f}, \qquad f \in \mathscr{H}_s^n. \tag{22}$$

The multipliers $c_{n,k,s}$ of the Alesker-Fourier transform can be computed in the even case (i.e., if s is even) by looking at Klain functions. In the odd case, there seems to be no easy way to compute them. We adapt ideas from [12], where the multipliers of the α-cosine transform were computed, to our situation. The main point is that the Alesker-Fourier transform is not only an $\mathrm{SO}(n)$-equivariant operator, but (if written in intrinsic terms) is equivariant under the larger group $\mathrm{GL}(n)$. Using elements from the Lie algebra $\mathfrak{gl}(n)$ allows us to pass from one irreducible $\mathrm{SO}(n)$-representation to another and to obtain a recursive formula for the constants $c_{n,k,s}$, which states that

$$\frac{c_{n,k,s+2}}{c_{n,k,s}} = -\frac{k+s}{n-k+s}. \tag{23}$$

This step requires extensive computations using differential forms, and we refer to [6] for the details.

Next, one can use induction over s,k,n to prove that

$$c_{n,k,s} = \mathbf{i}^s \frac{\Gamma\left(\frac{n-k}{2}\right)\Gamma\left(\frac{s+k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{s+n-k}{2}\right)}.$$

More precisely, in the even case, we may use as induction start the case $s=0$, which corresponds to intrinsic volumes, whose Alesker-Fourier transform is known by (17).

In the odd case, we use as induction start $s=3$. In order to compute $c_{n,k,3}$, we use a special case of a Crofton formula from [7] (see also Chap.) to compute the quotients $\frac{c_{n,k+1,3}}{c_{n,k,3}}$. This fixes all constants up to a scaling which may depend on n. More precisely,

$$c_{n,k,s} = \varepsilon_n \mathbf{i}^s \frac{\Gamma\left(\frac{n-k}{2}\right)\Gamma\left(\frac{s+k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{s+n-k}{2}\right)}, \tag{24}$$

where ε_n depends only on n. Using functorial properties of the Alesker-Fourier transform, we find that ε_n is independent of n. In the two-dimensional case, however, there is a very explicit description of the Alesker-Fourier transform (see also Example 4.14 (4)) which finally allows us to deduce that $\varepsilon_n = 1$ for all $n \geq 2$.

An alternative approach to determining the constants $c_{n,k,s}$ is to prove independently a Crofton formula for the tensor valuations Ψ_k^s, via the Crofton formula for area measures, as described before (see also Remark 4.6 in [10]). This point of view suggests to relate the Fourier operator for spherical valuations to the Fourier operators for spherical functions via the relation

$$\mathbb{F}(\bar{\mu}_{k,f}) = (2\pi)^{-\frac{d}{2}}\, \bar{\mu}_{d-k,I_k f},$$

for $f \in C^\infty(S^{d-1})$, where

$$\bar{\mu}_{k,f}(K) = \binom{d-1}{k}(2\pi)^{\frac{k}{2}} \int_{S^{d-1}} f(u)\, S_k(K,\mathrm{d}u),$$

is just a renormalization of $\mu_{k,f}(K)$.

4 Kinematic Formulas

In this section, we first describe the interplay between algebraic structures and kinematic formulas in general (i.e., for tensor valuations which are equivariant under a group G acting transitively on the unit sphere). Then we will specialize to the $\mathrm{O}(n)$-covariant case.

4.1 Relation Between Kinematic Formulas and Algebraic Structures

Let G be a subgroup of $\mathrm{O}(n)$ which acts transitively on the unit sphere. Then the space $\mathrm{TVal}^{s,G}(V)$ of G-covariant, translation invariant continuous $\mathrm{Sym}^s(V)$-valued valuations is finite-dimensional. Next we define two integral geometric operators. We start with the one for rotation sum formulas.

Let $\Phi \in \mathrm{TVal}^{s_1+s_2,G}(V)$. We define a bivaluation with values in the tensor product $\mathrm{Sym}^{s_1} V \otimes \mathrm{Sym}^{s_2} V$ by

$$a^G_{s_1,s_2}(\Phi)(K,L) := \int_G (\mathrm{id}^{\otimes s_1} \otimes g^{\otimes s_2})\Phi(K + g^{-1}L)\, \nu(\mathrm{d}g)$$

for $K, L \in \mathscr{K}(V)$, where G is endowed with the Haar probability measure ν (see [16]). (This notation is consistent with the case $V = \mathbb{R}^n$ and $G = \mathrm{O}(n)$.)

Let $\Phi_1, \dots, \Phi_{m_1}$ be a basis of $\mathrm{TVal}^{s_1,G}(V)$, and let $\Psi_1, \dots, \Psi_{m_2}$ be a basis of $\mathrm{TVal}^{s_2,G}(V)$. Arguing as in the classical Hadwiger argument (cf. [16, Theorem 4.3]), it can be seen that there are constants c^{Φ}_{ij} such that

$$a^G_{s_1,s_2}(\Phi)(K,L) = \sum_{i,j} c^{\Phi}_{ij}\, \Phi_i(K) \otimes \Psi_j(L)$$

for $K, L \in \mathscr{K}(V)$. The *additive kinematic operator* is the map

$$\begin{aligned} a^G_{s_1,s_2} : \mathrm{TVal}^{s_1+s_2,G}(V) &\to \mathrm{TVal}^{s_1,G}(V) \otimes \mathrm{TVal}^{s_2,G}(V) \\ \Phi &\mapsto \sum_{i,j} c^{\Phi}_{ij}\, \Phi_i \otimes \Psi_j, \end{aligned}$$

which is independent of the choice of the bases.

In view of intersectional kinematic formulas, we define a bivaluation with values in $\mathrm{Sym}^{s_1} V \otimes \mathrm{Sym}^{s_2} V$ by

$$k^G_{s_1,s_2}(\Phi)(K,L) := \int_{\bar{G}} (\mathrm{id}^{\otimes s_1} \otimes g^{\otimes s_2})\Phi(K \cap \bar{g}^{-1}L)\, \mu(\mathrm{d}\bar{g})$$

for $K, L \in \mathscr{K}(V)$, where $\bar{G}$ is the group generated by G and the translation group of V, endowed with the product measure μ of ν and a translation invariant Haar measure on V, and where g is the rotational part of $\bar{g}$. (Again this notation is consistent with the special case where $\bar{G} = \mathrm{G}_n$ is the motion group, $G = \mathrm{O}(n)$ and μ is the motion invariant Haar measure with its usual normalization as a 'product measure'.) Choosing bases and arguing as above, we find

$$k^G_{s_1,s_2}(\Phi)(K,L) = \sum_{i,j} d^{\Phi}_{ij}\, \Phi_i(K) \otimes \Psi_j(L) \tag{25}$$

for $K, L \in \mathscr{K}(V)$. Of course, the constants d^{Φ}_{ij} depend on the chosen bases and on Φ, but the operator, called *intersectional kinematic operator*,

$$\begin{aligned} k^G_{s_1,s_2} : \mathrm{TVal}^{s_1+s_2,G}(V) &\to \mathrm{TVal}^{s_1,G}(V) \otimes \mathrm{TVal}^{s_2,G}(V) \\ \Phi &\mapsto \sum_{i,j} d^{\Phi}_{ij}\, \Phi_i \otimes \Psi_j, \end{aligned}$$

is independent of these choices.

In the following, we explain the connection between these operators and then provide explicit examples.

For this we first lift the Poincaré duality maps to tensor-valued valuations. Let V be a Euclidean vector space with scalar product $\langle \cdot\,,\cdot \rangle$. For $s \leq r$ we define the contraction map by

$$\begin{aligned} \mathrm{contr} : V^{\otimes s} \times V^{\otimes r} &\to V^{\otimes (r-s)}, \\ (v_1 \otimes \ldots \otimes v_s, w_1 \otimes \ldots \otimes w_r) &\mapsto \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle \ldots \langle v_s, w_s \rangle w_{s+1} \otimes \cdots \otimes w_r, \end{aligned}$$

and linearity. This map restricts to a map $\mathrm{contr} : \mathrm{Sym}^s V \times \mathrm{Sym}^r V \to \mathrm{Sym}^{r-s} V$. In particular, if $r = s$, the map $\mathrm{Sym}^s V \times \mathrm{Sym}^s V \to \mathbb{R}$ is the usual scalar product on $\mathrm{Sym}^s V$, which will also be denoted by $\langle \cdot\,,\cdot \rangle$.

The trace map $\mathrm{tr} : \mathrm{Sym}^s V \to \mathrm{Sym}^{s-2} V$ is defined by restriction of the map $V^{\otimes s} \to V^{\otimes (s-2)}$, $v_1 \otimes \ldots \otimes v_s \mapsto \langle v_1, v_2 \rangle v_3 \otimes \ldots \otimes v_s$, for $s \geq 2$.

The scalar product on $\mathrm{Sym}^s V$ induces an isomorphism $q^s : \mathrm{Sym}^s V \to (\mathrm{Sym}^s V)^*$ and we set

$$\begin{aligned} \mathrm{pd}^s_c : \mathrm{TVal}^{s,\infty} = \mathrm{Val}^\infty \otimes \mathrm{Sym}^s V &\xrightarrow{\mathrm{pd}_c \otimes q^s} (\mathrm{Val}^\infty)^* \otimes (\mathrm{Sym}^s V)^* = (\mathrm{TVal}^{s,\infty})^*, \\ \mathrm{pd}^s_m : \mathrm{TVal}^{s,\infty} = \mathrm{Val}^\infty \otimes \mathrm{Sym}^s V &\xrightarrow{\mathrm{pd}_m \otimes q^s} (\mathrm{Val}^\infty)^* \otimes (\mathrm{Sym}^s V)^* = (\mathrm{TVal}^{s,\infty})^*. \end{aligned}$$

From Proposition 2.7 it follows easily that

$$\mathrm{pd}^s_m = (-1)^s\, \mathrm{pd}^s_c\,. \tag{26}$$

Finally, we write

$$m, c : \mathrm{TVal}^{s_1,\infty}(V) \otimes \mathrm{TVal}^{s_2,\infty}(V) \to \mathrm{TVal}^{s_1+s_2,\infty}(V)$$

for the maps corresponding to the product and the convolution. Moreover, we write m_G, c_G for the restrictions of these maps to the corresponding spaces of G-covariant tensor valuations.

Theorem 2.10. *Let G be a compact subgroup of $\mathrm{O}(n)$ acting transitively on the unit sphere. Then the diagram*

$$\begin{array}{ccc}
\mathrm{TVal}^{s_1+s_2,G} & \xrightarrow{a^G_{s_1,s_2}} & \mathrm{TVal}^{s_1,G} \otimes \mathrm{TVal}^{s_2,G} \\
\Big\downarrow{\scriptstyle \mathrm{pd}_c^{s_1+s_2}} & & \Big\downarrow{\scriptstyle \mathrm{pd}_c^{s_1} \otimes \mathrm{pd}_c^{s_2}} \\
\left(\mathrm{TVal}^{s_1+s_2,G}\right)^* & \xrightarrow{c_G^*} & \left(\mathrm{TVal}^{s_1,G}\right)^* \otimes \left(\mathrm{TVal}^{s_2,G}\right)^* \\
\Big\downarrow{\scriptstyle \mathbb{F}^*} & & \Big\downarrow{\scriptstyle \mathbb{F}^* \otimes \mathbb{F}^*} \\
\left(\mathrm{TVal}^{s_1+s_2,G}\right)^* & \xrightarrow{m_G^*} & \left(\mathrm{TVal}^{s_1,G}\right)^* \otimes \left(\mathrm{TVal}^{s_2,G}\right)^* \\
\Big\uparrow{\scriptstyle \mathrm{pd}_m^{s_1+s_2}} & & \Big\uparrow{\scriptstyle \mathrm{pd}_m^{s_1} \otimes \mathrm{pd}_m^{s_2}} \\
\mathrm{TVal}^{s_1+s_2,G} & \xrightarrow{k^G_{s_1,s_2}} & \mathrm{TVal}^{s_1,G} \otimes \mathrm{TVal}^{s_2,G}
\end{array}$$

commutes and encodes the relations between product, convolution, Alesker-Fourier transform, intersectional and additive kinematic formulas.

This diagram allows us to express the additive kinematic operator in terms of the intersectional kinematic operator, and vice versa, with the Fourier transform as the link between these operators.

Corollary 2.11. *Intersectional and additive kinematic formulas are related by the Alesker-Fourier transform in the following way:*

$$a^G = \left(\mathbb{F}^{-1} \otimes \mathbb{F}^{-1}\right) \circ k^G \circ \mathbb{F},$$

or equivalently

$$k^G = (\mathbb{F} \otimes \mathbb{F}) \circ a^G \circ \mathbb{F}^{-1}.$$

This follows by looking at the outer square in Theorem 2.10, by carefully taking into account the signs coming from (26).

4.2 Some Explicit Examples of Kinematic Formulas

We start with a description of a Crofton formula for tensor valuations. Combining the connection between Crofton formulas and the product of valuations (see [4, (2) and (16)]) and the explicit formulas for the product of tensor valuations given in (20), we obtain

$$\int_{\mathrm{A}(n,n-l)} \Phi_k^s(K\cap E)\,\mu_{n-l}(\mathrm{d}E) = \begin{bmatrix} n \\ l \end{bmatrix}^{-1} \left(\Phi_k^s \cdot \Phi_l^0\right)(K)$$
$$= \begin{bmatrix} n \\ l \end{bmatrix}^{-1} \binom{k+l}{k} \frac{kl}{k+l} \sum_{a=0, 2a\neq s-1}^{\lfloor \frac{s}{2} \rfloor} \frac{1}{(4\pi)^a a!}$$
$$\times \sum_{m=0}^{a} (-1)^{a-m} \binom{a}{m} \frac{\omega_{s-2m+k+l}}{\omega_{s-2m+k}\,\omega_l} Q^a \Phi_{k+l}^{s-2a}.$$

After simplification of the inner sum by means of Zeilberger's algorithm, we obtain the Crofton formula in the Φ-basis which was obtained in [6].

Theorem 2.12. *If $k,l \geq 0$ with $k+l \leq n$ and $s \in \mathbb{N}_0$, then*

$$\int_{\mathrm{A}(n,n-l)} \Phi_k^s(K\cap E)\,\mu_{n-l}(\mathrm{d}E) = \begin{bmatrix} n \\ l \end{bmatrix}^{-1} \binom{k+l}{k} \frac{kl}{2(k+l)} \frac{1}{\Gamma\left(\frac{k+l+s}{2}\right)}$$
$$\times \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \frac{\Gamma\left(\frac{l}{2}+j\right)\Gamma\left(\frac{k+s}{2}-j\right)}{(4\pi)^j j!} Q^j \Phi_{k+l}^{s-2j}(K).$$

The result is also true in the cases $k,l \in \{0,n\}$, if the right-hand side is interpreted properly; see the comments after [6, Theorem 3]. The same is true for the next result.

Comparing the trace-free part of this formula (or by inversion), we deduce the Crofton formula for the Ψ-basis, in which the result has a particularly convenient form.

Corollary 2.13. *If $k,l \geq 0$ and $k+l \leq n$, then*

$$\int_{\mathrm{A}(n,n-l)} \Psi_k^s(K\cap E)\,\mu_{n-l}(\mathrm{d}E) = \frac{\omega_{s+k+l}}{\omega_{s+k}\,\omega_l} \binom{k+l}{k} \frac{kl}{k+l} \begin{bmatrix} n \\ l \end{bmatrix}^{-1} \Psi_{k+l}^s(K).$$

Alternatively, as observed in [10], Corollary 2.13 can be deduced from (12), and then Theorem 2.12 can be obtained as a consequence.

Thus, having now a convenient Crofton formula for tensor valuations, we deduce from Hadwiger's general integral geometric theorem an intersectional kinematic formula in the Ψ-basis.

Theorem 2.14. *Let $K,M \in \mathscr{K}^n$ and $j \in \{0,\dots,n\}$. Then*

$$\int_{\mathrm{G}_n} \Psi_j^s(K\cap gM)\,\mu(\mathrm{d}g) = \sum_{k=j}^{n} \frac{\omega_{s+k}}{\omega_{s+j}\,\kappa_{k-j}} \binom{k-1}{j-1} \begin{bmatrix} n \\ k-j \end{bmatrix}^{-1} \Psi_k^s(K)\,V_{n-k+j}(M).$$

Let us now prove some more refined intersectional kinematic formulas. In principle, we could also use Corollary 2.11 to find the intersectional kinematic formulas once we know the additive formulas. The problem is that (15) only gives us the value of a_{s_1,s_2} on the basis element $\phi_j^{s_1+s_2}$, but not on multiples of such basis elements with powers of the metric tensors. However, such terms appear naturally in the Fourier transform.

We therefore use Theorem 2.10 with $V=\mathbb{R}^n$ and $G=\mathrm{O}(n)$, more precisely the lower square in the diagram.

In (20) we have computed the product of two tensor valuations. For fixed (small) ranks s_1, s_2, the formula simplifies and can be evaluated in a closed form. For instance, if $1 \leq k, l$ with $k+l \leq n$ and $s_1 = s_2 = 3$, we get

$$\begin{aligned} \Phi_k^3 \cdot \Phi_l^3 = & \frac{(k+1)(l+1)\Gamma\left(\frac{k+l+1}{2}\right)}{\pi^{\frac{5}{2}}(k+l+4)(k+l+2)(k+l)\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{l}{2}\right)} \cdot \\ & \cdot \left(-32\Phi_{k+l}^6\pi^3 + 8Q\Phi_{k+l}^4\pi^2 - Q^2\Phi_{k+l}^2\pi + \frac{1}{12}Q^3\Phi_{k+l}^0\right). \end{aligned} \tag{27}$$

Let us next work out the vertical arrows in the diagram of Theorem 2.10, that is, the Poincaré duality pd_m. Again, this is a computation involving differential forms. The result (see [6, Corollary 5.3]) is

$$\langle \mathrm{pd}_m^s(\Phi_k^s), \Phi_{n-k}^s \rangle = (-1)^s \frac{1-s}{\pi^s s!^2} \binom{n}{k} \frac{k(n-k)}{4} \frac{\Gamma\left(\frac{k+s}{2}\right)\Gamma\left(\frac{n-k+s}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}. \tag{28}$$

We now explain how to compute the intersectional kinematic formula $k_{3,3}^{\mathrm{O}(n)}$ with this knowledge.

Since $\Phi_m^1 \equiv 0$, it is clear that there is a formula of the form

$$k_{3,3}^{\mathrm{O}(n)}(\Phi_i^6) = \sum_{k+l=n+i} a_{n,i,k} \Phi_k^3 \otimes \Phi_l^3$$

with some constants $a_{n,i,k}$ which remain to be determined. Fix k, l with $k+l=n+i$. Using (28), we find

$$\begin{aligned} \langle \mathrm{pd}_m^3\, \Phi_k^3, \Phi_{n-k}^3 \rangle &= \frac{1}{72\pi^3}\binom{n}{k} k(n-k) \frac{\Gamma\left(\frac{k+3}{2}\right)\Gamma\left(\frac{n-k+3}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}, \\ \langle \mathrm{pd}_m^3\, \Phi_l^3, \Phi_{n-l}^3 \rangle &= \frac{1}{72\pi^3}\binom{n}{l} l(n-l) \frac{\Gamma\left(\frac{l+3}{2}\right)\Gamma\left(\frac{n-l+3}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}, \end{aligned}$$

and therefore

$$\begin{aligned} & \langle (\mathrm{pd}_m^3 \otimes \mathrm{pd}_m^3) \circ k_{3,3}^{\mathrm{O}(n)}(\Phi_i^6), \Phi_{n-k}^3 \otimes \Phi_{n-l}^3 \rangle \\ & \quad = a_{n,i,k} \frac{1}{72\pi^3}\binom{n}{k} k(n-k) \frac{\Gamma\left(\frac{k+3}{2}\right)\Gamma\left(\frac{n-k+3}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \\ & \qquad \cdot \frac{1}{72\pi^3}\binom{n}{l} l(n-l) \frac{\Gamma\left(\frac{l+3}{2}\right)\Gamma\left(\frac{n-l+3}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}. \end{aligned}$$

On the other hand, by (27) and (28),

$$\begin{aligned}
&\langle m^*_{\mathrm{O}(n)} \circ \mathrm{pd}^6_m(\Phi^6_i), \Phi^3_{n-k} \otimes \Phi^3_{n-l}\rangle = \langle \mathrm{pd}^6_m(\Phi^6_i), \Phi^3_{n-k} \cdot \Phi^3_{n-l}\rangle \\
&= \frac{(n-k+1)(n-l+1)\Gamma\left(\frac{n-i+1}{2}\right)}{\pi^{\frac{5}{2}}(n-i+4)(n-i+2)(n-i)\Gamma\left(\frac{n-l}{2}\right)\Gamma\left(\frac{n-k}{2}\right)} \cdot \\
&\quad \cdot \left\langle \mathrm{pd}^6_m(\Phi^6_i), -32\Phi^6_{n-i}\pi^3 + 8Q\Phi^4_{n-i}\pi^2 - Q^2\Phi^2_{n-i}\pi + \frac{1}{12}Q^3\Phi^0_{n-i} \right\rangle \\
&= \frac{1}{207360} \frac{(k-n-1)(i-k-1)\Gamma\left(\frac{n+1}{2}\right)(i+1)(i-1)(i-3)}{\pi^5\Gamma\left(\frac{i+1}{2}\right)\Gamma\left(\frac{n-k}{2}\right)\Gamma\left(\frac{k-i}{2}\right)}.
\end{aligned}$$

From this, the explicit value of $a_{n,i,k}$ given in the theorem follows. Comparing these expressions, we find that

$$a_{n,i,k} = \frac{(i+1)(i-1)(i-3)}{40\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{i+1}{2}\right)} \frac{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{l}{2}\right)}{(k+1)(l+1)}.$$

We summarize the result in the following theorem.

Theorem 2.15. *Let* $K, M \in \mathscr{K}^n$ *and* $i \in \{0, \dots, n-1\}$. *Then*

$$\begin{aligned}
&\int_{\mathrm{G}_n} (\mathrm{id}^{\otimes 3} \otimes g^{\otimes 3})\Phi^6_i(K \cap g^{-1}M)\,\mu(\mathrm{d}g) \\
&\qquad = \frac{(i+1)(i-1)(i-3)}{40\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{i+1}{2}\right)} \sum_{k+l=n+i} \frac{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{l}{2}\right)}{(k+1)(l+1)} \Phi^3_k(K) \otimes \Phi^3_l(M).
\end{aligned}$$

The same technique can be applied to all bidegrees, but it seems hard to find a closed formula which is valid simultaneously in all cases.

References

1. Alesker, S., Continuous rotation invariant valuations on convex sets. *Ann. of Math.* **149** (1999), 977–1005.
2. Alesker, S., Description of continuous isometry covariant valuations on convex sets. *Geom. Dedicata* **74** (1999), 241–248.
3. Alesker, S., Bernig, A., Schuster, F., Harmonic analysis of translation invariant valuations. *Geom. Funct. Anal.* **21** (2011),751–773.
4. Bernig, A., Algebraic integral geometry. In: Global differential geometry. Springer Proc. Math. **17**, Springer, Berlin (2012), 107–145. arXiv:1004.3145
5. Bernig, A., Fu., J.H.G., Convolution of convex valuations. *Geom. Dedicata*, **123** (2006),153–169.
6. Bernig, A., Hug, D., Kinematic formulas for tensor valuations. *J. Reine Angew. Math.* (to appear), arxiv:1402.2750.
7. Hug, D., Schneider, R., Schuster, R., Integral geometry of tensor valuations. *Adv. in Appl. Math.*, **41** (2008), 482–509.

8. Goodey, P., Weil, W., Sums of sections, surface area measures and the general Minkowski problem. *J. Diff. Geom.* **97** (2014), 477–514.
9. Goodey, P., V. Yaskin, M. Yaskina, Fourier transforms and the Funk-Hecke theorem in convex geometry. *J. Lond. Math. Soc.* **80** (2009), 388–404.
10. Goodey, P., Hug, D., Weil, W., Kinematic formulas for area measures. *Indiana Univ. Math. J.* (to appear), arXiv:1507.03353.
11. McMullen, P., Isometry covariant valuations on convex bodies. *Rend. Circ. Mat. Palermo, Ser. II, Suppl.* **50** (1997), 259–271.
12. Ólafsson, G., Pasquale, A., The Cos^λ and Sin^λ transforms as intertwining operators between generalized principal series representations of $\mathrm{SL}(n+1,\mathbb{K})$. *Adv. Math.*, **229** (2012), 267–293.
13. Schneider, R., *Convex bodies: the Brunn-Minkowski theory*, volume **151** of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, expanded edition, 2014.
14. Schneider, R., Weil, W., Stochastic and Integral Geometry. Probability and Its Applications. Springer, Berlin, 2008.
15. Schuster, F., Wannerer, T., Minkowski Valuations and Generalized Valuations. *J. of the EMS* (to appear), arXiv:1507.05412.
16. Wannerer, T., Integral geometry of unitary area measures. *Adv. Math.* **263** (2014), 1–44.
17. Weil, W., On surface area measures of convex bodies. *Geom. Dedicata* **9** (1980), 299–306.