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A class of tests for exponentiality against HNBUE alternatives

Bernhard Klar

Institut für Mathematische Stochastik, Universität Karlsruhe, Englerstrasse 2, 76128 Karlsruhe, Germany

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Abstract

This paper presents tests of exponentiality against HNBUE alternatives. The new class of test statistics is based on the difference between the integrated distribution function $\Psi(t)=\int_t^{\infty}(1-F(x))\,\mathrm{d}x$ and its empirical counterpart. As special cases, the class includes the asymptotically most powerful test for exponentiality against the Makeham alternative and the first nonzero component of Neyman's smooth test of fit for the exponential distribution.

Keywords: Goodness-of-fit test; Integrated distribution function; Exponential distribution; Life distribution; HNBUE alternatives; Pitman efficiency

1. Introduction

The problem of testing exponentiality against *harmonic new better than used in expectation* (HNBUE) alternatives gained much interest in recent years (Aly, 1992; Basu and Ebrahimi, 1985; Hendi et al., 1998; Kochar and Deshpande, 1985; Kochar and Gupta, 1988; Jammalamadaka and Lee, 1998; Klefsjö, 1983; Singh and Kochar, 1986). Here, a distribution F with support $[0,\infty)$ and finite mean $\mu=\int_0^{\infty}\overline{F}(x)\,\mathrm{d}x$, where $\overline{F}=1-F$, is said to be HNBUE if $\int_t^{\infty}\overline{F}(x)\,\mathrm{d}x\leqslant\mu\exp(-t/\mu)$ for every $t\geqslant 0$. By means of the *integrated distribution function* (idf)

$$\Psi_F(t):=E_F(X-t)^+=\int_t^{\infty}\overline{F}(x)\,\mathrm{d}x$$

which is also known under the name *stop-loss transform*, the definition of the HNBUE property can be restated as

$$\Psi_F(t)\leqslant\Psi(t,1/\mu)\quad\text{for every } t\geqslant 0, \tag{1}$$

where $\Psi(t,\lambda)=\exp(-\lambda t)/\lambda$ denotes the idf of the exponential distribution with distribution function $F(t,\lambda)=1-\exp(-\lambda t)$ for $t\geqslant 0$. If the reversed inequality holds in (1) then F is called *harmonic new worse than used in expectation* (HNWUE).

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The HNBUE class of life distributions was introduced by Rolski (1975); there, in Theorem 1, an equivalent definition is given which explains the term HNBUE. Another characterization of the HNBUE property follows from Theorems 3.5 and 3.6 in Müller (1996) (see also Rolski, 1975): Let X be a random variable with distribution function F and let Y be exponentially distributed with parameter $1/\mu$. Then X is HNBUE if and only if X precedes Y in the convex order, i.e. if $Ef(X)\leqslant Ef(Y)$ for each convex function f such that $f(X)$ has finite expectation.

Deshpande et al. (1986) give an overview over the various classes of life distributions such as IFR, IFRA, NBU, NBUE, HNBUE or $\mathscr{L}$-distributions and the relationships among them.

In what follows, let $\overline{X}_n = n^{-1}\sum_{j=1}^n X_j$ be the arithmetic mean of a random sample $X_1,\dots,X_n$ of size n from the distribution F and let Ψ_n denote the empirical integrated distribution function, defined by

$$\Psi_n(t) = \int_t^\infty \overline{F}_n(x)\,\mathrm{d}x = \frac{1}{n}\sum_{i=1}^n (X_i - t)\,\mathbf{1}\{X_i > t\},$$

where $F_n(x) = n^{-1}\sum_{j=1}^n \mathbf{1}\{X_j \leqslant x\}$ is the empirical distribution function. In view of (1), it seems natural to base a test of

$$\mathrm{H}_0\colon F\in\mathscr{F} = \{F(\cdot,\lambda),\ \lambda > 0\}$$

against the alternative

$$\mathrm{H}_1\colon F\notin\mathscr{F} \text{ and } F \text{ is HNBUE}$$

on the empirical counterpart $\Psi_n(x) - \Psi(x, 1/\overline{X}_n)$ of the difference $\Psi_F(x) - \Psi(x, 1/\mu)$. Generalizing the approach of Jammalamadaka and Lee (1998), the test statistic proposed is the weighted integral

$$T_{n,a} = \hat{\lambda}_n^2 \int_0^\infty (\Psi_n(t) - \Psi(t,\hat{\lambda}_n))\exp(-a\hat{\lambda}_n t)\,\mathrm{d}t, \tag{2}$$

where $\hat{\lambda}_n = 1/\overline{X}_n$ and $a\geqslant 0$ is a nonnegative constant. It will be seen in Sections 3 and 4, that the choice of a has a pronounced influence on the power of the test. However, $a = 1$ (the case considered in Jammalamadaka and Lee, 1998) and $a = 0$ seem to be natural choices.

Since $\Psi_F(x) - \Psi(x, 1/\mu)$ is nonpositive for HNBUE alternatives, H_0 is rejected for large negative values of $T_{n,a}$; similarly, a test of exponentiality against HNWUE alternatives has an upper rejection region. Straightforward computation yields the alternative representations

$$T_{n,a} = \frac{1}{na^2}\sum_{j=1}^n \mathrm{e}^{-aY_j} - \frac{1}{a^2(1+a)} \tag{3}$$

for $a > 0$ and

$$T_{n,0} = \frac{1}{2n}\sum_{j=1}^n Y_j^2 - 1, \tag{4}$$

where $Y_j = X_j/\overline{X}_n$ for $j = 1,\dots,n$. Note that $T_{n,0} = \lim_{a\to 0} T_{n,a}$.

For $a = 1$, we obtain $T_{n,1} = n^{-1}\sum_{j=1}^n (\exp(-Y_j) - \frac{1}{2})$. Doksum and Yandell (1984) have shown that the test based on $T_{n,1}$ is asymptotically most powerful for testing H_0 against the Makeham distribution (see Section 3) within the class of level α tests irrespective of the value of the unknown parameter λ. Singh and Kochar (1986) established consistency of $T_{n,1}$ against continuous HNBUE alternatives. As aforesaid, Jammalamadaka and Lee (1998) also considered the test statistic $T_{n,1}$.

Interestingly, $\sqrt{n}\,T_{n,0}$ is the first nonzero component of Neyman's smooth test of fit for the exponential distribution (see, e.g., Koziol, 1987). Note that, up to one-to-one transformations, $T_{n,0}$ coincides with Greenwood's statistic $G_n = (1/n^2)\sum_{j=1}^n Y_j^2$ and with the sample coefficient of variation $\mathrm{CV}_n = S_n/\overline{X}_n$, where $S_n^2 = n^{-1}\sum_{j=1}^n (X_j - \overline{X}_n)^2$ denotes the sample variance. As shown in Doksum and Yandell (1984), a test based on $\sqrt{n}\,T_{n,0}$ is asymptotically most powerful for testing H_0 against the linear failure rate distribution. Kochar and Gupta (1988) remarked that a test based on CV_n is consistent against continuous HNBUE alternatives.

Various other tests for exponentiality against HNBUE alternatives use the total time on test (TTT) transform $t(p) = \int_0^{F^{-1}(p)} \overline{F}(s)\,\mathrm{d}s$ which is related to the idf by $t(p) + \Psi(F^{-1}(p)) = \mu(0 < p < 1)$. Similarly, $\Psi_n(X_{(i)}) = \overline{X}_n - t_n(i)/n$, where $0 = X_{(0)} \leqslant X_{(1)} \leqslant \cdots \leqslant X_{(n)}$ denotes the ordered sample of $X_1, \ldots, X_n$ and

$$t_n(i) = \sum_{j=1}^{i} (X_{(j)} - X_{(j-1)})(n - j + 1), \quad i = 1, \ldots, n$$

is the empirical total time on test. The above relationship follows from the representation

$$\Psi_n(t) = \frac{1}{n}\sum_{j=i+1}^{n} (X_{(j)} - X_{(j-1)})(n - j + 1) + \frac{n-i+1}{n}(X_{(i)} - t)$$

for the empirical idf; in particular,

$$\Psi_n(X_{(i)}) = \frac{1}{n}\sum_{j=i+1}^{n} (X_{(j)} - X_{(j-1)})(n - j + 1), \quad i = 1, \ldots, n.$$

Klefsjö (1983), who seems to be the first who considered tests of exponentiality against HNBUE alternatives, examined the statistics

$$Q_{1,v} = \sum_{j=1}^{n} \left(-\frac{1}{v} + v\left(1 - \frac{j}{n}\right)^{v-1} \right) X_{(j)}/t_n(n) \quad (v \geqslant 2), \tag{5}$$

$$Q_{2,v} = \sum_{j=1}^{n} \left(\sum_{k=1}^{v} \frac{1}{k} - v\left(\frac{j}{n}\right)^{v-1} \right) X_{(j)}/t_n(n) \quad (v \geqslant 2) \tag{6}$$

which are related to the TTT-transform. In particular, $Q_{1,2} = Q_{2,2}$ coincides with the statistic K^* of Hollander and Proschan (1975), which, in turn, is a linear function of the cumulative total time on test statistic $V = \sum_{i=1}^{n-1} t_n(i)/t_n(n)$ (see, e.g., Barlow and Campo, 1975). V is asymptotically equivalent to $T_{n,1}$ (see Singh and Kochar, 1986, Section 3); particularly, they have the same Pitman asymptotic relative efficiencies.

Further tests of H_0 against H_1 based on the empirical TTT-transform or on the normalized spacings $(n - j + 1)(X_{(j)} - X_{(j-1)})$ are considered in Aly (1992), Basu and Ebrahimi (1985), Kochar and Deshpande (1985) and Kochar and Gupta (1988).

2. Asymptotic properties of the test statistics

The representations of $T_{n,a}$ in (3) and (4), respectively, show that $T_{n,a}$ is scale-invariant; hence, we assume $\mu = 1$ in the following. It is well-known that $\sqrt{n}\,T_{n,0}$, the first nonzero component of Neyman's smooth test of fit, has a limiting unit normal distribution under the hypothesis of exponentiality (see, e.g. Koziol, 1987,

p. 22 or Henze and Klar, 1996, Section 4.2). The asymptotic distribution of $T_{n,a}$ for $a>0$ is given in the following theorem.

Theorem 2.1. *Under* H_0, *the limiting distribution of* $\sqrt{n}T_{n,a}$ is $\mathrm{N}(0,\sigma^2)$ *where*

$$\sigma^2=\frac{1}{a^4}E_\lambda\left(\mathrm{e}^{-aX}-\frac{1}{1+a}+\frac{a}{(1+a)^2}(X-1)\right)^2=\frac{1}{(1+a)^4(2a+1)}.$$

Proof. By the Mean Value Theorem

$$\sqrt{n}a^2\,T_{n,a}=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(\mathrm{e}^{-aX_j}-\frac{1}{1+a}\right)+\sqrt{n}(\hat{\lambda}_n-\lambda)\frac{1}{n}\sum_{j=1}^{n}(-aX_j\,\mathrm{e}^{-a\lambda^* X_j}),$$

where λ^* is between $\hat{\lambda}_n=1/\overline{X}_n$ and 1. Since the sum on the right converges stochastically to $E(-aX\exp(-aX)) = -a/(1+a)^2$, Slutzky's Lemma yields

$$\begin{aligned}\sqrt{n}\,a^2\,T_{n,a}&=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(\mathrm{e}^{-aX_j}-\frac{1}{1+a}\right)+\sqrt{n}\left(\frac{1}{\overline{X}_n}-1\right)\frac{-a}{(1+a)^2}+\varepsilon_n\\&=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(\mathrm{e}^{-aX_j}-\frac{1}{1+a}+\frac{a(X_{j-1})}{(1+a)^2}\right)-\frac{a}{(1+a)^2}\frac{\sqrt{n}(\overline{X}_n-1)^2}{\overline{X}_n}+\varepsilon_n,\end{aligned}$$

where $\varepsilon_n=o_P(1)$. The assertion now follows from the Central Limit Theorem and Slutzky's Lemma.

As a consequence, the asymptotic distribution of $T^*_{n,a}=\sqrt{n}(a+1)^2\sqrt{2a+1}T_{n,a}$ $(a>0)$ under the hypothesis of exponentiality is standard normal. Furthermore, the proof of Theorem 2.1 shows that $(T^*_{n,a},T^*_{n,b})$ has a limiting bivariate normal distribution under H_0 with covariance

$$\rho(a,b)=\frac{\sqrt{(2a+1)(2b+1)}}{a+b+1}. \tag{7}$$

Moreover, a slight modification of the proof yields the asymptotic normality of $\sqrt{n}\,(T_{n,a}-E(T_{n,a}))$ under a nondegenerate alternative with finite second moment. □

Remark 2.2. Due to the distributional equivalence between the Y_j and the normalized uniform spacings under H_0, Theorem 2.1 also follows from the results in Rao and Sethuraman (1975).

Remark 2.3. For testing a simple hypothesis (i.e. exponentiality with a fixed mean μ), one has to replace $\overline{X}_n$ in (2) by the hypothesized μ. Evaluating the integral leads to the test statistics

$$\tilde{T}_{n,a}=\frac{1}{na^2}\sum_{j=1}^{n}\left(\exp\left(\frac{-aX_j}{\mu}\right)+\frac{aX_j}{\mu}-\frac{a^2+a+1}{a+1}\right)$$

for $a>0$ and $\tilde{T}_{n,0}=(2n)^{-1}\sum_{j=1}^{n}(X_j/\mu)^2-1$. By the CLT, $\tilde{T}_{n,a}$ has a limiting normal distribution with mean zero and variance $\tau^2=(2a+5)/((a+1)^2(2a+1))$ for $a>0$. In the special case $a=1$, $\tau^2=7/12$. $\tilde{T}_{n,0}$ is asymptotically $\mathrm{N}(0,5)$-distributed.

Note that the derivation of the limiting null distribution of $T_{n,1}$ given in Jammalamadaka and Lee (1998) is directed at the case of known μ since, in the proof of Theorem 2.1 of Jammalamadaka and Lee (1998), the empirical process is approximated by a standard brownian bridge. Accordingly, computing the asymptotic variance in (2.4) of Jammalamadaka and Lee (1998) gives the value $\frac{7}{12}$.

A proof along the lines of Jammalamadaka and Lee (1998) in case of a composite hypothesis has to be based on the estimated empirical process $\sqrt{n}(F_n(x) - F(x, \hat{\lambda}_n))$. The covariance function of the pertaining limiting Gaussian process is

$$\tilde{k}(s,t) = \min(s,t) - st - \phi(s)\,\phi(t),$$

where $\phi(t) = -(1-t)\log(1-t)$ (see Shorak and Wellner, 1986, Chapter 5). For $a = 1$, the variance of the limiting $\mathrm{N}(0, \sigma^2)$-distribution is then

$$\sigma^2 = \int_0^1 \int_0^1 \int_u^1 \int_v^1 \tilde{k}(s,t)\,\mathrm{d}F^{-1}(s)\,\mathrm{d}F^{-1}(t)\,\mathrm{d}u\,\mathrm{d}v = \frac{1}{48}$$

in accordance with Theorem 2.1 and the result at the end of Section 2 of Jammalamadaka and Lee (1998).

Let $a \geqslant 0$, $\alpha \in (0,1)$, and let $z_n(\alpha)$ denote the α-quantile of $T_{n,a}$ under H_0. Regarding consistency of the test which rejects the hypothesis of exponentiality if $T_n < z_n(\alpha)$, we have the following result.

Theorem 2.4. *The test based on $T_{n,a}$ is consistent against each HNBUE alternative.*

Proof. Let Ψ_A and μ_A denote the idf and the expectation of the alternative distribution. Now,

$$\int_0^\infty \Psi_n(t) - \Psi(t, \hat{\lambda}_n)\,\mathrm{d}t = \int_0^\infty \Psi_n(t) - \Psi(t, \lambda)\,\mathrm{d}t + \int_0^\infty \Psi(t, \lambda) - \Psi(t, \hat{\lambda}_n)\,\mathrm{d}t, \tag{8}$$

where $\lambda = 1/\mu_A$. By inequality (1), and since a distribution is uniquely determined by the idf (see, e.g., Müller, 1996), there exists some number $t_0 > 0$ with $\Psi_A(t_0) - \Psi(t_0, \lambda) \leqslant \delta < 0$. From the strong law of large numbers and Theorem II.2 of Pollard (1984),

$$\limsup_{n\to\infty} \int_0^\infty (\Psi_n(t) - \Psi(t, \lambda))\,\mathrm{d}t < 0 \quad \text{a.s.}$$

On the other hand,

$$\lim_{n\to\infty} \int_0^\infty (\Psi(t, \lambda) - \Psi(t, \hat{\lambda}_n))\,\mathrm{d}t = 0 \quad \text{a.s.},$$

such that $\limsup_{n\to\infty} \int_0^\infty (\Psi_n(t) - \Psi(t, \hat{\lambda}_n))\,\mathrm{d}t < 0$ a.s. in view of (8). This implies

$$\lim_{n\to\infty} \sqrt{n}\,T_{n,0} = -\infty \quad \text{a.s.}$$

and therefore $\lim_{n\to\infty} P(\sqrt{n}\,T_{n,0} \geqslant z_n(\alpha)) = 0$. A slight modification yields the same result for $T_{n,a}$, $a > 0$. □

It should be noted that the result holds without the usual assumption of an absolutely continuous alternative distribution (as in Aly, 1992; Klefsjö, 1983; Kochar and Deshpande, 1985; Singh and Kochar, 1986).

3. Asymptotic Pitman efficiency

In this section, we want to compare the power of the test statistics on the basis of the Pitman asymptotic relative efficiency (ARE). Let F_{ϑ_n} be a sequence of alternative distributions, where $\vartheta_n = \vartheta_0 + c/\sqrt{n}$ $(c > 0)$, and ϑ_0 corresponds to the exponential distribution. Since, for the alternatives considered below, the properly normalized test statistics $T_{n,a}$ also have a limiting normal distribution, asymptotic Pitman efficiency is given by

$$e_{F_\vartheta}(T_{n,a}) = \lim_{n\to\infty} \left.\frac{\mathrm{d}}{\mathrm{d}\vartheta} E_\vartheta(T_{n,a})\right|_{\vartheta=\vartheta_0} (\sigma^2(\vartheta_0))^{-1},$$

where $\sigma^2(\vartheta_0)$ is the null asymptotic variance of $T_{n,a}$ (see Klefsjö, 1983, p. 68). The ARE of a sequence of statistics $T_{n,a}$ with respect to another sequence $T_{n,b}$ is then given by $e_F(T_{n,a})/e_F(T_{n,b})$. We have calculated $e_F(T_{n,a})$ for linear failure rate, Makeham, Pareto, Weibull and gamma alternatives which are given by the distribution functions

$$F_1(x) = 1 - \exp(-(x + \vartheta x^2/2)) \quad \text{for } x \geqslant 0,\ \vartheta \geqslant 0,$$

$$F_2(x) = 1 - \exp(-(x + \vartheta(x + \mathrm{e}^{-x} - 1))) \quad \text{for } x \geqslant 0,\ \vartheta \geqslant 0,$$

$$F_3(x) = 1 - (1 + \vartheta x)^{-1/\vartheta} \quad \text{for } x \geqslant 0,\ \vartheta \geqslant 0,$$

$$F_4(x) = 1 - \exp(-x^{\vartheta}) \quad \text{for } x \geqslant 0,\ \vartheta > 0,$$

$$F_5(x) = \Gamma(\vartheta)^{-1} \int_0^x t^{\vartheta-1}\mathrm{e}^{-t}\,\mathrm{d}t \quad \text{for } x \geqslant 0,\ \vartheta > 0,$$

respectively. For F_1, F_2 and F_3, H_0 corresponds to $\vartheta = \vartheta_0 = 0$, and for F_4 and F_5, we have $\vartheta_0 = 1$. Calculations give the efficiencies

$$e_{F_1}(T_{n,a}) = (2a+1)/(a+1)^2 \quad \text{for } a \geqslant 0,$$

$$e_{F_2}(T_{n,a}) = (2a+1)/(4(a+2)^2) \quad \text{for } a \geqslant 0,$$

$$e_{F_3}(T_{n,a}) = (2a+1)/(a+1)^2 \quad \text{for } a \geqslant 0,$$

$$e_{F_4}(T_{n,a}) = (\log(a+1))^2(2a+1)/a^2 \quad \text{for } a > 0,\ e_{F_4}(T_{n,0}) = 1,$$

$$e_{F_4}(T_{n,a}) = ((a+1)\log(a+1) - a)^2(2a+1)/a^4 \quad \text{for } a > 0,\ e_{F_5}(T_{n,0}) = \tfrac{1}{4}.$$

$e_{F_1} = e_{F_3}$ is a decreasing function of a with maximal value $e_{F_1}(T_{n,0}) = 1$; e_{F_2} has a maximum at $a = 1$ with $e_{F_2}(T_{n,1}) = \frac{3}{36}$. The maximal value of e_{F_4} is 1.51 at $a = 2.16$; e_{F_5} reaches its maximum value of 0.58 at $a = 5.64$.

Interestingly, for all alternatives, $e_{F_j}(T_{n,a})$, $a \geqslant 1$, coincides with $e_{F_j}(Q_{1,a+1})$, where $Q_{1,v}$ is defined in (5) (for calculating the efficiencies of $Q_{1,v}$, see Klefsjö, 1983, p. 69).

4. A small sample study

A Monte Carlo study was performed to calculate critical points empirically for finite samples. Table 1 shows lower and upper critical values of $T^*_{n,a}$ for $a = 0, 0.5, 1, 3, 5$ and 10 for several significance levels and sample sizes. The entries in Table 1 are the 20%-trimmed means of 100 Monte Carlo simulations, each based on 10000 replications; here, we always used $\lambda = 1$.

Table 2 shows the critical values for the linear combination

$$T_n = (2 + 2\rho(0.5, 3))^{-1/2}(T^*_{n,0.5} + T^*_{n,3}),$$

Table 1
Estimated lower and upper critical values of $T_{n,a}$ for different values of a

a	n	$\alpha = 0.01$	0.025	0.05	0.10	0.90	0.95	0.975	0.99
0.0	10	−1.251	−1.169	−1.088	−0.982	0.582	1.020	1.480	2.098
0.0	20	−1.447	−1.331	−1.217	−1.071	0.823	1.323	1.842	2.564
0.0	50	−1.680	−1.511	−1.353	−1.156	1.038	1.544	2.063	2.776
0.0	100	−1.826	−1.619	−1.433	−1.199	1.140	1.636	2.119	2.755
0.0	200	−1.949	−1.710	−1.495	−1.233	1.199	1.667	2.114	2.677
0.0	500	−2.078	−1.801	−1.552	−1.258	1.250	1.686	2.093	2.589
0.0	1000	−2.139	−1.843	−1.582	−1.267	1.263	1.678	2.050	2.515
0.5	10	−1.784	−1.633	−1.487	−1.297	0.865	1.324	1.755	2.290
0.5	20	−1.946	−1.744	−1.557	−1.320	1.032	1.483	1.900	2.418
0.5	50	−2.098	−1.841	−1.608	−1.326	1.151	1.587	1.974	2.444
0.5	100	−2.168	−1.883	−1.627	−1.321	1.198	1.613	1.982	2.419
0.5	200	−2.219	−1.911	−1.636	−1.313	1.227	1.629	1.979	2.403
0.5	500	−2.264	−1.935	−1.643	−1.306	1.248	1.636	1.977	2.377
0.5	1000	−2.282	−1.945	−1.646	−1.299	1.259	1.638	1.974	2.363
1.0	10	−2.045	−1.846	−1.654	−1.412	0.972	1.410	1.802	2.274
1.0	20	−2.162	−1.905	−1.673	−1.390	1.084	1.502	1.875	2.326
1.0	50	−2.242	−1.944	−1.679	−1.362	1.167	1.568	1.928	2.342
1.0	100	−2.277	−1.954	−1.672	−1.342	1.200	1.590	1.932	2.341
1.0	200	−2.287	−1.953	−1.663	−1.324	1.232	1.612	1.949	2.348
1.0	500	−2.304	−1.959	−1.660	−1.311	1.248	1.625	1.953	2.336
1.0	1000	−2.313	−1.956	−1.656	−1.303	1.261	1.636	1.959	2.338
3.0	10	−2.159	−1.934	−1.712	−1.434	1.081	1.500	1.871	2.311
3.0	20	−2.238	−1.956	−1.701	−1.395	1.149	1.552	1.907	2.331
3.0	50	−2.278	−1.965	−1.683	−1.354	1.201	1.594	1.935	2.332
3.0	100	−2.304	−1.968	−1.674	−1.331	1.226	1.606	1.940	2.337
3.0	200	−2.314	−1.965	−1.664	−1.318	1.240	1.617	1.946	2.333
3.0	500	−2.322	−1.963	−1.657	−1.301	1.256	1.625	1.951	2.324
3.0	1000	−2.320	−1.962	−1.654	−1.296	1.264	1.634	1.955	2.334
5.0	10	−1.979	−1.813	−1.634	−1.385	1.136	1.570	1.953	2.421
5.0	20	−2.137	−1.885	−1.650	−1.357	1.183	1.596	1.959	2.395
5.0	50	−2.225	−1.925	−1.652	−1.330	1.221	1.616	1.967	2.384
5.0	100	−2.258	−1.937	−1.651	−1.317	1.240	1.629	1.969	2.364
5.0	200	−2.278	−1.942	−1.648	−1.306	1.252	1.633	1.967	2.358
5.0	500	−2.302	−1.952	−1.652	−1.302	1.264	1.641	1.970	2.347
5.0	1000	−2.310	−1.953	−1.646	−1.293	1.270	1.634	1.952	2.334
10.0	10	−1.534	−1.485	−1.413	−1.283	1.196	1.664	2.078	2.608
10.0	20	−1.884	−1.724	−1.548	−1.301	1.228	1.672	2.070	2.553
10.0	50	−2.095	−1.830	−1.588	−1.296	1.249	1.668	2.039	2.484
10.0	100	−2.174	−1.879	−1.612	−1.294	1.260	1.667	2.024	2.442
10.0	200	−2.227	−1.904	−1.624	−1.293	1.271	1.659	2.007	2.411
10.0	500	−2.266	−1.927	−1.634	−1.290	1.270	1.651	1.986	2.377
10.0	1000	−2.284	−1.935	−1.636	−1.288	1.274	1.654	1.985	2.370

where $\rho(0.5,3) = 2\sqrt{14}/9$ is defined in (7). T_n also has a limiting standard normal distribution. The entries in Table 2 are determined in the same way as in Table 1.

In order to examine the dependence of the power of the test on the weight function, we conducted a simulation study with the following alternative distributions: Weibull, Gamma, Linear failure rate and Pareto distribution with scale parameter 1 and shape parameter θ, denoted by $W(\theta)$, $\Gamma(\theta)$, $\mathrm{LFR}(\theta)$ and $\mathrm{Par}(\theta)$.

Table 2
Estimated lower and upper critical values of T_n

n	$\alpha = 0.01$	0.025	0.05	0.10	0.90	0.95	0.975	0.99
10	−2.040	−1.835	−1.642	−1.399	0.974	1.401	1.791	2.269
20	−2.146	−1.892	−1.662	−1.379	1.089	1.509	1.887	2.340
50	−2.220	−1.933	−1.670	−1.355	1.173	1.577	1.938	2.363
100	−2.259	−1.947	−1.667	−1.340	1.208	1.603	1.951	2.362
200	−2.284	−1.952	−1.664	−1.323	1.233	1.616	1.953	2.355
500	−2.296	−1.954	−1.657	−1.309	1.250	1.626	1.954	2.339
1000	−2.312	−1.954	−1.656	−1.303	1.263	1.637	1.961	2.340

Table 3
Empirical power of different tests, $\alpha = 0.05$, $n = 20$, 100 000 replications

Distribution	$T_{n,0}$	$T_{n,0.5}$	$T_{n,1}$	$T_{n,3}$	$T_{n,5}$	$T_{n,10}$	T_n	V	$Q_{1,3}$	$Q_{2,3}$
Exp(1)	5.04	5.03	5.04	5.01	4.99	5.01	4.98	5.01	5.05	5.05
$W(1.2)$	20.3	21.1	21.8	22.0	21.1	18.6	22.4	21.4	21.9	20.5
$W(1.5)$	61.2	63.8	65.4	64.6	60.4	50.7	66.6	64.3	64.8	61.8
$W(1.8)$	90.5	92.0	92.7	91.4	87.6	76.8	93.1	92.1	92.0	90.9
LFR(0.5)	18.5	18.4	18.2	16.0	14.0	11.6	17.5	18.2	17.0	18.4
LFR(1.0)	29.2	29.2	28.7	24.4	20.8	16.4	27.4	28.7	26.5	29.1
LFR(2.0)	43.7	43.7	43.0	36.5	30.9	23.2	41.1	43.0	39.9	43.5
LFR(3.0)	52.5	52.7	52.0	44.3	37.3	27.9	49.7	51.9	48.4	52.4
$\Gamma(1.5)$	26.3	28.1	30.0	33.1	33.1	30.2	32.0	29.2	31.5	27.0
$\Gamma(2.0)$	55.3	59.5	63.2	69.1	68.7	62.3	67.1	61.7	66.3	57.1
$\Gamma(2.5)$	77.6	82.0	85.4	90.1	89.7	84.0	88.7	84.5	88.1	80.0
Par(0.25)	27.4	28.0	27.1	22.1	18.8	14.8	25.0	26.9	23.8	27.6
Par(0.50)	53.6	55.7	55.6	49.4	43.9	35.4	53.4	55.4	51.7	55.8
Par(0.75)	73.1	75.7	76.3	72.2	67.6	58.9	74.9	76.2	74.1	76.1
Par(1.0)	84.8	87.2	87.8	85.8	82.8	76.4	87.1	87.8	86.8	87.5
Par(1.5)	95.1	96.4	96.9	96.6	95.7	93.5	96.8	96.9	96.9	96.6

Weibull and Gamma random numbers are generated by routines of the IMSL-library, LFR and Pareto random numbers by the inversion method.

The first seven columns of Tables 3 and 4 show the results of the tests based on $T^*_{n,a}$ for $a=0, 0.5, 1, 3, 5, 10$ and T_n for $n = 20$ and 50, respectively.

Besides the findings for $T^*_{n,a}$ and T_n, we provide the results of three other tests in the last three columns of Tables 3 and 4: The tests $Q_{1,3}$ and $Q_{2,3}$ recommended by Klefsjö (1983) (see (5) and (6)) and the cumulative total time on test statistic V. Again, the critical values are determined empirically.

The results indicate that the power of the tests based on $T^*_{n,a}$ depends heavily on a, as anticipated by the results of Section 3. Against Weibull alternatives, $T^*_{n,1}$ performs best among the first six tests; surprisingly, T_n outperforms all tests under consideration in this case. $T^*_{n,0}$ is the best test against LFR alternatives, whereas $T^*_{n,3}$ and $T^*_{n,5}$ turn out to be most powerful against Gamma distributions. For Pareto alternatives close to the null hypothesis ($\vartheta = 0.25$), $T^*_{n,0.5}$ outperforms all other tests; for larger values of ϑ, $T^*_{n,0.5}$, $T^*_{n,1}$, V and $Q_{2,3}$ exhibit similar power.

If nothing is known about the HNBUE (HNWUE) alternative, the test based on T_n can be recommended since it distributes its power more evenly over the range of alternatives.

Table 4
Empirical power of different tests, $\alpha = 0.05$, $n = 50$, 100 000 replications

Distribution	$T_{n,0}$	$T_{n,0.5}$	$T_{n,1}$	$T_{n,3}$	$T_{n,5}$	$T_{n,10}$	T_n	V	$Q_{1,3}$	$Q_{2,3}$
Exp(1)	5.00	5.05	5.03	5.03	4.96	5.00	4.99	5.04	5.05	5.01
$W(1.2)$	38.4	41.3	43.0	43.5	41.1	35.6	44.4	42.6	43.8	40.5
$W(1.5)$	94.3	96.1	96.8	96.4	94.7	88.8	97.3	96.6	96.7	95.7
$W(1.8)$	100.0	100.0	100.0	100.0	99.9	99.3	100.0	100.0	100.0	100.0
LFR(0.5)	38.6	38.0	36.5	29.1	23.9	17.9	34.5	36.6	32.7	37.6
LFR(1.0)	62.2	61.6	59.2	47.8	39.2	28.1	56.3	59.5	53.8	61.0
LFR(2.0)	83.1	82.8	81.0	69.6	58.9	42.5	78.2	81.2	75.9	82.3
LFR(2.5)	87.5	87.4	85.8	75.3	64.8	47.3	83.3	86.0	81.4	87.0
LFR(3.0)	90.7	90.6	89.5	80.0	69.9	51.9	87.3	89.6	85.6	90.3
$\Gamma(1.5)$	48.6	54.5	59.0	66.2	66.7	63.0	63.3	58.5	63.5	53.5
$\Gamma(2.0)$	87.3	92.5	95.1	97.8	97.9	96.5	96.9	95.0	97.0	92.3
$\Gamma(2.5)$	98.2	99.4	99.8	100.0	100.0	99.9	99.9	99.8	99.9	99.4
Par(0.25)	48.7	49.7	47.7	37.6	31.6	23.3	46.7	47.4	41.4	48.9
Par(0.50)	84.4	86.5	86.3	79.9	73.6	61.4	85.7	86.1	82.8	86.5
Par(0.75)	96.4	97.4	97.5	96.0	93.8	88.2	97.5	97.5	96.8	97.5
Par(1.0)	99.2	99.5	99.6	99.3	98.9	97.4	99.6	99.6	99.5	49.6
Par(1.5)	100.0	100.0	100.0	100.0	100.0	99.9	100.0	100.0	100.0	100.0

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