The joint distribution of Studentized residuals under elliptical distributions

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ABSTRACT: Scaled and Studentized statistics are encountered frequently, and they often play a decisive role in statistical inference and testing. For instance, taking the sample mean vector $\bar{\boldsymbol{X}} = \sum_{j=1}^{N} \boldsymbol{X}_j/N$ and the sample covariance matrix $S = \sum_{j=1}^{N}(\boldsymbol{X}_j - \bar{\boldsymbol{X}})(\boldsymbol{X}_j - \bar{\boldsymbol{X}})'/(N-1)$ for an iid sample $\{\boldsymbol{X}_j\}_{j=1}^{N}$, some statistics for testing normality of the underlying distribution consist of the scaled residuals (the Studentized residuals or the transformed samples), $\boldsymbol{Y}_j = S^{-1/2}(\boldsymbol{X}_j - \bar{\boldsymbol{X}})$ $(j = 1, 2, \ldots, N)$. In this paper, the distribution of the random matrix the columns of which consist of the scaled residuals is derived under elliptical distributions. Also exact distributions of Studentized statistics are discussed as application of the main result.

Keywords : Elliptical distribution, Left-spherical distribution, Scaled residuals, Spherical distribution, Studentized statistic.

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1. Introduction

Let $\boldsymbol{X}$ be a p-dimensional random vector distributed according to an *elliptical contoured distribution* (or briefly, *elliptical distribution*) which has a probability density function (pdf) of the form

$$f(\boldsymbol{x}) = K_p|\Lambda|^{-1/2}g((\boldsymbol{x}-\boldsymbol{\mu})'\Lambda^{-1}(\boldsymbol{x}-\boldsymbol{\mu})), \tag{1}$$

where $\boldsymbol{\mu} \in \mathbb{R}^p$, Λ is a positive definite matrix of order p, g is a real valued function, and K_p is a normalizing constant (see, for example, [8, Section 2.6.5], [12, Section 1.5]). Consequently, the characteristic function (cf) of $\boldsymbol{X}$ can be expressed as

$$\Psi(\boldsymbol{t}) = \exp(i\boldsymbol{t}'\boldsymbol{\mu})\psi(\boldsymbol{t}'\Lambda\boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{R}^p, \quad i = \sqrt{-1}. \tag{2}$$

If they exist, $\mathrm{E}[\boldsymbol{X}] = \boldsymbol{\mu}$ and $\Sigma = \mathrm{Cov}[\boldsymbol{X}] = -2\psi'(0))\Lambda \equiv c\Lambda > 0$.

Let $\{\boldsymbol{X}_j\}_{j=1}^N$ be an iid random sample, and let the sample mean vector and the sample covariance matrix be

$$\bar{\boldsymbol{X}} = \frac{1}{N}\sum_{j=1}^N \boldsymbol{X}_j, \tag{3}$$

$$S = \frac{1}{n}\sum_{j=1}^N (\boldsymbol{X}_j - \bar{\boldsymbol{X}})(\boldsymbol{X}_j - \bar{\boldsymbol{X}})', \quad n = N-1 \geq p, \tag{4}$$

respectively.

Mardia [13] has proposed measures of multivariate skewness and kurtosis, which are defined by

$$b_{1,p} = N\sum_{j,k=1}^N \left[(\boldsymbol{X}_j - \bar{\boldsymbol{X}})'(nS)^{-1}(\boldsymbol{X}_k - \bar{\boldsymbol{X}})\right]^3, \tag{5}$$

$$b_{2,p} = N\sum_{j=1}^N \left[(\boldsymbol{X}_j - \bar{\boldsymbol{X}})'(nS)^{-1}(\boldsymbol{X}_j - \bar{\boldsymbol{X}})\right]^2. \tag{6}$$

These can be used to test whether N observations $\boldsymbol{X}_j$ are drawn from a normal population, and many authors (see, for example, [17]) have discussed applications of (5) and (6) to test for sphericity and elliptical symmetry. Characteristic of the statistics (5) and (6) and many similar quantities is that both of them consist of transformed samples called the *scaled residuals* (see [16]) or the *Studentized residuals* (see [15]),

$$\boldsymbol{W}_j = S^{-1/2}(\boldsymbol{X}_j - \bar{\boldsymbol{X}}), \quad j = 1, \ldots, N, \tag{7}$$

where $S^{-1/2}$ denotes the inverse of a symmetric square root matrix of S (hereafter, we denote a symmetric square root of a $p\times p$ matrix A by $A^{1/2}$). As Fang and Liang [7] and Batsidis and Zografos [2] have mentioned, the transformed samples $\boldsymbol{W}_j$, $j = 1, \ldots, N$, are not independent, and $\boldsymbol{W}_j$'s distribution does not coincide with one of $\Sigma^{-1/2}(\boldsymbol{X}_j - \boldsymbol{\mu})$. However, if we would be able to obtain the exact joint distribution of $\boldsymbol{W}_j$,

$j = 1, \ldots N$, we could obtain a useful and powerful tool for testing not only elliptical symmetry but also various other statistical hypotheses.

Let $\{\boldsymbol{X}_j\}_{j=1}^N$ be iid sample drawn from $\mathrm{EC}_p(\mathbf{0}, I_p)$, a *spherical distribution*. Recently, Iwashita et al. [10] have considered the sampling distribution of *Student's t-type statistics*

$$T_{\boldsymbol{\alpha}}(\boldsymbol{Z}) = \frac{(p-1)^{1/2}\boldsymbol{\alpha}'\boldsymbol{Z}}{\sqrt{||\boldsymbol{Z}||^2 - (\boldsymbol{\alpha}'\boldsymbol{Z})^2}}, \quad ||\boldsymbol{Z}|| = \left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{1/2}, \quad \boldsymbol{\alpha}'\boldsymbol{\alpha} = 1, \quad \boldsymbol{\alpha} \in \mathbb{R}^p \tag{8}$$

where

$$\boldsymbol{Z} = \sqrt{N} S^{-1/2} \bar{\boldsymbol{X}}, \tag{9}$$

the *Studentized sample mean vector*, and they have obtained the asymptotic expansion of the distribution of $T_{\boldsymbol{\alpha}}(\boldsymbol{Z})$ as

$$\Pr[T_{\boldsymbol{\alpha}}(\boldsymbol{Z}) \in B] = \int_B t_{p-1}(x)dx + o(N^{-2}), \tag{10}$$

which holds uniformly over all Borel subsets B of $\mathbb{R}$, where $t_{p-1}(x)$ denotes the density of the t-distribution with $p-1$ degrees of freedom. In addition, they have conducted some numerical experiments for typical $\mathrm{EC}_p(\mathbf{0}, I_p)$ to investigate the accuracy of (10), and affirmed that the asymptotic expansion yields a high degree of precision.

It is a well-known fact that if $\boldsymbol{X} \sim \mathrm{EC}_p(\mathbf{0}, I_p)$, then

$$T_{\boldsymbol{\alpha}}(\boldsymbol{X}) = \frac{(p-1)^{1/2}\boldsymbol{\alpha}'\boldsymbol{X}}{\sqrt{||\boldsymbol{X}||^2 - (\boldsymbol{\alpha}'\boldsymbol{X})^2}} \tag{11}$$

has a t_{p-1} distribution (see [8, Theorem 2.5.8] and [12, Theorem 1.5.7]). In a similar vein, we expect that the statistic (8) is exactly distributed as t_{p-1}, and, furthermore, (9) is exactly $\mathrm{EC}_p(\mathbf{0}, I_p)$.

In this paper, we aim to show that the distribution of the $p \times N$ random matrix W

$$W = [\boldsymbol{W}_1, \boldsymbol{W}_2, \ldots, \boldsymbol{W}_N], \quad \boldsymbol{W}_j = S^{-1/2}(\boldsymbol{X}_j - \bar{\boldsymbol{X}}), \quad j = 1, \ldots, N, \tag{12}$$

has a *left-spherical distribution*, i.e., for every $p \times p$ orthogonal matrix H, HW has the same distribution of W (see [8, Definition 3.1.1]) when $\{\boldsymbol{X}_j\}_{j=1}^N$ are independently drawn from $\mathrm{EC}_p(\mathbf{0}, \Lambda)$ with pdf having the form (1), and that the Studentized sample mean vector (9) is exactly distributed as $\mathrm{EC}_p(\mathbf{0}, I_p)$.

In what follows, we write $\boldsymbol{X} \sim \mathrm{EC}_p(\boldsymbol{\mu}, \Lambda)$, when a random vector $\boldsymbol{X}$ has an elliptical distribution. Also, if we write $X \sim \mathrm{LS}_{p\times N}(\phi_X)$, then a $p \times N$ random matrix X has a left-spherical distribution with characteristic function $\phi_X(T'T)$, where T is a $p \times N$ matrix (see [8, Definitions 3.1.2 and 3.1.3]). Furthermore, if random matrices or random vectors X and Y have the same distribution, we write $X \stackrel{d}{=} Y$. Note that

Fang and Zhang [8, Chapter III] considered random matrices with dimension $N \times p$ throughout. However, for convenience, we use "$p \times N$-random matrices" instead.

In Section 2, we introduce an important proposition, and apply it to obtain a basic property of spherical distributions which is crucial in the sequel. In Section 3, we show that the Studentized sample mean vector $\boldsymbol{Z}$ has an $\mathrm{EC}_p(\mathbf{0}, I_p)$ distribution, and the random matrix W is $\mathrm{LS}_{p\times N}(\phi_W)$, by making use of the result in Section 2 and with the help of theory concerned with properties of random matrices which are distributed over $\mathcal{O}(p)$, the set of orthogonal $p \times p$ matrices. In the last section, we comment on an application of our results obtained in the previous sections.

2. Preliminaries

Let $\{\boldsymbol{X}_j\}_{j=1}^N$ be independent random copies of $\boldsymbol{X} \sim \mathrm{EC}_p(\mathbf{0}, I_p)$ with density (1) and characteristic function (2), and let $\bar{\boldsymbol{X}}$ and S be the sample mean vector and the sample covariance matrix defined by (3) and (4), respectively.

In this section, we show that the statistic $\boldsymbol{Z}$ defined by (9) is exactly distributed as $\mathrm{EC}_p(\mathbf{0}, I_p)$. First of all, we introduce the following proposition in Balakrishnan et al. [1].

Proposition 1. *Denote by S the sample covariance (4) and by $S^{1/2}$ its square root. In order to guarantee that S is non singular and all its eigenvalues are different, we assume that the $\boldsymbol{X}_j$ are absolutely continuous with respect to Lebesgue measure (see [5], [14]). Let M be a non singular $p \times p$ matrix, then $U = (MSM')^{-1/2}MS^{1/2}$ is an orthonormal matrix. If $M \in \mathcal{O}(p)$ (the orthogonal group), then $U = M$.*

Assuming the existence of the pdf of $\mathrm{EC}_p(\mathbf{0}, I_p)$, and by making use of the proposition above, we immediately obtain the following result.

Theorem 1. *Let $\{\boldsymbol{X}_j\}_{j=1}^N$ be N independent observations on $\boldsymbol{X} \sim \mathrm{EC}_p(\mathbf{0}, I_p)$ and let a $p \times N$ observation matrix be $X = [\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N]$.*

Set a $p \times N$ random matrix $W = [\boldsymbol{W}_1, \ldots, \boldsymbol{W}_N]$, where $\boldsymbol{W}_j = S^{-1/2}(\boldsymbol{X}_j - \bar{\boldsymbol{X}})$, $\bar{\boldsymbol{X}}$, S are the sample mean vector and the sample covariance matrix defined by (3) and (4), respectively, and $S^{-1/2}$ denotes a *symmetric square root of S^{-1}. Then rewriting W as*

$$W = S^{-1/2} X \left[I_N - \frac{1}{N} \boldsymbol{j}_N \boldsymbol{j}_N' \right], \quad \boldsymbol{j}_N = (1, \ldots, 1)' \in \mathbb{R}^N, \tag{13}$$

yields

$$W \sim \mathrm{LS}_{p \times N}(\phi_W), \tag{14}$$

where $\mathrm{LS}(\phi_W)$ denotes a left-spherical distribution of its characteristic function ϕ_W. Moreover, the Studentized mean vector $\boldsymbol{Z} = \sqrt{N} S^{-1/2} \bar{\boldsymbol{X}}$ can be expressed as

$$\boldsymbol{Z} = \frac{1}{\sqrt{N}} S^{-1/2} X \boldsymbol{j}_N, \tag{15}$$

hence

$$\boldsymbol{Z} \sim \mathrm{EC}_p(\mathbf{0}, I_p). \tag{16}$$

Consequently, the statistic

$$T_{\boldsymbol{\alpha}}(\boldsymbol{Z}) = \frac{(p-1)^{1/2} \boldsymbol{\alpha}' \boldsymbol{Z}}{\sqrt{||\boldsymbol{Z}||^2 - (\boldsymbol{\alpha}' \boldsymbol{Z})^2}},$$

where $||\boldsymbol{Z}|| = \left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{1/2}$, $\boldsymbol{\alpha}'\boldsymbol{\alpha} = 1$ and $\boldsymbol{\alpha} \in \mathbb{R}^p$, is exactly distributed as t-distribution with $p-1$ degrees of freedom.

Proof. Since $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_N$ are iid random sample drawn form $\mathrm{EC}_p(\mathbf{0}, I_p)$, $H\boldsymbol{X}_1, H\boldsymbol{X}_2, \ldots, H\boldsymbol{X}_N$ are independently $\mathrm{EC}_p(\mathbf{0}, I_p)$ for all $H \in \mathcal{O}(p)$. Therefore,

$$HX \stackrel{d}{=} X, \qquad HSH' \stackrel{d}{=} S,$$

and we have

$$S^{-1/2} X \stackrel{d}{=} (HSH')^{-1/2} HX. \tag{17}$$

On the other hand, according to Proposition 1,

$$(HSH')^{-1/2} HX = (HSH')^{-1/2} HS^{1/2} S^{-1/2} X = HS^{-1/2} X. \tag{18}$$

Hence, combining (17) and (18) yields

$$S^{-1/2} X \stackrel{d}{=} HS^{-1/2} X, \tag{19}$$

and therefore, W is distributed as $\mathrm{LS}_{p \times N}(\phi_W)$.

This, together with the representation of $\boldsymbol{Z}$ in (15), leads to $H\boldsymbol{Z} \stackrel{d}{=} \boldsymbol{Z}$, implying $\boldsymbol{Z} \sim \mathrm{EC}_p(\mathbf{0}, I_p)$. Hence, $T_{\boldsymbol{\alpha}}(\boldsymbol{Z})$ has a t_{p-1}-distribution (see, e.g., [12, Theorem 1.5.7]).

Remark 1. As a matter of fact, the result above does not guarantee $\boldsymbol{Z} \stackrel{d}{=} \boldsymbol{X}_j$, $j = 1, 2, \ldots, N$. For example, if we assume $p = 1$ and iid sample $\{\boldsymbol{X}_j\}_{j=1}^{N}$ is drawn from the standard normal distribution, then $\boldsymbol{Z}$ is distributed as t_{N-1}.

3. Exact distributions of W and $\boldsymbol{Z}$ under elliptical distributions

Let $\{\boldsymbol{X}_j\}_{j=1}^{2N}$ be independently distributed as $\mathrm{EC}_p(\mathbf{0}, I_p)$ with density defined by (1), and denote the $p \times N$ observation matrices by X_j, $j = 1, 2$, that is $X_j = [\boldsymbol{X}_{(j-1)N+1}, \boldsymbol{X}_{(j-1)N+2}, \ldots, \boldsymbol{X}_{jN}]$. Then the sample covariance matrices defined by (4) are rewritten as

$$S_j = \frac{1}{n} X_j \left[I_N - \frac{1}{N} \boldsymbol{j}_N \boldsymbol{j}_N' \right] X_j', \qquad \boldsymbol{j}_N = (1, \ldots, 1)' \in \mathbb{R}^N, \qquad j = 1, 2,$$

where $n = N - 1 \geq p$.

Consider the random p-vectors

$$\boldsymbol{Z}_j = S_j^{-1/2} X_j \boldsymbol{a}, \qquad j = 1, 2, \tag{20}$$

where $\boldsymbol{a}$ is a constant N-vector with $||\boldsymbol{a}|| = 1$. If we take $\boldsymbol{a} = (1/\sqrt{N})\boldsymbol{j}_N$, then (20) coincides with the Studentized mean vector $\boldsymbol{Z}$ defined by (9), and, by Theorem 1, $\boldsymbol{Z}_j$, $j = 1, 2$ are independently distributed as $\mathrm{EC}_p(\mathbf{0}, I_p)$.

Recalling Proposition 1, for a $p \times p$ symmetric matrix $\Lambda > 0$, define

$$H_j = (\Lambda^{1/2} S_j \Lambda^{1/2})^{-1/2} \Lambda^{1/2} S_j^{1/2}, \quad j = 1, 2. \tag{21}$$

Then, $H_j \in \mathcal{O}(p)$ and

$$H_j \boldsymbol{Z}_j = \sqrt{N} (\Lambda^{1/2} S_j \Lambda^{1/2})^{-1/2} \Lambda^{1/2} X_j \boldsymbol{a} \equiv \tilde{S}_j^{-1/2} Y_j \boldsymbol{a}, \qquad j = 1, 2 \tag{22}$$

where $Y_j = [\boldsymbol{Y}_{(j-1)N+1}, \boldsymbol{Y}_{(j-1)N+2}, \ldots, \boldsymbol{Y}_{jN}]$,

$$\tilde{S}_j = \frac{1}{n} Y_j \left[I_N - \frac{1}{N} \boldsymbol{j}_N \boldsymbol{j}_N' \right] Y_j', \qquad n = N - 1 \geq p, \qquad j = 1, 2,$$

and $\{\boldsymbol{Y}_j\}_{j=1}^{2N}$ are independently distributed as $\mathrm{EC}_p(\mathbf{0}, \Lambda)$.

Here we note that, at a first glance, the desired result

$$H_j \boldsymbol{Z}_j \sim \mathrm{EC}_p(\mathbf{0}, I_p), \qquad j = 1, 2,$$

seems to be completed by (22) because of $H_j \in \mathcal{O}(p)$, $j = 1, 2$. However, H_j ($j = 1, 2$) is a random matrix including the unknown parameter Λ. Hence, (22) is not sufficient to satisfy the definition of spherical distributions (see, [9, Definition 2.1], [12, Definition 1.5.1.]).

If $H_j \boldsymbol{Z}_j$, $j = 1, 2$ are independent having distributions $\mathrm{EC}_p(\boldsymbol{0}, I_p)$ with $\Pr[||\boldsymbol{Z}_j|| = 0] = 0$, then

$$\boldsymbol{U}_{(H_j;j)} = \frac{H_j \boldsymbol{Z}_j}{||H_j \boldsymbol{Z}_j||} = \frac{H_j \boldsymbol{Z}_j}{||\boldsymbol{Z}_j||}, \quad j = 1, 2, \tag{23}$$

are independently uniformly distributed on the unit sphere $\mathcal{S}$ in $\mathbb{R}^p$ (see [7]). Accordingly, we try to show the uniformity of $\boldsymbol{U}_{(H_j;j)}$, $j = 1, 2$. To start the proof, we present the following result due to Brown et al. [3] for uniformity of random variables.

Lemma 1. *Suppose $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent and that $P_{\boldsymbol{X}}$, the distribution of $\boldsymbol{X}$, has support $\mathcal{D}$. If $\rho(\boldsymbol{X}, \boldsymbol{Y}) = ||\boldsymbol{X} - \boldsymbol{Y}|| = \sqrt{(\boldsymbol{X} - \boldsymbol{Y})'(\boldsymbol{X} - \boldsymbol{Y})}$ is independent of $\boldsymbol{X}$, then $\boldsymbol{Y}$ is uniformly distributed on $\mathcal{D}$*

Consider a uniform random vector $\boldsymbol{V}$ on $\mathcal{S}$ which is independent of $\boldsymbol{U}_{(H_j;j)}$, $j = 1, 2$. According to Lemma 1, it is sufficient for the uniformity of $\boldsymbol{U}_{(H_j;j)}$ to prove the independence between $\boldsymbol{V}$ and $\tau(\boldsymbol{V}, \boldsymbol{U}_{(H_j;j)}) = \boldsymbol{V}'\boldsymbol{U}_{(H_j;j)} = 1 - (1/2)\rho(\boldsymbol{V}, \boldsymbol{U}_{(H(j);j)}))^2$, $j = 1, 2$. To this end, we need the next result concerning τ.

Lemma 2. *Let $\boldsymbol{V}$ be a uniform random vector on the unit sphere $\mathcal{S}$ in $\mathbb{R}^p$ which is independent of $\boldsymbol{U}_{(H_j;j)}$, $j = 1, 2$. Set $\boldsymbol{U}_{(I_p;j)} = \boldsymbol{Z}_j / ||\boldsymbol{Z}_j||$, $j = 1, 2$, then*

$$\tau(\boldsymbol{V}, \boldsymbol{U}_{(H_j;j)}) \stackrel{d}{=} \tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;j)}), \tag{24}$$

and $\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;j)})$, $j = 1, 2$, are mutually independent.

Proof. Under the assumption, the characteristic function of $\boldsymbol{V}$ has the form $\zeta(\boldsymbol{t}'\boldsymbol{t})$, $\boldsymbol{t} \in \mathbb{R}^p$. By making use of Fubini's theorem, we obtain

$$\begin{aligned}
\mathrm{E}[\exp(it\tau(\boldsymbol{V}, \boldsymbol{U}_{(H_j;j)}))] &= \mathrm{E}_{\boldsymbol{U}_{(H_j;j)}}[\mathrm{E}_{\boldsymbol{V}}[\exp(it\tau(\boldsymbol{V}, \boldsymbol{U}_{(H_j;j)}))|\boldsymbol{U}_{(H_j;j)}]] \\
&= \mathrm{E}_{\boldsymbol{U}_{(H_j;j)}}[\zeta(t^2)] = \zeta(t^2), \quad j = 1, 2, \quad t \in \mathbb{R}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathrm{E}[\exp(it\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;j)}))] &= \mathrm{E}_{\boldsymbol{U}_{(I_p;j)}}[\mathrm{E}_{\boldsymbol{V}}[\exp(it\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;j)}))|\boldsymbol{U}_{(I_p;j)}]] \\
&= \mathrm{E}_{\boldsymbol{U}_{(I_p;j)}}[\zeta(t^2)] = \zeta(t^2), \qquad j = 1, 2,
\end{aligned}$$

hence we have

$$\tau(\boldsymbol{V}, \boldsymbol{U}_{(H_j;j)}) \stackrel{d}{=} \tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;j)}), \qquad j = 1, 2.$$

Since $\boldsymbol{V}$ and $\boldsymbol{U}_{(I_p;j)}$ $(j = 1, 2)$ are mutually independent and are uniformly distributed on $\mathcal{S}$, they have a common characteristic function of the form $\zeta(\boldsymbol{t}'\boldsymbol{t})$, $\boldsymbol{t} \in \mathbb{R}^p$.

Let the joint characteristic function of $\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;1)})$ and $\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;2)})$ be $\Phi(s, t)$, $s, t \in \mathbb{R}$. Then,

$$\begin{aligned}
\Phi(s, t) &= \mathrm{E}[\exp\{is\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;1)}) + it\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;2)})\}] \\
&= \mathrm{E}_{\boldsymbol{V}}\Big[\mathrm{E}_{\boldsymbol{U}_{(I_p;1)}}[\exp(is\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;1)}))|\boldsymbol{V}]\mathrm{E}_{\boldsymbol{U}_{(I_p;2)}}[\exp(it\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;2)}))|\boldsymbol{V}]\Big] \\
&= \mathrm{E}_{\boldsymbol{V}}[\zeta(s^2)\zeta(t^2)|\boldsymbol{V}] = \zeta(s^2)\zeta(t^2),
\end{aligned}$$

which means $\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;1)})$ is independent of $\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;2)})$, and the proof is complete.

Using Lemma 2, we can now start proving the independence between $\boldsymbol{V}$ and $\tau(\boldsymbol{V}, \boldsymbol{U}_{(H_j;j)}), j = 1, 2$. For fixed $r \in \mathbb{R}$, $|r| \leq 1$, consider the variance of the conditional probability

$$\mathrm{Var}\left[\Pr[|\tau(\boldsymbol{V}, \boldsymbol{U}_{(H_1;1)})| \leq r, |\boldsymbol{V}]\right].$$

Consulting with the proof of Theorem 2 due to Brown et al. [3], under the condition $\boldsymbol{V} = \boldsymbol{v}$, Lemma 2 gives

$$\Pr[|\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;1)})| \leq r, |\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;2)})| \leq r|\boldsymbol{V}] = \Pr[|\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;1)})| \leq r|\boldsymbol{V}]\Pr[|\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;2)})| \leq r|\boldsymbol{V}],$$

and

$$\begin{aligned}
\mathrm{E}_{\boldsymbol{V}}\left[\Pr(|\tau(\boldsymbol{V}, \boldsymbol{U}_{(H_1;1)})| \leq r|\boldsymbol{V})^2\right] &= \mathrm{E}_{\boldsymbol{V}}\left[\Pr(|\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;1)})| \leq r|\boldsymbol{V})^2\right] \\
&= \mathrm{E}_{\boldsymbol{V}}[\Pr(|\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;1)})| \leq r, |\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;2)})| \leq r|\boldsymbol{V})] \\
&= \Pr(|\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;1)})| \leq r, |\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;2)})| \leq r) \\
&= \Pr(|\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;1)})| \leq r)^2. \qquad (25)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Pr(|\tau(\boldsymbol{V}, \boldsymbol{U}_{(I_p;1)})| \leq r)^2 &= \Pr(|\tau(\boldsymbol{V}, \boldsymbol{U}_{(H_1;1)})| \leq r)^2 \\
&= \Big[\mathrm{E}_{\boldsymbol{V}}[\Pr(|\tau(\boldsymbol{V}, \boldsymbol{U}_{(H_1;1)})| \leq r|\boldsymbol{V})\Big]^2. \qquad (26)
\end{aligned}$$

Combining (25) and (26) yields

$$\mathrm{Var}\left[\Pr(|\tau(\boldsymbol{V}, \boldsymbol{U}_{(H_1;1)})| \leq r|\boldsymbol{V})\right] = 0,$$

which shows that $\tau(\boldsymbol{V}, \boldsymbol{U}_{(H_1;1)})$ (and likewise $\tau(\boldsymbol{V}, \boldsymbol{U}_{(H_2;2)})$) is independent of $\boldsymbol{V}$. Hence, by Lemma 1, $\boldsymbol{U}_{(H_j;j)}$, $j = 1, 2$, have independent uniform distributions on $\mathcal{S}$. In view of Theorem 1 in Section 2, we obtain that $H_j\boldsymbol{Z}_j/||\boldsymbol{Z}_j|| \stackrel{d}{=} \boldsymbol{Z}_j/||\boldsymbol{Z}_j||$ and $H_j\boldsymbol{Z}_j/||\boldsymbol{Z}_j||$ is independent of $||\boldsymbol{Z}_j||$ ($j = 1, 2$)(see [8, p.57, Corollary 1]). Summarizing the above leads to the desired result.

Theorem 2. *Let* $\{\boldsymbol{X}_j\}_{j=1}^N$ *be independent random copies of* $\boldsymbol{X} \sim \mathrm{EC}_p(\boldsymbol{0}, \Lambda)$ *and* $X = [\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N]$ *be the observation matrix.*

Then the statistic

$$\boldsymbol{Z} = S^{-1/2}X\boldsymbol{a}, \qquad \boldsymbol{a} \in \mathbb{R}^N, \qquad ||\boldsymbol{a}|| = 1, \tag{27}$$

is exactly distributed as a spherical distribution $\mathrm{EC}_p(\boldsymbol{0}, I_p)$*, where* S *is the sample covariance matrix defined by (4). Taking* $\boldsymbol{a} = (1/\sqrt{N})\boldsymbol{j}_N$*, then* $\boldsymbol{Z}$ *becomes the Studentized mean vector and the statistic* $T_{\boldsymbol{\alpha}}(\boldsymbol{Z})$ *defined by (8) is exactly distributed as t-distribution with* $p-1$ *degrees of freedom.*

Let $\{\boldsymbol{X}_j\}_{j=1}^N$ and $\{\boldsymbol{Y}_j\}_{j=1}^N$ be drawn from $\mathrm{EC}_p(\boldsymbol{0}, I_p)$ and $\mathrm{EC}_p(\boldsymbol{0}, \Lambda)$, respectively, and denote the pertaining sample matrices by $X = [\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N]$ and $Y = [\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_N]$.

To condense Theorems 1 and 2, it holds

$$S_{\boldsymbol{Y}}^{-1/2}Y\boldsymbol{a} \stackrel{d}{=} S_{\boldsymbol{X}}^{-1/2}X\boldsymbol{a} \sim \mathrm{EC}_p(\boldsymbol{0}, I_p) \tag{28}$$

for all $\boldsymbol{a} \in \mathbb{R}^N, ||\boldsymbol{a}|| = 1$, where $S_{\boldsymbol{X}}$, $S_{\boldsymbol{Y}}$ are the sample covariance matrices based on the respective samples. Hence, if we take $\boldsymbol{a} = (1, 0, 0, \ldots, 0)', (0, 1, 0, \ldots, 0)', \ldots, (0, 0, 0, \ldots, 1)'$ by turn, then we have the following important result.

Corollary 1. *Suppose* $\{\boldsymbol{X}_j\}_{j=1}^N$ *are iid random sample drawn from* $\mathrm{EC}_p(\boldsymbol{\mu}, \Lambda)$*, where* Λ *is a* $p \times p$ *positive definite constant matrix and the* $p \times N$ *observation matrix* $X = [\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_N]$.

Then the random $p \times N$ *matrix*

$$W = [\boldsymbol{W}_1, \ldots, \boldsymbol{W}_N], \qquad \boldsymbol{W}_j = S^{-1/2}(\boldsymbol{X}_j - \bar{\boldsymbol{X}}), \qquad j = 1, \ldots, N,$$

is exactly distributed as a left-spherical distribution, that is,

$$W = S^{-1/2}X\left[I_N - \frac{1}{N}\boldsymbol{j}_N\boldsymbol{j}_N'\right] \sim \mathrm{LS}_{p\times N}(\phi_W), \tag{29}$$

where $\bar{\boldsymbol{X}}$ *and* S *are the sample mean vector and the sample covariance matrix defined by (3) and (4), respectively.*

4. Application

In this section, we suggest a test procedure based on probability plots for elliptical symmetry as discussed in Li et al. [11] as an application of our results and make small scale numerical experiments.

Let $\{\boldsymbol{X}_j\}_{j=1}^N$ be an iid sample composed of p-dimensional random vectors. Partition the sample $\{\boldsymbol{X}_j\}_{j=1}^N$ into K groups, each having size N_k $(\geq p+1)$, $k = 1, \ldots, K$, that is,

$$\{\boldsymbol{X}_j^{(k)}\}_{j=1}^{N_k}, \quad \sum_{k=1}^{K} N_k = N. \tag{30}$$

Now, consider the statistic, for each N_k-sample,

$$\boldsymbol{R}_k = S_{(k)}^{-1/2}\left[\boldsymbol{X}_1^{(k)}, \ldots, \boldsymbol{X}_{N_k}^{(k)}\right]\left[I_{N_k} - \frac{1}{N_k}\boldsymbol{j}_{N_k}\boldsymbol{j}'_{N_k}\right]\boldsymbol{\eta}_{N_k}, \quad k = 1, \ldots, K, \tag{31}$$

where $S_{(k)}$ denotes the sample covariance matrix defined by (4) based on $\{\boldsymbol{X}_j^{(k)}\}_{j=1}^{N_k}$ and $\boldsymbol{\eta}_{N_k}$ is any N_k-unit vector linearly independent of $\boldsymbol{j}_{N_k} = (1, \ldots, 1)' \in \mathbb{R}^{N_k}$.

According to the results in Section 3, if the iid sample $\{\boldsymbol{X}_j\}_{j=1}^N$ is drawn from $\mathrm{EC}_p(\boldsymbol{\mu}, \Lambda)$, then $\boldsymbol{R}_1, \ldots, \boldsymbol{R}_K$ are independently distributed as $\mathrm{EC}_p(\boldsymbol{0}, I_p)$. Therefore, for instance, the statistics $T_{\boldsymbol{\alpha}}(\boldsymbol{R}_1), \ldots, T_{\boldsymbol{\alpha}}(\boldsymbol{R}_K)$, where $T_{\boldsymbol{\alpha}}(\cdot)$ is defined by (11), are mutually independent and distributed as t_{p-1}.

Hence, by utilizing the testing procedures discussed in Li et al. [11], we are able to assess elliptical symmetry based on $\boldsymbol{R}_k$ (or $T_{\boldsymbol{\alpha}}(\boldsymbol{R}_k)$) without large sample approximation.

In practice, by making use of $T_{\boldsymbol{\alpha}}(\boldsymbol{R}_k)$, we present some plots for simulated data sets which are generated, by Monte Carlo methods, with employing the following 2 distributions;

(1) p-variate symmetric Kotz type distribution with density function

$$c_p|\Sigma|^{-1/2}((\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu}))\exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right], \quad \boldsymbol{\mu} \in \mathbb{R}^p, \quad \Sigma = \Sigma' \in \mathbb{R}^{p\times p},$$

(2) p-variate skew-normal distribution with density function

$$2\phi_p(\boldsymbol{x}; \Sigma)\Phi(\boldsymbol{\beta}'\boldsymbol{x}), \quad \boldsymbol{\beta} \in \mathbb{R}^p, \quad \Sigma = \Sigma' \in \mathbb{R}^{p\times p},$$

where $\phi_p(\boldsymbol{x}; \Sigma)$ and $\Phi(x)$ denote the pdf of the p-dimensional multivariate normal distribution (see, for instance, [4, Definition 1.5.1]).

We set the parameters as follows: $p = 3$, $N = 100$, $K = 25$, $N_k = 4$ $(k = 1, \ldots, 25)$, $\boldsymbol{\eta}_{N_k} = (1/2, -1/2, 1/2, -1/2)'$, $\Sigma = \mathrm{diag}(4, 3, 2)$, $\boldsymbol{\mu} = (0, 0, 0)'$ and $\boldsymbol{\beta} = (2, -3, -1)'$.

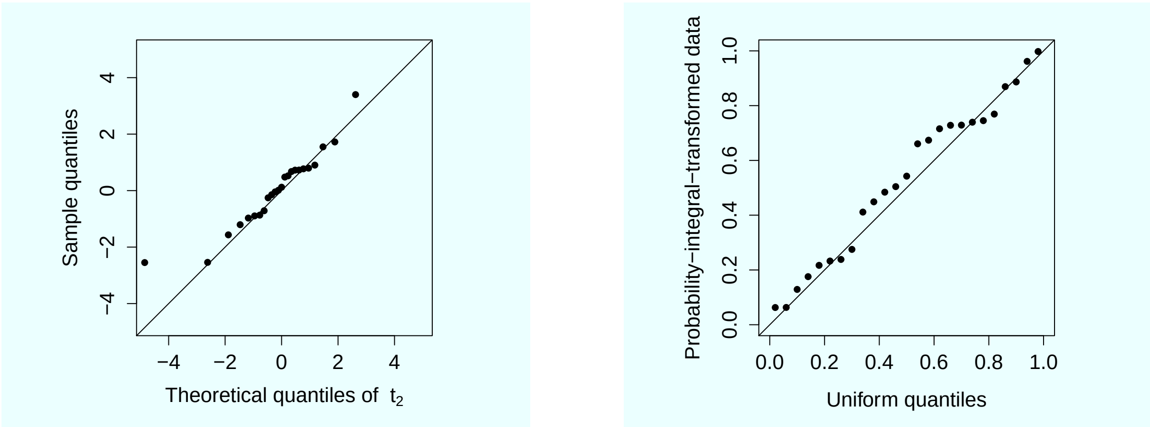

Figure 1: Q-Q Plot and P-P Plot under Kotz-type distribution

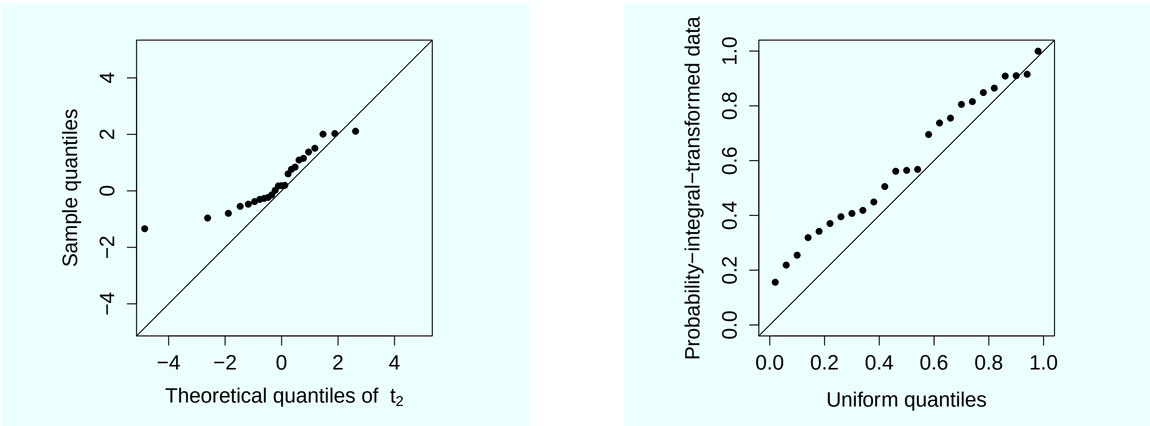

Figure 2: Q-Q Plot and P-P Plot under Skew-normal distribution

In judging the Q-Q plots for the t_2-distribution (left hand side in Figures 1 and 2), one has to be aware of the rather large variability of the smallest and largest order statistics. Hence, we additionally show P-P plots (right hand side in Figures 1 and 2). The plots for the Kotz distribution cling to the 45° through the origin, whereas the plots under skew-normality show some deviations from this line.

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References

[1] N. Balakrishnan, M. R. Brito, A. J. Quiroz, A vectorial notion of skewness and its use in testing for multivariate symmetry. Comm. Statist. Theor. Meth. 36 (2007) 1757–1767.

[2] A. Batsidis, K. Zografos, A necessary test of fit of specific elliptical distributions based on an estimator of Song's measure. J. Multivariate Anal. **113** (2013) 91–105.

[3] T. C. Brown, D. I. Cartwright, G. K. Eagleson, Correlations and characterizations of the uniform distribution. Austral. J. Statist. 28 (1986) 89–96.

[4] A. Dalla Valle, The skew-normal distribution, in; M. G. Genton (Ed.), Skew-Elliptical Distributions and Their Applications: A Journey Beyond Normality, Chapman & Hall/CRC, Boca Raton, FL, 2004, pp. 3–24.

[5] M. L. Eaton, M. D. Perlman, The non-singularity of generalized sample covariance matrices. Ann. Statist. 1 (1973) 710–717.

[6] K. T. Fang, H. F. Chen, On the spectral decompositions of spherical matrix distributions and some of their subclasses. J. Math. Res. Expo. 6 (1986) 147–156.

[7] K. T. Fang, J. J. Liang, Test of spherical and elliptical symmetry, in: S. Kotz, C. B. Read, D. L. Banks (Eds.), Encyclopedia of Statistical Sciences, (Update), Vol. 3, Wiley, New York, 1999, pp. 686–691.

[8] K. T. Fang, Y. T. Zhang, Generalized Multivariate Analysis, Science Press Beijing and Springer-Verlag, Berlin, 1990.

[9] K. T. Fang, S. Kotz, K. W. Ng, Symmetric Multivariate and Related Distributions. Chapman and Hall, London, New York, 1990.

[10] T. Iwashita, Y. Kakizawa, T. Inoue, T. Seo, An asymptotic expansion of the distribution of Student's t-type statistic under spherical distributions. Statist. Probab. Lett. 79 (2009) 1935–1942.

[11] R. Z. Li, K. T. Fang, L. X. Zhu, Some Q-Q probability plots to test spherical and elliptical symmetry. J. Comput. Graph. Statist. 6 (1997) 435–450.

[12] R. J. Muirhead, Aspects of Multivariate Statistical Theory, Wiley, New York, 1982.

[13] K. V. Mardia, Measures of multivariate skewness and kurtosis with applications. Biometrika 51 (1970) 519–530.

[14] M. Okamoto, Distinctness of the eigenvalues of a quadratic form in a multivariate sample. Ann. Statist. 1 (1973) 763–765.

[15] S. Pynnönen, Distribution of an arbitrary linear transformation of internally Studentized residuals of multivariate regression with elliptical errors. J. Multivariate Anal. 107 (2012) 42-52.

[16] N. J. H. Small, Plotting squared radii. Biometrika 65 (1978) 657–658.

[17] K. Zografos, On Mardia's and Song's measures of kurtosis in elliptical distributions. J. Multivariate Anal. 99 (2008) 858–879.

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