A note on the $\mathcal{L}$-class of life distributions

B. Klar *

Universität Karlsruhe

Abstract

This paper first recalls some stochastic orderings useful for studying the $\mathcal{L}$-class and the Laplace order in general. We use these orders to show that the higher moments of a $\mathcal{L}$-class distribution need not exist. Using simple sufficient conditions for the Laplace ordering, we give examples of distributions in the $\mathcal{L}$- and $\mathcal{L}_\alpha$-class. Moreover, we present explicit sharp bounds on the survival function of a distribution belonging to the $\mathcal{L}$-class of life distributions. The results reveal that the $\mathcal{L}$-class should not be seen as a more comprehensive class of aging distributions but rather as a large class of life distributions on its own.

Key words: Laplace order, s-(increasing) convex order, s-(increasing) concave order, $\mathcal{L}_\alpha$-class, reliability bounds, life distribution.

AMS 1991 Subject Classifications: Primary 90B25; Secondary: 60E15, 44A10

1 Introduction

A distribution function F with support $[0,\infty)$ and finite mean $\mu = \int_0^\infty \bar{F}(x)dx$, where $\bar{F} = 1 - F$, is said to belong to the $\mathcal{L}$-class of life distributions ($F \in \mathcal{L}$) if

$$\int_0^\infty e^{-sx}\bar{F}(x)\,dx \quad \geq \quad \frac{\mu}{1+s\mu} \quad \text{for all } s \geq 0. \tag{1}$$

*Institut für Mathematische Stochastik, Universität Karlsruhe, Englerstr. 2, 76128 Karlsruhe, Germany.

The class $\mathcal{L}$ was introduced by Klefsjö (1983). By means of the Laplace transform $L_F(s) = E_F e^{-sX}$, (1) can be restated as

$$L_F(s) \ \leq \ L(s, 1/\mu) \quad \text{for all } s \geq 0, \tag{2}$$

where $L(s, \lambda) = \lambda/(\lambda + s)$ denotes the Laplace transform of the exponential distribution with distribution function $F(t, \lambda) = 1 - \exp(-\lambda\, t)$ for $t \geq 0$. From (2), a distribution belongs to the $\mathcal{L}$-class if it dominates the exponential distribution with the same mean in the Laplace transform order (Stoyan (1983), p. 22). In this case, we write $X \geq_L Y$, where the random variable (rv) X has distribution function F, and Y is exponentially distributed with mean μ.

We would have to point out that the above definition of the Laplace ordering is adopted from Stoyan (1983). It differs from the definition used by Klefsjö (1983), where the reverse inequality is required in (2). One reason why we prefer the above definition is the following result (see, e.g., Reuter and Riedrich (1981), or Alzaid, Kim, and Proschan (1991)): $X \leq_L Y$ if and only if $Ef(X) \leq Ef(Y)$ for each non-negative function f having a completely monotone derivative, provided the expectations exist.

For the following definitions, see Rolski (1976), Fishburn (1980a), Kaas and Hesselager (1995) and Denuit, Lefevre, and Shaked (1998). We use the terminology of Denuit et al. (1998). Let X and Y be positive random variables. X is said to be smaller than Y in the s-increasing convex order ($X \leq_{s-icx} Y$) if

$$E(X - t)_+^{s-1} \ \ \leq \ \ E(Y - t)_+^{s-1} \ < \ \infty \quad \text{for all } t \geq 0. \tag{3}$$

X is smaller than Y in the s-increasing concave order $X \leq_{s-icv} Y$ if

$$E(t - X)_+^{s-1} \ \ \geq \ \ E(t - Y)_+^{s-1} \quad \text{for all } t \geq 0. \tag{4}$$

If, in addition to (3), $EX^k = EY^k$ for $k = 1, 2, \ldots, s-1$, then X is said to be smaller than Y in the s-convex order (written $X \leq_{s-cx} Y$). Likewise, if, in addition to (4), $EX^{s-1} < \infty, EY^{s-1} < \infty$, and $EX^k = EY^k$ for $k = 1, 2, \ldots, s-1$, then X is said to be smaller than Y in the s-concave order (written $X \leq_{s-cv} Y$). Hence, $X \leq_{s-cx} Y$ ($X \leq_{s-cv} Y$) implies $X \leq_{s-icx} Y$ ($X \leq_{s-icv} Y$).

There are further connections between the different order relations. For example, if $X \leq_{s-cx} Y$, then $X \leq_{s-cv} Y$ when s is odd, and $Y \leq_{s-cv} X$ when s is even. Moreover,

$X \leq_{s-icx} Y$ ($X \leq_{s-icv} Y$) implies $X \leq_{(s+1)-icx} Y$ ($X \leq_{(s+1)-icv} Y$), provided the moments exist.

Note that, when $s = 1$ and 2, the orders are known as stochastic dominance, convex and concave orders, as well as increasing convex and concave orders. In actuarial science, the increasing convex orders are called n-th stop-loss orders (Kaas and Hesselager (1995)).

Since $L_X(t) \geq L_Y(t)$ for $t > 0$ if $X \leq_{s-icv} Y$ (Rolski (1976), Corollary 2 of Theorem 2.1), and, hence, $X \leq_{s-icv} Y$ implies $X \leq_L Y$, these order relations can be utilized for studying the $\mathcal{L}$-class of life distribution.

For example, let X be in the harmonic new better than used in expectation (HNBUE) class of life distributions, satisfying $\int_t^\infty \bar{F}(x)\,dx \leq \mu \exp(-t/\mu)$ for every $t \geq 0$ (Rolski (1975)). Equivalently, $X \leq_{2-cx} Y$ (or $X \geq_{2-cv} Y$), where Y is exponentially distributed with mean μ, which implies the well-known fact that the $\mathcal{L}$-class is larger than the HNBUE class.

The paper is organized as follows. In Section 2 we give an example of a $\mathcal{L}$-class distribution with infinite third moment. Section 3 is devoted to examples of distributions in the $\mathcal{L}$-class. To this end, we provide simple sufficient conditions for the s-increasing concave order. In Section 4 we present an explicit upper bound on the survival function $S(t)$ of a distribution in the $\mathcal{L}$-class; this bound is sharp for $t > 2\,EX$ and differs substantially from the corresponding bound for the HNBUE class. Section 5 concludes the paper.

2 Finiteness of moments

In contrast to cases of other well-known life distribution families like the HNBUE class mentioned above, not much is known about the existence of moments of distributions in the $\mathcal{L}$-class.

One well-known result is that each distribution $F \in \mathcal{L}$ has a finite second moment; furthermore, the coefficient of variation (CV) is not greater than 1 (Mitra, Basu, and Bhattacharjee (1995), Bhattacharjee and Sengupta (1996), Lin (1998a), Lin and Hu (2000)). However, the exponential distribution is not characterized within the $\mathcal{L}$-class by the property that the coefficient of variation is one (contrary to the HNBUE class).

In this section, we give an example of a $\mathcal{L}$-class distribution with infinite third moment. This result is in sharp contrast to the HNBUE class, since distributions which are HNBUE have finite moments of all (positive) orders (Klefsjö (1982)).

A real-valued function ϕ on $[0, \infty)$ is said to have n sign changes if there exists a disjoint partition $I_1 < I_2 < \ldots < I_{n+1}$ of $[0, \infty)$ such that ϕ has opposite signs on subsequent intervals I_j and I_{j+1} and $\int_{I_j} \phi(t)dt \neq 0$ for all j.

The next theorem (Rolski (1976), Theorem 2.3, see also Denuit et al. (1998), Theorem 4.3) gives conditions which imply the s-convex (s-concave) orders.

Theorem 2.1 *Let U and V be positive random variables with distribution functions G and H, respectively. Further, let $EU^k = EV^k$, $k = 1, \ldots, s-1$.*

(i) If $G - H$ has exactly $s-1$ sign changes, and if in some interval after the last sign change the function $G - H$ is greater than zero, then $U \leq_{s-cx} V$.

(ii) If $G - H$ has exactly $s-1$ sign changes, and if in some interval after the last sign change the function $G - H$ is greater than zero for s odd and is less than zero for s even, then $U \leq_{s-cv} V$.

In particular, if a rv X has expected value μ and variance μ^2, and its distribution function F crosses $1-\exp(t/\mu)$ twice with $1-\exp(t/\mu) > F(t)$ after the second crossing, then $Y \leq_{3-cv} X$, where $Y \sim \exp(1/\mu)$. Hence, X belongs to the $\mathcal{L}$-class.

We use this fact to construct a $\mathcal{L}$-class distribution with infinite third moment.

Example 2.1 Let X be a positive random variable with survival function $\bar{F}$ defined by

$$\bar{F}(t) = \begin{cases} 1, & t \leq \frac{81}{100} \\ c, & \frac{81}{100} < t \leq 3 \\ \frac{d}{t^3}, & t > 3, \end{cases}$$

where $c = 9361/179361$ and $d = 124/91$. Elementary calculations yield $EX = Var(X) = 1$. Since $\exp(-81/100) < 1$, $\exp(-3) < c < \exp(-81/100)$ and $\exp(-t) < d/t^3$ for $t > 3$, F crosses $1 - \exp(-t)$ twice and $1 - \exp(-t) > F(t)$ for $t > 3$. Therefore, $X \in \mathcal{L}$, but $EX^3 = \infty$.

Hence, the question posed by Lin and Hu (2000) wether or not each life distribution $F \in \mathcal{L}$ possesses finite moments of all orders has to be answered in the negative.

Remark 2.2 If, for arbitrary life distributions F and G, and for some $t_0 \in [0, \infty)$, $\bar{F}(t) \geq \bar{G}(t)$ if $t \leq t_0$ and $\bar{F}(t) \leq \bar{G}(t)$ if $t > t_0$, where $\bar{F} = 1 - F$, Mitra et al. (1995) said that $\bar{F}$ crosses $\bar{G}$ from above. They showed that $F \in \mathcal{L}$ if $\bar{F}$ crosses $\exp(-t/\mu_F)$ from above. Theorem 2.1 with $s = 2$ shows that, in this case, X is even HNBUE.

A related notion is used in actuarial science: X is said to be less dangerous than Y, if $EX \leq EY$ and $\bar{F}_X$ crosses $\bar{F}_Y$ from above; this fact implies $X \leq_{2-icx} Y$ (Karlin and Novikoff (1963)).

3 Examples of $\mathcal{L}_\alpha$-class distributions

Many commonly used life distributions like the gamma or the Weibull distribution with shape parameter greater than 1 are HNBUE and, hence, also belong to the $\mathcal{L}$-class. To show that the $\mathcal{L}$-class is strictly larger than the HNBUE class, Klefsjö (1983) used the two point distribution with $P(X = 0.3) = 0.3$ and $P(X = 3) = 0.7$ (for a proof that $X \in \mathcal{L}$, see Lin and Hu (2000)).

Bhattacharjee and Sengupta (1996) gave an example of a two-point distribution with $CV = 1$ (hence, the distribution is not HNBUE) that belongs to the $\mathcal{L}$-class (see Example 3.1 below).

Likewise, the distribution in Example 2.1 belongs to the $\mathcal{L}$-class, but is not HNBUE since its expectation and its variance are 1.

To obtain more natural examples of distributions from the $\mathcal{L}$-class (besides the aforementioned HNBUE distributions), we relax the equality condition on the $(s-1)$-th moment in Theorem 2.1 to an inequality (see Kaas and Hesselager (1995), Theorem 2.3 for a corresponding statement for the s-increasing convex or $(s-1)$-th degree stop loss order).

Using this result, it is also possible to obtain examples of distributions belonging to the $\mathcal{L}_\alpha$-class (Lin (1998b)), which is defined as follows. If X dominates the gamma distribution $\Gamma(\alpha, \beta)$ (with density $\beta^\alpha x^{\alpha-1} e^{-\beta x}/\Gamma(\alpha)$ for $x > 0$, expectation α/β, and variance α/β^2) with $\alpha > 0, \beta = \alpha/EX$ in the Laplace order, then X belongs to the $\mathcal{L}_\alpha$-class. Equivalently, $L_X(s) \leq (1 + s/\beta)^{-\alpha}$ for $s \geq 0$, where L_X denotes the Laplace transform of X. Note that the $\mathcal{L}_1$-class and the $\mathcal{L}$-class coincide. Obviously, $\mathcal{L}_\alpha \subset \mathcal{L}_{\alpha'}$ for $0 < \alpha' < \alpha$. In particular, each distribution in the $\mathcal{L}_\alpha$-class with $\alpha > 1$ belongs to the $\mathcal{L}$-class.

Theorem 3.1 *Let U and V be positive random variables with distribution functions G and H, respectively. Further, let $EU^j = EV^j, j = 1, \ldots, s-2$ and $(-1)^{s-1}EU^{s-1} \geq (-1)^{s-1}EV^{s-1}$. Then each of the following conditions is sufficient for $U \leq_{s-icv} V$.*

(i) $S[G - H] \leq s - 1$ with $G \geq H$ before the first sign change, where $S[G - H]$ denotes

the number of sign changes of $G - H$.

(ii) $S[\log(g/h)] \leq s$ with $g \geq h$ before the first sign change, where G and H are assumed to be absolutely continuous with densities g and h, respectively.

PROOF: Define $\kappa_0(x) = G(x), \kappa_j(x) = \int_0^x \kappa_{j-1}(t)dt$ for all $x \geq 0$ and $j = 1, 2, \ldots, n-1$, and λ_j the same for H. Then

$$j!\,\kappa_j(x) \quad = \quad E\,(x-U)_+^j, \quad x \geq 0,\ j = 1, 2, \ldots$$

One has to show that $\Delta_j(x) = \kappa_j(x) - \lambda_j(x) \geq 0$ for $x \geq 0$. To this end, assume that $S[\Delta_{j-1}] = h$ for some positive integer h, with opposite signs on subsequent intervals $\tilde{I}_1 < \tilde{I}_2 < \ldots < \tilde{I}_{h+1}$. As in the proof of Theorem 2.3 of Kaas and Hesselager (1995), one can see that Δ_j can have at most h sign changes, one occurring on each of the intervals $\tilde{I}_2, \ldots, \tilde{I}_{h+1}$. But if a sign change occurs on $\tilde{I}_{h+1}$, the monotonicity of Δ_j on $\tilde{I}_{h+1}$ implies that $\lim_{x\to\infty} \Delta_j(x) \neq 0$, and in particular that $\lim_{x\to\infty} \Delta_{s-1}(x) < 0$, which contradicts the assumption that $E[U^j - V^j] = 0$ for $j = 1, \ldots, s-2$ and $(-1)^{s-1}E[U^{s-1} - V^{s-1}] \geq 0$. Hence $S[\Delta_j] \leq \max\{0, S[\Delta_{j-1}]\}$ for $j = 1, \ldots, s-1$, and consequently $S[\Delta_{s-1}] = 0$. The assumption $G \geq H$ on I_1 then implies $\Delta_{s-1}(x) \geq 0$, which proves (i).

As to the second assertion, one only has to note that (ii) is a sufficient condition for (i). ■

Remark 3.2 Suppose the moments of G and H through order s are finite and $U \leq_{s-icv} V$. Then Fishburn (1980b) showed that either $G \equiv H$, or for some $k \leq s$, $EU^j = EV^j$, $(j = 1, 2, \ldots, k-1)$ and $(-1)^k EU^k > (-1)^k EV^k$. Hence, if $EU^j = EV^j, j = 1, \ldots, s-2$, then the inequality $(-1)^{s-1}EU^{s-1} \geq (-1)^{s-1}EV^{s-1}$ in Theorem 3.1 is necessary for $U \leq_{s-icv} V$.

Example 3.1 Consider a random variable X with $P(X = 3/10) = 25/29$ and $P(X = 7/4) = 4/29$. Bhattacharjee and Sengupta (1996), Example 3.1, proved that $X \in \mathcal{L}$. Since $EX = EX^2 = 1/2$, X is not HNBUE (Lin and Hu (2000)).

In order to show that $X \in \mathcal{L}$ using Theorem 3.1, let Y be exponentially distributed with mean 1/2. Then, $EY = EX, EY^2 = EX^2$ and $EY^3 = 3/4 \leq EX^3 = 61/80$. The difference of the distribution functions $F_Y - F_X$ has three sign changes; clearly, $F_Y - F_X > 0$ before the first sign change. By Theorem 3.1(i), $Y \leq_{4-icv} X$, and, consequently, $X \in \mathcal{L}$.

Example 3.2 Let $IG(\mu, \lambda)$ denote the inverse Gaussian distribution with parameters $\mu > 0$ and $\lambda > 0$, which has density

$$f(x) = \sqrt{\frac{\lambda}{2\pi}}\, x^{-3/2} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right), \qquad x > 0.$$

The expectation and variance of $IG(\mu, \lambda)$ are μ and μ^3/λ, respectively.

Proposition 3.3 *Let $X \sim IG(\mu, \lambda)$ and $Y \sim \Gamma(\alpha, \beta)$ with $EX = EY$ and $Var(X) \leq Var(Y)$. Then $Y \leq_{3-icv} X$.*

PROOF: By rescaling if necessary, we may take $\mu = 1$, so $X \sim IG(1, \lambda)$ and $Y \sim \Gamma(\alpha, \alpha)$ with $\lambda \geq \alpha$. For some constant c, we have

$$\log \frac{f_Y(x)}{f_X(x)} = c + \left(\alpha + \frac{1}{2}\right) \log x + \frac{\lambda}{2x} - \left(\alpha - \frac{\lambda}{2}\right) x,$$

which tends to $+\infty$ for $x \downarrow 0$ and has no more than three sign changes (see Kaas and Hesselager (1995), p. 197). By Theorem 3.1(ii), the assertion follows. ■

As a consequence, each $IG(\mu, \lambda)$-distribution with $\lambda \geq \mu\alpha$ belongs to the class $\mathcal{L}_\alpha$ $(\alpha > 0)$. In particular, when $\lambda \geq \mu$ (or $CV = \sqrt{\mu/\lambda} \leq 1$), then $IG(\mu, \lambda) \in \mathcal{L}$. Henze and Klar (2001) obtained this result using the Laplace transform of $IG(\mu, \lambda)$. If $\lambda < \mu$ then $CV > 1$, and consequently $IG(\mu, \lambda) \notin \mathcal{L}$.

Example 3.3 Let $LN(\nu, \tau^2)$ denote the lognormal distribution with parameters $\nu \in \mathcal{R}$ and $\tau^2 > 0$, which has density

$$f(x) = \frac{1}{x\tau\sqrt{2\pi}} \exp\left(-\frac{(\log x - \nu)^2}{2\tau^2}\right), \qquad x > 0.$$

The expectation and variance of $LN(\nu, \tau^2)$ are $e^{\nu+\tau^2/2}$ and $e^{2\nu+\tau^2}(e^{\tau^2} - 1)$.

Proposition 3.4 *Let $X \sim LN(\nu, \tau^2)$ and $Y \sim \Gamma(\alpha, \beta)$ with $EX = EY$ and $Var(X) \leq Var(Y)$. Then $Y \leq_{3-icv} X$.*

PROOF: Again, we take $\mu = 1$, so $X \sim LN(-\tau^2/2, \tau^2)$ and $Y \sim \Gamma(\alpha, \alpha)$, where necessarily $\tau^2 \leq \log(1 + 1/\alpha)$. Here, we obtain

$$\log \frac{f_Y(x)}{f_X(x)} = c + \left(\alpha + \frac{1}{2}\right) \log x - \alpha x + \frac{\log^2 x}{2\tau^2},$$

which tends to $+\infty$ for $x \downarrow 0$ and has at most three sign changes (see Kaas and Hesselager (1995), p. 197). Hence, the assertion follows. ■

By Proposition 3.4, each $LN(\nu, \tau^2)$-distribution with $\tau^2 \leq \log(1 + 1/\alpha)$ belongs to the $\mathcal{L}_\alpha$-class.

Example 3.4 In the last example, we consider the Birnbaum-Saunders distribution $BS(\gamma, \delta)$ with parameters $\gamma, \delta > 0$, which has density

$$f(x) = \frac{\exp(\gamma^{-2})}{2\gamma\sqrt{2\pi\delta}}\, x^{-3/2}(x+\delta)\exp\left\{-\frac{1}{2\gamma^2}\left(\frac{x}{\delta}+\frac{\delta}{x}\right)\right\}, \qquad x > 0$$

(Johnson, Kotz, and Balakrishnan (1995), p. 651). The expectation and variance of $BS(\gamma, \delta)$ are $\delta(\gamma^2/2 + 1)$ and $\delta^2\gamma^2(5\gamma^2/4 + 1)$.

We deal only with the case $\alpha = 1$; without restriction, we take the means equal to 1. Hence, let $X \sim BS(\gamma, (\gamma^2/2 + 1)^{-1})$ and $Y \sim \Gamma(1, 1)$. If $\gamma \leq 1$, $Var(X) \leq Var(Y)$. Now,

$$\log\frac{f_Y(x)}{f_X(x)} = c + \frac{2-3\gamma^2}{4\gamma^2}x + \left(\gamma^2(\gamma^2+2)x\right)^{-1} + \frac{3}{2}\log x - \log\left(x(\gamma^2+2)+2\right)$$

which tends to $+\infty$ for $x \downarrow 0$. The first derivative is a rational function in x. The denominator is positive for $x > 0$. The nominator is a third degree polynomial; some computations show that it has at most two zeros for $x \geq 0$, so the function itself has no more than three sign changes. By Theorem 3.1(ii), we have $Y \leq_{3-icv} X$. Therefore, the Birnbaum-Saunders distribution belongs to the $\mathcal{L}$-class, provided $\gamma \leq 1$ (i.e. the coefficient of variation does not exceed 1).

4 Explicit reliability bounds for the $\mathcal{L}$-class

Chaudhuri, Deshpande, and Dharmadhikari (1996) gave the following explicit lower bound on the survival function of a distribution F belonging to the $\mathcal{L}$-class. If F has mean μ, and $F \in \mathcal{L}$, then $\bar{F}(t) \geq 1 - (t/\mu)\exp(1 - t/\mu)$ for $0 \leq t \leq \mu$.

Sengupta (1995) obtained the following implicit, but sharp upper and lower bounds. Suppose $F \in \mathcal{L}$ with mean μ. Then $\alpha_t \leq \bar{F}(t) \leq 1$ if $t \leq \mu$ and $0 \leq \bar{F}(t) \leq 1 - \alpha_t$ if $t > \mu$, where

$$\alpha_t = \inf\left\{\alpha : f_{\alpha,t/\mu}(s) \geq 0\ \forall s > 0\right\} \tag{5}$$

and

$$f_{\alpha,u}(s) \;=\; e^{su} - (1+s)\left(1-\alpha+\alpha e^{-s(1-u)/\alpha}\right).$$

We use this result to derive explicit bounds on survival functions belonging to the $\mathcal{L}$-class.

Theorem 4.1 *Let $F \in \mathcal{L}$ with mean μ. Then the following bounds hold:*

(i) $\bar{F}(t) \geq 1 - \dfrac{1}{(t/\mu)^2 - 2t/\mu + 2}$, *if $t \leq \mu$. The bound is sharp for $2-\sqrt{2} \leq t/\mu \leq 1$.*

(ii) $\bar{F}(t) \leq \dfrac{1}{(t/\mu)^2 - 2t/\mu + 2}$, *if $t > \mu$. The bound is sharp for $t/\mu \geq 2+\sqrt{2}$.*

PROOF: Fix t and put $u = t/\mu$. A Taylor series expansion around $s = 0$ yields

$$\begin{aligned} f_{\alpha,u}(s) &= \left(\frac{u^2}{2} - \frac{(u-1)^2}{2\alpha} - (u-1)\right) s^2 + O(s^3) \\ &= c_1(u,\alpha)\, s^2 + O(s^3), \end{aligned}$$

say. Now, $f_{\alpha,u}(s) \geq 0$ for all $s > 0$ only if $c_1(u,\alpha) \geq 0$. Hence,

$$\alpha_t \;\geq\; \frac{(u-1)^2}{1+(u-1)^2} \;=:\; \alpha_t^* \;\text{ (say)}, \tag{6}$$

which yields the bounds in (i) and (ii). Another series expansion around $s = 0$ gives

$$f_{\alpha_t^*,u}(s) \;=\; c_2(u)\, s^3 + O(s^4),$$

where $c_2(u) = (u^2-4u+2)/(6(u-1)) \geq 0$ for $u \in I_1 = [2-\sqrt{2}, 1]$ or $u \in I_2 = [2+\sqrt{2}, \infty)$. Furthermore, using the inequality $e^x \geq 1 + x$ ($x \in \mathbb{R}$), we see that the second derivative

$$f''_{\alpha_t^*,u}(s) \;=\; e^{su}\left[u^2 - \left(\left(u^2-2u+2\right)s + u^2\right) e^{-s\frac{u-2}{u-1}}\right]$$

is non-negative for $u \in I_1 \cup I_2$. Hence, if $u \in I_1 \cup I_2$, $f_{\alpha_t^*,u}(s) \geq 0$ for all $s > 0$. In view of (5) and (6), the lower (upper) bound is sharp for $u \in I_1$ ($u \in I_2$). ■

Remark 4.2 (i) The explicit lower bound $1 - (t/\mu)\exp(1 - t/\mu)$ mentioned above is better than the bound in (i) in the range $0 \leq t/\mu \leq 0.415$.

(ii) The upper bound μ/t, which applies to any survival function with mean μ, is better than the bound in (ii) in the range $\mu < t < 2\mu$.

(iii) Klefsjö (1982) obtained the sharp upper bound $e^{1-t/\mu}$ $(t > \mu)$ on survival functions in the HNBUE class. The difference between this bound and the upper bound in (ii) also indicates that the $\mathcal{L}$-property is considerably weaker than the HNBUE notion.

In the rest of this section, we consider the class $\mathcal{L}_\alpha$ of life distributions. For a distribution with mean μ belonging to $\mathcal{L}_\alpha$, Lin and Hu (2000) gave the functional lower bound $\bar{F}(t) \geq 1 - \left(\frac{t}{\mu}\exp(1 - t/\mu)\right)^\alpha$ for all $t \leq \mu$.

If the distributions F and G have the same mean μ and $F \geq_L G$, Sengupta (1995) obtained the bounds $\delta_t \leq \bar{F}(t) \leq 1$ if $t \leq \mu$ and $0 \leq \bar{F}(t) \leq 1 - \delta_t$ if $t > \mu$, where

$$\delta_t \;=\; \inf\left\{\delta : \inf_{r>0}\left[e^{rt}\left(1 - r\int_0^\infty (1 - G(x))e^{-rx}dx\right) + \delta\left(1 - e^{-r(\mu-t)/\delta}\right) - 1\right] \geq 0\right\}.$$

If G is the $\Gamma(\alpha, \alpha/\mu)$-distribution with Laplace transform $L_G(r) = (1 + \mu r/\alpha)^{-\alpha}$, we obtain (by using $\int_0^\infty (1 - G(x))e^{-rx}dx = (1 - L_G(r))/r$ and putting $\mu r = s$)

$$\delta_t \;=\; \inf\left\{\delta : f_{\delta,t/\mu}(s) \geq 0 \;\forall s > 0\right\},$$

where $f_{\delta,u}(s) = e^{su} - (1 + s/\alpha)^\alpha\left(1 - \delta + \delta e^{-s(1-u)/\delta}\right)$. Utilizing this result, we obtain the next theorem. Its proof is omitted, since the reasoning closely follows the first part of the proof of Theorem 4.1.

Theorem 4.3 *Let $F \in \mathcal{L}_\alpha$ with mean μ. Then the following bounds hold:*

(i) $\bar{F}(t) \geq 1 - \left(\alpha\,(t/\mu - 1)^2 + 1\right)^{-1}$, *if* $t \leq \mu$.

(ii) $\bar{F}(t) \leq \left(\alpha\,(t/\mu - 1)^2 + 1\right)^{-1}$, *if* $t > \mu$.

We conjecture that, similar as for $\alpha = 1$, the bound in (i) is sharp for $\frac{1+\alpha-\sqrt{1+\alpha}}{\alpha} \leq \frac{t}{\mu} \leq 1$, and the bound in (ii) is sharp for $\frac{t}{\mu} \geq \frac{1+\alpha+\sqrt{1+\alpha}}{\alpha}$.

5 Conclusion

The inequalities (1) and (2), which define the $\mathcal{L}$-class of life distributions, correspond to several models in reliability and maintenance (see Klefsjö (1983), Alzaid et al. (1991)). This fact explains the great deal of attention received by the $\mathcal{L}$-class in the literature. As remarked by Klefsjö (1983), a further reason for the importance of the $\mathcal{L}$-class lies in the fact that it may be easier to determine the Laplace transform than the corresponding survival function explicitly.

Often, the $\mathcal{L}$-class is viewed as the largest class in a chain of classes of life distributions that describe positive aging. As stated by Kochar and Deshpande (1985), the essential property of distributions belonging to these classes is that the residual performance of a unit having already survived up to time t is 'inferior' in some stochastic sense than the performance of a fresh unit. This general definition of positive aging applies for classes like increasing failure rate, increasing failure rate average, new better than used, decreasing mean residual life, new better than used in expectation or harmonic new better than used in expectation. However, there doesn't seem to exist a corresponding characterization for the class $\mathcal{L}$.

The $\mathcal{L}$-class contains distributions which do not belong to the aforementioned aging classes, but are connected to some notion of aging. For example, the inverse Gaussian and the Birnbaum-Saunders distribution are typical examples of fatigue failure models (see, e.g., Bhattacharyya and Fries (1982)). The hazard rates of both distributions increase from zero at time $t = 0$ until they attain a maximum at some critical time and then decrease to a non-zero asymptotic value.

On the other hand, the distribution in Example 2.1 with hazard rate decreasing to zero with the rate $1/t$ as $t \to \infty$, or the lognormal distribution with hazard rate decreasing to zero, show that the $\mathcal{L}$-class also includes distributions which are not well suited for describing positive aging.

Hence, the $\mathcal{L}$-class should not be seen as a more comprehensive class of aging distributions but rather as a large class of life distributions on its own.

Acknowledgement

The author would like to thank Alfred Müller for helpful discussions.

References

Alzaid, A., Kim, J., & Proschan, F. (1991). Laplace ordering and its applications. *J. Appl. Prob.*, *28*, 116–130.

Bhattacharjee, A., & Sengupta, D. (1996). On the coefficient of variation of the $\mathcal{L}$- and $\bar{\mathcal{L}}$-classes. *Statist. & Prob. Lett.*, *27*, 177–180.

Bhattacharyya, G., & Fries, A. (1982). Fatigue failure models - Birnbaum-Saunders vs. inverse Gaussian. *IEEE Trans. Reliab.*, *R-31*, 439–441.

Chaudhuri, G., Deshpande, J., & Dharmadhikari, A. (1996). A lower bound on the L-class of life distributions and its applications. *Calcutta Statist. Assoc. Bull.*, *46*, 269–274.

Denuit, M., Lefevre, C., & Shaked, M. (1998). The s-convex orders among real random variables, with applications. *Math. Inequal. Appl.*, *1*, 585–613.

Fishburn, P. (1980a). Continua of stochastic dominance relations for unbounded probability distributions. *Journal of Mathematical Economics*, *7*, 271–285.

Fishburn, P. (1980b). Stochastic dominance and moments of distributions. *Math. Oper. Res.*, *5*, 94–100.

Henze, N., & Klar, B. (2001). Testing exponentiality against the L-class of life distributions. *Mathematical Methods of Statistics*, *10*, 232–246.

Johnson, N., Kotz, S., & Balakrishnan, N. (1995). *Continuous univariate distributions* (Vol. 2, 2nd ed.). New York: Wiley.

Kaas, R., & Hesselager, O. (1995). Ordering claim size distributions and mixed Poisson probabilities. *Insurance: Mathematics and Economics*, *17*, 193–201.

Karlin, S., & Novikoff, A. (1963). Generalized convex inequalities. *Pacific J. of Mathematics*, *13*(1), 1251–1279.

Klefsjö, B. (1982). The HNBUE and HNWUE classes of life distributions. *Naval Res. Logist. Quart.*, *29*, 331–344.

Klefsjö, B. (1983). A useful ageing property based on the Laplace transform. *J. Appl. Prob.*, *20*, 615-626.

Kochar, S., & Deshpande, J. (1985). On exponential scores statistic for testing against positive aging. *Statist. & Prob. Lett.*, *3*, 71–73.

Lin, G. (1998a). Characterizations of the $\mathcal{L}$-class of life distributions. *Statist. & Prob. Lett.*, *40*, 259–266.

Lin, G. (1998b). On weak convergence within the $\mathcal{L}$-like classes of life distributions. *Sankhya,Ser. A*, *60*, 176–183.

Lin, G., & Hu, C. (2000). A note on the $\mathcal{L}$-class of life distributions. *Sankhya,Ser. A*, *62*, 267–272.

Mitra, M., Basu, S., & Bhattacharjee, M. (1995). Characterizing the exponential law under Laplace order domination. *Calcutta Statist. Assoc. Bull.*, *45*, 171–178.

Reuter, H., & Riedrich, T. (1981). On maximal sets of functions compatible with a partial ordering for distribution functions. *Math. Operationsforsch. Statist. Ser. Optim.*, *12*, 597–605.

Rolski, T. (1975). Mean residual life. *Bull. Internat. Statist. Inst.*, *46*, 266–270.

Rolski, T. (1976). Order relations in the set of probability distributions and their applications in queuing theory. *Dissertationes Math.*, *132*, 3–47.

Sengupta, D. (1995). Reliability bounds fo the $\mathcal{L}$-class and Laplace order. *J. Appl. Prob.*, *32*, 832–835.

Stoyan, D. (1983). *Comparison models for queues and other stochastic models.* Wiley.