On a test for exponentiality against Laplace order dominance

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Abstract

In a recent paper, Basu and Mitra (2002) introduced a class of tests for exponentiality against the nonparametric class $\mathcal{L}$ of life distributions. The test statistics are fractional sample moments. In a simulation study, the tests show a remarkable behavior: when performed on a nominal level of 5%, each of the tests always accepted the null hypothesis of exponentiality if the underlying data came from an exponential distribution. Furthermore, power of the tests was very low compared to other procedures designed for the same testing situation. It is the aim of this paper to show that the reasons for this behavior are an incorrectly stated asymptotic distribution, a slow convergence of the finite sample distributions of the test statistics to their limit distribution, and, to a minor extent, the dependence of the proposed tests on the choice of parameter values.

To this end, we derive the limit distributions of the test statistics in case of a general underlying distribution and the local approximate Bahadur efficiency of the tests against several parametric families of alternatives to exponentiality. Additionally, the finite sample behavior of the tests is examined by means of a simulation study.

Further, we enlarge the proposed class of tests by extending some characterization of exponentiality within the $\mathcal{L}$-class.

Key words: $\mathcal{L}$-class, Laplace transform, Exponential distribution, characterization, goodness-of-fit test, fractional moment.

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1 Introduction

A distribution function F with support $[0, \infty)$ and finite mean $\mu = \int_0^\infty \overline{F}(x)dx$, where $\overline{F} = 1 - F$, is said to belong to the $\mathcal{L}$-class of life distributions ($F \in \mathcal{L}$) if

$$\int_0^\infty e^{-sx}\overline{F}(x)dx \;\; \geq \;\; \frac{\mu}{1+s\mu} \quad \text{for all } s \geq 0. \tag{1}$$

The class $\mathcal{L}$ was introduced by Klefsjö (1983). By means of the Laplace transform $L_F(s) = E_F e^{-sX}$, (1) can be restated as

$$L_F(s) \;\; \leq \;\; L(s, 1/\mu) \quad \text{for all } s \geq 0, \tag{2}$$

where $L(s, \lambda) = \lambda/(\lambda + s)$ denotes the Laplace transform of the exponential distribution with distribution function $F(t, \lambda) = 1 - \exp(-\lambda t)$ for $t \geq 0$. The $\mathcal{L}$-class is strictly larger than the harmonic new better than used in expectation (HNBUE) class of life distributions , satisfying $\int_t^\infty \overline{F}(x)dx \leq \mu \exp(-t/\mu)$ for every $t \geq 0$.

The following characterization of exponentiality within the $\mathcal{L}$-class can be found in Cai and Wu (1997) in case of $-1 < \alpha < 0$, and in Lin (1998) and Bhattacharjee (1999) in case of $\alpha \in (-1, 0) \cup (0, 1)$.

1.1 Proposition *Let $X \in \mathcal{L}$. Then X is exponential if and only if $EX^\alpha = \Gamma(\alpha+1)(EX)^\alpha$, for some $\alpha \in (-1, 0) \cup (0, 1)$.*

Writing

$$\theta_\alpha = EX^\alpha/(EX)^\alpha \quad (= \Gamma(\alpha+1) \;\text{ under exponentiality}), \tag{3}$$

one has to distinguish the following cases. For an $\mathcal{L}$-class alternative, if $\alpha \in (-1, 0)$, then $\theta_\alpha < \Gamma(\alpha+1)$; but if $\alpha \in (0, 1)$, then $\theta_\alpha > \Gamma(\alpha+1)$.

For $\alpha \in (-1, 0)$, Proposition 1.1 also follows from Theorem 2.1 in Basu and Mitra (2002). Note that they use $EX^{-r} = \pi/(\Gamma(r)(EX)^r \sin(r\pi))$ $(0 < r < 1)$ under exponentiality; this coincides with (3) since $\Gamma(r)\Gamma(1-r) = \pi/\sin(r\pi)$ for $0 < r < 1$.

In view of Proposition 1.1, a test for exponentiality which is consistent against all alternatives from the $\mathcal{L}$-class can be based on the fractional sample moments $n^{-1}\sum_{j=1}^n X_j^\alpha$, where $X_1, \ldots, X_n$ is a random sample of size n from F. Alternatively, one can use the scale invariant statistic

$$m_\alpha = \frac{1}{n}\sum_{j=1}^n Y_j^\alpha \qquad\qquad (\alpha \in (-1, 0) \cup (0, 1)),$$

where $Y_j = X_j/\overline{X}_n$ and $\overline{X}_n = n^{-1}\sum_{j=1}^n X_j$ is the sample mean.

Basu and Mitra (2002) considered the case $\alpha \in (-1/2, 0)$. Tests of exponentiality based on m_α for $\alpha \in (0,1)\cup(1,\infty)$ have been studied by Lee, Locke, and Spurrier (1980). They showed that the tests are consistent against IFRA and DFRA alternatives.

Tests for exponentiality against the $\mathcal{L}$-class have been previously proposed by Chaudhuri (1997) and by Henze and Klar (2001).

Differing from the case $\alpha \in (-1,0)\cup(0,1)$, it is well-known that the exponential distribution is not characterized within the $\mathcal{L}$-class by the property that $EX^\alpha = \Gamma(\alpha+1)(EX)^\alpha$ for $\alpha = 2$. Hence, the question arises wether Proposition 1.1 is still valid for $\alpha \in (1,2)$.

In the next section, we answer this question in the affirmative. In Section 3, we state the asymptotic behavior of test statistics based on m_α. Section 4 is devoted to the calculation of local approximate Bahadur efficiencies of the proposed tests of exponentiality with respect to four families of alternative distributions from the class $\mathcal{L}$. Finally, in Section 5, the finite sample behavior of the tests is examined by means of a simulation study.

2 A characterization of exponentiality within the $\mathcal{L}$-class

The following lemma expresses the fractional moments by means of the Laplace transform and the integer moments.

2.1 Lemma *Let X be a positive random variable with distribution F. Then, for $k = 0, 1, \ldots$ and $r \in (0,1)$,*

$$EX^{r+k-1} = \frac{(-1)^k \prod_{j=0}^{k-1}(r+j)}{\Gamma(1-r)} \int_0^\infty s^{-r-k}\left(L_F(s) - \sum_{j=0}^{k-1}\frac{(-1)^j s^j \mu_j}{j!}\right) ds,$$

where $\mu_j = EX^j$ for integer j.

Proof: Integration by parts yields

$$\int_0^\infty s^{-(r+k)}\left(e^{-sx} - \sum_{j=0}^{k-1}\frac{(-1)^j (sx)^j}{j!}\right) ds$$
$$= \frac{1}{r+k-1}\int_0^\infty s^{-(r+k-1)}(-x)\left(e^{-sx} - \sum_{j=0}^{k-2}\frac{(-1)^j (sx)^j}{j!}\right) ds$$
$$= \ldots$$

$$= \frac{(-1)^k x^k}{(r+k-1)\cdots r} \int_0^\infty s^{-r} e^{-sx} ds = \frac{(-1)^k x^{r+k-1} \Gamma(1-r)}{\prod_{j=0}^{k-1}(r+j)}.$$

Using Fubini's theorem, we obtain

$$EX^{r+k-1} = \frac{(-1)^k \prod_{j=0}^{k-1}(r+j)}{\Gamma(1-r)} \int_0^\infty s^{-r-k} \int_0^\infty \left(e^{-sx} - \sum_{j=0}^{k-1} \frac{(-1)^j (sx)^j}{j!} \right) dF(x)\, ds,$$

whence the assertion follows. ■

The cases $k = 0$ (which remains valid for arbitrary negative exponents) and $k = 1$ can be found in Hoffmann-Jørgensen (1994), page 303, and Lin (1998).

2.2 Corollary *Proposition 1.1 holds for $\alpha \in (1, 2)$ as well.*

Proof: Let $X \in \mathcal{L}$, and assume that Z has an exponential distribution with the same mean μ as X. Writing $\alpha = r + 1$ with $r \in (0, 1)$, and using Lemma 2.1 for $k = 2$, we obtain

$$EX^{1+r} - EZ^{1+r} = \frac{r(r+1)}{\Gamma(1-r)} \int_0^\infty s^{-r-2} \left(L_F(s) - L(s, 1/\mu) \right) ds.$$

The condition $EX^{1+r} = \Gamma(r+2)\mu^{1+r}$ forces $L_F(s) = L_G(s)$ a.e. on $(0, \infty)$, and the uniqueness theorem for Laplace transforms yields the assertion. ■

As in case of $\alpha \in (-1, 0)$, $\theta_\alpha < \Gamma(\alpha + 1)$ for $\alpha \in (1, 2)$ for any alternative from the $\mathcal{L}$-class.

3 The asymptotic distribution of the test statistics

In the following, let $\alpha \in (-1/2, 0) \cup (0, 2)$. Then, if X has an exponential distribution, $EX^{2\alpha} < \infty$, and the central limit theorem yields

$$\sqrt{n} \left(\frac{1}{n\mu^\alpha} \sum_{j=1}^n X_j^\alpha - \Gamma(\alpha+1) \right) \xrightarrow{\mathcal{D}} N\left(0, \Gamma(2\alpha+1) - \Gamma^2(\alpha+1)\right). \qquad (4)$$

Hence, a test of

$$H_0: \ F \in \mathcal{E} = \{F(\cdot, \lambda), \ \lambda > 0\}$$

against the alternative

$$H_1: \ F \in \mathcal{L} \quad \text{and} \quad F \notin \mathcal{E}$$

can be based on

$$M_{n,\alpha} = \frac{1}{n}\sum_{j=1}^{n} Y_j^{\alpha} - \Gamma(\alpha+1). \tag{5}$$

However, $\sqrt{n}\, M_{n,\alpha}$ has not the limit distribution given in (4), as stated by Basu and Mitra (2002). We have to take into account that μ is replaced by the estimator $\overline{X}_n$.

Since $M_{n,\alpha}$ is scale-invariant, we assume $\mu = 1$ in the following.

3.1 Theorem *Assume $X_1, \ldots, X_n$ is a random sample of a nonnegative nondegenerate random variable X with $EX_1^{2\alpha} < \infty$. Then, as $n \to \infty$,*

$$\sqrt{n}\left(\frac{1}{n}\sum_{j=1}^{n} Y_j^{\alpha} - EX_1^{\alpha}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2),$$

where

$$\sigma^2 = E\left(X^{\alpha} - E\left(X^{\alpha}\right) - \alpha\, E\left(X^{\alpha}\right)(X-1)\right)^2. \tag{6}$$

Under H_0, we have $\sqrt{n}\, M_{n,\alpha} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_0^2)$*, where*

$$\sigma_0^2 = \Gamma(2\alpha+1) - (r^2+1)\Gamma^2(\alpha+1). \tag{7}$$

Proof: Notice that

$$\sqrt{n}\left(\frac{1}{n}\sum_{j=1}^{n} Y_j^{\alpha} - EX_1^{\alpha}\right) = U_{n,1} + U_{n,2},$$

where

$$U_{n,1} = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(X_j^{\alpha} - EX_1^{\alpha}\right), \qquad U_{n,2} = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(Y_j^{\alpha} - X_j^{\alpha}\right).$$

A Taylor expansion of the function $g(t) = (X_j/t)^{\alpha}$ around $t = 1$ yields

$$\begin{aligned} U_{n,2} &= \sqrt{n}\left(\frac{1}{\overline{X}_n} - 1\right)\frac{1}{n}\sum_{j=1}^{n}(-\alpha)\, X_j^{\alpha} \;+\; o_P(1) \\ &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}(X_j - 1)\,\alpha\, EX_1^{\alpha} \;+\; o_P(1), \end{aligned}$$

whence, by the Central limit theorem and Slutsky's lemma, $U_{n,1} + U_{n,2} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$, where σ^2 is given in (6).

The formula (7) for σ^2 in case of H_0 follows from straightforward calculations. ■

As a consequence, the asymptotic distribution of $M^*_{n,\alpha} = \sqrt{n}\, M_{n,\alpha}/\sigma_0^2$ is standard normal under the hypothesis of exponentiality.

In view of Proposition 1.1 and Corollary 2.2, and the subsequent discussion, one has the following result.

3.2 Proposition *A one-sided test with lower rejection region based on* $M_{n,\alpha}$, $\alpha \in (-1,0)\cup(1,2)$, *is consistent against each alternative from the class* $\mathcal{L}$.

Similarly, a test for exponentiality rejecting H_0 *for large positive values of* $M_{n,\alpha}$, $\alpha \in (0,1)$, *is consistent against each* $\mathcal{L}$*-class alternative.*

Next, we want to show that the test statistics are integrals of a weighted difference between the empirical Laplace transform

$$L_n(t) = \int_0^\infty e^{-tX} dF_n(x) = \frac{1}{n}\sum_{j=1}^n e^{-tX_j}$$

and the Laplace transform of a fitted exponential distribution. Using

$$\overline{X}_n^{-\alpha}\int_0^\infty s^{-(\alpha+1)}\left(L_n(t) - L(t,1/\overline{X}_n)\right) ds = \int_0^\infty t^{-(\alpha+1)}\left(n^{-1}\sum_{j=1}^n e^{-tY_j} - \frac{1}{1+t}\right) dt,$$

integration by parts as in Lemma 2.1 and $n^{-1}\sum_{j=1}^n Y_j = 1$, one obtains

$$M_{n,\alpha} = C_\alpha \int_0^\infty s^{-(\alpha+1)}\left(L_n(t) - L(t,1/\overline{X}_n)\right) ds,$$

where

$$\begin{aligned} C_\alpha &= 1/\left(\Gamma(-\alpha)\overline{X}_n^\alpha\right) && (-1<\alpha<0),\\ C_\alpha &= -\alpha/\left(\Gamma(1-\alpha)\overline{X}_n^\alpha\right) && (0<\alpha<1),\\ C_\alpha &= \alpha(\alpha-1)/\left(\Gamma(2-\alpha)\overline{X}_n^\alpha\right) && (1<\alpha<2). \end{aligned}$$

Hence, the test statistics are based on the empirical counterpart $L_n(x) - L(x,1/\overline{X}_n)$ of $L_F(x) - L(x,1/\mu)$ which is a natural approach in view of (2).

4 Local approximate Bahadur efficiency

In this section, we investigate the efficiency of the tests for exponentiality based on $M_{n,\alpha}$ against several one-parametric families of distributions from the class $\mathcal{L}$. In each case, the parameter space, denoted by Θ, is some subinterval of $(0,\infty)$. Depending on the specific alternative family, the unit exponential distribution corresponds either to the parameter value $\vartheta_0 = 1$ or to the value $\vartheta_0 = 0$. As in Henze and Klar (2001), our measure of efficiency is the local approximate Bahadur slope (see , e.g., Nikitin (1995), p. 10).

The approximate Bahadur slope $c^*(\cdot,\alpha)$ of the sequence $(M^*_{n,\alpha})$ of test statistics is given by

$$c^*(\vartheta,\alpha) = \left[\frac{1}{\sigma_0}\left\{E_\vartheta\left(\frac{X}{\mu(\vartheta)}\right)^\alpha \; - \; E_{\vartheta_0}\left(X^\alpha\right)\right\}\right]^2.$$

We now consider the local behavior of $c^*(\vartheta,\alpha)$ as $\vartheta \to \vartheta_0$. Under some regularity assumptions which hold for the examples in this section, one obtains

$$c^*(\vartheta,\alpha) \sim \frac{l_\alpha^2}{\sigma_0^2}(\vartheta-\vartheta_0)^2 \qquad \text{as } \vartheta \to \vartheta_0,$$

where

$$l_\alpha = \int_0^\infty x^\alpha \frac{\partial}{\partial\vartheta} f(x,\vartheta)\Big|_{\vartheta=\vartheta_0} dx \; - \; \mu'(\vartheta_0)\,\alpha\,\Gamma(\alpha+1),$$

and σ_0^2 is given in (7). For details, see Henze and Klar (2001), Section 3. Our measure of asymptotic local efficiency of $M_{n,\alpha}$ is

$$e_{F_\vartheta}(M_{n,\alpha}) \; = \; \frac{l_\alpha^2}{\sigma_0^2}.$$

We have calculated $e_F(M_{n,\alpha})$ for linear failure rate, Makeham, Weibull and gamma alternatives. These are given by the distribution functions

$$\begin{aligned}
F_\vartheta^{(1)}(x) &= 1-\exp\left(-\left(x+\vartheta x^2/2\right)\right) \quad \text{for } x\ge 0, \vartheta \ge 0,\\
F_\vartheta^{(2)}(x) &= 1-\exp\left(-\left(x+\vartheta\left(x+e^{-x}-1\right)\right)\right) \quad \text{for } x\ge 0, \vartheta \ge 0,\\
F_\vartheta^{(3)}(x) &= 1-\exp\left(-x^\vartheta\right) \quad \text{for } x\ge 0, \vartheta > 0,\\
F_\vartheta^{(4)}(x) &= \Gamma(\vartheta)^{-1}\int_0^x t^{\vartheta-1}e^{-t}\,dt \quad \text{for } x\ge 0, \vartheta > 0,
\end{aligned}$$

respectively. For $F_\vartheta^{(1)}$ and $F_\vartheta^{(2)}$, H_0 corresponds to $\vartheta=\vartheta_0=0$, and for $F_\vartheta^{(3)}$ and $F_\vartheta^{(4)}$, we have $\vartheta_0 = 1$.

Calculations give the efficiency

$$e_{F^{(1)}}(M_{n,\alpha}) = \left(\alpha(\alpha-1)\Gamma(\alpha+1)/2\right)^2/\sigma_0^2$$

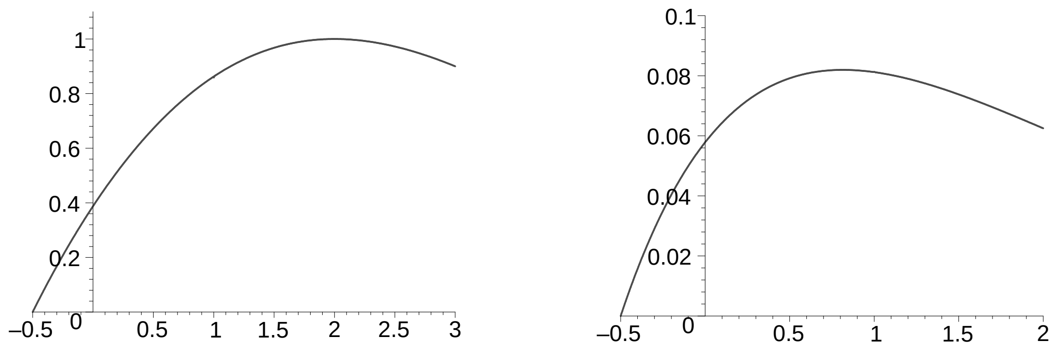

Figure 1: Local approximate Bahadur efficiency of $M_{n,\alpha}$ against LFR (left) and Makeham alternatives (right)

for $\alpha > -1/2$, where $\sigma_0^2 (= \sigma_0^2(\alpha))$ is given in (7). Figure 1 (left) shows $e_{F^{(1)}}$ for $-1/2 < \alpha < 3$. $e_{F^{(1)}}$ has a maximum value at $\alpha^* = 2$ with $e_{F^{(1)}}(M_{n,\alpha^*}) = 1$. It is well known that the test based on $M_{n,2}$ is asymptotically most powerful for testing H_0 against the linear failure rate distribution.

Next, we have

$$e_{F^{(2)}}(M_{n,\alpha}) = \left((1 - 2^{-\alpha} - \alpha/2)\Gamma(\alpha+1)\right)^2 / \sigma_0^2$$

for $\alpha > -1/2$. $e_{F^{(2)}}$ has a maximum value at $\alpha^* = 0.815$ with $e_{F^{(2)}}(M_{n,a^*}) = 0.082$. There is little difference between this value and 1/12, which is the Pitman efficiency of the asymptotically most powerful test of exponentiality against the Makeham distribution (Doksum and Yandell (1984)). Figure 1 (right) shows the local approximate Bahadur efficiencies of $M_{n,\alpha}$ against Makeham alternatives for $-1/2 < \alpha < 2$.

For $F_\vartheta^{(3)}$, we obtain the efficiency

$$e_{F^{(3)}}(M_{n,\alpha}) = \left((\alpha - \alpha\gamma - 1 - \alpha\Psi(\alpha))\Gamma(\alpha+1)\right)^2 / \sigma_0^2$$

for $\alpha > -1/2$, where $\gamma \approx 0.577$ is Euler's constant and Ψ denotes the digamma function. The maximum value of $e_{F^{(3)}}(M_{n,\alpha})$ is 1.63 at $\alpha^* = 0.285$.

The last efficiency is

$$e_{F^{(4)}}(M_{n,\alpha}) = \left((\gamma - \alpha + \Psi(\alpha) - \alpha^{-1})\Gamma(\alpha+1)\right)^2 / \sigma_0^2$$

for $\alpha > -1/2$. $e_{F^{(4)}}$ increases for $-1/2 < \alpha < 0$ and decreases for $\alpha > 0$; its maximum value is 0.645. Figure 2 displays the efficiencies of $M_{n,\alpha}$ against Weibull and Gamma

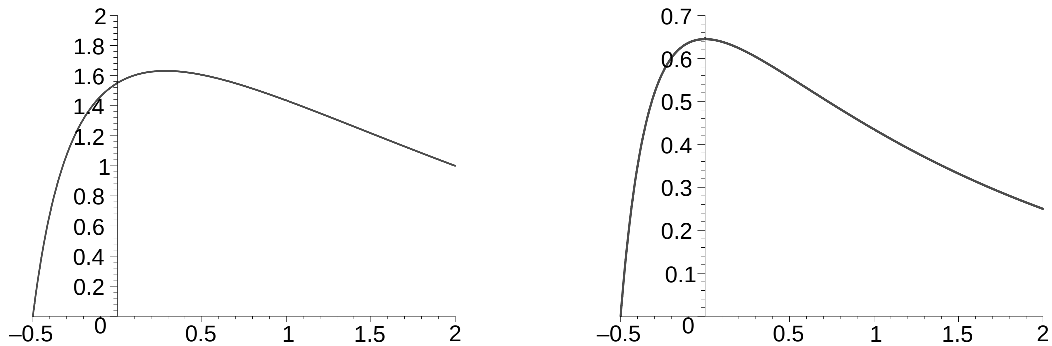

Figure 2: Local approximate Bahadur efficiency of $M_{n,\alpha}$ against Weibull (left) and gamma alternatives (right)

alternatives for $\alpha \in (-1/2, 2)$.

Obviously, local approximate Bahadur efficiency heavily depends on the value of α. However, the shape of the curves in Figures 1 and 2 is always similar. For each of the four distributions, efficiency goes to zero if α tends to -1/2. For increasing values of α, efficiency increases to a maximum which is attained between zero and two, and decreases again to zero for higher values of α.

5 Simulations

This section presents the results of two Monte Carlo studies. The first simulation study was conducted in order to obtain critical points of the statistics under discussion which is necessary due to the slow convergence of the finite sample distributions of the test statistics to their limit distribution.

Tables 1 to 4 show the p-quantiles of $M^*_{n,\alpha} = \sqrt{n} M_{n,\alpha}/\sigma_0$ under exponentiality for several sample sizes and $p = 0.05, 0.10, 0.90$ and 0.95, respectively. The exponents for $M^*_{n,\alpha}$ were chosen to be $\alpha = -0.4, -0.25, -0.1, 0.25, 0.5, 0.75, 1.25, 1.5, 1.75$ and 2.0. The entries in Tables 1 to 4 are based on 100000 replications; here, we always used $\lambda = 1$.

The speed of convergence to the asymptotic values is generally quite low, and it differs for different values of α. Whereas the asymptotic quantiles could be used for a test based on $M^*_{n,\alpha}$ with $\alpha = 0.25, 0.5$ or 0.75 for sample sizes $n \geq 100$, the speed of convergence decreases for larger as well as for smaller values of α. The finite sample quantiles are not symmetric around 0.

A second simulation study has been conducted to examine the dependence of the power of the tests on the exponent. As alternative distributions from the $\mathcal{L}$-class, we used

α	-0.4	-0.25	-0.1	0.25	0.5	0.75	1.25	1.5	1.75	2.0
$n = 10$	-0.82	-1.20	-1.41	-1.48	-1.42	-1.36	-1.42	-1.31	-1.21	-1.09
$n = 20$	-0.93	-1.31	-1.49	-1.56	-1.52	-1.48	-1.49	-1.41	-1.31	-1.21
$n = 30$	-0.99	-1.36	-1.52	-1.58	-1.55	-1.55	-1.52	-1.45	-1.37	-1.28
$n = 50$	-1.06	-1.42	-1.57	-1.61	-1.58	-1.58	-1.56	-1.50	-1.43	-1.35
$n = 100$	-1.13	-1.48	-1.59	-1.62	-1.60	-1.61	-1.59	-1.55	-1.49	-1.43
$n = 200$	-1.21	-1.52	-1.60	-1.63	-1.62	-1.63	-1.62	-1.58	-1.54	-1.50
$n = 500$	-1.29	-1.56	-1.63	-1.63	-1.63	-1.63	-1.62	-1.61	-1.58	-1.56
$n = 1000$	-1.34	-1.58	-1.63	-1.64	-1.64	-1.64	-1.64	-1.62	-1.59	-1.59

Table 1: Empirical 5%-quantiles of $M^*_{n,\alpha}$ based on 100000 replications

α	-0.4	-0.25	-0.1	0.25	0.5	0.75	1.25	1.5	1.75	2.0
$n = 10$	-0.74	-1.07	-1.23	-1.04	-0.98	-0.92	-1.24	-1.16	-1.08	-0.98
$n = 20$	-0.82	-1.13	-1.26	-1.13	-1.09	-1.05	-1.27	-1.22	-1.14	-1.07
$n = 30$	-0.86	-1.16	-1.28	-1.16	-1.13	-1.12	-1.28	-1.23	-1.18	-1.11
$n = 50$	-0.90	-1.19	-1.29	-1.20	-1.17	-1.15	-1.29	-1.25	-1.22	-1.15
$n = 100$	-0.95	-1.21	-1.28	-1.23	-1.21	-1.20	-1.30	-1.27	-1.24	-1.20
$n = 200$	-1.00	-1.23	-1.29	-1.24	-1.23	-1.23	-1.30	-1.28	-1.26	-1.23
$n = 500$	-1.06	-1.26	-1.29	-1.26	-1.25	-1.24	-1.29	-1.29	-1.27	-1.27
$n = 1000$	-1.09	-1.26	-1.29	-1.27	-1.26	-1.26	-1.29	-1.29	-1.26	-1.27

Table 2: Empirical 10%-quantiles of $M^*_{n,\alpha}$ based on 100000 replications

α	-0.4	-0.25	-0.1	0.25	0.5	0.75	1.25	1.5	1.75	2.0
$n = 10$	0.71	1.01	1.09	1.37	1.39	1.36	0.80	0.73	0.65	0.58
$n = 20$	0.83	1.12	1.18	1.36	1.37	1.34	0.99	0.94	0.88	0.84
$n = 30$	0.89	1.15	1.20	1.35	1.36	1.35	1.06	1.02	0.98	0.93
$n = 50$	0.95	1.20	1.22	1.33	1.35	1.33	1.13	1.11	1.08	1.05
$n = 100$	1.02	1.25	1.25	1.33	1.34	1.33	1.19	1.17	1.16	1.15
$n = 200$	1.07	1.26	1.26	1.31	1.32	1.32	1.24	1.23	1.21	1.21
$n = 500$	1.10	1.27	1.26	1.31	1.31	1.30	1.25	1.24	1.25	1.24
$n = 1000$	1.14	1.29	1.27	1.29	1.30	1.29	1.26	1.26	1.26	1.26

Table 3: Empirical 90%-quantiles of $M^*_{n,\alpha}$ based on 100000 replications

α	-0.4	-0.25	-0.1	0.25	0.5	0.75	1.25	1.5	1.75	2.0
$n = 10$	1.23	1.59	1.63	1.60	1.62	1.58	1.26	1.18	1.10	1.01
$n = 20$	1.37	1.67	1.68	1.64	1.65	1.60	1.46	1.41	1.37	1.33
$n = 30$	1.40	1.67	1.67	1.65	1.65	1.64	1.53	1.50	1.49	1.44
$n = 50$	1.46	1.71	1.67	1.64	1.66	1.63	1.56	1.58	1.56	1.56
$n = 100$	1.51	1.72	1.67	1.66	1.66	1.64	1.63	1.62	1.62	1.65
$n = 200$	1.55	1.70	1.66	1.66	1.66	1.64	1.65	1.65	1.66	1.68
$n = 500$	1.55	1.69	1.65	1.65	1.66	1.65	1.65	1.65	1.66	1.68
$n = 1000$	1.59	1.69	1.65	1.64	1.66	1.64	1.64	1.65	1.67	1.67

Table 4: Empirical 95%-quantiles of $M^*_{n,\alpha}$ based on 100000 replications

the Weibull, Gamma and Linear failure rate distribution with scale parameter 1 and shape parameter ϑ, denoted by $W(\vartheta)$, $\Gamma(\vartheta)$ and $LFR(\vartheta)$, respectively. Furthermore, we included the inverse Gaussian distribution $IG(1,\lambda)$ and the Lognormal distribution $LN(0,\tau^2)$ which also belong to the $\mathcal{L}$-class if the coefficient of variation is equal or smaller than one (Klar (2002)). The values $\lambda = 1, 1.2, 1.5, 2.0$ and 3.0 correspond to a coefficient of variation of 1,0.91,0.82,0.71 and 0.58, respectively. The values $\tau^2 = \log 2, 0.6, 0.5, 0.4$ and 0.3 correspond to a coefficient of variation of 1,0.91,0.81,0.70 and 0.59. Notice that $M^*_{n,2}$ is not consistent against $IG(1,1)$ and $LN(0,\log 2)$.

The first ten columns of Table 5 show power estimates of the tests based on $M^*_{n,\alpha}$ for $n = 30$. The last column contains the results of the test statistic

$$T_{n,1} = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{Y_j+1} - e\,E_1(1)$$

as an example of the statistics introduced in Henze and Klar (2001). All entries are the percentages of 10000 Monte Carlo samples that resulted in rejection of H_0, rounded to the nearest integer. The nominal level of the test is $\alpha = 0.05$.

The main conclusions that can be drawn from the simulation results are the following:

1. Using the empirical quantiles, all tests maintain their nominal level very closely. However, if one uses the asymptotic critical values instead, the percentage of rejection under exponentiality is $0\%, 1\%, 3\%$ and 1% for $\alpha = -0.4, -0.25, -0.1$ and 1.75, respectively. Hence, one should not use asymptotic critical values for $n = 30$.

2. The power of the tests based on $M^*_{n,\alpha}$ depends to a certain extent on α.

α	-0.4	-0.25	-0.1	0.25	0.5	0.75	1.25	1.5	1.75	2.0	$T_{n,1}$
$Exp(1)$	5	5	5	5	5	5	5	5	5	5	5
$W(1.1)$	14	14	14	15	15	14	15	14	14	13	14
$W(1.3)$	45	48	49	50	51	50	49	48	46	45	50
$W(1.5)$	77	80	82	84	85	84	84	81	80	79	84
$W(1.7)$	93	95	96	97	97	97	97	96	96	96	97
$W(2.0)$	99	100	100	100	100	100	100	100	100	100	100
$\Gamma(1.2)$	16	18	17	17	16	15	15	14	14	14	16
$\Gamma(1.5)$	46	48	48	47	46	42	40	38	36	34	43
$\Gamma(2.0)$	86	88	88	88	86	84	79	76	73	70	83
$\Gamma(2.5)$	98	99	99	99	98	97	95	94	92	90	97
$\Gamma(3.0)$	100	100	100	100	100	100	99	99	98	97	100
$LFR(0.5)$	16	18	20	21	23	24	26	26	26	26	23
$LFR(1.0)$	25	28	30	35	37	38	41	41	41	41	39
$LFR(2.0)$	38	41	44	50	55	57	60	60	61	60	57
$LFR(3.0)$	45	51	53	62	65	68	71	71	72	71	67
$LFR(5.0)$	55	60	65	72	77	78	82	82	82	81	79
$IG(1,1.0)$	66	63	57	45	38	30	23	20	17	15	31
$IG(1,1.2)$	85	82	78	65	56	49	37	33	29	26	49
$IG(1,1.5)$	96	95	94	86	80	72	60	54	49	47	74
$IG(1,2.0)$	100	100	100	98	96	93	86	82	77	74	94
$IG(1,3.0)$	100	100	100	100	100	100	99	98	97	96	100
$LN(0,\log 2)$	71	68	65	53	46	39	31	28	25	22	40
$LN(0,0.6)$	87	85	81	72	65	57	47	43	37	35	59
$LN(0,0.5)$	97	96	95	89	84	78	68	63	58	55	81
$LN(0,0.4)$	100	100	99	98	96	93	87	84	79	76	95
$LN(0,0.3)$	100	100	100	100	100	99	98	96	95	93	100

Table 5: Empirical power of the tests based on $M^*_{n,\alpha}$ and $T_{n,1}$, $\alpha = 0.05$, $n = 30$, 10000 replications

Against Weibull alternatives, $M^*_{n,0.25}$ and $M^*_{n,0.5}$ perform best. $M^*_{n,\alpha}$ with values of α around zero are most powerful against Gamma distributions. Values of α round about two are best suited to safeguard against LFR alternatives. $M^*_{n,-0.4}$ outperforms all tests under consideration in case of an inverse Gaussian as well as in case of a Lognormal distribution.

3. The results for $W(\vartheta)$, $\Gamma(\vartheta)$ and $LFR(\vartheta)$ are in good agreement with the local approximate Bahadur efficiencies derived in Section 4.
4. The test based on $T_{n,1}$ behaves nearly identical with the test based on $M^*_{n,0.75}$.
5. If nothing is known about the $\mathcal{L}$-class alternative, the tests based on $M^*_{n,\alpha}$ with values of α around zero can be recommended since they distribute their power more evenly over the range of alternatives.

To compare our results with the simulation study in Basu and Mitra (2002), consider for example the power of the tests against the $W(1.5)$-alternative for $\alpha = -0.1$ and -0.4. Performed with the incorrect standardization of Basu and Mitra (2002), the power is 21% and 0% for $\alpha = -0.1$ and -0.4, respectively.

If one uses the correct standardization together with the asymptotic critical values, one obtains 76% and 0.2% for $\alpha = -0.1$ and -0.4. While a percentage of rejection of 76% for $\alpha = -0.1$ is comparable with 82% in Table 5, this is definitely not the case for $\alpha = -0.4$.

Hence, the reason for the poor performance of the proposed tests in Tables II and III of Basu and Mitra (2002) lies partly in the incorrect limiting distribution, but the sharp decline of power of the tests against Weibull and gamma alternatives if α tends to $-1/2$ observed by them is explained by the very slow convergence of the empirical quantiles to their asymptotic values for $\alpha = -0.4$. In comparison with this observation, the actual decrease of power for $\alpha = -0.4$ compared with $\alpha = -0.1$ is quite small.

Performed with the empirical critical values, the tests based on the fractional sample moments are competitive procedures for testing exponentiality against the $\mathcal{L}$-class of life distributions.

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