Tests for exponentiality against the $\mathcal{M}$ and $\mathcal{LM}$-classes of life distributions

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Abstract

This paper studies tests for exponentiality against the nonparametric classes $\mathcal{M}$ and $\mathcal{LM}$ of life distributions introduced by Klar and Müller (2003). The test statistics are integrals of the difference between the empirical moment generating function of given data and the moment generating function of a fitted exponential distribution. We derive the limit distributions of the test statistics in case of a general underlying distribution and the local approximate Bahadur efficiency of the procedures against several parametric families of alternatives to exponentiality. The finite sample behavior of the tests is examined by means of a simulation study. Finally, the tests under discussion are applied to two data sets, and we discuss the applicability of the tests under random censorship.

Key words: $\mathcal{M}$-class, $\mathcal{LM}$-class, moment generating function order, Exponential distribution, goodness-of-fit test, local approximate Bahadur efficiency.

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1 Introduction

Notions of positive aging play an important role in reliability theory, survival analysis and other fields. Therefore a multitude of classes of distributions describing aging have been introduced in the literature. Likewise, a large number of tests for exponentiality against special classes of life distributions have been proposed. The rationale for using

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such tests instead of omnibus tests for exponentiality is well-known: omnibus goodness-of-fit tests distribute their power over a rather small set of alternatives. Therefore, if one has some knowledge about the class of distributions which may occur, it is reasonable to use tests that are well adapted to detect the possible alternatives. On the other hand, it may be even more dangerous to overly restrict the set of possible alternatives. Hence, using a fairly large class of aging distributions seems to be a reasonable compromise.

One of the more commonly used classes is the so called $\mathcal{L}$-class. It was introduced by Klefsjö (1983) as a large class of distributions that contains most of previously known classes like IFR, DMRL, NBU, NBUE or HNBUE. Tests for exponentiality against the $\mathcal{L}$-class have been proposed by Chaudhuri (1997), Henze and Klar (2001), Basu and Mitra (2002) and Klar (2003). However, Klar (2002) gives an example of a distribution belonging to class $\mathcal{L}$ having an infinite third moment and a hazard rate that tends to zero as time approaches infinity. This example leads to serious doubts whether the $\mathcal{L}$-class should be considered as a reasonable notion of positive aging.

Therefore, in Klar and Müller (2003), a new so-called $\mathcal{M}$-class of life distributions is presented. It is defined similarly as the $\mathcal{L}$-class; only the ordering of the Laplace transforms is replaced by the ordering of the moment generating functions. The $\mathcal{M}$-class also contains the above listed classes of life distributions. However, it does not have the property of the $\mathcal{L}$-class mentioned above. In fact, if F is an absolutely continuous distribution with hazard rate r and mean μ belonging to the $\mathcal{M}$-class, then

$$\limsup_{t\to\infty} r(t) \geq \frac{1}{\mu}$$

(Klar and Müller (2003), Corollary 3.1).

Hence, if the data at hand come from an aging distribution, it seems worthwhile to consider tests for exponentiality against the $\mathcal{M}$-class. The formal definition of this class is as follows.

1.1 Definition *a) A non-negative random variable X with mean $\mu = EX > 0$ and distribution function F is said to be in the $\mathcal{M}$-class if*

$$Ee^{tX} = \int_0^\infty e^{tx} F(\mathrm{d}x) \leq \frac{1}{1-\mu t} \qquad \text{for all } 0 \leq t < 1/\mu. \tag{1}$$

Here, $M(t, 1/\mu) = (1-\mu t)^{-1}$ is the moment generating function of an exponential distribution with mean μ.

b) X is said to be in the $\mathcal{L}$-class, if

$$Ee^{-tX} = \int_0^\infty e^{-tx} F(\mathrm{d}x) \leq \frac{1}{1+\mu t} \qquad \text{for all } t \geq 0.$$

c) X is said to be in the $\mathcal{LM}$-class, if it is in the $\mathcal{L}$-class and in the $\mathcal{M}$-class. For the $\mathcal{LM}$-class the inequality in (1) *must hold for all $-\infty < t < 1/\mu$.*

This paper proposes tests for exponentiality against the classes $\mathcal{M}$ and $\mathcal{LM}$. The procedures are based on the empirical moment generating function

$$M_n(t) \;=\; \int_0^\infty e^{tX} dF_n(x) \;=\; \frac{1}{n}\sum_{i=1}^n e^{tX_i},$$

of a random sample $X_1, \dots, X_n$ of size n from F, where $F_n(x) = n^{-1}\sum_{j=1}^n \mathbf{1}\{X_j \le x\}$ is the empirical distribution function.

Let $M_F(t) = E[e^{tX}]$ denote the moment generating function of X, and write $F(t, \lambda) = 1 - e^{-\lambda t}$, $t \ge 0$, for the distribution function of an exponential distribution with mean $\mu = 1/\lambda$. In view of (1), it seems natural to base a test of

$$H_0: \; F \in \mathcal{E} = \{F(\cdot, \lambda), \; \lambda > 0\}$$

against the alternative

$$H_1: \; F \in \mathcal{M} \quad \text{and} \quad F \notin \mathcal{E}$$

or

$$H_2: \; F \in \mathcal{LM} \quad \text{and} \quad F \notin \mathcal{E}$$

on the empirical counterpart $M_n(t) - M(t, 1/\overline{X}_n)$ of $M_F(t) - M(t, 1/\mu)$. Here, $\overline{X}_n = 1/n\sum_{i=1}^n X_i$ denotes the sample mean.

As a class of test statistics for a test against H_1 we propose $(T_{n,a})$, where

$$T_{n,a} \;=\; \overline{X}_n \int_0^{a/\overline{X}_n} \left(M_n(t) - M(t, 1/\overline{X}_n)\right)\, dt, \tag{2}$$

and $a \in (0, 1/2)$ is a positive constant. Since, for $t \in [0, 1/\mu)$, $M_F(t) - M(t, 1/\mu)$ is nonpositive for alternatives from the $\mathcal{M}$-class, H_0 is rejected for large *negative* values of $T_{n,a}$. The statistic $T_{n,a}$ takes the form

$$T_{n,a} \;=\; \frac{1}{n}\sum_{j=1}^n \frac{\exp(aY_j) - 1}{Y_j} \;+\; \log(1-a), \tag{3}$$

where $Y_j = X_j/\overline{X}_n, \; 1 \le j \le n$.

A class of test statistics for a test against H_2 is $(\tilde{T}_{n,a})_{0<a<1/2}$, where

$$\begin{aligned} \tilde{T}_{n,a} &= \overline{X}_n \int_{-a/\overline{X}_n}^{a/\overline{X}_n} \left(M_n(t) - M(t, 1/\overline{X}_n) \right) \, dt \\ &= \frac{1}{n} \sum_{j=1}^{n} \frac{\exp(aY_j) - \exp(-aY_j)}{Y_j} \; + \; \log\left(\frac{1-a}{1+a} \right). \end{aligned} \tag{4}$$

1.2 Remark If Y_j is zero, then $(\exp(aY_j) - 1)/Y_j$ in (3) has to be replaced by its limiting value a. Similarly, $(\exp(aY_j) - \exp(-aY_j))/Y_j$ has to be replaced by $2a$. Note that it is possible that an $\mathcal{M}$-class distribution has a point mass at zero. In contrast, one has $X > 0$ almost surely for distributions in class $\mathcal{L}$ or $\mathcal{LM}$.

A test statistic based on $\sup_t |M_n(t) - M(t, 1/\overline{X}_n)|$ has been studied in Csörgő and Welsh (1989) as omnibus tests for exponentiality. However, there don't exist simple computational formulas for this and corresponding one sided statistics, contrary to the usual supremum statistics based on the empirical distribution function. For this reason, we prefer integral test statistics as defined above.

The paper is organized as follows. In Section 2 we state the asymptotic behavior of the statistics $T_{n,a}$ and $\tilde{T}_{n,a}$ as $a \to 0$ and derive their limit distributions in case of a general underlying distribution. A test for exponentiality rejecting H_0 for large negative values of $T_{n,a}$ and $\tilde{T}_{n,a}$ is seen to be consistent against each fixed alternative from the class $\mathcal{M}$ and $\mathcal{LM}$, respectively. Section 3 is devoted to the calculation of local approximate Bahadur efficiencies of the proposed tests of exponentiality with respect to some standard families of alternative distributions from the class $\mathcal{LM}$. In Sections 4, we present the results of a simulation study. The tests are applied to two data sets in Section 5. In the last section, we discuss the applicability of the tests under random censorship.

2 The asymptotic distribution of the test statistics

Our first result shows that both $T_{n,a}$ and $\tilde{T}_{n,a}$, when suitably scaled, approach the same limit if the parameter a tends to 0.

2.1 Proposition *For fixed n, we have*

$$T_n \;\equiv\; \lim_{a \to 0} \left(\frac{2T_{n,a}}{a^3} + \frac{2}{3} \right) \;=\; \lim_{a \to 0} \left(\frac{\tilde{T}_{n,a}}{a^3} + \frac{2}{3} \right) \;=\; \frac{1}{3n} \sum_{j=1}^{n} Y_j^2 .$$

PROOF: Series expansions yield

$$\lim_{a\to 0}\left(\frac{T_{n,a}}{a^3}+\frac{1}{3}\right) = \lim_{a\to 0}\left(\frac{1}{n}\sum_{j=1}^{n}\frac{\exp(aY_j)-1}{a^3Y_j}+\frac{\log(1-a)}{a^3}+\frac{1}{3}\right)$$
$$= \lim_{a\to 0}\left(\frac{1}{n}\sum_{j=1}^{n}\left(\frac{1}{a^2}+\frac{Y_j}{2a}+\frac{Y_j^2}{6}+O(a)\right)-\frac{1}{a^3}\left(a+\frac{a^2}{2}+\frac{a^3}{3}+O(a^4)\right)+\frac{1}{3}\right)$$
$$= \frac{1}{n}\sum_{j=1}^{n}\frac{Y_j^2}{6}.$$

The proof of the second assertion is similar. ■

Up to one-to-one transformations, T_n coincides with Greenwood's statistic $G_n = 1/n^2\sum_{j=1}^{n}Y_j^2$ (Greenwood (1946)), with the sample coefficient of variation $CV_n = S_n/\overline{X}_n$, where $S_n^2 = n^{-1}\sum_{j=1}^{n}(X_j-\overline{X}_n)^2$ denotes the sample variance, and with the first nonzero component of Neyman's smooth test of fit for exponentiality (see, e.g., Koziol (1987)); it is asymptotically most powerful for testing H_0 against the linear failure rate distribution (Doksum and Yandell (1984)).

However, since the exponential distribution is not characterized within the $\mathcal{LM}$-class by the property that $V(X) = (EX)^2$ (Klar and Müller (2003)), a test based on T_n is not consistent against the $\mathcal{M}$- or $\mathcal{LM}$-class.

It is well-known that $\sqrt{n}(3T_n/2-1)$ has a limiting unit normal distribution under H_0. The next theorem gives the asymptotic distribution of $T_{n,a}$ for $0 < a < 1/2$. Since the representation of $T_{n,a}$ in (3) shows that $T_{n,a}$ is scale-invariant, we assume $\mu = 1$ in the following. For simplicity, we further assume $P(X_j > 0) = 1$. Otherwise, expressions like $(\exp(ax)-1)/x$ have to be replaced by $(\exp(ax)-1)/x\cdot I\{x>0\}+a\cdot I\{x=0\}$.

2.2 Theorem *Assume $X_1,\ldots,X_n$ is a random sample of a nonnegative nondegenerate random variable X with $E\exp(cX) < \infty$ for $c < 1$. Further, let $0 < a < 1/2$. Then, as $n\to\infty$,*

$$\sqrt{n}\left(\frac{1}{n}\sum_{j=1}^{n}\frac{\exp(aY_j)-1}{Y_j}-E\left(\frac{\exp(aX)-1}{X}\right)\right)\xrightarrow{\mathcal{D}}\mathcal{N}(0,\sigma^2),$$

where

$$\sigma^2 = E\left(\kappa_1(X-1)+\frac{\exp(aX)-1}{X}-\mu_1\right)^2 \tag{5}$$

and

$$\kappa_1 = E\left(\frac{(1-aX)\exp(aX)-1}{X}\right), \qquad \mu_1 = E\left(\frac{\exp(aX)-1}{X}\right). \tag{6}$$

Under H_0, we have $\sqrt{n}\,T_{n,a} \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma_0^2)$, where

$$\sigma_0^2 = (1-2a)\log(1-2a) - 2\left(1-a+\frac{a}{1-a}\right)\log(1-a) - 2(\log(1-a))^2 - \frac{a^2}{(1-a)^2}. \tag{7}$$

PROOF: Notice that

$$\sqrt{n}\left(\frac{1}{n}\sum_{j=1}^{n}\frac{\exp(aY_j)-1}{Y_j} - E\left(\frac{\exp(aX)-1}{X}\right)\right) = U_{n,1}+U_{n,2},$$

where

$$U_{n,1} = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(\frac{\exp(aX_j)-1}{X_j} - \mu_1\right), \quad U_{n,2} = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(\frac{\exp(aY_j)-1}{Y_j} - \frac{\exp(aX_j)-1}{X_j}\right).$$

A Taylor expansion of the function $g(t) = (\exp(aX_j/t)-1)/(X_j/t)$ around $t=1$ yields

$$\begin{aligned} U_{n,2} &= \sqrt{n}\left(\overline{X}_n - 1\right)\frac{1}{n}\sum_{j=1}^{n}\frac{(1-aX_j)\exp(aX_j)-1}{X_j} \;+\; o_P(1) \\ &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}(X_j-1)\,\kappa_1 \;+\; R_n \;+\; o_P(1), \end{aligned}$$

where

$$R_n = \sqrt{n}\left(\overline{X}_n - 1\right)\left(\frac{1}{n}\sum_{j=1}^{n}\frac{(1-aX_j)\exp(aX_j)-1}{X_j} - \kappa_1\right).$$

Since $R_n \xrightarrow{P} 0$, we obtain

$$U_{n,1}+U_{n,2} = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(\frac{\exp(aX_j)-1}{X_j} - \mu_1 \;+\; (X_j-1)\,\kappa_1\right) \;+\; o_P(1).$$

By the Central limit theorem and Slutsky's lemma, $U_{n,1}+U_{n,2} \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma^2)$, where σ^2 is given in (5). The formula (7) for σ^2 in case of H_0 follows from straightforward calculations. ■

As a consequence, the asymptotic distribution of $T^*_{n,a} = \sqrt{n}\,T_{n,a}/\sigma_0$ is standard normal under the hypothesis of exponentiality.

The next result states the asymptotic distribution of $\tilde{T}_{n,a}$. The proof follows the reasoning given above and will thus be omitted.

2.3 Theorem *Assume $X_1, \dots, X_n$ is a random sample of a nonnegative nondegenerate random variable X with $E\exp(cX) < \infty$ for $c < 1$. Further, let $0 < a < 1/2$. Then, as $n \to \infty$,*

$$\sqrt{n}\left(\frac{1}{n}\sum_{j=1}^{n}\frac{\exp(aY_j) - \exp(-aY_j)}{Y_j} - \mu_2\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tilde{\sigma}^2),$$

where

$$\mu_2 = E\left(\frac{\exp(aX) - \exp(-aX)}{X}\right),$$
$$\kappa_2 = E\left(\frac{(1-aX)\exp(aX) - (1+aX)\exp(-aX)}{X}\right)$$

and

$$\tilde{\sigma}^2 = E\left(\kappa_2(X-1) + \frac{\exp(aX) - \exp(-aX)}{X} - \mu_2\right)^2.$$

Under H_0, we have $\sqrt{n}\,\tilde{T}_{n,a} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tilde{\sigma}_0^2)$, *where*

$$\begin{aligned}\tilde{\sigma}_0^2 &= \frac{4a}{1-a^2}\log\frac{1+a}{1-a} - 2\left(\log\frac{1+a}{1-a}\right)^2 - \frac{4a^2}{(1-a^2)^2} \\ &\quad + (1-2a)\log(1-2a) + (1+2a)\log(1+2a). \end{aligned} \tag{8}$$

We now consider the behavior of $T_{n,a}$ and $\tilde{T}_{n,a}$, $0 < a < 1/2$, under fixed alternatives to H_0, with the aim of showing the consistency of the corresponding tests. To this end, suppose that the distribution of X is from the $\mathcal{M}$-class; in particular, X has finite positive expectation μ. We then have $M(t, 1/\overline{X}_n) \to M(t, 1/\mu)$ almost surely, and by Fatou's Lemma, it follows that

$$\liminf_{n\to\infty}\frac{-T_{n,a}}{n} \;\geq\; \mu\int_0^{a/\mu}\left(M(t,1/\mu) - E[\exp(tX)]\right)dt \tag{9}$$

almost surely. Since the behaviour of the moment generating function on any interval $[0, a]$, $0 < a < 1/\mu$, completely specifies H_0, the right-hand side of (9) is positive if the distribution of X comes from class $\mathcal{M}$ but is not exponential. Hence, a test that rejects H_0 for small values of $T_{n,a}$ is consistent against any such alternative.

The same reasoning yields the consistency of the test based on $\tilde{T}_{n,a}$ against alternatives from the $\mathcal{LM}$-class.

Note that the property of consistency of both tests continues to hold under the condition $EX < \infty$ and $Ee^{t_0X} = \infty$ where $t_0 < a/\mu$, since, in this case, we have $M_n(t) \to \infty$ almost surely for $t \geq t_0$. An example is the Lognormal distribution which is used in the simulation study in Section 4 as alternative distribution.

3 Local approximate Bahadur efficiency

In this section, we investigate the efficiency of the tests for exponentiality based on $T_{n,a}$ and $\tilde{T}_{n,a}$ against several one-parametric families of distributions from the class $\mathcal{LM}$. In each case, the distributions have a density function $f(x, \vartheta)$, and the parameter space, denoted by Θ, is some subinterval of $(0, \infty)$. Depending on the specific alternative family, the unit exponential distribution corresponds either to the parameter value $\vartheta_0 = 1$ or to the value $\vartheta_0 = 0$. As in Henze and Klar (2001), our measure of efficiency is the local approximate Bahadur slope (see, e.g., Nikitin (1995), Section 1.2).

First, note that $P_{\vartheta_0}(T^*_{n,a} \leq t) = P_{\vartheta_0}(\sqrt{n}T_{n,a}/\sigma_0 \leq t) \to \Phi(t)$ as $n \to \infty$, where Φ denotes the standard normal distribution function. Further, $\log(1 - \Phi(t)) \sim t^2/2,\ t \to \infty$. Since

$$\frac{T^*_{n,a}}{\sqrt{n}} \xrightarrow{P} \frac{1}{\sigma_0}\left\{E_\vartheta\left(\frac{\exp(aX/\mu(\vartheta)) - 1}{X/\mu(\vartheta)}\right) - E_{\vartheta_0}\left(\frac{\exp(aX) - 1}{X}\right)\right\}$$

under P_ϑ, where $\mu(\vartheta) = E_\vartheta[X]$, the approximate Bahadur slope $c^*(\cdot, a)$ of the sequence $(T^*_{n,a})$ of test statistics is (Nikitin (1995), p. 10)

$$c^*(\vartheta, a) = \left[\frac{1}{\sigma_0}\left\{E_\vartheta\left(\frac{\exp(aX/\mu(\vartheta)) - 1}{X/\mu(\vartheta)}\right) - E_{\vartheta_0}\left(\frac{\exp(aX) - 1}{X}\right)\right\}\right]^2.$$

We now consider the local behavior of $c^*(\vartheta, a)$ as $\vartheta \to \vartheta_0$, assuming the one-parametric family of alternative distributions to be sufficiently regular to allow a Taylor expansion of order two of $c^*(\vartheta, a)$ with respect to ϑ. Moreover, differentiation of $E_\vartheta[(\exp(aX/\mu(\vartheta)) - 1)/(X/\mu(\vartheta))]$ may be done under the integral sign. These assumptions hold for each of the four families of distributions considered later in this section. After straightforward calculations, one obtains

$$c^*(\vartheta, a) \sim \frac{(l_{F_\vartheta}(T_{n,a}))^2}{\sigma_0^2}(\vartheta - \vartheta_0)^2 \quad \text{as } \vartheta \to \vartheta_0,$$

where

$$l_{F_\vartheta}(T_{n,a}) = \int_0^\infty \frac{\exp(aX) - 1}{X} \frac{\partial}{\partial\vartheta} f(x, \vartheta)\bigg|_{\vartheta=\vartheta_0} dx - \mu'(\vartheta_0)\left(\frac{a}{1-a} + \log(1-a)\right),$$

and σ_0^2 is given in (7). Here, $\mu'(\vartheta_0) = \frac{\partial}{\partial\vartheta}\mu(\vartheta)|_{\vartheta=\vartheta_0}$. Then, the local approximate Bahadur efficiency of $T_{n,a}$ is given by

$$e_{F_\vartheta}(T_{n,a}) = \frac{(l_{F_\vartheta}(T_{n,a}))^2}{\sigma_0^2}.$$

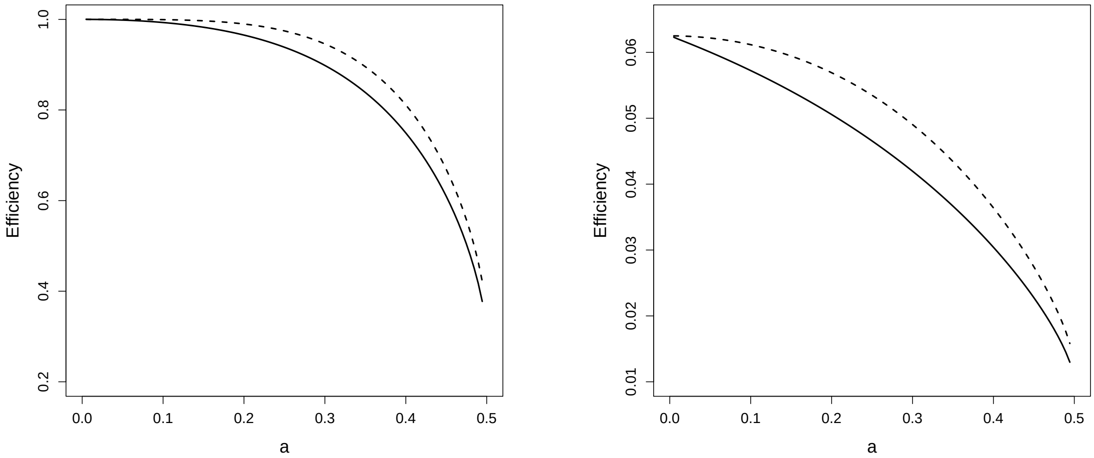


Figure 1: Local approximate Bahadur efficiency of $T_{n,a}$ (solid line) and $\tilde{T}_{n,a}$ (dashed) against LFR (left) and Makeham alternatives (right)

Similarly, we have $e_{F_\vartheta}(\tilde{T}_{n,a}) = \left(\tilde{l}_{F_\vartheta}(T_{n,a})\right)^2 / \tilde{\sigma}_0^2$, where

$$\begin{aligned} \tilde{l}_{F_\vartheta}(T_{n,a}) &= \int_0^\infty \frac{\exp(aX) - \exp(-aX)}{X} \frac{\partial}{\partial\vartheta} f(x,\vartheta)\bigg|_{\vartheta=\vartheta_0} dx \\ &\quad - \mu'(\vartheta_0)\left(\frac{2a}{1-a^2} - \log\frac{1+a}{1-a}\right), \end{aligned}$$

and $\tilde{\sigma}_0^2$ is given in (8). We have calculated $e_F(T_{n,a})$ and $e_{F_\vartheta}(\tilde{T}_{n,a})$ for linear failure rate (LFR), Makeham, Weibull and gamma alternatives. All of them belong to the much narrower class of IFR distributions. The pertaining distribution functions are

$$\begin{aligned} &F_\vartheta^{(1)}(x) = 1 - \exp\left(-\left(x + \vartheta x^2/2\right)\right) \quad \text{for } x \ge 0, \vartheta \ge 0, \\ &F_\vartheta^{(2)}(x) = 1 - \exp\left(-\left(x + \vartheta\left(x + e^{-x} - 1\right)\right)\right) \quad \text{for } x \ge 0, \vartheta \ge 0, \\ &F_\vartheta^{(3)}(x) = 1 - \exp\left(-x^\vartheta\right) \quad \text{for } x \ge 0, \vartheta > 0, \\ &F_\vartheta^{(4)}(x) = \Gamma(\vartheta)^{-1} \int_0^x t^{\vartheta-1} e^{-t}\, dt \quad \text{for } x \ge 0, \vartheta > 0, \end{aligned}$$

respectively. For $F_\vartheta^{(1)}$ and $F_\vartheta^{(2)}$, H_0 corresponds to $\vartheta = \vartheta_0 = 0$, and for $F_\vartheta^{(3)}$ and $F_\vartheta^{(4)}$, we have $\vartheta_0 = 1$.

Calculations give

$$l_{F^{(1)}}(T_{n,a}) = \frac{a}{1-a} - \frac{a^2}{2(1-a)^2} + \log(1-a)$$

and

$$l_{F^{(1)}}(\tilde{T}_{n,a}) = \frac{2a}{1-a^2} - \frac{2a^3}{(1-a^2)^2} - \log\frac{1+a}{1-a}$$

for $0 < a < 1/2$. The efficiencies $e_{F^{(1)}}(T_{n,a})$ and $e_{F^{(1)}}(\tilde{T}_{n,a})$ are plotted in Figure 1 (left). They have a maximum value at $a^* = 0$ with $e_{F^{(1)}} = 1$. As noted above, the test based on T_n is asymptotically most powerful for testing H_0 against the linear failure rate distribution.

Next, we obtain

$$l_{F^{(2)}}(T_{n,a}) = 2\log\frac{2-a}{2} - \frac{3}{2}\log(1-a) - 2\log 2 - \frac{a}{2(1-a)}$$

and

$$l_{F^{(2)}}(\tilde{T}_{n,a}) = 2\log\frac{2-a}{2+a} + \frac{3}{2}\log\frac{1+a}{1-a} - \frac{a}{1-a^2}$$

for $0 < a < 1/2$. $e_{F^{(2)}}$ has a maximum value at $a^* = 0$ with $e_{F^{(2)}}(T_{n,a^*}) = e_{F^{(2)}}(\tilde{T}_{n,a^*}) = 1/16$. Note that $1/12$ is the efficiency of the asymptotically most powerful test of exponentiality against the Makeham distribution (Doksum and Yandell (1984)). Figure 1 (right) shows the local approximate Bahadur efficiencies of $T_{n,a}$ and $\tilde{T}_{n,a}$ against Makeham alternatives.

For $F_\vartheta^{(3)}$, closed form expressions for $l_{F^{(3)}}(T_{n,a})$ and $l_{F^{(3)}}(\tilde{T}_{n,a})$ do not exist; we have

$$l_{F^{(3)}}(T_{n,a}) = \int_0^\infty \frac{e^{ax}-1}{x}\left(1+(1-x)\log(x)\right)e^{-x}\,dx + (1-\gamma)\left(\frac{a}{1-a} + \log(1-a)\right)$$

and

$$l_{F^{(3)}}(\tilde{T}_{n,a}) = \int_0^\infty \frac{e^{ax}-e^{-ax}}{x}\left(1+(1-x)\log(x)\right)e^{-x}\,dx + (1-\gamma)\left(\frac{2a}{1-a^2} - \log\frac{1+a}{1-a}\right)$$

for $0 < a < 1/2$, where $\gamma \approx 0.577$ is Euler's constant. The maximum value of $e_{F^{(3)}}$ is 1 at $a^* = 0$.

For the family of gamma alternatives, the efficiencies are

$$l_{F^{(4)}}(T_{n,a}) = \int_0^\infty \frac{e^{ax}-1}{x}\left(\log x + \gamma\right)e^{-x}\,dx - \frac{a}{1-a} - \log(1-a)$$

and

$$l_{F^{(4)}}(\tilde{T}_{n,a}) = \int_0^\infty \frac{e^{ax}-e^{-ax}}{x}\left(\log x + \gamma\right)e^{-x}\,dx - \frac{2a}{1-a^2} + \log\frac{1+a}{1-a}$$

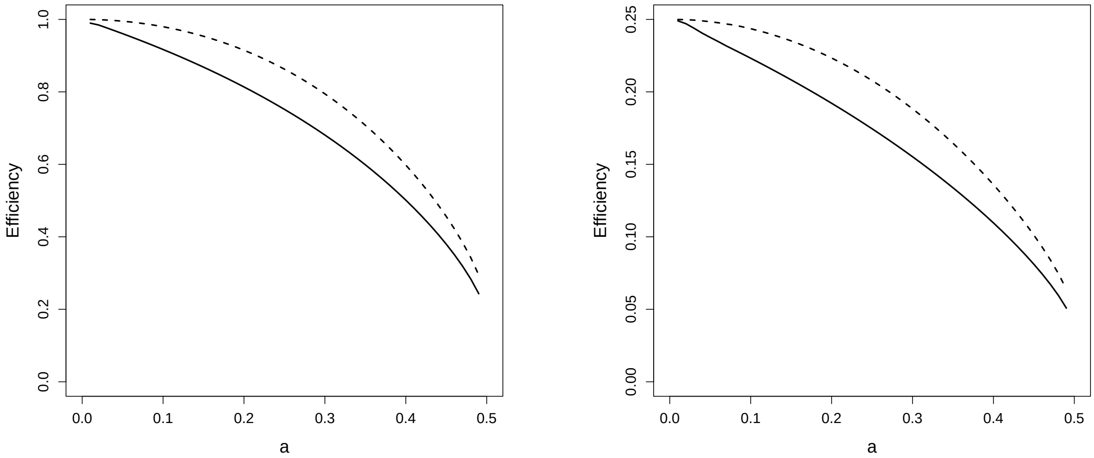


Figure 2: Local approximate Bahadur efficiency of $T_{n,a}$ (solid line) and $\tilde{T}_{n,a}$ (dashed) against Weibull (left) and gamma alternatives (right)

for $0 < a < 1/2$. Again, $e_{F^{(4)}}$ has a maximum value at 0; here, $e_{F^{(4)}}(T_n) = e_{F^{(4)}}(\tilde{T}_{n,0}) = 1/4$. Figure 2 displays the efficiencies of $T_{n,a}$ and $\tilde{T}_{n,a}$ against Weibull and Gamma alternatives for $a \in (0, 1/2)$.

Obviously, local approximate Bahadur efficiency heavily depends on the value of a. The shape of the curves in Figures 1 and 2 is always similar: for each of the four distributions, efficiency decreases if a tends to 1/2. However, for fixed positive value of a, the efficiency of $\tilde{T}_{n,a}$ is always above the efficiency of $T_{n,a}$.

The decrease of Bahadur efficiency with increasing a can be explained as follows. Since the tail behavior of a distribution is reflected on the behavior of its moment generating function around zero, choosing a small value of a renders the test statistics powerful against distributions with markedly different tail behaviour than those in $\mathcal{E}$. This is the case for the IFR distribution used in this section. Hence, tests looking only at the mean/variance ratio as T_n are well suited for these particular alternatives.

4 Simulations

This section presents the results of two Monte Carlo studies. The first simulation study was conducted in order to obtain critical points of the statistics under discussion which is necessary due to the slow convergence of the finite sample distributions of the test statistics to their limit distribution.

Tables 1 and 2 show the p-quantiles of $T^*_{n,a} = \sqrt{n}T_{n,a}/\sigma_0$ and $\tilde{T}^*_{n,a} = \sqrt{n}\tilde{T}_{n,a}/\tilde{\sigma}_0$

a	$T^*_{n,a}$						$\tilde{T}^*_{n,a}$					
	0.05	0.1	0.2	0.3	0.4	0.45	0.05	0.1	0.2	0.3	0.4	0.45
$n = 10$	-1.05	-0.99	-0.89	-0.76	-0.60	-0.49	-1.08	-1.07	-1.00	-0.88	-0.70	-0.58
$n = 20$	-1.18	-1.13	-1.02	-0.88	-0.70	-0.59	-1.21	-1.19	-1.13	-1.00	-0.81	-0.68
$n = 30$	-1.24	-1.20	-1.09	-0.95	-0.77	-0.65	-1.28	-1.26	-1.19	-1.07	-0.88	-0.73
$n = 50$	-1.32	-1.28	-1.17	-1.04	-0.85	-0.72	-1.35	-1.34	-1.27	-1.15	-0.95	-0.81
$n = 100$	-1.40	-1.36	-1.27	-1.14	-0.96	-0.82	-1.43	-1.41	-1.36	-1.24	-1.05	-0.90
$n = 200$	-1.46	-1.44	-1.36	-1.24	-1.05	-0.91	-1.49	-1.48	-1.42	-1.32	-1.14	-0.99
$n = 500$	-1.53	-1.50	-1.44	-1.35	-1.17	-1.02	-1.55	-1.54	-1.51	-1.41	-1.25	-1.09
$n = 1000$	-1.57	-1.55	-1.50	-1.41	-1.25	-1.10	-1.58	-1.57	-1.53	-1.47	-1.31	-1.17

Table 1: Empirical 5%-quantiles of $T^*_{n,a}$ and $\tilde{T}^*_{n,a}$ based on 100000 replications

a	$T^*_{n,a}$						$\tilde{T}^*_{n,a}$					
	0.05	0.1	0.2	0.3	0.4	0.45	0.05	0.1	0.2	0.3	0.4	0.45
$n = 10$	-0.95	-0.90	-0.89	-0.70	-0.55	-0.46	-0.97	-0.96	-0.90	-0.80	-0.64	-0.53
$n = 20$	-1.04	-1.00	-0.91	-0.79	-0.64	-0.54	-1.06	-1.05	-0.99	-0.89	-0.73	-0.61
$n = 30$	-1.08	-1.04	-0.96	-0.85	-0.69	-0.58	-1.11	-1.09	-1.04	-0.94	-0.78	-0.66
$n = 50$	-1.13	-1.10	-1.02	-0.91	-0.75	-0.64	-1.15	-1.14	-1.09	-0.99	-0.83	-0.71
$n = 100$	-1.18	-1.16	-1.08	-0.98	-0.83	-0.71	-1.20	-1.19	-1.14	-1.06	-0.91	-0.78
$n = 200$	-1.21	-1.20	-1.14	-1.05	-0.90	-0.78	-1.23	-1.22	-1.18	-1.11	-0.97	-0.84
$n = 500$	-1.25	-1.23	-1.18	-1.12	-0.99	-0.87	-1.25	-1.25	-1.23	-1.16	-1.04	-0.92
$n = 1000$	-1.26	-1.25	-1.22	-1.16	-1.04	-0.93	-1.26	-1.26	-1.24	-1.20	-1.09	-0.97

Table 2: Empirical 10%-quantiles of $T^*_{n,a}$ and $\tilde{T}^*_{n,a}$ based on 100000 replications

under exponentiality for several sample sizes and $p = 0.05$ and 0.10, respectively. The parameter a was chosen to be $a = 0.05, 0.1, 0.2, 0.3, 0.4$ and 0.45. The entries in Tables 1 and 2 are based on 100000 replications; here, we always used $\mu = 1$.

The speed of convergence to the asymptotic values is generally quite low, and it differs for different values of a. The asymptotic quantiles should not be used for a test based on $T^*_{n,a}$ or $\tilde{T}^*_{n,a}$ even for the sample size $n = 1000$. The finite sample quantiles are not symmetric around 0.

A second simulation study has been conducted to examine the dependence of the power of the tests on the exponent. As alternative distributions from the $\mathcal{LM}$-class, we used the Weibull, Gamma and Linear failure rate distribution with scale parameter 1 and shape parameter ϑ, denoted by $W(\vartheta)$, $\Gamma(\vartheta)$ and $LFR(\vartheta)$, respectively. A further alternative is the inverse Gaussian distribution $IG(\mu, \lambda)$ with mean $\mu = 1$, denoted by $IG(\lambda)$ which belongs to the $\mathcal{L}$-class for $\lambda \geq 1$; it belongs to the $\mathcal{M}$-class (and, hence, to the $\mathcal{LM}$-class) if $\lambda \geq 2$.

In addition, we included the Birnbaum-Saunders distribution $BS(\gamma, \delta)$ with $\delta = (\gamma^2/2 + 1)^{-1}$ (and, hence, a mean value of one), denoted by $BS(\gamma)$. A simple method of generating $BS(\gamma, \delta)$ distributed random variables is to generate a standard normal variate, Z, and then use the formula

$$X = \delta \left(1 + \gamma^2 Z^2/2 + \gamma Z \left(\gamma^2 Z^2/4 + 1\right)^{1/2}\right)$$

(Rieck (2003)). The Birnbaum-Saunders distribution $BS(\gamma)$ belongs to the $\mathcal{L}$-class for $\gamma \leq 1$; it is a member of the class $\mathcal{M}$ if $\gamma \leq \sqrt{2/3}$ (Klar and Müller (2003)).

We finally considered the uniform distribution $U(0, 1)$ which is symmetric and, hence, in the $\mathcal{M}$-class (Corollary 3.2 in Klar and Müller (2003)) and the Lognormal distribution $LN(0, \tau^2)$. The latter is not in the $\mathcal{M}$-class, but it belongs to the $\mathcal{L}$-class if $\tau^2 < \log 2$ (Klar (2002)). By the remark at the end of Section 2, the tests are consistent against this alternative.

Table 3 shows power estimates of the tests based on $T^*_{n,a}$ and $\tilde{T}^*_{n,a}$ for $n = 30$. All entries are the percentages of 10000 Monte Carlo samples that resulted in rejection of H_0, rounded to the nearest integer. The nominal level of the test is $\alpha = 0.05$.

The first three lines of Table 4 show three examples of power estimates of "contiguous" alternatives for $n = 500$.

The results for the binomial distribution $Bin(1, 1/2)$ are given in the last line of Table 4. This distribution is symmetric and, hence, in the $\mathcal{M}$-class. However, since $V(X) = (EX)^2 = 1/4$, a test based on T^*_n or on $T^*_{n,a}$ for small values of a can not detect this alternative. The values in Table 4 are computed as in Table 3.

The main conclusions that can be drawn from the simulation results are the following:

1. The tests based on $T^*_{n,a}$ and $\tilde{T}^*_{n,a}$ behave fairly similar, whereby the power of the tests depends only to a small extent on a. An exception is the binomial distribution.

2. For all distributions used in the simulation apart from the binomial distribution, power decreases for increasing values of a.

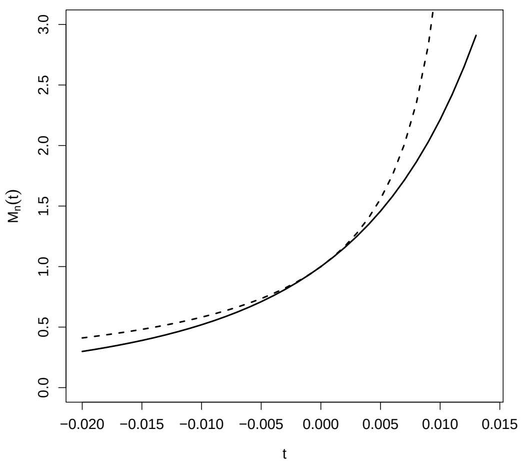
3. Despite the strong decrease of local approximate Bahadur efficiency for $W(\vartheta)$, $\Gamma(\vartheta)$ and $LFR(\vartheta)$ as a tends to $1/2$, the results in Table 4 indicate that the actual loss of power is small, even for $n = 500$.

	$T^*_{n,a}$						$\tilde{T}^*_{n,a}$					
a	0.05	0.1	0.2	0.3	0.4	0.45	0.05	0.1	0.2	0.3	0.4	0.45
$Exp(1)$	5	5	5	5	5	5	5	5	5	5	5	5
$W(1.3)$	44	45	44	43	43	42	45	44	45	44	43	45
$W(1.5)$	78	77	77	77	76	75	78	79	79	78	77	78
$W(1.7)$	95	95	95	95	94	93	96	95	95	95	94	95
$\Gamma(1.5)$	34	35	33	33	33	31	34	35	34	34	33	34
$\Gamma(2.0)$	70	69	69	68	67	64	70	71	70	69	69	69
$\Gamma(3.0)$	97	97	97	96	96	95	97	97	97	97	96	97
$LFR(0.5)$	26	25	25	26	25	25	25	26	25	25	25	26
$LFR(2.0)$	61	60	60	61	59	58	60	60	60	60	59	60
$LFR(5.0)$	81	81	81	81	80	79	81	82	81	81	80	82
$IG(1.2)$	26	26	25	24	23	22	27	26	26	25	25	25
$IG(1.5)$	46	44	43	42	41	39	46	45	45	45	43	44
$IG(2.0)$	73	71	70	69	67	66	72	73	73	72	70	70
$IG(3.0)$	95	95	94	93	92	92	96	96	95	95	94	95
$BS(1.2)$	1	1	1	1	1	1	1	2	2	2	1	2
$BS(1)$	10	8	9	8	9	8	9	9	9	9	9	9
$BS(\sqrt{2/3})$	39	38	37	37	34	32	38	38	37	37	36	37
$BS(0.7)$	70	69	67	66	65	63	70	71	70	69	67	69
$U(0,1)$	98	98	98	98	98	98	97	97	98	98	98	98
$LN(0,0.6)$	35	34	33	32	31	30	36	36	35	34	33	34
$LN(0,0.4)$	76	74	74	72	70	69	76	76	75	74	73	74

Table 3: Empirical power of the tests based on $T^*_{n,a}$ and $\tilde{T}^*_{n,a}$, $\alpha = 0.05$, $n = 30$, 10000 replications

	$T^*_{n,a}$						$\tilde{T}^*_{n,a}$					
a	0.05	0.1	0.2	0.3	0.4	0.45	0.05	0.1	0.2	0.3	0.4	0.45
$W(1.1)$	70.9	69.7	67.9	65.1	64.1	61.3	71.9	71.1	70.7	68.4	66.2	65.1
$\Gamma(1.2)$	67.4	66.3	65.4	60.7	58.7	56.8	69.0	68.5	66.7	65.4	61.2	59.8
$LFR(0.2)$	87.4	87.6	87.3	85.8	86.4	85.7	87.2	86.6	86.8	86.8	86.2	85.7
$Bin(0,1)$	6.8	11.0	25.4	40.9	62.2	72.2	5.3	6.0	7.1	13.1	21.8	31.0

Table 4: Empirical power of the tests based on $T^*_{n,a}$ and $\tilde{T}^*_{n,a}$, $\alpha = 0.05$, $n = 500$, 10000 replications



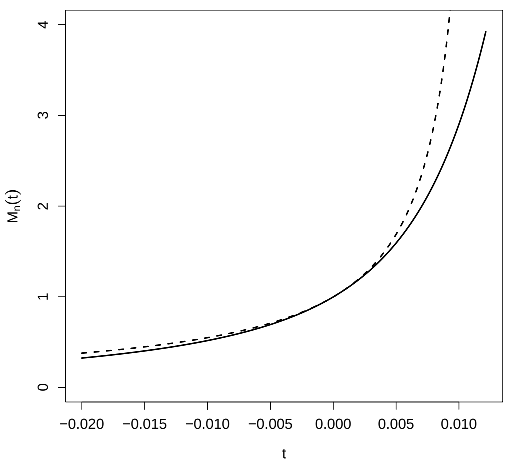


Figure 3: Left: Empirical moment generating function of data set 1 (solid line) and moment generating function under exponentiality (dashed).
Right: The same plots for data set 2.

4. In the majority of cases, the power of $\tilde{T}^*_{n,a}$ in Table 4 is higher than the power of $T^*_{n,a}$ for a fixed value of a which coincides with the results of Section 3.

5 Examples

We applied the tests under discussion to two data sets. The first example was also considered by Pavur et al. (1992). The results recorded in the following table are the number of revolutions (in millions) to failure of $n = 23$ ball bearings in a life test study.

17.88	28.92	33.00	41.52	42.12	45.60	48.48	51.84
51.96	54.12	55.56	67.80	68.64	68.64	69.88	84.12
93.12	98.64	105.12	105.84	127.92	128.04	173.40	

Mean value and standard deviation of the data are 71.8 and 38.2, which results in a coefficient of variation of 0.53. The empirical moment generating function lies below the moment generating function under exponentiality, given by $(1 - 71.8\,t)^{-1}$ (see Figure 3, left). Therefore, it is plausible to assume that the distribution underlying the data belongs to the $\mathcal{LM}$-class.

The second data set consists of $n = 16$ intervals in operating hours between successive failures of airconditioning equipment in a Boeing 720 aircraft (see Edgeman et al. (1988)).

102	209	14	57	54	32	67	59	134	152	27	14	230	66	61	34

The mean value of 82.0 and the standard deviation of 66.3 yield a coefficient of variation of 0.81. The plot on the right-hand side of Figure 3 looks similar as the plot on the left; again, it is plausible to assume that the underlying distribution belongs to the $\mathcal{LM}$-class.

To assess wether the data could come from exponential distributions, we applied the tests based on $T^*_{n,a}$ and $\tilde{T}^*_{n,a}$ for several values of a to the two data sets. The values of the test statistics are given in Table 5.

Comparing the values for the first data set with the 5%-quantiles for $n = 30$ in Table 1, we can see that all tests reject the hypotheses of exponentiality at the level 0.05.

However, the values for the second data set are always larger than the 10%-quantiles for $n = 10$ in Table 2. Hence, the hypotheses of exponentiality is not rejected at the 10%-level.

Therefore, we can retain the exponential model for the second data set, whereas it is necessary to look for other parametric distributions from class $\mathcal{LM}$ for a better description of the data in the first example.

6 The test statistics under random censorship

In this section, we discuss how the test statistics for testing for exponentiality against H_1 have to be modified in the case of randomly censored data. Tests for exponentiality against HNBUE alternatives under random censorship have been proposed by Aly (1992) and Hendi, Al-Nachawati, Montasser, and Alwasel (1998).

Let $X_1, X_2, \dots$ be a sequence of nonnegative independent lifetimes with d.f. F. Along with the X-sequence, let $Y_1, Y_2, \dots$ be a sequence of independent censoring random variables with d.f. G also being independent of the X's. We assume that F and G are continuous. We observe the censored lifetimes $Z_i = \min(X_i, Y_i)$ together with $\delta_i =$

	$T^*_{n,a}$				$\tilde{T}^*_{n,a}$			
a	0.1	0.2	0.3	0.4	0.1	0.2	0.3	0.4
Data set 1 ($n = 23$)	-1.59	-1.41	-1.19	-0.93	-1.71	-1.60	-1.40	-1.11
Data set 2 ($n = 16$)	-0.73	-0.67	-0.59	-0.48	-0.76	-0.72	-0.66	-0.54

Table 5: Values of $T^*_{n,a}$ and $\tilde{T}^*_{n,a}$ for the two data sets

$1_{\{X_i \le Y_i\}}$ indicating the cause of death.

The nonparametric maximum likelihood estimate of F is given by the Kaplan-Meier product-limit estimator $\hat{F}_n$ defined by

$$1 - \hat{F}_n(t) = \prod_{i=1}^{n} \left(1 - \frac{\delta_{[i:n]}}{n-i+1}\right)^{1_{\{Z_{i:n} \le t\}}}.$$

Here $Z_{1:n} \le \ldots \le Z_{n:n}$ are the ordered Z-values, and $\delta_{[i:n]}$ is the concomitant of the i-th order statistic, that is, $\delta_{[i:n]} = \delta_j$ if $Z_{i:n} = Z_j$.

If φ is integrable, one has the representation

$$\int \varphi(x) d\hat{F}_n(x) = \sum_{i=1}^{n} W_{in}\, \varphi(Z_{i:n}),$$

where for $1 \le i \le n$,

$$W_{in} = \frac{\delta_{[i:n]}}{n-i+1} \prod_{j=1}^{i-1} \left(\frac{n-j}{n-j+1}\right)^{\delta_{[j:n]}}.$$

In particular,

$$\hat{M}_n(t) = \int e^{tx} d\hat{F}_n(x) = \sum_{i=1}^{n} W_{in}\, e^{Z_{i:n}}$$

is a plausible estimate of the moment generating function $M(t)$ under random censorship. Using the results of Stute and Wang (1993), it follows that $\hat{M}_n(t) \to \int_{\{x<\tau_H\}} e^{tx} d\hat{F}_n(x)$ *a.s.* where $\tau_H = \inf\{x : H(x) = 1\}$ and H is the d.f. of the Z's, i.e. $(1 - H) = (1 - F)(1 - G)$.

Consider first the special case of Type 1 censoring, i.e. $Y_i = t_0, 1 \le i \le n$. Then, $\hat{M}_n(t)$ is in general not a consistent estimator of $M(t)$ since it is impossible to observe the right tail of F. Hence, one could only test if the observable part of F corresponds to an exponential distribution; the moment generating function and the $\mathcal{M}$-class are not suitable for this purpose.

We assume therefore $\tau_H = \infty$ in the following. If $F(t) = F(t, \lambda)$ is the exponential distribution, the maximum likelihood estimator of λ under random censorship is given by (see, e.g., Kalbfleisch and Prentice (1980), Section 3)

$$\hat{\lambda}_n = \frac{\sum_{i=1}^{n} \delta_i}{\sum_{i=1}^{n} Z_i}.$$

λ_1	p	$n=25$	$n=100$	$n=400$	$n=1600$	$n=6400$	$n=25600$
	0.05	-2.62	-2.56	-2.41	-2.20	-1.99	-1.87
	0.1	-2.05	-1.99	-1.86	-1.67	-1.56	-1.49
	0.25	-0.34	-0.50	-0.69	-0.86	-0.90	-0.90
5	0.5	-0.17	-0.23	-0.28	-0.30	-0.29	-0.25
	0.75	-0.03	0.04	0.12	0.21	0.31	0.40
	0.9	0.15	0.34	0.54	0.73	0.90	1.00
	0.95	0.30	0.55	0.83	1.09	1.26	1.37
	0.05	-0.77	-1.12	-1.47	-1.59	-1.62	-1.65
	0.1	-0.56	-0.84	-1.11	-1.26	-1.30	-1.31
	0.25	-0.39	-0.56	-0.68	-0.73	-0.76	-0.74
25	0.5	-0.20	-0.22	-0.20	-0.15	-0.12	-0.07
	0.75	0.03	0.19	0.35	0.50	0.55	0.61
	0.9	0.32	0.67	1.00	1.14	1.20	1.24
	0.95	0.58	1.03	1.38	1.58	1.63	1.63
	0.05	-0.88	-1.18	-1.42	-1.49	-1.57	-1.59
	0.1	-0.76	-0.99	-1.16	-1.21	-1.25	-1.23
	0.25	-0.56	-0.67	-0.73	-0.71	-0.74	-0.67
100	0.5	-0.30	-0.25	-0.20	-0.11	-0.08	-0.04
	0.75	0.08	0.28	0.43	0.56	0.59	0.65
	0.9	0.57	0.94	1.11	1.27	1.25	1.32
	0.95	0.98	1.39	1.57	1.69	1.71	1.71

Table 6: Empirical p-quantiles of $\sqrt{n}\hat{T}_{n,0.25}/\hat{\sigma}_n$ based on 10000 replications, where $\hat{\sigma}_n^2$ is the empirical variance of $\hat{T}_{n,0.25}$

Hence, a suitable modification of the test statistics $T_{n,a}$ for testing for exponentiality against H_1 is

$$\begin{aligned}\hat{T}_{n,a} &= 1/\hat{\lambda}_n \int_0^{a\hat{\lambda}_n} \Big(\hat{M}_n(t) - M(t, \hat{\lambda}_n)\Big)\ dt \\ &= \sum_{j=1}^{n} W_{in} \frac{e^{aU_{i:n}} - 1}{U_{i:n}} \ + \ \log(1-a),\end{aligned}$$

where $a \in (0, 1/2)$ and $U_{i:n} = \hat{\lambda}_n Z_{i:n},\ 1 \le j \le n$.

Table 6 shows the p-quantiles of $\sqrt{n}\hat{T}_{n,0.25}/\hat{\sigma}_n$, where $\hat{\sigma}_n^2$ is the empirical variance of $\hat{T}_{n,0.25}$, for increasing sample sizes. The distributions of X_1 and Y_1 are exponential with parameter λ_1 and 1, respectively; hence, the censored portion of the data is $1/(\lambda_1 + 1)$. The entries in Tables 6 are based on 10000 replications.

Looking at the critical values, they seem to converge to the quantiles of a standard normal distribution. This is not unexpected in view of the central limit theorem under random censorship (Stute (1995)). However, the limiting variance does not only depend on a as in the case without censoring, but also on the d.f.'s F and G. Furthermore, convergence is very slow, depending on the degree of censoring. To perform the test in practice, estimating the unknown variance of $\hat{T}_{n,a}$, presumably by some resampling procedure, would be necessary.

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