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A note on a measure of asymmetry

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Abstract A recently proposed measure of asymmetry (Patil, Patil, and Bagkavos, 2012) is analyzed in detail. Several examples illustrate the peculiar behavior of this measure η as a measure of asymmetry or skewness. These findings are supported by theoretical considerations. Specifically, η is revealed to be a measure of similarity with the exponential distribution rather than an asymmetry measure. To illustrate this, we consider a related goodness of fit test for exponentiality. Moreover, we show that the partly erratic behavior of η also has a negative impact on the estimation of the measure.

Keywords asymmetry measure, skewness, correlation, characterization of exponentiality, goodness of fit test.

1 Introduction

Symmetry of a distribution or density function is an important concept in probability theory and statistics. A random variable X with cumulative distribution function F is symmetric about θ if $X - \theta \sim \theta - X$, or, equivalently, if $F(\theta - x) = 1 - F(x + \theta), x \in \mathbb{R}$.

Clearly, then the task is to quantify deviations from symmetry; to this end, a sizeable number of asymmetry or skewness measures has been proposed in the literature. It seems to be a matter of definition if there is any difference between skewness and asymmetry. If skewness is simply defined as the deviation of symmetry measured by the classical moment based measure, the standardized third central moment $\gamma = E[(X - EX)^3]/\operatorname{Var}(X)^{3/2}$, then there is a difference. This measure is strongly affected by the distributional tails, so "that it is difficult to estimate accurately in practice when the distribution is markedly skew" (Hosking, 1990).

Hence, in view of this measure, Patil, Patil, and Bagkavos (2012) have a point in writing "However, skewness is primarily influenced by the tail behavior of a density

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function, and the skewness coefficients are designed to capture this behavior. Thus they do not calibrate asymmetry in the density curves."

However, if the question "What is skewness" is answered with "Skewness is asymmetry, plain and simple." as done by Arnold and Groeneveld (1993), then the above statement is untenable. Defining skewness as the degree of deviation from symmetry is the common approach in the statistical literature on the topic (see, e.g., MacGillivray (1986); Benjamini and Krieger (1996); Critchley and Jones (2008)), and in this case, skewness and asymmetry are just synonyms.

A typical example from this perspective is the quantile based "skewness measure"

$$b_1 = \frac{q_X(3/4) + q_X(1/4) - 2q_X(1/2)}{q_X(3/4) - q_X(1/4)}$$

(see Yule (1912), Bowley (1920)), or the general version

$$b_2(\tau) = \frac{q_X(1-\tau) + q_X(\tau) - 2q_X(1/2)}{q_X(1-\tau) - q_X(\tau)},$$

where $q_X(\tau)$ denotes the τ-quantile of the distribution of a random variable X. Since a distribution is symmetric if and only if $q_X(1-\tau) - q_X(1/2) = q_X(1/2) - q_X(\tau)$ for each $\tau \in (0, 1/2)$, it is apparent that these measures capture all kind of deviations from symmetry.

In what follows, let X be a random variable with distribution function F and probability density function f satisfying $\operatorname{Var}(f(X)) < \infty$. Then, Patil, Patil, and Bagkavos (2012) define a measure of asymmetry as

$$\eta = \begin{cases} -\operatorname{Corr}(f(X), F(X)) & \text{if } 0 < \operatorname{Var}(f(X)) < \infty, \\ 0 & \text{if } \operatorname{Var}(f(X)) = 0. \end{cases} \tag{1}$$

Note that Patil, Patil, and Bagkavos (2012) additionally assume that the density is continuous in the interior of its support; this requirement is typically met by standard distributions, but does not seem to be necessary in the definition.

Assume that X has a symmetric distribution. Taking, without loss of generality, zero as the center of symmetry, we have $X \sim -X$. Then, $F(x) = 1 - F(-x)$ and $f(x) = f(-x)$ for every $x \in \mathbb{R}$. Hence,

$$\begin{aligned} \operatorname{Cov}(f(X), F(X)) &= \operatorname{Cov}(f(-X), 1 - F(-X)) \\ &= \operatorname{Cov}(f(X), 1 - F(X)) \\ &= -\operatorname{Cov}(f(X), F(X)), \end{aligned}$$

which implies $\operatorname{Cov}(f(X), F(X)) = 0$. For an analytical proof of this fact, which is the foundation of the definition of η in (1), see Patil, Patil, and Bagkavos (2012).

Partlett and Patil (2017) suggest several test statistics based on an estimator of η to test for symmetry about an unspecified center. Zhang, Zhang, Zhu and Lu (2018) proposed generalizations of η given by $\eta_k = -\operatorname{Corr}(f^k(X), F(X))$ if $0 < \operatorname{Var}(f^k(X)) < \infty$ for $k > 0$ as measures of asymmetry. Further measures of asymmetry related to η are introduced in Patil, Bagkavos and Wood (2014); these measures are used as basis for tests of symmetry in Bagkavos, Patil and Wood (2016). Xu, Li and Chevapatrakul (2016) adopt η to examine the relation between return asymmetry and expected stock returns.

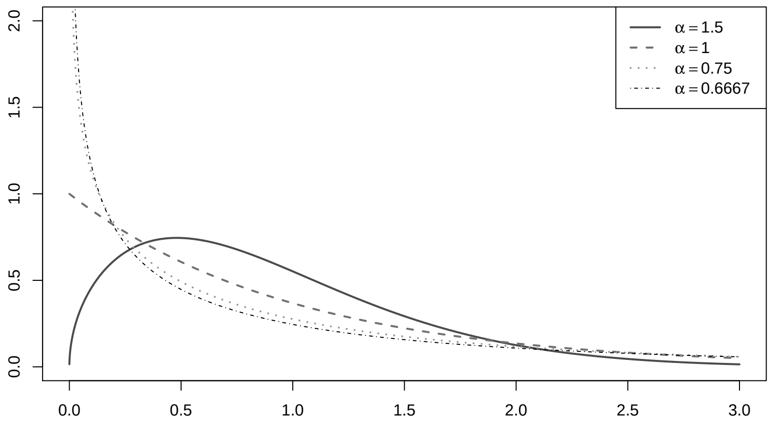

Fig. 1 Densities of Weibull distributions with scale parameter 1 and shape parameters α.

Patil, Patil, and Bagkavos (2012) claim that "the proposed measure does an admirable job of capturing the visual impression of asymmetry of a continuous density function." It is the aim of this work to show that this strong claim is not vindicable in general. To this end, we first give several examples where the measure η shows a peculiar behavior as measure of asymmetry. We explain these findings in more detail in section 3. Examples of possible effects in estimating η and a goodness of fit test for exponentiality related to η are treated in sections 4 and 5.

2 Examples of counterintuitive behavior

In this section, we give several examples showing the peculiar behavior of the measure η for Weibull and beta distributions.

Example 1. Consider first the (standard) Weibull distribution with probability density function $f(x) = \alpha x^{\alpha-1}\exp(-x^{\alpha})$ for $x > 0$, $\alpha > 0$.

(i) First, we compare the densities for shape parameters $\alpha = 1.5, \alpha = 1$ and $\alpha = 0.75$, which are plotted in Figure 1. Clearly, the density for $\alpha = 1.5$ is the most symmetric one, followed by the decreasing exponential density ($\alpha = 1$). The density for $\alpha = 0.75$ is also decreasing, but markedly more asymmetric.
The pertaining values of η are $0.732, 1$, and 0.732, respectively. Hence, the exponential distribution has the highest value, whereas the values for the first and third density are significantly smaller and coincide!
For comparison, we also state the values for the quantile and the moment based measures b_1 and γ. These are $0.139, 0.262, 0.375$ for b_1 and $1.072, 2.0, 3.121$ for γ, i.e. they are increasing with decreasing values of α in both cases.

(ii) Second, we consider the Weibull distribution with shape parameter $\alpha = 0.6667$. Here, we obtain $\eta = 0.017$, i.e. the value is zero for practical purposes.
Values of η near zero are also obtained for shape parameters around 3.5, indicating symmetry (Patil, Patil, and Bagkavos (2012)). For these values, the

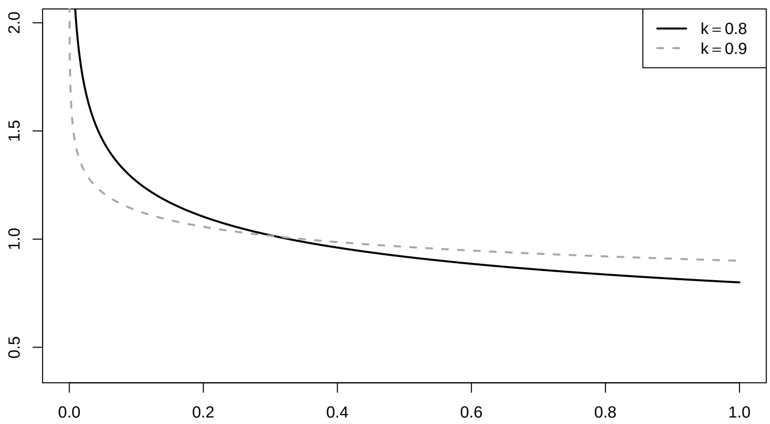

Fig. 2 Densities of beta distributions with density $f(x) = kx^{k-1}, 0 < x < 1$, for $k = 0.8$ and $k = 0.9$.

shape of the Weibull density is indeed close to symmetric; and other skewness measures also deliver very small values.

Here, the situation is totally different, as Figure 1 shows. The density is decreasing and strongly asymmetric (the most asymmetric one of the 4 densities in Figure 1!). The values of $b_1 = 0.428$ and $\gamma = 3.802$ show that this asymmetry is clearly detected by the quantile and moment based measures. However, the very small value of η can certainly not be interpreted as indication of symmetry. Note further, that the densities for $\alpha = 0.75$ and $\alpha = 0.6667$ have a similar visual appearance, however, the pertaining values of η are very different.

It should be emphasized that the behaviour of η shown in (i) and (ii) does not depend on the fact that the density of the Weibull distribution is unbounded at the origin for $\alpha < 1$. One can repeat the foregoing examples, replacing the Weibull distribution by a left truncated Weibull distribution without changing the qualitative behavior.

Example 2. Here, we consider a certain class of beta distributions with densities $f(x) = kx^{k-1}$ for $0 < x < 1, k > 0$.

(i) Consider first the two densities in Figure 2 with shape parameters $k = 0.8$ and $k = 0.9$. Without doubt, the first is more asymmetric than the latter: Visually, the (probability) mass for $k = 0.9$ is relatively more evenly spread than the mass for $k = 0.8$, suggesting the asymmetry for $k = 0.9$ is less than the asymmetry for $k = 0.8$. Also, the density with $k = 0.9$ is smaller in the convex transform order, introduced and analyzed by van Zwet (1964) and Oja (1981), indicating a lesser degree of asymmetry or skewness (Groeneveld and Meeden, 1984). This is confirmed by decreasing values 0.065 and 0.029 of b_1 and $0.197, 0.092$ of γ for $k = 0.8$ and 0.9.
However, the values of η are 0.700 for $k = 0.8$ and 0.809 for $k = 0.9$, hence indicating a more asymmetric density for $k = 0.9$, contrary to expectations.

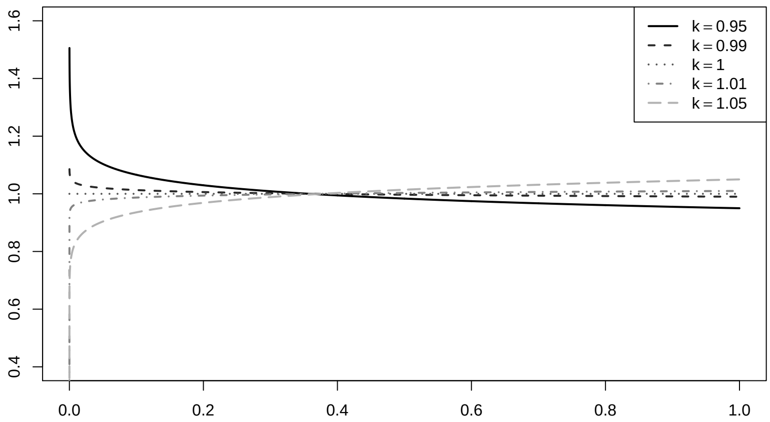

Fig. 3 Densities of beta distributions with density $f(x) = kx^{k-1}, 0 < x < 1$, for values of k around 1.

(ii) Even more peculiar is the behavior of η for values around 1. The five densities in Figure 3 are all close to symmetric; for $k = 1$, we get the uniform density, the prototype of a symmetric distribution. The values of η, b_1 and γ are as follows:

k	0.95	0.99	1	1.01	1.05
η	0.841	0.862	0.000	-0.870	-0.885
b_1	0.014	0.003	0.000	-0.003	-0.012
γ	0.045	0.009	0.000	-0.009	-0.042

As one would expect, b_1 and γ take on values close to zero, indicating nearly symmetric distributions. The asymmetry measure η is zero by definition for $k = 1$, since $\text{Var}(f(X)) = 0$. For the other values of k, η takes on absolute values close to 0.85, hence indicating strong assymmetry, and jumps from positive to negative values for k smaller and larger than 1, respectively.

(iii) Next, consider the densities in Figure 4 with shape parameters $k = 2, 3, 5, 7$. Compared to the first density, which belongs to a half triangular distribution, the other densities are more and more asymmetric. All are increasing, resulting in negative values for all measures. The specific values are given in the following table:

k	2	3	5	7
η	-0.980	-0.992	-0.997	-0.999
b_1	-0.132	-0.175	-0.210	-0.225
γ	-0.566	-0.861	-1.183	-1.361

Most notably, η takes on values close to the minimum possible value -1 in each case, even though the shapes of the four densities are quite different. The measure η does not seem to be able to capture these differences, in contrast to b_1 and γ, where the absolute values are doubled from $k = 2$ to $k = 7$.

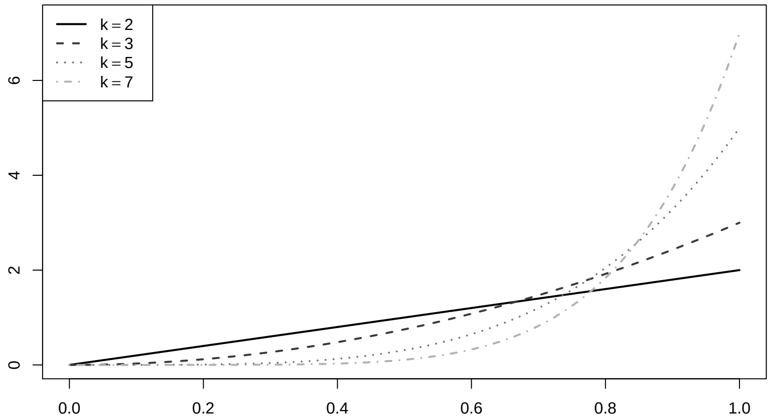

Fig. 4 Densities of beta distributions with density $f(x) = kx^{k-1}, 0 < x < 1$, for $k = 2, 3, 5, 7$.

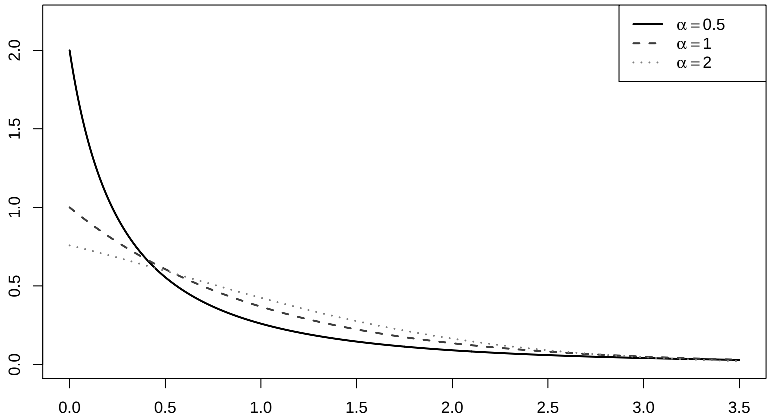

Fig. 5 Densities of the generalized exponential distribution of Nadarajah and Haghighi with different shape parameters α and scale parameter λ chosen such that the expected value is 1.

Example 3. As final example, we consider a generalization of the exponential distribution introduced by Nadarajah and Haghighi (2011). Cumulative distribution function, probability density function and quantile function are given by

$$F(t) = 1 - \exp[1 - (1 + \lambda t)^{\alpha}], \quad t > 0,$$
$$f(t) = \alpha\lambda(1 + \lambda t)^{\alpha-1} \exp[1 - (1 + \lambda t)^{\alpha}], \quad t > 0,$$
$$Q(p) = [(1 - \log(1 - p))^{1/\alpha} - 1]/\lambda, \quad 0 < p < 1,$$

respectively, where $\lambda > 0$ and $\alpha > 0$ are scale and shape parameters, respectively. The probability densities are always bounded and have its mode at zero. For $\alpha = 1$, the distribution coincides with the exponential distribution.

The densities for shape parameters $\alpha = 0.5, \alpha = 1$ and $\alpha = 2$ are plotted in Figure 5. There, we haven chosen the scale parameter λ in such a way that

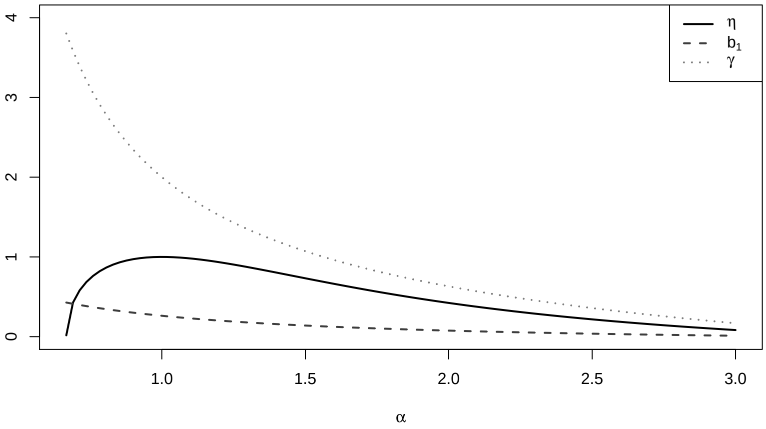

Fig. 6 Asymmetry measure $\eta(\alpha)$ for the Weibull distribution, together with measures b_1 and γ.

the expected value is 1 in all three cases. Visually, the asymmetry or skewness is maximal for $\alpha = 0.5$ and minimal for $\alpha = 2$.

This is confirmed by decreasing values $0.40, 0.26, 0.188$ for γ and $4.87, 2, 1.25$ for b_1 for $\alpha = 0.5, 1, 2$. However, the pertaining values of η are $0.978, 1.0, 0.988$. Hence, the exponential distribution has again the highest value, whereas the values for the first and third density are both smaller, albeit to a very small extend.

3 Theoretical explanations

For the Weibull distribution with shape parameter α and scale parameter 1, the measure η exists for $\alpha > 2/3$ and can be explicitly computed as

$$\eta(\alpha) = \sqrt{12}\,\Gamma(2-1/\alpha)\left(\frac{1}{3^{2-1/\alpha}} - \frac{1}{2^{3-1/\alpha}}\right)\left(\frac{\Gamma(3-2/\alpha)}{3^{3-2/\alpha}} - \frac{\Gamma(2-1/\alpha)^2}{2^{4-2/\alpha}}\right)^{-1/2}$$

(see Patil, Patil, and Bagkavos (2012)). Its visual appearance as a function of the shape parameter can be seen in Figure 6, together with the skewness measures b_1 and γ.

As remarked by Patil, Patil, and Bagkavos (2012), we obtain $\eta(1) = 1$ for the exponential distribution with density $f(x) = \exp(-x), x > 0$, since then $f(x) = 1 - F(x)$ holds for $x > 0$. But more can be said: It is well known that the exponential distribution is characterized in the class of continuous life distributions by having a constant failure rate $h(x) = f(x)/(1-F(x))$ for all $x > 0$ (see, e.g. Johnson, Kotz and Balakrishnan (1995)). Likewise, a suitably shifted exponential distribution is the only continuous distribution with constant failure rate on $[a, \infty)$, say. Further, if the support of F is unbounded from below, or if the support is a finite interval, then there doesn't exist a distribution with non-increasing failure rate (Barlow, Marshall and Proschan, 1963). Hence, the exponential distribution is essentially the only distribution with constant failure rate (on the interior of its support), which is in turn equivalent to $\mathrm{Corr}(f(X),(1-F(X))) = -\mathrm{Corr}(f(X),F(X)) = 1$.

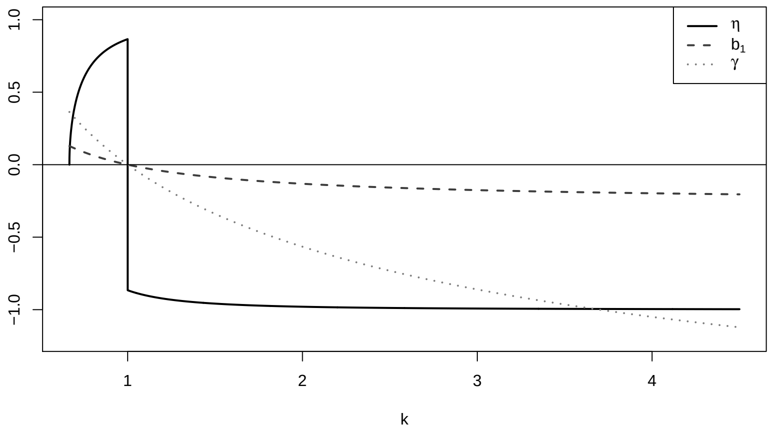

Fig. 7 Asymmetry measure $\eta(k)$ for the beta distribution with density $f(x) = kx^{k-1}, 0 < x < 1$, together with measures b_1 and γ.

It follows that the exponential distribution is the most asymmetric one judged by the measure η, i.e. it is the only distribution that attains the maximal value $\eta = 1$. Looking at it this way, a value of η close to 1 does not indicate strong asymmetry, but indicates a strong similarity to the exponential distribution.

These facts explain the strange behavior of η in Example 1(i): If α decreases from 1, the densities become more and more dissimilar compared to the exponential density, leading to smaller values of η, even if the densities become more and more asymmetric. In the limit $\alpha \to 2/3$, η even converges to zero, as exemplified in Example 1(ii). Since the densities are decreasing, η stays positive, but the increasing dissimilarity leads to values near zero indicating symmetry if η is used as a measure of asymmetry!

Summarizing these points, it seems quite questionable wether the exponential density is a reasonable starting point as a maximally asymmetric density.

For the beta distribution with parameter k considered in Example 2, the measure η again exists for $k > 2/3$, and is given by

$$\eta(k) = \operatorname{sgn}(1-k)\,\frac{\sqrt{3k(3k-2)}}{3k-1}, \quad \text{where} \quad \operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

A plot of η as a function of k is shown in Figure 7, together with the skewness measures b_1 and γ. As written above, $\eta(1) = 0$ by definition, since $\operatorname{Var}(f(X)) = 0$. However, η does not converge to zero for $k \to 1$, as one would expect. Instead,

$$\lim_{k \to 1-} \eta(k) = +\sqrt{3}/2 \approx 0.866,$$
$$\lim_{k \to 1+} \eta(k) = -\sqrt{3}/2 \approx -0.866.$$

Hence, for distributions which are indistinguishable from the (symmetric) uniform distribution, η takes on values close to the upper and lower bounds ± 1. What's more, the sign changes in an arbitrary small neighbourhood of 1.

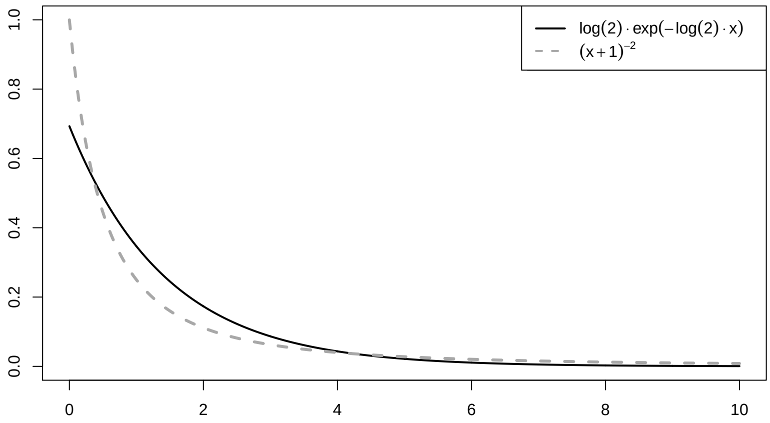

Fig. 8 Exponential density and density $(x+1)^{-2}, x > 0$, both scaled to median 1.

Further, for $2/3 < k < 1$, $\eta(k)$ is increasing in k, indicating an increasing asymmetry in k, whereas the visual impression is a decreasing asymmetry, as Figure 3 clearly shows. As explained in section 2, this behavior shows that η violates the convex transform order of van Zwet.

This was also recognized by Patil, Patil, and Bagkavos (2012, section 3.6), giving as example the comparison of a standard exponential density $g(x) = \exp(-x), x > 0$, with the density $h(x) = (x+1)^{-2}, x > 0$. There, this violation was rather seen as a positive feature of η, since h "is relatively more evenly spread than the mass under" g. Note, however, that the two distributions have a totally different tail behavior; even the expected value of h does not exist. If we scale both distribution in such a way that both medians equal 1 (i.e. we have to take an exponential distribution with rate log(2)), the densities have two crossing points, and it is not clear which probability mass is more evely spread (see Figure 8).

In the above example of beta distributions with $k < 1$, where the support is bounded and the distributions are well comparable, the violation of the van Zwet ordering raises serious doubts on the applicability of η as asymmetry measure.

For $k \to 2/3$, $\eta(k) \to 0$, even though the densities are decreasing and strongly asymmetric - a similar behavior as in the case of Weibull distributions for $\alpha \to 2/3$.

For $k > 1$, $\eta(k)$ is decreasing with $\lim_{k\to\infty} \eta(k) = -1$, which can easily be explained: A reasoning analogous to Example 1 shows that η takes on the minimal value -1 solely for the negative exponential distribution, but $\lim_{k\to\infty} k\,\mathrm{Beta}(k,1)$ converges to this distribution.

Rather unpleasantly, however, the values of η are all very close to the minimal value of -1 for $k > 2$, as Figure 7 reveals. This indicates a comparable degree of asymmetry, whereas the visual impression is quite different (see Figure 4).

In contrast, the measures b_1 and γ show none of these peculiarities, as Figure 7 shows.

For the generalized exponential distribution considered in Example 3, η exists for all values of α. It is computed numerically and plotted as a function of α in Figure 9, together with the skewness measures b_1 and γ.

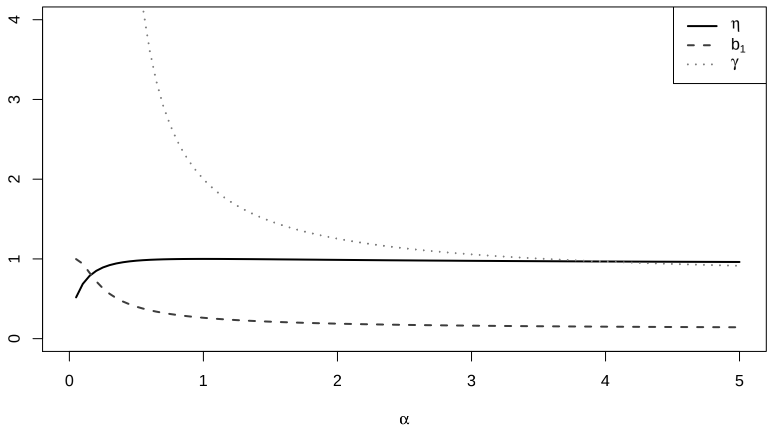

Fig. 9 Asymmetry measure $\eta(\alpha)$ for the generalized exponential distribution from Example 3, together with measures b_1 and γ.

The behavior of η is explained similar as in Example 1: If α departs from 1, the densities differ more and more from the exponential density, leading to decreasing values of η, even if the densities are more asymmetric. In the limit $\alpha \to 0$, η even seems to converge to zero. For $\alpha > 1$, the decrease of η with increasing values of α is extremely slow, and nearly invisible in Figure 9. Again, for practical use, η can not distinguish between distributions with quite different shape. Hence, Example 3 shows that the previously mentioned peculiarities of η as a measure of asymmetry also occur for very well behaved, bounded densities.

4 Performance of the empirical counterpart

Among the three estimators for η proposed in Patil, Patil, and Bagkavos (2012), the most intuitive and most efficient one (see Partlett and Patil, 2017) seems to be

$$\hat{\eta} = -\frac{\sum_{i=1}^n U_i V_i - n\bar{U}_n\bar{V}_n}{\sqrt{\left(\sum_{i=1}^n U_i^2 - n\bar{U}_n^2\right)\left(\sum_{i=1}^n V_i^2 - n\bar{V}_n^2\right)}},$$

where $U_i = \hat{f}_n(X_i)$, $V_i = \hat{F}_n(X_i)$, $i = 1, ..., n$, and $\bar{U}_n = 1/n\sum_{i=1}^n U_i$, $\bar{V}_n = 1/n\sum_{i=1}^n V_i$. Here,

$$\hat{f}_n(x) = (nh)^{-1}\sum_{i=1}^n K\left(\frac{x - X_i}{h}\right), \quad x \in \mathbb{R},$$

denotes a kernel density estimator with kernel K and bandwidth h and $\hat{F}_n(x) = 1/n\sum_{i=1}^n \mathbb{1}\{X_i \leq x\}$, $x \in \mathbb{R}$, is the empirical distribution function. In Partlett and Patil (2017), slightly different but asymptotically equivalent estimators for f and F are utilized. Here, we analyze the behavior of $\hat{\eta}$ for the Weibull and beta distribution and compare it to the corresponding theoretical values of η. According

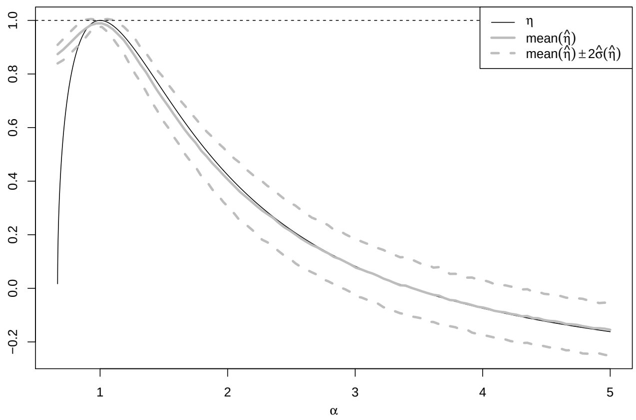

Fig. 10 Mean values of the empirical asymmetry measure $\hat{\eta}$ for Weibull distributions with shape parameter α together with empirical two-sigma regions.

to the central limit theorem by Partlett and Patil (2017, Theorem 1(i)), $\hat{\eta}$ is a consistent estimator of η.

For our simulations, we used the Gaussian kernel K for computational reasons. However, usage of the Epanechnikov kernel, which satisfies assumption A2 in Partlett and Patil (2017), leads to nearly identical results. Additionally, we used the bandwidth $h = n^{-3/8}$ (where n denotes the sample size) instead of the usual rate $h \sim n^{-1/5}$ in order to satisfy assumption A3 in Partlett and Patil (2017). This actually slightly improves the estimation of η (while the direct estimation of f is worsened considerably). All simulations were carried out with the R software (R Core Team, 2019), using the R packages evmix (Hu and Scarrott, 2018) for density estimation. Sample size was set to $n = 1000$, and all results are based on 1000 replications.

First, we consider the Weibull distribution with scale parameter 1; the corresponding mean values of $\hat{\eta}$ within a band of two standard errors are depicted in Figure 10. Boundary issues in the density estimation were resolved with an implementation of the simple reflection method of Jones (1993, p. 137) as done in Patil, Patil, and Bagkavos (2012). For $\alpha \geq 1$, the means of the estimated values are very close to the theoretical values of η, but the variance is increasing in α. However, the estimation fails for $\alpha < 1$, i.e. if the distribution is actually more asymmetric but less similar to the exponential distribution as expounded in section 3. The values of $\hat{\eta}$ get slightly closer to η if the sample size is increased, but for $n = 20000$, we still observed mean values of $\hat{\eta}(0.6667)$ of around 0.7 while $\eta(0.6667) = 0$.

Second, we consider the class of beta distributions as defined in Example 2. The results of the simulation are depicted in Figure 11. For density estimation, we used a reflection method applicable to the reflection on two boundaries taking into account the bounded support of the beta distribution; specifically, we used the so-called beta kernels as proposed by Chen (1999) and implemented in evmix (Hu and Scarrott, 2018). Once again, for relatively large parameter values ($k \geq 1.4$), the means of the estimated values are very close to the theoretical function $\eta(k)$, now even with a small variance.

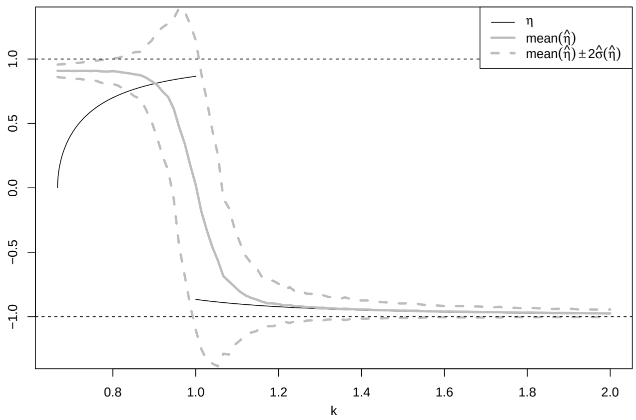

Fig. 11 Mean values of the empirical asymmetry measure $\hat{\eta}$ for the class of beta distributions considered in Example 2 together with empirical two-sigma regions.

For values of k around 1, the function $\eta(k)$ with its gap in 1 is smoothed by the means of the estimated values. This smoothing goes along with a very high variance of $\hat{\eta}$; the two-sigma region in Figure 11 includes the entire intervall $[-1, 1]$. Then, for $k \to 2/3$, the distance between η and $\hat{\eta}$ grows further with the mean of the estimated values flattening out at about 0.9 and with decreasing variance, while η rapidly tends toward 0.

Overall, we conclude that while $\hat{\eta}$ estimates η well in some areas, there are large differences in others, especially when η itself behaves counterintuitively. While increasing the sample size seems to improve the performance of $\hat{\eta}$ slightly for the Weibull distribution, its bad behavior seems to be independent of n for the beta distribution.

5 A related goodness of fit test for exponentiality

The considerations in section 3 suggest that an empirical counterpart of η could be used as basis for a goodness of fit test for exponentiality. In constructing such tests, a nonparametric estimate is typically compared with the theoretical counterpart under the hypothesis. Here, specifically, one could use a nonparametric density estimate $\hat{f}$, evaluated at the sample point $x_i, i = 1, \ldots, n$, and compare it with the distribution function $F(x_i, \hat{\lambda})$ of the exponential distribution with estimated parameter. Alternatively, one transforms x_i by the exponential density with estimated parameter, and uses the empirical distribution function as nonparametric estimate. We take the second route: Define $z_i = f(x_i, \hat{\lambda}) = \hat{\lambda} \exp(-\hat{\lambda} x_i), i = 1, \ldots, n$, with $\hat{\lambda} = n / \sum_{i=1}^n x_i$, and $p_i = i/(n+1), i = 1, \ldots, n$ (here, we use the denominator $n+1$ instead of n in the nonparametric estimate of the distribution function at x_i).

Then, the test statistic is defined by

$$T = -\hat{\rho}\left((z_1, \ldots, z_n), (p_1, \ldots, p_n)\right),$$

where $\hat{\rho}$ denotes the empirical correlation coefficient of Pearson. Note that the points (z_i, p_i) correspond, up to a affine transformation, to the plotting positions in a P-P plot for exponentiality. Hence, the test based on T is equivalent to a test based on the squared correlation coefficient for the points on a standardized P-P plot as discussed in Gan, Koehler and Thompson (1991).

Since small values of T speak against the hypothesis of exponentiality, the test is a one-tailed test with lower rejection region. Its simulated critical value based on 10^6 replications for $\alpha = 0.05$ and sample size $n = 50$ is 0.9856.

As competitors to the correlation test, we used tests based on the empirical distribution function, namely the Kolomgorov-Smirnov (KS) and the Cramér-von Mises test (CvM), the test of Epps and Pulley (EP) derived from the empirical characteristic function, and the exponentiality test of Cox and Oakes (CO) (see, e.g., Henze and Meintanis (2005) for a description of the tests).

Empirical critical values of size $\alpha = 0.05$ for all tests were obtained from 10^6 replications. With these critical values, the power of the tests was simulated based on samples of size $n = 50$ from the following alternatives to exponentiality: Weibull distributions $W(\alpha)$, beta distributions $Beta(k, l)$ with $k = 1$, and lognormal distributions $LN(\mu, \sigma^2)$ with $\mu = 0$. Here again, values are based on 10^6 replications. As in section 4, all simulations are done with the R software (R Core Team, 2019), now using the R packages gof.test (Faraway, Marsaglia, Marsaglia and Baddeley, 2017) and exptest (Novikov, Pusev and Yakovlev, 2013).

	T	KS	CvM	EP	CO
$Exp(3)$	0.050	0.050	0.050	0.051	0.051
$W(0.7)$	0.478	0.735	0.799	0.828	0.902
$W(1.4)$	0.300	0.645	0.748	0.799	0.809
$W(1.6)$	0.499	0.913	0.966	0.982	0.984
$Beta(1,1)$	0.956	0.928	0.984	0.984	0.902
$Beta(1,2)$	0.458	0.471	0.596	0.673	0.504
$Beta(1,5)$	0.109	0.123	0.142	0.165	0.128
$LN(0,1.0)$	0.329	0.248	0.293	0.168	0.140
$LN(0,1.3)$	0.685	0.690	0.735	0.762	0.653
$LN(0,1.5)$	0.869	0.909	0.934	0.950	0.923

Table 1 Percentage of Monte Carlo samples of size $n = 50$ declared significant by the different tests, nominal level $\alpha = 0.05$.

From Table 1, we see that the test based on T works as expected. Distributions similar in shape to the exponential distribution as the $W(1.4)$ or the $Beta(1, 5)$ distribution are detected with low probability compared to densities which differs sharply from the exponential, like the $Beta(1, 1)$ or the $LN(0, 1.5)$ densities.

However, compared to the other tests, the power is somewhat lower in most cases; only for the lognormal distribution with parameters $\mu = 0$ and $\sigma^2 = 1$, its power exceeds all competitors.

6 Conclusions

We have demonstrated by several examples that the measure η proposed by Patil, Patil, and Bagkavos (2012) has severe shortcomings when used as a measure of asymmetry. First, it behaves counterintuitively for highly skewed distributions.

This is explained by the fact that η is rather a measure for the similarity to the exponential distribution than a measure of skewness or asymmetry. Second, it completely fails for distributions close to the uniform. This is connected to the definition of the measure η which assumes that the variance of the random variable $f(X)$ is positive; otherwise, the measure is set somewhat artificially to zero. Third, it may yield values close to zero for totally asymmetric distributions, which is again connected to its definition: η vanishes for high values of the variance of $f(X)$. Further, η takes values close to the maximum of 1 for distributions with very different shapes. Hence, it is difficult to differentiate between skew distributions on basis of η.

Connected to these findings is the fact that η violates the skewness order of van Zwet (1964) based on convex transformations. In our opinion, this does not necessarily disqualify a measure from the outset. However, "It is of course easy to recognize symmetric distributions but not so easy to decide whether one non symmetric distribution is more unsymmetric than another." (Arnold and Groeneveld, 1993). When comparing distributions, the visual impression depends strongly on location and scale of the involved distributions, despite the fact that asymmetry measures are invariant to location and scale. Therefore, a formal concept of asymmetry is very helpful, as discussed in detail in Oja (1981) and MacGillivray (1986), and a measure without such a formal substantiation has to be assessed with special care to be acceptable for practical applications.

The peculiar behavior of η is reflected by its empirical counterpart, as shown in section 4. Taking into consideration that quite a large number of suitable measures of skewness and asymmetry has been proposed over time, we think that there are much better alternatives, and, hence, η should not be used as a measure of asymmetry.

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