A Treatment of Multivariate Skewness, Kurtosis and Related Statistics

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This paper gives a unified treatment of the limit laws of different measures of multivariate skewness and kurtosis which are related to components of Neyman's smooth test of fit for multivariate normality. The results are also applied to other multivariate statistics which are built up in a similar way as the smooth components. Special emphasis is given to the case that the underlying distribution is elliptically symmetric.

1. INTRODUCTION

Suppose $X_1, \ldots, X_n$ are independent observations on a d-dimensional random column vector X with expectation $E(X) = \mu$ and nonsingular covariance matrix $T = E[(X-\mu)(X-\mu)']$, where the prime denotes transpose. Let $\tilde{X} = T^{-1/2}(X-\mu)$ denote the standardized vector with $E[\tilde{X}] = 0$ and $E[\tilde{X}\tilde{X}'] = I_d$, the unit matrix of order d. Let

$$Z_j \;=\; (z_{1j}, \ldots, z_{dj})' \;=\; S_n^{-1/2}(X_j - \bar{X}_n), \quad j = 1, \ldots, n,$$

where $\bar{X}_n = n^{-1}\sum_{j=1}^n X_j$ and $S_n = n^{-1}\sum_{j=1}^n (X_j - \bar{X}_n)(X_j - \bar{X}_n)'$ are the sample mean vector and the sample covariance matrix of $X_1, \ldots, X_n$, respectively. We assume that S_n is nonsingular with probability one. This condition is satisfied if X has a density with respect to Lebesgue measure and $n \geq d+1$ (Eaton and Perlman (1973)). Writing $\tilde{Y}$ is an independent copy of $\tilde{X} = (\xi_1, \ldots, \xi_d)'$, Mardia (1970) introduced the affine-invariant skewness measure

$$\beta_{1,d} \;=\; E(\tilde{X}'\tilde{Y})^3 \qquad (1)$$

AMS 1991 subject classification numbers: primary 62H10, secondary 62G20.
Key words and phrases: multivariate skewness, multivariate kurtosis, Neyman's smooth test, components, multivariate normality, elliptically symmetric distributions.

$$= \sum_{r=1}^{d} \left(E[\xi_r^3]\right)^2 + 3\sum_{r\neq s} \left(E[\xi_r^2\xi_s]\right)^2 + 6 \sum_{1\le r<s<t\le d} \left(E[\xi_r\xi_s\xi_t]\right)^2 ,$$

and the multivariate sample skewness

$$b_{1,d} \;=\; \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[(X_i-\bar{X}_n)'S_n^{-1}(X_j-\bar{X}_n)\right]^3 \;=\; \frac{1}{n^2}\sum_{i,j=1}^{n}\left(Z_i'Z_j\right)^3, \quad (2)$$

and he proposed to use $b_{1,d}$ for testing the hypothesis H_0 that the distribution of X is nondegenerate d-variate normal. $b_{1,d}$ is closely related to the first nonzero component of Neyman's smooth test of fit for multivariate normality, introduced by Koziol (1987).

The pertaining test statistics are built up as follows. First, a system of orthonormal multivariate polynomials is defined. To this end, suppose H_k are the normalized Hermite polynomials; H_k is a polynomial of degree k, orthonormal on the (univariate) standard normal distribution. In particular,

$$H_1(x) = x, \quad H_2(x) = (x^2-1)/\sqrt{2},$$
$$H_3(x) = (x^3-3x)/\sqrt{3!}, \quad H_4(x) = (x^4-6x^2+3)/\sqrt{4!} \qquad (3)$$

Multivariate polynomials are then defined by

$$L_{k_1,\ldots,k_d}(\tilde{X}) \;=\; H_{k_1}(\xi_1)\cdots H_{k_d}(\xi_d), \quad k_1,\ldots,k_d\in\mathcal{N}_0.$$

Let $\{L_r\}$ denote the sequence arising from an arbitrary ordering. Since, under H_0,

$$E[L_r(\tilde{X})L_s(\tilde{X})] = \prod_{i=1}^{d} E[H_{k_i}(\xi_i)H_{m_i}(\xi_i)] = \prod_{i=1}^{d}\delta_{k_i m_i} = \delta_{rs}, \qquad (4)$$

where δ_{ij} denotes Kronecker's delta, the sequence is orthonormal. Usually, the polynomials are ordered by their degree $k = k_1+\ldots+k_d$; particularly, $L_0 = 1$. The smooth test of order k for multivariate normality rejects H_0 for large values of

$$\hat{\Psi}_{n,k}^2 = \sum_r \hat{V}_{n,r}^2, \qquad \hat{V}_{n,r} = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} L_r(Z_j), \qquad (5)$$

where summation is over all polynomials of degree at most k. Summing only over all polynomials of degree k yields the kth smooth component $\hat{U}_{n,k}^2$.

Since the standardized values Z_j are used in the definition of $\hat{V}_{n,r}$, $\hat{V}_{n,r} = 0$ for each polynomial L_r of degree one or two; hence, the first two components are zero (see, e.g., Rayner and Best (1989), p. 102). Consequently,

$$\hat{\Psi}^2_{n,k} = \hat{U}^2_{n,3} + \ldots + \hat{U}^2_{n,k}$$

for $k \geq 3$. The first nonzero component $\hat{U}^2_{n,3}$ consists of $H_3(x_j)$, $H_2(x_j)$ $H_1(x_k)$ and $H_1(x_j)$ $H_1(x_k)$ $H_1(x_l)$, where j, k, l are different integers in the range $\{1, \ldots, d\}$. This gives

$$\hat{U}^2_{n,3} = \frac{1}{n} \left\{ \sum_r \frac{1}{6} \left(\sum_i (z^3_{ri} - 3z_{ri}) \right)^2 + \sum_{r \neq s} \frac{1}{2} \left(\sum_i (z^2_{ri} - 1) z_{si} \right)^2 + \sum_{r<s<t} \left(\sum_i (z_{ri} z_{si} z_{ti}) \right)^2 \right\}$$

and hence, by comparison with (2), the identity (Koziol (1987))

$$\frac{n}{6}\, b_{1,d} = \hat{U}^2_{n,3}. \tag{6}$$

An alternative affine-invariant measure of multivariate skewness was introduced by Móri et al. (1993); they proposed

$$\tilde{b}_{1,d} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} Z'_i\, Z_j\, \|Z_i\|^2\, \|Z_j\|^2 \tag{7}$$

with population counterpart $\tilde{\beta}_{1,d} = \|E(\tilde{X}\|\tilde{X}\|^2)\|^2 = E[(\tilde{X}'\tilde{Y})(\tilde{X}'\tilde{X})(\tilde{Y}'\tilde{Y})]$. $\tilde{b}_{1,d}$ was further examined in Henze (1997).

To derive the limit null distribution of Mardia's skewness measure, several different approaches have been utilized. Mardia (1970) showed that $b_{1,d}$ is asymptotically equivalent under H_0 to a quadratic form of a normal vector; Koziol (1987) used the theory of empirical processes and weak convergence arguments to establish the appropriate distribution theory under multivariate normality. Rayner and Best (1989) derived the limit law of $b_{1,d}$ under H_0 as score statistics (concerning difficulties of this approach see Mardia and Kent (1991), p. 356; Kallenberg et al. (1997), p. 45). Baringhaus and Henze (1992) represented $b_{1,d}$ as V-statistic and utilized the appertaining distribution theory to study the asymptotic behavior of Mardia's skewness measure under arbitrary distributions. They showed that in the special case of an elliptical distribution, the limit law is a weighted sum of two independent χ^2-variates.

The present paper gives a unified treatment of the limit laws of both skewness measures and other statistics like multivariate kurtosis which are closely related to components of Neyman's smooth test of fit for multivariate normality.

In Section 2 we state a general result about the limit distribution of statistics which are built up similarly as the components of a smooth test. Using these findings, the limit laws of $\beta_{1,d}$ and $\tilde{\beta}_{1,d}$ are derived if the underlying distribution is elliptically symmetric. We point out that the skewness measure $\tilde{b}_{1,d}$, albeit being asymptotically distribution-free under elliptical symmetry, is not well-balanced in a certain sense. Hence, we propose a new skewness measure similar to Mardia's skewness, but asymptotically distribution-free under elliptical symmetry.

In Section 3, we consider measures of multivariate kurtosis and the fourth component of Neyman's smooth test for multivariate normality. The limit law of Mardia's kurtosis measure and of the fourth component under elliptical distributions is examined in detail.

Higher order variants of multivariate skewness and kurtosis are considered in Section 4. Since these statistics do not consist of orthogonal polynomials, the necessary computations are more involved. A kurtosis measure introduced by Koziol (1989) and higher order analogues are closely examined.

2. THE LIMIT DISTRIBUTION OF SOME MEASURES OF MULTIVARIATE SKEWNESS

We first derive the asymptotic distribution of the random vector $\hat{V}_n = (\hat{V}_{n,1}, \ldots, \hat{V}_{n,r})'$, where $\hat{V}_{n,s} = n^{-1/2} \sum_{j=1}^{n} L_s(Z_j)$ as in (5) and the multivariate polynomials L_s are of degree at least 3, but at most k ($k \geq 3$). Put $\vartheta = (\vartheta_1, \ldots, \vartheta_s)' = (\mu, T^{-1})$ ($s = (d^2 + 3d)/2$), and write $L_s(X, \vartheta)$ instead of $L_s(\tilde{X})$ to indicate the dependence of $L_s(\tilde{X})$ on ϑ. Furthermore, let $\nabla_\vartheta h(x; \vartheta)$ denote the $(r \times s)$-matrix with entries $\partial h_i(x; \vartheta)/\partial \vartheta_j$, where $h(x; \vartheta) = (L_1(x; \vartheta), \ldots, L_k(x; \vartheta))'$.

Since the multivariate skewness, and hence the first nonzero component $\hat{U}_{n,3}^2$, consists of products $Z_i' Z_j$, it is affine-invariant, i.e.,

$$\hat{U}_{n,3}^2(X_1, \ldots, X_n) \;=\; \hat{U}_{n,3}^2(b + BX_1, \ldots, b + BX_n)$$

for each nonsingular $(d \times d)$-matrix B and each $b \in \mathcal{R}^d$. Since the same property holds for each of the statistics studied in this paper, we always assume $\mu = 0$ and $T = I_d$. Assuming further that $E\|X\|^{2k} < \infty$, we have

$\sqrt{n}(\bar{X}_n - \mu) = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i + o_P(1)$ and

$$\sqrt{n}(S_n^{-1} - I_d) \;=\; -\frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i X_i' - I_d) + o_P(1) \tag{8}$$

(see, e.g., Baringhaus and Henze (1992)), and hence $\sqrt{n}(\hat{\vartheta}_n - \vartheta) = \frac{1}{\sqrt{n}} \sum_{i=1}^n l(X_i) + o_P(1)$, where $\hat{\vartheta}_n = (\bar{X}_n, S_n^{-1})$ and the function l satisfies $E_P[l(X)] = 0$.

Theorem 2.1.

a)Assume that $E\|X\|^{2k} < \infty$. *Let* $\tau = (\tau_1, \ldots, \tau_r)' = (E[L_1(\tilde{X})], \ldots, E[L_r(\tilde{X})])'$. *Then*

$$\hat{V}_n - \tau \;\xrightarrow{\mathcal{D}}\; \mathcal{N}_r(0, \Sigma), \tag{9}$$

where the covariance matrix Σ *is given by*

$$\Sigma \;=\; E\left[(v_1(X,\vartheta), \ldots, v_r(X,\vartheta))\,(v_1(X,\vartheta), \ldots, v_r(X,\vartheta))'\right] - \tau\,\tau', \tag{10}$$

and $v_i(x,\vartheta) \;=\; L_i(x,\vartheta) \;+\; E[\nabla_\vartheta L_i(X,\vartheta)]\, l(x,\vartheta)$ *for* $i = 1, \ldots, r$.

b)Let X *have distribution* $P \in \mathcal{P}_0^k$, *where* $\mathcal{P}_0^k$ *is the set of probability distributions on* $\mathcal{R}^d$ *defined by*

$$\mathcal{P}_0^k \;:=\; \{P \;:\; E_P[L_j(\tilde{X})] = 0 \;\text{ for each polynomial } L_j \text{ of degree less than or equal to } k,\;\; E_P\|X\|^{2k} < \infty\}. \tag{11}$$

Then the covariance matrix in (9) takes the form

$$\Sigma \;=\; E\left[\Big(L_1(\tilde{X}), \ldots, L_r(\tilde{X})\Big)\,\Big(L_1(\tilde{X}), \ldots, L_r(\tilde{X})\Big)'\right]. \tag{12}$$

Proof. a) follows by a series expansion similar as in Theorem 2.1 of Klar (2000). Under the hypothesis H_0 of multivariate normality, $E_\vartheta[\nabla_\vartheta h(X;\vartheta)] = -C_\vartheta$, where C_ϑ is the $(r \times s)$-matrix with entries

$$c_{ij} \;=\; E_\vartheta\left[h_i(X;\vartheta)\;\frac{\partial \log f(X;\vartheta)}{\partial \vartheta_j}\right]$$

(see Klar (2000), Theorem 2.1). Now, some computation yields $c_{ij} = 0$ $(i = 1, \ldots, r, j = 1, \ldots, s)$ (see Rayner and Best (1989), p. 101). Alternatively, this can be verified along the lines of the proof of Theorem 3.1 in

Klar (2000). Hence, b) follows under H_0. The general case $P \in \mathcal{P}_0^k$ can be treated similarly as in Theorem 2.3 of Klar (2000), noting that $\partial L_s/\partial\vartheta_j$ is a polynomial of degree at most k. ∎

Using Theorem 2.1, it is possible to derive the limit law of statistics which consist of polynomials L_s. For example, the asymptotic distribution of a component of the smooth test of fit depends on whether $\tau = 0$ or $\tau \neq 0$. If $\tau = 0$, it is well-known that

$$\hat{U}_{n,k}^2 \xrightarrow{\mathcal{D}} \sum_{j=1}^{r} N_j^2, \tag{13}$$

where $(N_1, \ldots, N_r)' \sim \mathcal{N}_r(0, \Sigma)$. The limit law is a weighted sum of independent chi-squared random variables, the weights being the eigenvalues of Σ.

In particular, if X has some nondegenerate d-dimensional normal distribution $\mathcal{N}_d$, equation (4) shows that $\mathcal{N}_d \in \mathcal{P}_0^k$ and $\tau = 0$. Using Theorem 2.1b) and (4) again yields $\Sigma = I_d$. Hence, the asymptotic distribution of $\hat{U}_{n,k}^2$ and $\hat{\Psi}_{n,k}^2$ under $\mathcal{N}_d$ is χ_ν^2, where ν equals the number of polynomials which are used to build up $\hat{U}_{n,k}^2$ and $\hat{\Psi}_{n,k}^2$, respectively. Since there are $\binom{k+d-1}{k}$ polynomials of degree k, $\hat{U}_{n,3}^2$ has a limit χ^2-distribution with $\binom{d+2}{3}$ degrees of freedom under $\mathcal{N}_d$; the limit law of $\hat{U}_{n,4}^2$ is χ_ν^2 with $\nu = \binom{d+3}{4}$, and $\hat{\Psi}_{n,k}^2$ has a limit chi-squared distribution with

$$\binom{d+2}{3} + \binom{d+3}{4} + \ldots + \binom{k+d-1}{k} = \binom{k+d}{k} - \binom{2+d}{2}$$

degrees of freedom. However, if $\tau \neq 0$,

$$\sqrt{n}\left(\frac{\hat{U}_{n,k}^2}{n} - \tau'\tau\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, 4\sum_{i,j=1}^{r} \sigma_{ij}\,\tau_i\,\tau_j\right), \tag{14}$$

where σ_{ij} denote the entries of Σ (see, e.g., Serfling (1980), Corollary 3.3).

Remark: The above results show that a test for multivariate normality based on the components $\hat{U}_{n,j}^2$ of order at most k is not consistent against distributions of $\mathcal{P}_0^k$. An analogous remark applies to each of the statistics examined in the following sections.

2.1. The limit distribution of Mardia's skewness measure

In this subsection, we derive the asymptotic distribution of $\hat{U}_{n,3}^2$ (and, hence, of Mardia's measure $b_{1,d}$) by means of Theorem 2.1 if the underlying distribution P^X is elliptically symmetric.

A d-dimensional random vector X has a spherically symmetric distribution (or simply spherical distribution) if $H\,X \stackrel{\mathcal{D}}{=} X$ for every orthogonal $(d \times d)$-matrix H. The distribution of X is elliptically symmetric (or simply elliptical) with parameters $\mu \in \mathcal{R}^d$ and $\Delta \in \mathcal{R}^{d \times d}$ if there is a random $(k \times 1)$-vector Y having a spherical distribution and a $(k \times d)$-matrix A of rank k such that $\Delta = A'A$ and $X \stackrel{\mathcal{D}}{=} \mu \;+\; A'\,Y$.

PROPOSITION 2.1. *(see Fang et al. (1989), p. 72) Let $X = (X_1, \ldots, X_d)'$ have an elliptically symmetric distribution. Let $s_1, \ldots, s_d$ be nonnegative integers, and put $s = s_1 + \ldots + s_d$. Then*

$$E\left[\prod_{i=1}^{d} X_i^{s_i}\right] \;=\; \begin{cases} E(\|X\|^s)\;\left(\dfrac{2}{d}\right)^{[l]} \displaystyle\prod_{i=1}^{d} \dfrac{(2l_i)!}{4^{l_i}(l_i)!} \;, & \textit{if } s_i = 2l_i,\ l_i \in \mathcal{N}_0,\ i = 1, \ldots, d,\ s = 2l; \\ 0 \;, & \textit{if at least one of the } s_i \textit{ is odd} \end{cases}$$

where $a^{[l]} = a(a+1)\cdots(a+l-1)$.

COROLLARY 2.1. *Let $\mu_{s_1,\ldots,s_d} = E\left[\prod_{i=1}^{d} X_i^{s_i}\right]$, where zeroes are suppressed in the notation since the order of the s_i is irrelevant. Then, from Proposition 2.1,*

$$\mu_4 \;=\; 3\,\mu_{22}; \quad \mu_6 \;=\; 5\,\mu_{42} \;=\; 15\,\mu_{222};$$
$$\mu_8 \;=\; 7\,\mu_{62} \;=\; \frac{35}{3}\,\mu_{44} \;=\; 35\,\mu_{422} \;=\; 105\,\mu_{2222}.$$

If P^X is elliptical with parameters μ and Δ, and if $E\|X\|^2 < \infty$, then $E[X] = \mu$, $Cov(X) = E[R^2]\,\Delta/rank(\Delta)$. If Δ is positive definite and $E(R^2) > 0$, the standardized vector $\tilde{X} = [Cov(X)]^{-1/2}(X - \mu)$ satisfies $E[\tilde{X}] = 0$ and $E[\tilde{X}\tilde{X}'] = I_d$. Hence, $\tilde{X}$ has a spherically symmetric distribution with $E\|\tilde{X}\|^2 = d$. The mixed moments of $\tilde{X}$ are given in 2.1; in particular, $\beta_{1,d}$ in (1) satisfies

$$\beta_{1,d} \;=\; E[(\tilde{X}'\tilde{Y})^3] \;=\; 0. \tag{15}$$

Hence, elliptically symmetric distributions belong to the class $\mathcal{P}_0^3$ of distributions with $\beta_{1,d} = 0$. The covariance matrix Σ can be computed by equation (12). The third component consists of $L_r(\tilde{X}) = H_3(\xi_r)$, $L_{rs}(\tilde{X}) = H_2(\xi_r)H_1(\xi_s)$ $(r \neq s)$ and $L_{rst}(\tilde{X}) = H_1(\xi_r)H_1(\xi_s)H_1(\xi_t)$ $(r, s, t \in \{1, \ldots, d\}, r < s < t)$, where the Hermite polynomials H_j are given in (2). We obtain the following entries in the covariance matrix:

$$\begin{aligned}
\sigma^1 &= E[L_r^2(\tilde{X})] = \frac{1}{6}E[(\xi_1^3 - 3\xi_1)^2] = \frac{1}{6}(\mu_6 - 6\mu_4 + 9),\\
\sigma^{12} &= E[L_r(\tilde{X})L_{sr}(\tilde{X})] = \frac{1}{2\sqrt{3}}(\mu_{42} - 3\mu_{22} - \mu_4 + 3),\\
\sigma^2 &= E[L_{rs}^2(\tilde{X})] = \frac{1}{2}E[(\xi_1^2 - 1)^2\xi_2^2] = \frac{1}{2}(\mu_{42} - 2\mu_{22} + 1),\\
\sigma^{22} &= E[L_{rs}(\tilde{X})L_{ts}(\tilde{X})] = \frac{1}{2}(\mu_{222} - 2\mu_{22} + 1),\\
\sigma^3 &= E[L_{rst}^2(\tilde{X})] = \mu_{222}.
\end{aligned}$$

The remaining entries vanish by Proposition 2.1. Arranging the polynomials in the form

$$\begin{array}{l}
L_1,\, L_{21},\, L_{31}, \ldots, L_{d1},\\
L_2,\, L_{12},\, L_{32}, \ldots, L_{d2},\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
L_d,\, L_{1d},\, L_{2d}, \ldots, L_{d-1,d},\\
L_{123},\, L_{124},\, L_{125}, \ldots, L_{d-2,d-1,d}\ ,
\end{array}$$

the correlation between polynomials of different rows is zero, i.e. the covariance matrix partitions into $d+1$ block diagonal matrices.

For simplification, we consider in the following the covariance matrix of the polynomials pertaining to $6\,\hat{U}_{n,3}^2$. Using Corollary 2.1 and putting $u = 6\sigma^{22} = \mu_6/5 - 2\mu_4 + 3$, $v = 6\sigma^2 = 3/5\,\mu_6 - 2\mu_4 + 3$ and hence $6\sigma^1 = 2u + v$, $6\sigma^{12} = \sqrt{3}u$, the $(d \times d)$-covariance matrix corresponding to one of the first d rows takes the form

$$\Sigma_{1,d} = \left(\begin{array}{c|cccc}
2u+v & \sqrt{3}u & \sqrt{3}u & \cdots & \sqrt{3}u\\
\hline
\sqrt{3}u & v & u & \cdots & u\\
\sqrt{3}u & u & \ddots & \ddots & \vdots\\
\vdots & \vdots & \ddots & \ddots & u\\
\sqrt{3}u & u & \cdots & u & v
\end{array}\right). \tag{16}$$

The characteristic equation of $\Sigma_{1,d}$ can be written equivalently as

$$\det\left(\begin{array}{cc|ccc} 2u+v-\lambda & (d-1)\sqrt{3}u & \sqrt{3}u & \cdots & \sqrt{3}u \\ \sqrt{3}u & (d-2)u+v-\lambda & u & \cdots & u \\ \hline 0 & 0 & v-u-\lambda & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & v-u-\lambda \end{array}\right) = 0.$$

Hence, two eigenvalues are solutions of $(2u+v-\lambda)((d-2)u+v-\lambda) - 3(d-1)u^2 = 0$,which yields $\lambda_1 = v-u = 2\mu_6/5$ and $\lambda_2 = (d+1)u+v = (d+4)\mu_6/5 - 2(d+2)\mu_4 + 3(d+2)$. Furthermore, $\lambda_3 = v-u = 2\mu_6/5$ is an eigenvalue of multiplicity $d-2$.

The matrix pertaining to the last row is a diagonal matrix with entries $6\sigma^3$; hence, it has the eigenvalue $\lambda_4 = 6\mu_{222} = 2\mu_6/5$ with multiplicity $\binom{d}{3}$.

By (13), the limit distribution of $6\,\hat{U}^2_{n,3}$ is a weighted sum of $\binom{d+2}{3}$ independent χ^2_1 random variables. Since only two different weights occur, we have the following result:

Theorem 2.2. *Let X have an elliptical distribution with expectation μ and nonsingular covariance matrix T such that $E[\{(X-\mu)'T^{-1}(X-\mu)\}^3] < \infty$; hence, $P^X \in \mathcal{P}^3_0$. Then*

$$6\,\hat{U}^2_{n,3} \xrightarrow{\mathcal{D}} \alpha_1\chi^2_{\nu_1} + \alpha_2\,\chi^2_{\nu_2}$$

as $n \to \infty$, where

$$\alpha_1 = \frac{2}{5}\mu_6, \qquad \nu_1 = d(d-1) + \binom{d}{3} = \frac{d}{6}(d-1)(d+4),$$
$$\alpha_2 = \frac{d+4}{5}\mu_6 - 2(d+2)\,\mu_4 + 3(d+2), \qquad \nu_2 = d,$$

and $\chi^2_{\nu_i}$ are independent chi-squared random variables with ν_i degrees of freedom.

Remark 2. 1. Putting $r_{2k} = E(\tilde{X}'\tilde{X})^k$ and noting that, for elliptical distributions, $r_4 = \mu_4\,d(d+2)/3, r_6 = \mu_6\,d(d+2)(d+4)/15$ (cp. Theorem 4.2 below), we have

$$\alpha_1 = \frac{6r_6}{d(d+2)(d+4)}, \quad \alpha_2 = \frac{3}{d}\left(\frac{r_6}{d+2} - 2r_4 + d(d+2)\right).$$

In view of (6), Theorem 2.2 corresponds to Theorem 2.2 in Baringhaus and Henze (1992) which was proved in a different way under the additional assumption $P(X=\mu)=0$.

2.2. The skewness measure of Móri, Rohatgi and Székely

The skewness measure of Móri et al. (1993) in (7) can be written as

$$\begin{aligned}\tilde{b}_{1,d} &= \frac{1}{n^2}\left\{\sum_{r=1}^{d}\left(\sum_i z_{ri}^3\right)^2 + 2\sum_{r\neq s}\left(\sum_i z_{ri}^3\sum_j z_{rj}z_{sj}^2\right)\right.\\ &\qquad \left. +\sum_{r\neq s}\left(\sum_i z_{ri}^2 z_{si}\right)^2 + \sum_{r\neq s\neq t}\left(\sum_i z_{ri}z_{si}^2\sum_j z_{rj}z_{tj}^2)\right)\right\}.\end{aligned}$$

Defining $\hat{V}_r = (1/\sqrt{n})\sum_{i=1}^n L_r(Z_i)$ and $\hat{V}_{rs} = (1/\sqrt{n})\sum_{i=1}^n L_{rs}(Z_i)$ with L_r and L_{rs} given in Subsection 2.1, $\tilde{b}_{1,d}$ takes the form

$$n\,\tilde{b}_{1,d} = 6\sum_r \hat{V}_r^2 + 4\sqrt{3}\sum_{r\neq s}\hat{V}_r\hat{V}_{sr} + 2\sum_{r\neq s}\hat{V}_{rs}^2 + 2\sum_{r\neq s\neq t}\hat{V}_{sr}\hat{V}_{tr}.$$

Furthermore, putting $W_r = (\hat{V}_r, \hat{V}_{1r}, \ldots, \hat{V}_{r-1,r}, \hat{V}_{r+1,r}, \ldots, \hat{V}_{dr})'$, it follows that

$$n\,\tilde{b}_{1,d} = W_1' AW_1 + \ldots + W_d' AW_d, \tag{17}$$

where the $(d\times d)$-matrix A has the entries $a_{11}=6, a_{1k}=a_{k1}=2\sqrt{3}$ $(1<k\leq d)$ and $a_{kl}=2$, otherwise. By (17), the asymptotic distribution of $n\tilde{b}_{1,d}$ is readily obtained if the underlying distribution is elliptically symmetric.

Theorem 2.3. *Under the assumptions of Theorem 2.2, we have*

$$n\,\tilde{b}_{1,d} \xrightarrow{\mathcal{D}} \frac{(d+2)\,\alpha_2}{3}\,\chi_d^2\,,$$

where $\alpha_2 = (d+4)\mu_6/5 - 2(d+2)\mu_4 + 3(d+2)$ *as in Theorem 2.2.*

Proof. W_r and W_s $(r\neq s)$ are uncorrelated and hence asymptotically independent (cp. Subsection 2.1). The covariance matrix of W_r is $\Sigma_{1,d}/6$ with $\Sigma_{1,d}$ given by (16). We therefore consider the quadratic form $W_1'AW_1$ which is asymptotically distributed as a weighted sum of independent χ_1^2 random variables, the weights being the eigenvalues of the $(d\times d)$-matrix $\Sigma_{1,d}\,A/6$.

Note that $A = 2\, e_d\, e_d'$, where $e_d = (\sqrt{3}, 1, \ldots, 1)' \in \mathcal{R}^d$ is the eigenvector of $\Sigma_{1,d}$ pertaining to the eigenvalue α_2. Consequently, $\Sigma_{1,d}\, A = 2\alpha_2\, e_d\, e_d'$. Hence, $(d+2)\alpha_2/3$ is a simple eigenvalue of $\Sigma_{1,d}\, A/6$ with eigenvector e_d, and 0 is an eigenvalue of multiplicity $d-1$. ▮

Remark 2. 2. a) Under the additional assumption $P(X = \mu) = 0$, Theorem 2.3 was proved in Henze (1997) by a completely different reasoning.
b) The above proof yields the representation

$$n\, \tilde{b}_{1,d} \;=\; 2 \sum_{r=1}^{d} (W_r'\, e_d)^2. \tag{18}$$

Hence, the skewness statistic of Móri et al. uses the projections of the W_r on the eigenvector e_d.
c) For a distribution with $\tilde{\beta}_{1,d} > 0$, Corollary 3.3 in Serfling (1980) yields

$$\sqrt{n}(\tilde{b}_{1,d} \;-\; \tilde{\beta}_{1,d}) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \tau'\, (A + diag(A))'\, \Sigma\, (A + diag(A))\tau\right).$$

2.3. A new measure of skewness

Theorem 2.2 shows that $b_{1,d}$ is not asymptotically distribution-free within the class of elliptical distributions. The skewness of Móri et al. can be modified to obtain an asymptotically distribution-free statistic, but this property is achieved by projection of the vectors W_r into a particular direction (see (18)). Therefore, one may ask whether there is a skewness measure which gives equal weights to all polynomials of order three as does Mardia's skewness in case of a normal distribution, and at the same time being asymptotically distribution-free within the whole class of elliptically symmetric distributions. The previous subsections show that this will be the case for the statistic

$$V \;=\; 6\; W_1'\, \Sigma_{1,d}^{-1}\, W_1 + \ldots + 6\; W_d'\, \Sigma_{1,d}^{-1}\, W_d + \frac{6}{\alpha_1} \sum_{r \neq s \neq t} \hat{V}_{rst},$$

where $\hat{V}_{rst} = (1/\sqrt{n}) \sum_{i=1}^{n} L_{rst}(Z_i)$. V has a limit chi-squared distribution with $\binom{d+2}{3}$ degrees of freedom if P^X is elliptical. With the notations of

Subsection 2.1, we have

$$\Sigma_{1,d}^{-1} = \frac{1}{c}\left(\begin{array}{c|cccc} (d-2)u+v & -\sqrt{3}u & -\sqrt{3}u & \cdots & -\sqrt{3}u \\ \hline -\sqrt{3}u & du+v & -u & \cdots & -u \\ -\sqrt{3}u & -u & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -u \\ -\sqrt{3}u & -u & \cdots & -u & du+v \end{array}\right),$$

where $c = (v-u)\,((d+1)u+v) = \alpha_1\,\alpha_2$. Hence,

$$\begin{aligned} V &= \frac{6}{\alpha_1\alpha_2}\sum_{r=1}^{d} W_r' \left(\alpha_2\, I_d - \frac{u}{2}A\right) W_r + \frac{6}{\alpha_1}\sum_{r\neq s\neq t} \hat{V}_{rst} \\ &= \frac{6}{\alpha_1}\hat{U}_{n,3}^2 - \frac{3\,u}{\alpha_1\,\alpha_2} n\,\tilde{b}_{1,d}, \end{aligned}$$

where $u = (\alpha_2 - \alpha_1)/(d+2)$. Since the moments μ_4 and μ_6 figuring in the definition of α_1 and α_2 are unknown, they have to be replaced by the corresponding empirical moments. This yields the following result.

Theorem 2.4. *Let X have an elliptically symmetric distribution with expectation μ and nonsingular covariance matrix T such that $E[\{(X-\mu)'T^{-1}(X-\mu)\}^3] < \infty$. Let $\hat{r}_{2k} = \frac{1}{n}\sum_{i=1}^{n}(Z_i'\,Z_i)^k$ and*

$$\hat{\alpha}_1 = \frac{6\hat{r}_6}{d(d+2)(d+4)}, \quad \hat{\alpha}_2 = \frac{3}{d}\left(\frac{\hat{r}_6}{d+2} - 2\hat{r}_4 + d(d+2)\right).$$

Then

$$n\,\tilde{\tilde{b}}_{1,d} := \frac{n}{\hat{\alpha}_1} b_{1,d} - \frac{3\,(\hat{\alpha}_2 - \hat{\alpha}_1)}{(d+2)\,\hat{\alpha}_1\,\hat{\alpha}_2}\, n\,\tilde{b}_{1,d}$$

has a limit chi-squared distribution with $\binom{d+2}{3}$ degrees of freedom.

Remark: A test for elliptical symmetry based on $n\,\tilde{\tilde{b}}_{1,d}$ is consistent against all distributions with $\beta_{1,d} > 0$.

3. MULTIVARIATE KURTOSIS AND THE FOURTH COMPONENT OF NEYMAN'S SMOOTH TEST

Mardia (1970) introduced the measure of multivariate kurtosis

$$b_{2,d} \;=\; \frac{1}{n}\sum_{i=1}^{n}(Z_i'Z_i)^2 \;=\; \frac{1}{n}\left\{\sum_{r=1}^{d}\sum_{i} z_{ri}^4 + \sum_{r\neq s}\sum_{i} z_{ri}^2 z_{si}^2\right\},$$

which is an estimator of $\beta_{2,d} = E[(\tilde{X}'\tilde{X})^2] = E\|\tilde{X}\|^4$. Hence, $b_{2,d}$ only examines the fourth moment of $\|\tilde{X}\|$. Koziol (1989) proposed the alternative kurtosis measure

$$\begin{aligned} b_{2,d}^* \;&=\; \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(Z_i'Z_j)^4 \\ &=\; \frac{1}{n^2}\left\{\sum_{r=1}^{d}\left(\sum_{i} z_{ri}^4\right)^2 + 4\sum_{r\neq s}\left(\sum_{i}(z_{ri}^3 z_{si})\right)^2\right. \\ &\quad +3\sum_{r\neq s}\left(\sum_{i}(z_{ri}^2 z_{si}^2)\right)^2 + 6\sum_{r\neq s\neq t}\left(\sum_{i}(z_{ri}^2 z_{si} z_{ti})\right)^2 \\ &\quad \left. +24\sum_{1\le r<s<t<u\le d}\left(\sum_{i}(z_{ri}z_{si}z_{ti}z_{ui})\right)^2\right\} \end{aligned} \qquad (19)$$

with population counterpart $\beta_{2,d}^* = E[(\tilde{X}'\tilde{Y})^4]$. In contrast to Mardia's kurtosis measure, $b_{2,d}^*$ is a next higher degree analogue of $b_{1,d}$.

To derive a connection of these measures with a component of Neyman's smooth test for multivariate normality, consider the polynomials

$$\begin{gathered} H_4(y_j),\ H_3(y_j)H_1(y_k),\ H_2(y_j)H_2(y_k), \\ H_2(y_j)H_1(y_k)H_1(y_l),\ H_1(y_j)H_1(y_k)H_1(y_l)H_1(y_m) \end{gathered}$$

of degree 4 and the pertaining fourth component

$$\begin{aligned} \hat{U}_{n,4}^2 \;&=\; \frac{1}{n}\left\{\frac{1}{24}\sum_{r}\left(\sum_{i}(z_{ri}^4-3)\right)^2 + \frac{1}{6}\sum_{r\neq s}\left(\sum_{i}(z_{ri}^3 z_{si}\right)^2\right. \\ &\quad +\frac{1}{8}\sum_{r\neq s}\left(\sum_{i}(z_{ri}^2 z_{si}^2-1)\right)^2 + \frac{1}{4}\sum_{r\neq s\neq t}\left(\sum_{i}(z_{ri}^2 z_{si} z_{ti})\right)^2 \\ &\quad \left. + \sum_{r<s<t<u}\left(\sum_{i}(z_{ri}z_{si}z_{ti}z_{ui})\right)^2\right\}. \end{aligned}$$

Since $\sum_i z_{ri} = 0$ and $\sum_i z_{ri}^2 = 1$, one obtains the algebraic identity

$$\hat{U}_{n,4}^2 = \frac{n}{24}\left(b_{2,d}^* - 6\, b_{2,d} + 3\, d(d+2)\right)$$

(Koziol (1989)). Moreover,

$$\begin{aligned}\sqrt{n}\,(b_{2,d} - d(d+2)) &= \sum_{r=1}^{d} \frac{1}{\sqrt{n}} \sum_i (z_{ri}^4 - 3) + \sum_{r \neq s} \frac{1}{\sqrt{n}} \sum_i (z_{ri}^2 z_{si}^2 - 1) \\ &= \frac{\sqrt{24}}{n} \sum_{r=1}^{d} \sum_i L_r(Z_i) + \frac{4}{\sqrt{n}} \sum_{r<s} \sum_i L_r^s(Z_i), \quad (20)\end{aligned}$$

where the polynomials $L_r(z_1, \ldots, z_d) = (z_r^4 - 6z_r^2 + 3)/\sqrt{24}$ and $L_r^s(z_1, \ldots, z_d) = (z_r^2 - 1)\,(z_s^2 - 1)/2$ belong to the building blocks of $\hat{U}_{n,4}^2$.

After centering, the individual terms in (19) are asymptotically normal (cp. (9)); due to the different weights in (19), the covariance matrix $\hat{\Sigma}$ differs from Σ. Hence, the limit law of $\sqrt{n}(b_{2,d} - \beta_{2,d})$ is $\mathcal{N}(0, e'\hat{\Sigma}e)$ with $e = (1, \ldots, 1)'$. If $\tau = 0$, then

$$\beta_{2,d} = \sum_{j=1}^{d} E[\xi_j^4] + \sum_{i \neq j} E[\xi_i^2 \xi_j^2] = 3d + d(d-1) = d(d+2).$$

Using the general results of Section 2, one has to consider two cases to derive the asymptotic distribution of $\hat{U}_{n,4}^2$. If $\tau = 0$, the limit law of $\hat{U}_{n,4}^2$ is a weighted sum of χ_1^2-distributed random variables and, in particular, a chi-squared distribution with $\binom{d+3}{4}$ degrees of freedom under normality. If $\tau \neq 0$, the limit distribution of $(\hat{U}_{n,4}^2/\sqrt{n} - \sqrt{n}\tau'\tau)$ is normal.

3.1. The limit law of Mardia's kurtosis measure under elliptical distributions

In this subsection, we obtain the asymptotic distribution of $\sqrt{n}(b_{2,d} - d(d+2))$ if the underlying distribution P is elliptically symmetric. Now, if $P \notin \mathcal{P}_0^4$ (see (10)), i.e. $\tau \neq 0$, then $\beta_{2,d} \neq d(d+2)$ and, by the above results, $b_{2,d}$ tends to infinity. Hence, we consider elliptical distributions $P \in \mathcal{P}_0^4$. In principle, $\hat{\Sigma}$ can be computed as in (12), taking into account the different weights in (19). Using Corollary 2.1, an elliptical distribution is in $\mathcal{P}_0^4$ if μ_4 takes the 'normal' value 3. $\hat{\Sigma}$ has the entries

$$\begin{aligned}\hat{\sigma}^{11} &= 24E[L_r^2(\tilde{X})] = E[(\xi_r^4 - 6\xi_r^2 + 3)^2] = \mu_8 - 12\mu_6 + 99, \\ \hat{\sigma}^{12} &= 24E[L_r(\tilde{X})L_s(\tilde{X})] = E[(\xi_r^4 - 6\xi_r^2 + 3)(\xi_s^4 - 6\xi_s^2 + 3)] \\ &= \mu_{44} - 12\mu_{42} + 36\mu_{22} + 6\mu_4 - 27 = 3\mu_8/35 - 12\mu_6/5 + 27,\end{aligned}$$

$$\begin{aligned}
\hat\sigma^{21} &= 16E[(L_r^s(\tilde X))^2] \;=\; 4(3\mu_8/35 - 4\mu_6/5 + 7),\\
\hat\sigma^{22} &= 16E[L_r^s(\tilde X)L_r^t(\tilde X)] \;=\; 4(\mu_8/35 - 8\mu_6/15 + 5),\\
\hat\sigma^{23} &= 16E[L_r^s(\tilde X)L_t^u(\tilde X)] \;=\; 4(\mu_8/105 - 4\mu_6/15 + 3),\\
\hat\sigma^{31} &= 4\sqrt{24}E[L_r(\tilde X)L_r^s(\tilde X)] \;=\; 2(\mu_8/7 - 12\mu_6/5 + 21),\\
\hat\sigma^{32} &= 4\sqrt{24}E[L_r(\tilde X)L_s^t(\tilde X)] \;=\; 2(\mu_8/35 - 4\mu_6/5 + 9),
\end{aligned}$$

where $r, s, t, u \in \{1, \ldots, d\}$, $r < s < t < u$. Now, $\hat\sigma^{11}$ appears d times in $\hat\Sigma$, $\hat\sigma^{12}$ $d(d-1)$ times, $\hat\sigma^{21}$ $\binom{d}{2}$ times, $\hat\sigma^{22}$ $2(d-2)\binom{d}{2}$ times, $\hat\sigma^{23}$ $\binom{d}{2}\binom{d-2}{2}$ times, $\hat\sigma^{31}$ $4\binom{d}{2}$ times and, finally, $\hat\sigma^{32}$ $2(d-2)\binom{d}{2}$ times. Summing over all terms yields the variance

$$\begin{aligned}
e'\hat\Sigma e &= \mu_8\frac{d}{105}(d^3 + 12d^2 + 44d + 48) - \mu_6\frac{4d}{15}(d^3 + 8d^2 + 20d + 16)\\
&\quad + d(3d^3 + 20d^2 + 44d + 32).
\end{aligned}$$

Replacing μ_{2k} ($k = 2, 3, 4$) by $r_{2k} = E(X'X)^k$ as in Subsection 2.1 (for elliptical distributions, $r_8 = \frac{\mu_8}{105}d(d+2)(d+4)(d+6)$, see 4.2 below), $e'\hat\Sigma e$ takes the form

$$e'\hat\Sigma e \;=\; r_8 - 4(d+2)r_6 + d(d+2)^2(3d+8).$$

This is the result of Henze (1994a), Example 3.3, letting $\mu_4 = 3$. Under multivariate normality, $e'\hat\Sigma e = 8d(d+2)$, which is the well-known result of Mardia (1970).

Remark: If $P \notin \mathcal{P}_0^4$, the limit law of $\sqrt{n}(b_{2,d} - \beta_{2,d})$ could be determined in a similar way using Theorem 2.1a) and (14) (regarding the necessary computation to obtain the entries of Σ in this case, see Section 4).

3.2. The limit distribution of the fourth component under elliptical symmetry

As second example in this section, we derive the limit law of $24\hat U_{n,4}^2$ if the underlying distribution is elliptically symmetric. Again, we consider elliptical distributions $P \in \mathcal{P}_0^4$ (i.e. with $m_4 = 3$), since otherwise $\hat U_{n,4}^2$ tends to infinity (in this case, one could determine the asymptotic distribution of $\sqrt{n}(\hat U_{n,k}^2/n - \tau'\tau)$ using (14)). As in Subsection 2.1, we have to determine the eigenvalues of the covariance matrix in (12), multiplied with the factor 24. Besides the polynomials L_r and L_r^s, $\hat U_{n,4}^2$ consists of $L_{rs}(z_1, \ldots, z_d) = (z_r^3 - 3z_r)z_s/6$ ($r \neq s$), $L_{rst}(z_1, \ldots, z_d) = (z_r^2 - 1)z_s z_t/2$ ($r \neq s, t$, $s < t$) and $L_{rstu}(z_1, \ldots, z_d) = z_r z_s z_t z_u$ ($r < s < t < u$). First note that the covariance of any polynomial which is part of $b_{2,d}$ and the remaining polynomials is zero since it solely consists of moments which are zero by Proposition 2.1; we have, e.g.,

$$E[(\tilde X^4 - 6\tilde X^2 + 3)(\tilde X^2 - 1)\tilde Y\tilde Z] \;=\; \mu_{611} - 7\mu_{411} + 9\mu_{211} - 3\mu_{11} \;=\; 0.$$

For the same reason, the correlation of L_{rstu} and any other polynomial is zero. Hence, the covariance matrix can be decomposed into three parts.

The matrix pertaining to the polynomials L_{rstu} is a $\binom{d}{4} \times \binom{d}{4}$−diagonal matrix with entries $24\mu_{2222} = 24\mu_8/105$. Hence, $\alpha_1 = 8\mu_8/35$ is an eigenvalue of multiplicity $\binom{d}{4}$.

The second matrix has the nonzero entries

$$\begin{aligned}
\sigma^{41} &= 24E[L_{rs}^2(\tilde{X})] \;=\; 4(\mu_8/7 - 6\mu_6/5 + 9),\\
\sigma^{42} &= 24E[L_{rs}(\tilde{X})L_{sr}(\tilde{X})] \;=\; 4(3\mu_8/35 - 6\mu_6/5 + 9),\\
\sigma^{43} &= 24E[L_{rst}^2(\tilde{X})] \;=\; 12(\mu_8/35 - 2\mu_6/15 + 1),\\
\sigma^{44} &= 24E[L_{rst}(\tilde{X})L_{st}(\tilde{X})] \;=\; 4\sqrt{3}(\mu_8/35 - 2\mu_6/5 + 3),
\end{aligned}$$

where $r, s, t \in \{1, \ldots, d\}, r \neq s \neq t \neq r$. Arranging the polynomials in the form

$$\begin{array}{l}
L_{12},\, L_{21},\, L_{312},\, L_{412}, \ldots, L_{d12},\\
L_{13},\, L_{31},\, L_{213},\, L_{413}, \ldots, L_{d13},\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
L_{1d},\, L_{d1},\, L_{21d},\, L_{31d}, \ldots, L_{d-1,1,d},\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
L_{d-1,d},\, L_{d,d-1},\, L_{1,d-1,d},\, L_{2,d-1,d}, \ldots, L_{d-2,d-1,d}\ ,
\end{array}$$

the matrix under consideration splits into $\binom{d}{2}$ matrices

$$\left(\begin{array}{cc|ccc}
\sigma^{41} & \sigma^{42} & \sigma^{44} & \cdots & \sigma^{44}\\
\sigma^{42} & \sigma^{41} & \sigma^{44} & \cdots & \sigma^{44}\\
\hline
\sigma^{44} & \sigma^{44} & \sigma^{43} & & 0\\
\vdots & \vdots & & \ddots & \\
\sigma^{44} & \sigma^{44} & 0 & & \sigma^{43}
\end{array}\right). \tag{21}$$

Here, the diagonal matrix has dimension $d-2$. For the matrix in (21), $\alpha_2 = \sigma^{43}$ is an eigenvalue of multiplicity $d-3$. The remaining eigenvalues are those solutions of the equation $\big((\sigma^{41} - \lambda)(\sigma^{43} - \lambda) - (\sigma^{44})^2(d-2)\big)^2 = \big(\sigma^{42}(\sigma^{43} - \lambda) - (\sigma^{44})^2(d-2)\big)^2$ which differ from σ^{43}. This yields the additional eigenvalue $\alpha_1 = 8\mu_8/35$ and the two eigenvalues

$$\alpha_{3,4} \;=\; \frac{22}{35}\mu_8 - \frac{28}{5}\mu_6 + 42 \pm \frac{2}{35}\sqrt{24d-23}(\mu_8 - 14\mu_6 + 105)$$

which depend on the dimension d.

Computing the eigenvalues of the third matrix is more involved since all entries are nonzero; up to constant factors due to the different weighting, the matrix corresponds to the covariance matrix of $b_{2,d}$: defining $\sigma^{11} = \hat{\sigma}^{11}, \sigma^{12} = \hat{\sigma}^{12}, \sigma^{21} = 3\hat{\sigma}^{21}/2, \sigma^{22} = 3\hat{\sigma}^{22}/2, \sigma^{23} = 3\hat{\sigma}^{23}/2, \sigma^{31} = (3/2)^{1/2}\hat{\sigma}^{31}$ and $\sigma^{32} = (3/2)^{1/2}\hat{\sigma}^{32}$, and replacing $\hat{\sigma}^{ij}$ in the covariance matrix of Subsection 3.1 by the corresponding σ^{ij} yields the third matrix which we denote by Σ_3. For a proof of the following lemma, see Klar (1998), Lemma 1.3.10.

Lemma 3.1. *Σ_3 has the following eigenvalues: $\alpha_1 = 8\mu_8/35$ is eigenvalue of multiplicity $\binom{d}{2}$ (where $\binom{1}{2} = 0$);*

$$\alpha_5 \;=\; 4\left(\frac{(d+6)}{35}\mu_8 - \frac{2d+8}{5}\mu_6 + 3(d+4)\right)$$

is an eigenvalue of multiplicity $d-1$; finally,

$$\alpha_6 \;=\; \frac{d^2+10d+24}{35}\mu_8 - \frac{4(d^2+6d+8)}{5}\mu_6 + 3(3d^2+14d+16)$$

is a simple eigenvalue.

Summarizing all results, we obtain the following theorem.

Theorem 3.1. *Let X have an elliptically symmetric distribution with expectation μ and nonsingular covariance matrix T. Assume $\tilde{X} \in \mathcal{P}_0^4$, i.e. $E[(\tilde{X}'\tilde{X})^4] < \infty$ and $m_4 = E[\xi_1^4] = 3$, where $\tilde{X} = (\xi_1, \ldots, \xi_d)' = T^{-1/2}(X-\mu)$. Then*

$$24\,\hat{U}_{n,4}^2 \;\xrightarrow{\mathcal{D}}\; \sum_{i=1}^{6} \alpha_i\,\chi^2_{\nu_i}\,,$$

where

$$\alpha_1 = 8\mu_8/35, \qquad \nu_1 = \binom{d}{4} + 2\binom{d}{2},$$

$$\alpha_2 = 12(\mu_8/35 - 2\mu_6/15 + 1), \qquad \nu_2 = (d-3)\binom{d}{2},$$

$$\alpha_{3,4} = \frac{22}{35}\mu_8 - \frac{28}{5}\mu_6 + 42 \pm \frac{2}{35}\sqrt{24d-23}(\mu_8 - 14\mu_6 + 105),\ \nu_{3,4} = \binom{d}{2},$$

$$\alpha_5 = 4\left(\frac{(d+6)}{35}\mu_8 - \frac{2d+8}{5}\mu_6 + 3(d+4)\right), \qquad \nu_5 = d-1,$$

$$\alpha_6 = \frac{d^2+10d+24}{35}\mu_8 - \frac{4(d^2+6d+8)}{5}\mu_6 + 3(d^2+14d+16),\ \nu_6 = 1$$

and $\chi^2_{\nu_i}$ *are independent chi-squared random variables with* ν_i *degrees of freedom.*

Remark: Under multivariate normality, we obtain $\alpha_i = 24$ for $i = 1, \ldots, 6$; since $\sum_{i=1}^{6} \nu_i = \binom{d+3}{4}$, $\hat{U}^2_{n,4}$ has a chi-squared distribution with $\binom{d+3}{4}$ degrees of freedom.

4. THE LIMIT LAW OF VARIANTS OF MULTIVARIATE SKEWNESS AND KURTOSIS

In this section, similar methods as in Sections 2 and 3 are used to treat other statistics such as $b^*_{2,d}$ in (18) which are not directly related to components of the smooth test of fit for multivariate normality, but which are direct higher degree analogues of $b_{1,d}$.

The results are again based on Theorem 2.1 a) which makes no use of the orthogonality of the polynomials (see the remark after 2.1). However, the computation of the covariance matrix Σ pertaining to the polynomials which build up the statistics is more involved: whereas the examples in 2 and 3 make use of the fact that Σ can be computed by (12) not only under the hypothesis of normality but also in the class $\mathcal{P}_0^k$, one always has to compute the covariance matrix in case of nonorthogonal polynomials (i.e. even under the parametric hypothesis) using equation (10).

In the following, we determine the limit distribution of the statistics

$$b_{k00} = \frac{1}{n^2} \sum_{i,j=1}^{n} \left((X_i - \bar{X}_n)' S_n^{-1} (X_j - \bar{X}_n) \right)^k = \frac{1}{n^2} \sum_{i,j=1}^{n} \left(Z_i' Z_j \right)^k ,$$

where k is a positive integer. Again we assume that S_n is nonsingular with probability 1. An alternative notation is

$$b_{k00} = \sum_{\substack{k_1,\ldots,k_d \geq 0 \\ k_1+\ldots+k_d=k}} \binom{k}{k_1 \cdots k_d} \left(\frac{1}{n} \sum_{i=1}^{n} \prod_{r=1}^{d} z_{ri}^{k_r} \right)^2 .$$

The population counterpart of b_{k00} is

$$\begin{aligned} \beta_{k00} &= E[((X-\mu)' T^{-1} (Y-\mu))^k] \\ &= \sum_{\substack{k_1,\ldots,k_d \geq 0 \\ k_1+\ldots+k_d=k}} \binom{k}{k_1 \cdots k_d} \left(E[\prod_{r=1}^{d} \xi_r^{k_r}] \right)^2 , \end{aligned}$$

where $\xi_1, \ldots, \xi_d$ are the components of $\tilde{X} = T^{-1/2}(X - \mu)$. Since b_{k00} is affine-invariant, assume without restriction $\mu = 0$ and $T = I_d$. Again, $E\|X\|^{2k} < \infty$.

The parameterization used in Section 2 is no longer convenient since it requires the partial derivatives of $T^{-1/2}$ with respect to the elements of T^{-1}. Besides the expected value μ, we therefore use the elements $t_{ij}^{-1/2}$ of $T^{-1/2}$ as parameters. The linear representation, which is now required explicitly, is easily found: using (8) and

$$\sqrt{n}(S_n^{-1} - I_d) \;=\; \sqrt{n}(S_n^{-1/2} - I_d)\,(S_n^{-1/2} + I_d),$$

we obtain

$$\sqrt{n}(S_n^{-1/2} - I_d) \;=\; -\frac{1}{2\sqrt{n}} \sum_{i=1}^{n} (X_i X_i' - I_d) + o_P(1).$$

Hence, $l_{ij}(x, \vartheta) = -(x_i x_j - \delta_{ij})/2$. Now, it is not difficult to compute the necessary partial derivatives. Noting that $\partial \xi_r / \partial \mu_i \,|_{(\mu,T)=(0,I_d)} = -t_{ri}^{-1/2}\Big|_{(0,I_d)} = -\delta_{ri}$, we obtain

$$\left. \frac{\partial \xi_1^{k_1} \cdots \xi_d^{k_d}}{\partial \mu_i} \right|_{(0,I_d)} \;=\; -k_i\, \xi_1^{k_1} \cdots \xi_{i-1}^{k_{i-1}}\, \xi_i^{k_i - 1}\, \xi_{i+1}^{k_{i+1}} \cdots \xi_d^{k_d} \tag{22}$$

for $i = 1, \ldots, d$ if $k_i \geq 1$. Using $\partial \xi_r / \partial t_{ij}^{-1/2}\Big|_{(0,I_d)} = \delta_{ir}\, \xi_j$ yields the derivatives

$$\left. \frac{\partial \xi_1^{k_1} \cdots \xi_d^{k_d}}{\partial t_{ij}^{-1/2}} \right|_{(0,I_d)} \;=\; k_i\, \xi_1^{k_1} \cdots \xi_i^{k_i - 1} \cdots \xi_j^{k_j + 1} \cdots \xi_d^{k_d}, \tag{23}$$

for $i, j = 1, \ldots, d$, if $k_i \geq 1$. Therefore all quantities required to compute the covariance matrix Σ are known. Arranging the building blocks $\binom{k}{k_1 \cdots, k_d}^{1/2} x_1^{k_1} \cdots x_d^{k_d}$ of b_{k00} in an arbitrary order and denoting them by $h_l(x)$, $l = 1, \ldots, d^k$, defining $\tau = (\tau_1, \ldots, \tau_{d^k})$ by $\tau_l = E[h_l(\tilde{X})]$ and letting

$$\begin{aligned} v_l(x) \;=\; & h_l(x) \;+\; \sum_{i=1}^{d} E\left(\frac{\partial h_l(\tilde{X})}{\partial \mu_i} \right) x_i \\ & - \frac{1}{2} \sum_{i,j=1}^{d} E\left(\frac{\partial h_l(\tilde{X})}{\partial t_{ij}^{-1/2}} \right) (x_i\, x_j - \delta_{ij}), \end{aligned} \tag{24}$$

we can compute Σ using (10).

Regarding the limit distribution, one has to distinguish two cases as in Section 2. If $\tau = 0$ (which is only possible for odd k), the asymptotic distribution is a weighted sum of independent chi-squared distributed random variables:

$$n\, b_{k00} \;\xrightarrow{\mathcal{D}}\; \sum_{j=1}^{d^k} \lambda_j\, \chi_1^2(j),$$

where the weights are the eigenvalues of Σ. Note that the statistic b_{300} coincides with Mardia's skewness $b_{1,d}$. For higher odd values of k, Henze, Gutjahr and Folkers (1999) determined the weights λ_j under elliptical symmetry.

If $\tau \neq 0$ (which is always the case for non-degenerate distributions if k is even), it follows from (14), using $\tau'\tau = \beta_{k00}$,

$$\sqrt{n}\,(b_{k00} - \beta_{k00}) \;\xrightarrow{\mathcal{D}}\; \mathcal{N}\,(0, 4\tau'\Sigma\tau)\,. \tag{25}$$

To examine the case $\beta_{k00} > 0\,(k \geq 3)$ more closely, we express the variance $\sigma^2 = 4\tau'\Sigma\tau$ of the limit distribution in a different way. To this end, let $h = (h_1, \ldots, h_{d^k})'$ and $v = (v_1, \ldots, v_{d^k})'$. Further define

$$h_{1,k}(x) \;:=\; \tau'\, h(x) \;=\; E[(x'X)^k]. \tag{26}$$

Combining (25) and (10) gives

$$\begin{aligned} \sigma^2 \;&=\; 4\,\tau'\, E[v(\tilde{X})\, v(\tilde{X})']\,\tau \;-\; 4\,\tau'\,(\tau\,\tau')\,\tau \\ &=\; E[(2\tau' v(\tilde{X}))\;(2\tau' v(\tilde{X}))'] \;-\; 4\,\beta_{k00}^2. \end{aligned} \tag{27}$$

Besides $h_{1,k}(x)$, the product $\tau' v(x)$ consists of terms like $x_i \sum_l \tau_l E[\partial h_l(\tilde{X})$ $/\partial \mu_i]$ and $(x_i x_j - \delta_{ij}) \sum_l \tau_l E[\partial h_l(\tilde{X})/\partial t_{ij}^{-1/2}]$. Using (22) and (23), a comparison of the coefficients shows that the sums are given by

$$\sum_{l=1}^{d^k} E[h_l(\tilde{X})]\; E\left[\frac{\partial h_l(\tilde{X})}{\partial \mu_i}\right] \;=\; (-k)\, E[(\tilde{X}'\,\tilde{Y})^{k-1}\eta_i]$$

and

$$\sum_{l=1}^{d^k} E[h_l(\tilde{X})]\; E\left[\frac{\partial h_l(\tilde{X})}{\partial t_{ij}^{-1/2}}\right] \;=\; k\, E[\xi_i(\tilde{X}'\,\tilde{Y})^{k-1}\eta_j].$$

In view of these equations and with the definitions

$$\begin{aligned} a_k &= E\left[(\tilde{X}'\tilde{Y})^{k-1}\tilde{Y}'\right], \\ B_k &= (b_{ij})_{1\le i,j\le d} = E\left[\tilde{X}(\tilde{X}'\tilde{Y})^{k-1}\tilde{Y}'\right], \\ u_k &= (2, -k\,b_{11}, -k\,b_{12}, \ldots, -k\,b_{1d}, -k\,b_{21}, \ldots, -k\,b_{dd}, -2k\,a_k')', \\ Z_k &= (h_{1,k}(\tilde{X}) - \beta_{k00}, \xi_1^2 - 1, \xi_1\xi_2, \ldots, \xi_1\xi_d, \xi_2\xi_1, \ldots, \xi_d^2 - 1, \tilde{X}')', \end{aligned} \tag{28}$$

the asymptotic variance in (26) can be written as $\sigma^2 = u_k' E[Z_k\, Z_k']u_k$. Summarizing, we have the following result.

THEOREM 4.1. *Let the random vector X with expectation μ and nonsingular covariance matrix T satisfy $E[\{(X-\mu)'T^{-1}(X-\mu)\}^k] = E[(\tilde{X}'\tilde{X})^k] < \infty$. Assume that the empirical covariance matrix S_n is nonsingular with probability 1, and that $\beta_{k00} > 0$. Then*

$$\sqrt{n}\,(b_{k00} - \beta_{k00}) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, u_k' E[Z_k\, Z_k']u_k\right),$$

where the $(1 + d^2 + d)$-dimensional vectors u_k and Z_k are defined in (27).

REMARK: In the cases $k = 3$, $k = 4$ and k odd, this is the assertion of Theorem 3.2 in [1], Theorem 2.2 in [7] and Theorem 4.3 in [9], respectively. In these papers, the proofs are based on the theory of V-statistics, which entails the additional requirement that the support of P^X has positive Lebesgue-measure.

EXAMPLE 4.1. In the univariate case $d = 1$, the quantities in (26) and (27) are

$$h_{1,k}(x) = \mu_k\, x^k, \quad a_k = \mu_{k-1}\mu_k, \quad B_k = \mu_k^2,$$
$$u_k = (2, -k\mu_k^2, -2k\mu_{k-1}\mu_k)', \quad Z_k = (\mu_k\tilde{X}^k - \mu_k^2, \tilde{X}^2 - 1, \tilde{X})'$$

and consequently

$$E[Z_k\, Z_k'] = \begin{bmatrix} \mu_k^2(\mu_{2k} - \mu_k^2) & \mu_k(\mu_{k+2} - \mu_k) & \mu_k\,\mu_{k+1} \\ \mu_k(\mu_{k+2} - \mu_k) & \mu_4 - 1 & \mu_3 \\ \mu_k\,\mu_{k+1} & \mu_3 & 1 \end{bmatrix}.$$

This yields

$$\begin{aligned} u_k' E[Z_k\, Z_k']u_k &= 4\mu_k^2\Big(\mu_{2k} - k\,\mu_k\,\mu_{k+2} + \frac{k^2}{4}\mu_4\,\mu_k^2 - \frac{(k-2)^2}{4}\mu_k^2 \\ &\quad + k^2\,\mu_3\,\mu_{k-1}\,\mu_k - 2k\,\mu_{k-1}\,\mu_{k+1} + k^2\,\mu_{k-1}^2\Big). \end{aligned}$$

In particular, for $k=3$,

$$u_3' E[Z_3\, Z_3'] u_3 \;=\; 4\,\mu_3^2\,(\mu_6 + 9 + 9\mu_3^2(\mu_4 - 1)/4 - 6\mu_4 - 3\mu_3\mu_5 + 11\mu_3^2)$$

(cp. Baringhaus and Henze (1992a), Example 3.3). After normalizing, we obtain the well-known result of Gastwirth and Owens (1977).

In the remainder of this section, we derive the variance of the limit normal distribution of $\sqrt{n}(b_{2k,0,0} - \beta_{2k,0,0})\,(k \geq 2)$ under elliptical symmetry. As a special case, the asymptotic distribution of Koziol's kurtosis measure $b_{2,d}^* = b_{400}$ (cp. Section 2) is obtained. Under elliptical symmetry, $\beta_{2k,0,0}$ has a simple representation.

Theorem 4.2. *Let $X = (X_1, \ldots, X_d)'$ have a spherical distribution with $E\|X\|^{2k} < \infty$, and let Y be an independent copy of X. Then*

$$r_{2k} \;=\; E[(X'X)^k] \;=\; \mu_{2k}\,\frac{d(d+2)\cdots(d+2k-2)}{1\cdot 3\cdot 5\cdots(2k-1)},$$

where $\mu_{2k} = E[X_1^{2k}]$. Furthermore,

$$\beta_{2k,0,0} \;=\; E[(X'Y)^{2k}] \;=\; r_{2k}^2\,\frac{1\cdot 3\cdot 5\cdots(2k-1)}{d(d+2)\cdots(d+2k-2)},$$

and

$$B_{2k} \;=\; E[X\,(X'\,Y)^{2k-1}\,Y'] \;=\; r_{2k}^2\,\frac{1\cdot 3\cdot 5\cdots(2k-1)}{d^2(d+2)\cdots(d+2k-2)}\,I_d.$$

Proof. Assume $d > 1$. If $N \sim \mathcal{N}_d(0, I_d)$, it is well-known that

$$E\|N\|^s \;=\; \frac{\Gamma((d+s)/2)\,2^{s/2}}{\Gamma(d/2)}$$

and hence $E[(N'N)^k] \;=\; \Gamma(d/2+k)\,2^k/\,\Gamma(d/2) \;=\; d(d+2)\cdots(d+2(k-1))$. Using Theorem 2.1, one obtains $\mu_{2k}/\mu_{2k}^N \;=\; r_{2k}/r_{2k}^N$, where μ_{2k}^N and r_{2k}^N denote the corresponding quantities for N. Since $\mu_{2k}^N = 1\cdot 3\cdot 5\cdots(2k-1)$, the first assertion follows.

To show the remaining parts, let U be uniformly distributed on $S_d = \{x \in \mathcal{R}^d : \|x\| = 1\}$. S_d has the surface area $A = 2\pi^{d/2}/\Gamma(d/2)$. If $u_1 \in \mathcal{R}^d$ with $\|u_1\| = 1$, Lemma 2.5.1 of Fang and Zhang (1990) yields

$$E[\,(u_1'U)^{2k}\,] \;=\; \frac{\Gamma(d/2)\,\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(k+\frac{d}{2})} \;=\; \prod_{j=0}^{k-1}\frac{2j+1}{d+2j}\,.$$

Since P^X is spherically symmetric, the decomposition $X \stackrel{\mathcal{D}}{=} RU$, where $R \stackrel{\mathcal{D}}{=} \|X\|$ and R, U are independent, gives

$$h_{1,2k}(z) \;=\; E[(z'X)^{2k}] \;=\; \|z\|^{2k} E[R^{2k}] \prod_{j=0}^{k-1} \frac{2j+1}{d+2j}. \tag{29}$$

Hence, the second assertion follows. In view of Proposition 2.1, it is not difficult to see that $B_{2k} = b_{11}\, I_d$. Using $trace(B_{2k}) = \beta_{2k,0,0}$, we obtain $b_{11} = \beta_{2k,0,0}/d$. ▮

In view of Theorem 2.1, the vector a_{2k} defined in (27) is zero. Putting

$$\delta = \prod_{j=0}^{k-1} (2j+1)/(d+2j), \tag{30}$$

Theorem 4.2 yields

$$u_k \;=\; (2, -2k\,\delta\, r_{2k}^2/d \cdot e_1', -2k\,\delta\, r_{2k}^2/d \cdot e_2', \ldots, -2k\,\delta\, r_{2k}^2/d \cdot e_d', 0')',$$

where $e_j = (0,\ldots,1,0,\ldots,0)'$ denotes the jth unit vector in $\mathcal{R}^d$. Writing $L_{2k} = h_{1,2k}(\tilde X) - \beta_{2k,0,0}$ and $W_j = (\xi_j\xi_1, \ldots, \xi_j\xi_{j-1}, \xi_j^2 - 1, \xi_j\xi_{j+1}, \ldots, \xi_j\xi_d)'$ for $j = 1,\ldots,d$, the vector Z_{2k} in (27) can be written as $Z_{2k} = (L_{2k}, W_1', \ldots, W_d', \tilde X')'$. Using (29) and Theorem 4.2, we obtain $L_{2k} \;=\; \delta\, r_{2k}\,(\|\tilde X\|^{2k} - r_{2k})$, which yields $E[L_{2k}\tilde X] = 0,\; E[L_{2k}^2] = \delta^2\, r_{2k}^2\,(r_{4k} - r_{2k}^2)$ and

$$E[L_{2k}\, W_j] \;=\; \delta\, r_{2k}\,(\frac{r_{2k+2}}{d} - r_{2k})\; e_j \quad (j = 1,\ldots,d).$$

Furthermore, $E[\tilde X W_j'] = O$ $(j = 1,\ldots,d)$, where O is the zero matrix of order d. Defining $(d \times d)$-matrices

$$B_{ij} \;=\; E[W_i\, W_j'] \;=\; (b_{k,l}^{(i,j)})_{1\le k,l\le d} \quad (i,j = 1,\ldots,d)$$

and putting $\rho = \delta\, r_{2k}(\frac{r_{2k+2}}{d} - r_{2k})$, the matrix $E[Z_{2k}Z_{2k}']$ of order $1 + d^2 + d$ can be written as

$$E[Z_{2k}Z_{2k}'] \;=\; \begin{bmatrix} E[L_{2k}^2] & \rho e_1' & \rho e_2' & \cdots & \rho e_d' & 0' \\ \rho e_1' & B_{11} & B_{12} & \cdots & B_{1d} & O \\ \rho e_2' & B_{21} & B_{22} & \cdots & B_{2d} & O \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \rho e_d' & B_{d1} & B_{d2} & \cdots & B_{dd} & O \\ 0 & O & O & \cdots & O & I_d \end{bmatrix}.$$

Noting that $b_{i,i}^{(i,i)} = 3r_4/(d(d+2)) - 1\,(i = 1,\dots,d)$ and $b_{i,j}^{(i,j)} = r_4/(d(d+2)) - 1\,(i,j = 1,\dots,d,\ i \neq j)$, $\sigma_{2k}^2 = u_{2k}' E[Z_{2k}Z_{2k}']u_{2k}$ takes the form

$$\sigma_{2k}^2 \;=\; 4\delta^2 r_{2k}^2 \left(r_{4k} - (k-1)^2 r_{2k}^2 - \frac{2k}{d} r_{2k} r_{2k+2} + \frac{k^2}{d^2} r_4 r_{2k}^2 \right), \quad (31)$$

where δ is defined in (30). Summarizing, we have the following result.

THEOREM 4.3. *Let X be elliptically symmetric with expectation μ and nonsingular covariance matrix T. Further, let $E[\{(X-\mu)'T^{-1}(X-\mu)\}^k] < \infty$, and assume that the empirical covariance matrix S_n is nonsingular with probability 1. Then*

$$\sqrt{n}(b_{2k,0,0} - \beta_{2k,0,0}) \;\xrightarrow{\mathcal{D}}\; \mathcal{N}\left(0, \sigma_{2k}^2\right),$$

where σ_{2k}^2 is given by (31). In particular,

$$\begin{aligned} &\sqrt{n}(b_{2,d}^* - \beta_{2,d}^*) = \sqrt{n}(b_{400} - \beta_{400}) \\ &\xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{36 r_4^2}{d^2(d+2)^2}\left(r_8 + 4\frac{r_4}{d}\left(\frac{r_4^2}{d} - r_6\right) - r_4^2\right)\right). \end{aligned}$$

REMARK: The result requires neither that the support of P^X has positive Lebesgue-measure nor that $P(X = 0) = 0$ as assumed in Henze (1994b), Corollary 3.1, for the case $k = 2$.

COROLLARY 4.1. *If X has a non-degenerate normal distribution, then*

$$\sqrt{n}\left(b_{2k,0,0} - \prod_{j=0}^{k-1} (2j+1)\,(d+2j) \right) \;\xrightarrow{\mathcal{D}}\; \mathcal{N}\left(0, \sigma_{2k}^2\right),$$

where

$$\sigma_{2k}^2 \;=\; 4 \prod_{j=0}^{k-1} (2j+1)^2 (d+2j) \left[\prod_{j=k}^{2k-1} (d+2j) - \frac{d+2k^2}{d} \prod_{j=0}^{k-1} (d+2j) \right].$$

Acknowledgements. This work is based on a part of the author's doctoral thesis at the University of Karlsruhe, written under the supervision of Professor Norbert Henze, whose guidance is gratefully appreciated.

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