A Test Procedure for Uniformity on the Stiefel Manifold Based on Projection

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ABSTRACT : This paper proposes a new procedure to test uniformity on the Stiefel manifold. As well as the theoretical analysis of the test procedure, numerical experiments are conducted to illustrate usage and efficiencies through the power under alternative hypotheses.

Keywords: Anderson-Darling test; Goodness of fit test; Kolmogorov-Smirnov test; Spherical distribution; Stiefel manifold; Uniform distribution

MSC: primary 62H11, secondary 62H15

1. Introduction

Let X be a $p \times r\,(p \geq r)$ random matrix which satisfies $X'X = I_r$, where A' denotes the transpose of the matrix A, and I_d is the $d \times d$ identity matrix. Consider the testing problem of the null hypothesis

$$H_0 : X \text{ are uniformly distributed over } V_r(\mathbb{R}^p), \tag{1}$$

where $V_r(\mathbb{R}^p)$ stands for the *Stiefel manifold* of orthonormal r-frames in the *Euclidean space* $\mathbb{R}^p$.

To address the hypothesis (1), Jupp (2001) reviewed the Rayleigh test and proposed a procedure based on the modified Rayleigh's statistic defined by

$$S^* = \left(1 - \frac{1}{2N} + \frac{1}{2(pr+2)N} S\right) S, \tag{2}$$

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to test uniformity under the matrix Fisher distribution, where $S = Np\,\mathrm{tr}(\bar{X}'\bar{X})$, $\bar{X} = (1/N)\sum_{j=1}^{N} X_j$, and $X_1, \ldots, X_N$ are N independent random copies of X. Furthermore, Arnold and Jupp (2013) provided statistics for tests of uniformity and location of the orthogonal axial frames on the quotient manifold $V_r(\mathbb{R}^p)/\mathbb{Z}_2^r$, where $\mathbb{Z}_2^r = \{(\varepsilon_1, \ldots, \varepsilon_r) : \varepsilon_j = \pm 1\}$, and gave examples of location tests for certain parametric models.

In the $r = 1$ case, the Stiefel manifold $V_1(\mathbb{R}^p)$ is equivalent to the unit sphere $\mathbb{S}^{p-1}$ in $\mathbb{R}^p$, thus replacing X with $\boldsymbol{X}$, a p-dimensional random unit vector, the null hypothesis (1) corresponds to

$$H_0^* : \boldsymbol{X} \text{ are uniformly distributed on } \mathbb{S}^{p-1}. \tag{3}$$

For this hypothesis, for example, Fang et al. (1993) proposed a necessary test for sphericity based on non-parametric goodness of fit Wilcoxon-type statistic for two-sample problem by utilizing the well-known fact that if $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$ are independent and uniformly distributed on $\mathbb{S}^{p-1}$, then for each constant $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{S}^{p-1}$, all random variables $\boldsymbol{a}_l'\boldsymbol{X}_j$'s have the same distribution.

On the other hand, Cai et al. (2013) considered the asymptotic behavior of the pairwise angles between vectors which are randomly and uniformly distributed on $\mathbb{S}^{p-1}$, and observed the empirical distributions of the maximum and the minimum of angles to test for spherical symmetry (for characteristics of spherical symmetry, see, e.g., Fang and Zhang ((1990, Sections 5.5 and 5.6))). In terms of the power of tests for uniformity on $\mathbb{S}^{p-1}$, Figueiredo (2007) compared some typical procedures, namely, Rayleigh's test, Giné's test and Ajne's test through numerical experiments in some dimensions. These procedures hypothesize the von Mises-Fisher distribution as alternative distribution against the uniform distribution in the sphere.

An interesting approach for testing for uniformity on $\mathbb{S}^{p-1}$ was invented by Cuesta-Albertos et al. (2009). Let $\{\boldsymbol{X}_j\}_{j=1}^{N}$ be an independent random sample from $\mathbb{S}^{p-1}$. Take another random sample $\{\boldsymbol{U}_l\}_{l=1}^{k}$ of $\mathbb{S}^{p-1}$, independent from $\boldsymbol{X}_j$'s. Under these conditions, Cuesta-Albertos et al. (2009) introduced random variables $Z_{lj} = \boldsymbol{U}_l'\boldsymbol{X}_j$, the random projections of $\boldsymbol{X}_j$ on $\boldsymbol{U}_l$, to test the null hypothesis H_0^* with employing the *Kolmogorov-Smirnov statistic* defined by

$$D_{N;l} = \sup_z |F_{N;l}(z) - F_0(z)|\,, \quad l = 1, \ldots, k, \tag{4}$$

and performed numerical experiments, where $F_{N;l}$ and F_0 are the empirical and the population distribution function of Z_{lj}, respectively (see, e.g., Gibbons and Chakraborti (2010, Section 4.3)).

In this paper, we develop a new test procedure for uniformity on the Stiefel manifold patterned after the methodology devised by Cuesta-Albertos et al. (2009). In Section 2, we prepare for some results concerned with matrix spherical distributions which includes the uniform distribution on the Stiefel manifold $V_r(\mathbb{R}^p)$

to provide the probability density function (pdf) and the cumulative distribution function (cdf) of Z_{lj}. On the basis of pdf and cdf of Z_{lj}, we describe the procedure to test the null hypothesis H_0^* in substitution for H_0, and run a simulation study to examine the consistency and the power of our test in Section 3.

2. Test statistic and its distribution

Let X be a $p \times r$ random matrix uniformly distributed on the Stiefel manifold $V_r(\mathbb{R}^p)$ in $\mathbb{R}^p$, that is, it holds that $X'X = I_r$. Then the following proposition is given.

Proposition. *Let a $p \times r$ random matrix X be uniformly distributed over the Stiefel manifold $V_r(\mathbb{R}^p)$, $p \geq r$. For a fixed vector $\boldsymbol{a} \in \mathbb{S}^{r-1}(= V_1(\mathbb{R}^r))$ the p-dimensional random vector $X\boldsymbol{a}$ is uniformly distributed on the unit sphere $\mathbb{S}^{p-1}$ in $\mathbb{R}^p$.*

Proof. According to (i) of Lemma 3.1.3 in Fang and Zhang (1990, p. 95), if a $p \times r$ random matrix X is uniformly distributed over $V_r(\mathbb{R}^p)$, then, fixed $\boldsymbol{a} \in \mathbb{S}^{r-1}$, $X\boldsymbol{a}$ is uniformly distributed over $V_1(\mathbb{R}^p)$, namely, $\mathbb{S}^{p-1}$.

Henceforth if an m-dimensional unit vector $\boldsymbol{U}$ is uniformly distributed on the sphere $\mathbb{S}^{m-1}$ on $\mathbb{R}^m$, we will write $\boldsymbol{U} \sim \mathbb{S}^{m-1}$.

Remark 1. Following (ii) of Lemma 3.1.3 by Fang and Zhang (1990, p. 94), it is obvious that if we take a r-dimensional random unit vector $\boldsymbol{V}$ instead of $\boldsymbol{a}$, independent from X, the assertion in the Proposition holds since $X\boldsymbol{V} \stackrel{d}{=} X\boldsymbol{a}$.

By making use of the result above, it easily follows that if $\{X_j\}_{j=1}^N$ is a uniform random sample drawn from $V_r(\mathbb{R}^p)$, then the N random vectors $\{X_j\boldsymbol{a}\}_{j=1}^N$ are all independent and uniformly distributed on $\mathbb{S}^{p-1}$. Hence, hereafter, we would like to focus on a problem of testing the null hypothesis H_0^* defined by (3). On applying Proposition 1 and Lemma 2 in Iwashita and Klar (2014), we obtain the following result which plays an important part in constructing our testing procedure for H_0^*.

Theorem. *Suppose $\{X_j\}_{j=1}^N$ is a random sample from the uniform distribution over the Stiefel manifold $V_r(\mathbb{R}^p)$ and $\boldsymbol{U}$, independent from X_j's, is a p-dimensional random vector drawn from some continuous distribution with support on $\mathbb{S}^{p-1}$, the unit sphere in $\mathbb{R}^p$. Then, for any r-dimensional unit vector $\boldsymbol{a}$, respective*

N random variables

$$Z_j = \boldsymbol{U}' X_j \boldsymbol{a}, \quad j = 1, \ldots, N, \tag{5}$$

are independent each with probability density function

$$f_p(z) = \begin{cases} \dfrac{1}{\mathrm{B}\left(1/2, (p-1)/2\right)} (1-z^2)^{(p-3)/2}, & |z| \leq 1, \\ \\ 0, & |z| > 1, \end{cases} \tag{6}$$

and the cumulative distribution function

$$F_p(z) = \Pr[Z_j \leq z] = \begin{cases} 0, & z < -1, \\ \\ \dfrac{1}{2}\left[1 + \mathrm{sgn}(z)\, G_p(z^2)\right], & |z| \leq 1, \\ \\ 1, & z > 1, \end{cases} \tag{7}$$

where $\mathrm{B}(\alpha, \beta)$ *is the beta function with the parameters* (α, β)*, the sign function* $\mathrm{sgn}(x)$ *is defined as*

$$\mathrm{sgn}(x) = \begin{cases} -1, & x < 0, \\ \\ 0, & x = 0, \\ \\ 1, & x > 0, \end{cases} \tag{8}$$

and

$$G_p(u) = \frac{1}{\mathrm{B}\left(1/2, (p-1)/2\right)} \int_0^u x^{-1/2}(1-x)^{(p-3)/2} dx, \tag{9}$$

the cumulative distribution function of beta distribution with the parameters $(1/2, (p-1)/2)$*. Additionally, let* $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_N, \boldsymbol{U}_{N+1}, \ldots, \boldsymbol{U}_{kN}$*,* $k \in \mathbb{N}$*, be independent random copies of* $\boldsymbol{U}$*, then* kN *random variables*

$$Z_{lj} = \boldsymbol{U}'_{k(j-1)+l} X_j \boldsymbol{a}, \quad l = 1, \ldots, k; j = 1, \ldots, N, \tag{10}$$

are independent and have the same distribution as the Z_j*'s.*

Proof. Consulting with Lemma 2 in Iwashita and Klar (2014), Z_j are mutually independent and their common characteristic function, denoted by $\Psi(t)$, can be expressed as, for $j = 1, \ldots, N$,

$$\Psi(t) \equiv \mathrm{E}[\exp(itZ_j)] = {}_0F_1\left(\frac{p}{2}; -\frac{1}{4}t^2\right), \quad t \in \mathbb{R}, \quad i = \sqrt{-1}, \tag{11}$$

where ${}_0F_1(\cdot;\cdot)$ designates the generalized hypergeometric function.

On the other hand, a straightforward calculation based on (6) with making a change of variables from z to $\cos\theta$ $(0 \leq \theta \leq \pi)$ and use of Lemma 3.1 in Fang et al. (1990, p. 71) yields

$$\begin{aligned} \int_{-\infty}^{\infty} \exp(itz)\, f_p(z) dz &= \frac{1}{\mathrm{B}(1/2, (p-1)/2)} \int_0^{\pi} \exp(it\cos\theta) \sin^{p-2}\theta d\theta \\ &= {}_0F_1\left(\frac{p}{2}; -\frac{1}{4}t^2\right), \end{aligned} \tag{12}$$

which implies the density of Z_j can be expressed as (6).

We next consider the distribution function of $Y = Z_j^2$. Since the pdf $f_p(z)$ given by (6) is an even function, it holds, for $0 \leq y \leq 1$,

$$\Pr[Y \leq y] = \Pr[-\sqrt{y} \leq Z_j \leq \sqrt{y}] = 2\Pr[0 \leq Z_j \leq \sqrt{y}],$$

therefore the density of Y, $g_p(y) = (d/dy)\Pr[Y \leq y]$, is given by

$$\begin{aligned} g_p(y) &= \frac{d}{dy}(2\Pr[0 \leq Z_j \leq \sqrt{y}]) \\ &= \begin{cases} \dfrac{1}{\mathrm{B}(1/2, (p-1)/2)} y^{-1/2}(1-y)^{(p-3)/2}, & 0 \leq y \leq 1, \\ \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{13}$$

which reveals $Y = Z_j^2$ follows the beta distribution with the parameters $(1/2, (p-1)/2)$.

Hence we finally obtain the cumulative distribution function (cdf) $F_p(z) = \Pr[Z_j \leq z]$ for $|z| \leq 1$ as follows:

$$F_p(z) = \begin{cases} \dfrac{1}{2}\Pr[z^2 \leq Z_j^2] = \dfrac{1}{2}\left(1 - G_p(z^2)\right), & -1 \leq z \leq 0, \\ \\ 1 - \dfrac{1}{2}\Pr[z^2 \leq Z_j^2] = \dfrac{1}{2}\left(1 + G_p(z^2)\right), & 0 \leq z \leq 1, \end{cases} \tag{14}$$

where

$$G_p(u) = \frac{1}{\mathrm{B}(1/2, (p-1)/2)} \int_0^u x^{-1/2}(1-x)^{(p-3)/2} dx,$$

the proof is complete.

Remark 2. The results above hold when we set r-dimensional random unit vector $\boldsymbol{V}$, instead of a constant unit vector $\boldsymbol{a}$. Furthermore, if we substitute p-dimensional fixed unit vector $\boldsymbol{b}$ for $\boldsymbol{U}$ on (5), then the assertion also holds.

Remark 3. The results (6) and (7) essentially coincide with (3) and (4) given by Juan and Prieto (2001), respectively.

3. Numerical experiments

Constitute a $p \times r$ $(p \geq r)$ random matrix Q by a random sample $R_{11}, \dots, R_{1r}, R_{21}, \dots, R_{pr}$ from a random variable R on $\mathbb{R}$, where the distribution of R is absolutely continuous with respect to Lebesgue measure, and $\mathrm{E}[R] = 0$, $\mathrm{Var}[R] < \infty$, as

$$Q = \begin{bmatrix} R_{11} & R_{12} & \dots & R_{1r} \\ R_{21} & R_{22} & \dots & R_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ R_{p1} & R_{p2} & \dots & R_{pr} \end{bmatrix}. \tag{15}$$

Setting $X = Q(Q'Q)^{-1/2}$, where $A^{-1/2}$ symbolizes a square root matrix of A^{-1}, X is a random matrix having a distribution defined on the Stiefel manifold $V_r(\mathbb{R}^p)$, since $X'X = I_r$ (see, e.g., Gupta and Nagar (2000, p. 288)). In addition, if we hypothesize the random variable R possesses the standard normal distribution $N(0, 1)$, then the $p \times r$ random matrix X is uniformly distributed over $V_r(\mathbb{R}^p)$ (see, for details, Fang and Zhang (1990, Chapter III)).

For setting the alternatives, we should notice the results in Cuesta-Albertos et al. (2009). Cuesta-Albertos et al. (2009) showed through numerical experiments that the test based on Rayleigh's statistic could not demonstrate good performance in terms of the power for the uniformity on $\mathbb{S}^{p-1}$, $p = 2, 3$ when the underlying distribution was a scale normal projected model, that is, the data were generated by $\boldsymbol{Z}_b/||\boldsymbol{Z}_b||$, where $\boldsymbol{Z}_b \sim N_p(\boldsymbol{0}, \Sigma_b)$, $\boldsymbol{0} = (0, \dots, 0)' \in \mathbb{R}^p$ and Σ_b is a $p \times p$ matrix with $1 + b^2$ in the diagonal and $2b$ outside for $b \in \mathbb{R}$.

Taking account of the above, we perform some numerical experiments to assess the performance of our test procedure. We consider the following models as the underlying distribution of a random variable Y to generate $R = Y - \mathrm{E}[Y]$, i.e., $\mathrm{E}[R] = 0$, as follows;

M1 : The standard normal distribution $N(0, 1)$ (Null distribution)

M2 : Student's t-distribution with 5 degrees of freedom (Pseudo null distribution)

M3 : Exponential distribution of which density $f(y) = \exp(-y),\ y > 0$

M4-1, 2 : χ^2-distribution with 1 and 4 degrees of freedom

M5-1, 2 : Weibull distribution $Wei(\alpha, \beta)$ with parameters $(\alpha, \beta) = (0.5, 1)$ and $(2, 1)$ of which density has a form

$$f(y) = \frac{\beta}{\alpha}\left(\frac{y}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{y}{\alpha}\right)^{\beta}\right\}, \quad y > 0$$

M6 : The skew-normal distribution $SN(0, 1, 10)$ (the notation $SN(\xi, \sigma, \alpha)$ is used in Dalla Valle (2004, p. 5)).

For all models, we conduct experiments based on 10^4 replications with Type I error $\alpha = 0.05$ and set the fixed directions of projection (i.e., non-random projection) $\boldsymbol{b}'X\boldsymbol{a}$ with $\boldsymbol{a} = r^{-1/2}\boldsymbol{j}_r, r^{-1/2}\boldsymbol{l}_r$ and $\boldsymbol{b} = p^{-1/2}\boldsymbol{j}_p, p^{-1/2}\boldsymbol{l}_p$, where $\boldsymbol{j}_m$ is an m-dimensional column vector of ones and $\boldsymbol{l}_m = (1, -1, \ldots, (-1)^m)' \in \mathbb{R}^m$. The parameters N (i.e., the number of copies of X above), p and r are fixed as follows; for non-random projection, $N = 5, 20$, $p = 5(2)9$, $r = 2, 3, 4$, and $N = 20$, $p = 5(2)9$, $r = 2, 3, 4$ for random projection. Concerning the choice of the goodness of fit test procedures, we employ the *Anderson-Darling* test (A-D) (see, for example, Gibbons and Chakraborti (2010, pp.137–142)) in addition to *Kolmogorov-Smirnov* (K-S) test given in (4). For comparison, we also show the results on the *Rayleigh* test (Ray) by making use of (2).

All results are tabulated in Tables 1-6. From these numerical experiments, we can draw the following conclusions;

(1) Under the model M1, all test procedures succeed in holding the level of significance. However, there is no obvious difference between M1 and M2 regarding the empirical power.

(2) In most cases, power of the Anderson-Darling test was higher than for its competitors. Referring to Figures 1 and 2, describing (7), one may notice that Anderson-Darling's test is good at detecting the difference between the null and the empirical distribution due to their tail behavior (see, Anderson and Darling (1952)).

(3) In comparison of Tables 2 and 3, the performance due to projection based on fixed unit vectors $p^{-1/2}\boldsymbol{j}_p$ and $r^{-1/2}\boldsymbol{j}_r$ is better than the performance based on random projections. However, Table 6 shows the test procedure by using the fixed vectors $p^{-1/2}\boldsymbol{l}_p$ and $r^{-1/2}\boldsymbol{l}_r$ fails to distinguish between underlying distributions.

Finally, we will confirm the performance of our test procedure under a typical distribution on $V_r(\mathbb{R}^p)$, that is, a $p \times r$ random matrix Q has the von Mises-Fisher matrix distribution (see, for example, Khatri and Mardia (1977)) with setting the $p \times r$ parameter matrix $F = (f_{ij})$ of the distribution as follows:

Model F1-F3 : $f_{ij} = c \ (i = 1, \dots p; j = 1, \dots, r)$ with $c = 0.2$ (F1), 0.4 (F2), 0.6 (F3)

Model F4-F6 : $f_{ij} = c\,\delta_{ij} \ (i = 1, \dots p; j = 1, \dots, r)$ with $c = 1$ (F4), 2 (F5), 4 (F6), where δ_{ij} is the Kronecker symbol.

The results are given in Table 7. For cases (F1)-(F6), the test procedure based on A-D test works slightly better than the one based on K-S does. Also, in the case (F1)-(F3), the procedure based on A-D test shows results much better than the others. On the contrary, for the cases (F4)-(F6), the new tests are inferior to Rayleigh test.

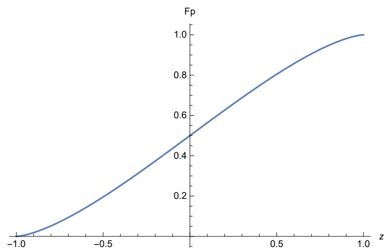

Figure 1. cdf of (7) in the case $p = 4$

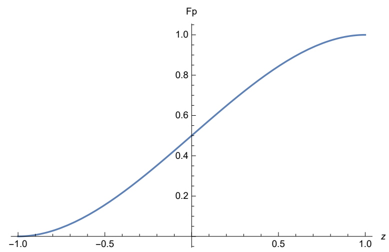

Figure 2. cdf of (7) in the case $p = 5$

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Table 1. Empirical power ($\times 10^2$) for non-random projection $\boldsymbol{b}'X\boldsymbol{a}$ when $\boldsymbol{a} = r^{-1/2}\boldsymbol{j}_r, \boldsymbol{b} = p^{-1/2}\boldsymbol{j}_p$ and $N = 5$

p	r	Stat	$N(0,1)$	t_5	$Exp(1)$	χ_1^2	χ_4^2	$Wei(0.5,1)$	$Wei(2,1)$	$SN(0,1,10)$
5	2	K-S	5.08	4.86	11.80	19.74	9.30	41.99	5.49	7.25
		A-D	5.02	4.75	15.54	27.20	10.74	56.83	5.90	8.19
		Ray	4.46	4.19	6.21	9.05	5.32	18.16	4.50	5.08
	3	K-S	4.98	4.71	15.48	25.92	10.38	51.94	5.68	7.57
		A-D	4.80	4.93	16.75	29.27	11.05	60.75	5.90	7.60
		Ray	3.86	3.65	5.94	7.58	4.58	15.58	3.96	4.09
	4	K-S	4.96	5.08	17.45	28.49	11.35	56.46	6.48	8.16
		A-D	4.89	5.21	16.88	28.86	10.87	61.24	6.11	7.75
		Ray	3.36	3.61	4.89	6.55	4.28	11.87	3.84	4.15
7	2	K-S	5.15	5.55	11.42	18.02	8.02	38.73	5.48	6.99
		A-D	5.02	5.39	15.70	27.30	10.46	56.78	5.99	8.41
		Ray	4.12	4.72	5.84	8.19	4.93	16.81	4.02	4.60
	3	K-S	5.22	5.35	14.84	23.88	9.31	51.34	6.50	7.31
		A-D	5.20	5.38	17.91	30.64	11.15	63.74	6.70	8.03
		Ray	3.94	4.02	5.44	7.30	4.45	16.07	4.08	4.64
	4	K-S	5.20	4.94	17.01	28.42	10.92	57.74	6.34	7.87
		A-D	5.26	5.02	18.81	32.56	11.70	67.04	6.37	8.08
		Ray	4.12	3.50	5.04	7.02	4.23	13.98	3.93	3.92
9	2	K-S	4.92	5.04	10.54	16.10	7.38	36.48	5.72	6.45
		A-D	4.83	4.93	15.16	26.36	9.77	55.67	6.52	7.42
		Ray	3.89	4.27	6.13	7.17	4.51	15.91	4.31	4.78
	3	K-S	5.30	5.13	13.85	22.03	9.10	48.86	5.86	7.12
		A-D	5.04	5.04	17.22	29.58	11.28	63.60	6.28	7.37
		Ray	4.04	3.91	5.70	7.08	4.55	15.04	4.22	4.43
	4	K-S	5.14	4.80	15.90	26.30	10.36	57.84	5.95	8.56
		A-D	4.94	4.58	19.44	32.60	11.68	69.99	6.25	9.09
		Ray	3.94	3.52	5.12	6.92	4.50	14.07	4.38	3.87

Table 2. Empirical power ($\times 10^2$) for non-random projection $\boldsymbol{b}'X\boldsymbol{a}$ when $\boldsymbol{a} = r^{-1/2}\boldsymbol{j}_r, \boldsymbol{b} = p^{-1/2}\boldsymbol{j}_p$ and $N = 20$

p	r	Stat	$N(0,1)$	t_5	$Exp(1)$	χ_1^2	χ_4^2	$Wei(0.5,1)$	$Wei(2,1)$	$SN(0,1,10)$
5	2	K-S	5.22	5.03	32.97	57.48	18.88	92.85	8.01	12.15
		A-D	4.94	4.77	43.55	71.59	24.55	97.49	8.90	15.24
		Ray	4.92	5.15	17.78	33.77	10.02	77.42	5.94	7.44
	3	K-S	4.99	4.85	46.66	73.36	27.45	98.17	10.02	17.08
		A-D	5.30	4.65	52.95	80.67	30.63	99.29	10.80	18.31
		Ray	5.02	5.30	17.76	35.63	10.92	80.68	5.87	8.02
	4	K-S	5.08	5.01	54.56	82.51	31.70	99.38	10.75	19.88
		A-D	4.79	5.18	57.83	85.36	32.81	99.72	11.08	20.95
		Ray	4.97	5.16	16.87	32.80	10.31	78.53	5.34	7.61
7	2	K-S	4.92	5.10	28.32	50.07	16.55	90.13	7.58	10.74
		A-D	4.67	4.97	39.97	67.68	23.32	97.11	8.58	14.25
		Ray	5.06	5.26	14.37	28.06	9.17	72.92	6.08	7.28
	3	K-S	5.60	4.97	41.60	68.87	23.85	97.78	8.99	14.84
		A-D	5.33	4.96	50.83	79.55	29.02	99.37	9.93	17.46
		Ray	5.22	4.75	15.60	29.73	9.38	77.63	5.62	7.22
	4	K-S	4.59	5.31	51.34	79.34	29.35	99.17	10.02	17.63
		A-D	4.80	4.92	58.40	85.66	33.36	99.80	11.23	19.91
		Ray	4.84	4.69	15.49	30.55	9.28	77.77	5.68	7.28
9	2	K-S	4.82	4.86	24.44	44.82	14.57	87.54	6.77	9.73
		A-D	5.07	5.30	36.82	64.12	21.39	96.70	7.75	12.81
		Ray	4.62	5.10	12.72	23.24	7.99	66.76	5.27	6.18
	3	K-S	5.10	5.33	36.85	62.72	20.96	96.51	7.92	12.94
		A-D	4.79	4.91	47.10	76.06	26.80	99.00	8.74	15.92
		Ray	5.59	4.85	12.82	25.52	8.94	71.88	5.29	6.59
	4	K-S	4.80	4.99	46.29	74.71	26.17	99.09	9.60	15.76
		A-D	4.87	4.90	55.38	84.13	31.51	99.81	10.52	18.83
		Ray	4.96	4.66	13.46	26.09	9.10	73.90	5.60	6.52

Table 3. Empirical power ($\times 10^2$) for random projection $\boldsymbol{U}'X\boldsymbol{V}$ when $\boldsymbol{U} \sim \mathbb{S}^{p-1}, \boldsymbol{V} \sim \mathbb{S}^{r-1}$ and $N = 20$

p	r	Stat	$N(0,1)$	t_5	$Exp(1)$	χ^2_1	χ^2_4	$Wei(0.5,1)$	$Wei(2,1)$	$SN(0,1,10)$
5	2	K-S	4.75	4.83	8.96	12.12	6.77	21.38	5.23	5.83
		A-D	4.80	5.04	9.54	12.63	7.21	21.93	5.32	5.95
	3	K-S	4.65	5.04	7.71	9.68	6.54	16.91	5.11	5.75
		A-D	4.62	5.01	8.21	10.60	6.41	17.50	5.63	5.89
	4	K-S	5.23	5.01	7.22	8.86	6.23	14.27	5.13	5.29
		A-D	5.32	4.98	7.25	8.84	6.37	14.94	5.19	5.40
7	2	K-S	4.77	4.78	7.37	9.57	5.82	17.41	5.30	5.73
		A-D	4.74	4.94	7.50	10.14	6.01	18.34	5.29	5.83
	3	K-S	5.09	5.08	7.07	8.40	5.71	15.12	5.13	5.41
		A-D	5.26	4.99	7.18	8.97	6.21	15.78	5.29	5.67
	4	K-S	5.03	4.96	6.72	8.33	5.96	12.55	4.93	5.37
		A-D	5.16	4.83	6.98	8.92	6.11	13.26	5.14	5.47
9	2	K-S	4.73	5.33	6.87	8.28	5.90	15.06	4.94	5.26
		A-D	4.70	5.13	6.93	9.31	5.92	15.58	5.15	5.73
	3	K-S	4.64	5.11	5.94	7.60	5.61	12.55	5.15	5.45
		A-D	4.86	5.12	6.32	8.08	5.53	13.83	5.18	5.40
	4	K-S	4.83	5.05	5.86	7.32	5.88	11.12	5.35	5.17
		A-D	4.79	5.06	6.34	8.11	5.88	11.90	5.70	5.12

Table 4. Empirical power ($\times 10^2$) for random projection $\boldsymbol{b}'X\boldsymbol{V}$ when $\boldsymbol{b} = p^{-1/2}\boldsymbol{j}_p, \boldsymbol{V} \sim \mathbb{S}^{r-1}$ and $N = 20$

p	r	Stat	$N(0,1)$	t_5	$Exp(1)$	χ^2_1	χ^2_4	$Wei(0.5,1)$	$Wei(2,1)$	$SN(0,1,10)$
5	2	K-S	4.81	4.89	17.11	29.53	10.44	55.15	5.87	8.23
		A-D	5.16	5.01	27.48	46.35	15.46	72.90	6.88	11.52
	3	K-S	4.84	5.01	16.06	26.03	10.91	45.72	6.36	8.38
		A-D	4.94	4.87	22.24	36.38	13.71	57.02	6.82	10.11
	4	K-S	4.97	5.25	15.04	23.74	10.48	39.64	6.23	8.35
		A-D	4.77	5.10	18.49	28.84	11.80	46.20	6.89	9.05
7	2	K-S	4.97	5.17	14.48	24.98	9.33	52.08	6.00	7.42
		A-D	4.89	4.99	25.37	44.71	14.27	75.29	6.70	10.04
	3	K-S	4.75	5.23	14.50	24.59	9.88	46.37	5.68	7.36
		A-D	4.71	5.25	22.41	37.59	13.37	62.13	6.52	9.09
	4	K-S	4.95	5.11	14.91	24.26	9.32	41.41	5.88	7.38
		A-D	4.96	5.28	19.44	32.70	12.23	52.44	6.38	8.79
9	2	K-S	5.08	5.35	12.82	21.69	8.43	50.53	5.66	7.01
		A-D	5.12	4.91	22.96	41.39	13.52	75.11	6.75	9.78
	3	K-S	5.11	5.18	13.93	22.88	8.97	45.24	6.25	6.67
		A-D	4.87	5.19	22.20	36.92	12.58	63.38	6.44	8.91
	4	K-S	5.07	4.96	14.13	22.18	9.08	41.97	6.08	6.84
		A-D	4.93	4.80	19.24	32.37	12.13	56.55	6.81	8.40

Table 5. Empirical power ($\times 10^2$) for random projection $\boldsymbol{U}'X\boldsymbol{a}$ when $\boldsymbol{a} = r^{-1/2}\boldsymbol{j}_r, \boldsymbol{U} \sim \mathbb{S}^{p-1}$ and $N = 20$

p	r	Stat	$N(0,1)$	t_5	$Exp(1)$	χ_1^2	χ_4^2	$Wei(0.5,1)$	$Wei(2,1)$	$SN(0,1,10)$
5	2	K-S	4.88	4.97	11.12	19.01	8.49	34.90	5.61	6.59
		A-D	5.01	5.10	11.87	20.09	8.91	36.28	5.60	6.91
	3	K-S	4.94	4.99	12.34	20.18	8.30	36.75	5.68	6.81
		A-D	5.03	5.33	14.12	21.98	9.26	38.22	5.73	7.17
	4	K-S	4.99	4.87	12.97	20.88	9.28	36.25	5.56	7.52
		A-D	4.92	5.03	14.51	23.13	10.16	38.54	5.64	7.91
7	2	K-S	4.73	5.11	9.02	13.40	6.62	27.31	5.47	6.01
		A-D	4.90	5.26	9.67	14.56	7.25	28.86	5.88	6.33
	3	K-S	5.06	4.58	10.39	15.73	7.74	31.33	5.86	5.83
		A-D	5.20	4.74	11.31	17.49	7.98	33.14	5.91	6.29
	4	K-S	5.12	4.69	11.14	17.19	8.38	32.31	5.45	6.62
		A-D	5.02	4.81	12.09	18.95	9.04	34.59	5.64	6.84
9	2	K-S	5.22	4.94	7.71	11.19	6.06	23.63	5.32	5.53
		A-D	4.97	4.78	8.27	12.32	6.29	24.40	5.60	5.69
	3	K-S	5.06	4.95	8.84	12.42	7.17	26.67	5.46	5.82
		A-D	5.16	5.00	9.29	14.09	7.34	28.47	5.48	6.02
	4	K-S	4.98	5.13	10.00	14.40	7.06	28.51	5.95	6.16
		A-D	5.09	4.89	10.80	15.80	7.41	30.65	6.04	6.22

Table 6. Empirical power ($\times 10^2$) for non-random projection $\boldsymbol{b}'X\boldsymbol{a}$ when $\boldsymbol{a} = r^{-1/2}\boldsymbol{l}_r, \boldsymbol{b} = p^{-1/2}\boldsymbol{l}_p$ and $N = 20$

p	r	Stat	$N(0,1)$	t_5	$Exp(1)$	χ_1^2	χ_4^2	$Wei(0.5,1)$	$Wei(2,1)$	$SN(0,1,10)$
5	2	K-S	4.88	4.91	5.03	4.95	5.31	4.95	5.15	5.10
		A-D	5.01	4.97	4.52	4.05	4.71	3.90	4.81	4.88
	3	K-S	4.94	4.60	5.05	4.75	5.05	5.75	5.03	5.28
		A-D	4.85	4.88	4.68	4.46	4.98	5.19	4.98	5.08
	4	K-S	5.02	5.22	5.12	4.59	5.01	4.21	5.10	4.71
		A-D	5.11	5.44	5.25	4.84	5.19	4.95	4.96	4.81
7	2	K-S	5.06	4.71	4.90	5.09	4.73	5.09	4.83	5.00
		A-D	4.88	4.98	4.78	4.42	4.66	4.33	4.86	4.94
	3	K-S	4.77	4.75	5.32	5.32	4.43	4.99	5.17	5.21
		A-D	4.51	4.70	4.79	4.94	4.44	4.76	4.95	4.80
	4	K-S	4.89	5.04	4.90	4.83	4.80	4.58	4.98	4.94
		A-D	4.68	5.12	4.61	4.55	4.82	4.21	4.85	4.86
9	2	K-S	5.00	5.35	4.93	4.54	4.82	5.05	4.61	5.04
		A-D	4.99	5.17	4.85	4.21	4.83	4.18	4.70	4.68
	3	K-S	4.96	4.83	5.02	4.99	5.01	4.82	5.01	4.70
		A-D	4.61	4.69	5.14	4.81	4.75	4.51	5.20	4.88
	4	K-S	5.15	5.09	4.58	4.97	5.01	4.85	4.98	4.68
		A-D	5.19	4.68	4.74	4.82	4.81	4.49	5.20	4.54

Table 7. Empirical power ($\times 10^2$) for non-random projection $\boldsymbol{b}'X\boldsymbol{a}$ when $\boldsymbol{a} = r^{-1/2}\boldsymbol{l}_r, \boldsymbol{b} = p^{-1/2}\boldsymbol{l}_p$ and $N = 20$

p	r	Stat	F1	F2	F3	F4	F5	F6
		K-S	20.07	60.38	90.28	19.63	56.26	97.15
	2	A-D	22.75	67.57	94.40	21.76	63.81	98.67
		Ray	10.04	30.59	64.90	41.36	97.88	100.00
		K-S	27.52	76.03	97.53	27.95	77.06	99.89
5	3	A-D	31.97	83.12	99.12	32.17	83.92	99.98
		Ray	10.67	37.42	75.89	52.74	99.82	100.00
		K-S	34.95	87.63	99.65	36.25	89.63	100.00
	4	A-D	40.45	92.68	99.96	42.07	93.82	100.00
		Ray	12.54	44.60	82.41	63.93	100.00	100.00
		K-S	20.40	59.21	90.25	11.90	33.53	81.09
	2	A-D	23.45	67.71	94.88	13.08	39.06	87.21
		Ray	9.12	26.47	60.12	24.15	86.70	100.00
		K-S	27.19	76.56	97.72	16.82	47.30	94.73
7	3	A-D	31.61	83.73	99.22	19.02	55.05	97.73
		Ray	10.18	33.00	69.86	31.92	95.71	100.00
		K-S	34.72	86.62	99.52	20.69	60.28	98.93
	4	A-D	40.60	92.61	99.89	23.85	68.66	99.69
		Ray	10.21	38.72	78.23	38.33	98.58	100.00
		K-S	19.92	59.43	90.78	9.24	22.28	61.95
	2	A-D	23.16	68.33	95.17	10.44	25.62	69.24
		Ray	8.69	23.65	54.11	16.97	69.06	99.99
		K-S	27.60	76.81	97.97	11.23	32.06	80.63
9	3	A-D	32.18	84.16	99.26	13.00	37.48	87.79
		Ray	9.41	29.62	66.06	20.65	83.37	100.00
		K-S	34.30	86.73	99.54	14.21	39.95	91.65
	4	A-D	40.89	92.62	99.83	15.78	46.98	95.76
		Ray	9.91	34.46	73.75	24.73	91.55	100.00