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BOUNDS ON TAIL PROBABILITIES OF DISCRETE DISTRIBUTIONS

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We examine several approaches to derive lower and upper bounds on tail probabilities of discrete distributions. Numerical comparisons exemplify the quality of the bounds.

1. INTRODUCTION

There has not been much work concerning bounds on tail probabilities of discrete distributions. In the work of Johnson, Kotz, and Kemp [9], for example, only two (quite bad) upper bounds on Poisson tail probabilities are quoted.

This article examines several more generally applicable methods to derive lower and upper bounds on tail probabilities of discrete distributions. Section 2 presents an elementary approach to obtain such bounds for many families of discrete distributions. Besides the Poisson, the binomial, the negative binomial, the logseries, and the zeta distribution, we consider the generalized Poisson distribution, which is frequently used in applications. Some of the bounds emerge in different areas in the literature. Despite the elementary proofs, the bounds reveal much of the character of the distributional tail and are sometimes better than other, more complicated bounds.

As shown in Section 3, some of the elementary bounds can be improved using similar arguments. In a recent article, Ross [11] derived bounds on Poisson tail probabilities using the importance sampling identity. Section 4 extends this method to the logseries and the negative binomial distribution. Finally, some numerical comparisons illustrate the quality of the bounds.

Whereas approximations to tail probabilities become less important due to an increase in computing power, bounds on the tails are often needed for theoretical

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reasons. For example, this work was motivated from the need of obtaining tightness conditions in the context of goodness-of-fit testing for discrete distributions (see Klar [10]). For other immediate applications, see Remarks 3 and 4 below.

2. ELEMENTARY LOWER AND UPPER BOUNDS

Let X be a random variable taking nonnegative integer values, and write $f_x = P(X = x)$ and $S_x = P(X \geq x)$ for the probability mass function (pmf) and the survivor function of X, respectively.

Proposition 1: *For the tail probabilities $P(X \geq n) = \sum_{x \geq n} f_x$, the following inequalities hold*:

(a) *Let X have a Poisson distribution with parameter $\vartheta > 0$ (i.e., $f_x = e^{-\vartheta}\vartheta^x/x!$, $x \geq 0$). Then, for each $n > \vartheta - 1$,*

$$f_n < P(X \geq n) < \left(1 - \frac{\vartheta}{n+1}\right)^{-1} f_n.$$

(b) *Suppose X has a negative binomial distribution with parameters r and p, where $r > 0$ and $0 < p < 1$; that is, $f_x = \binom{x+r-1}{x} p^r q^x (x \geq 0)$ with $q = 1 - p$.*

If $r > 1$ and $n \geq rq/p$, then

$$\frac{1}{p} f_n < P(X \geq n) < \left(1 - \frac{n+r}{n+1}\, q\right)^{-1} f_n.$$

If $r < 1$ and $n \geq rq/p$, then

$$\left(1 - \frac{n+r}{n+1}\, q\right)^{-1} f_n < P(X \geq n) < \frac{1}{p} f_n.$$

For $r > 1$ [$r < 1$], the lower [upper] bound holds for each $n \geq 0$.

(c) *Let X have a binomial distribution with parameters $m \in \mathbb{N}$ and $p (0 < p < 1)$; the pmf is given by $f_x = \binom{m}{x} p^x q^{m-x} (0 \leq x \leq m)$, where $q = 1 - p$. Then, for $mp \leq n \leq m$,*

$$f_n \leq P(X \geq n) \leq \frac{(n+1)q}{n+1-(m+1)p} f_n.$$

(d) *For the logseries distribution with parameter ϑ, $0 < \vartheta < 1$, the pmf is given by $f_x = a(\vartheta)\vartheta^x/x$, $x \geq 1$, where $a(\vartheta) = -(\log(1 - \vartheta))^{-1}$. Here, for $n \geq 1$,*

$$\left(1 - \frac{n\vartheta}{n+1}\right)^{-1} f_n < P(X \geq n) < (1 - \vartheta)^{-1} f_n.$$

(e) *Suppose X has a generalized Poisson distribution with parameters* ϑ *and* λ, *where* $\vartheta > 0$ *and* $0 < \lambda < 1$. *The pmf of X is given by* $f_x = \vartheta(\vartheta + x\lambda)^{x-1}\exp(-\vartheta - x\lambda)/x!$ *for* $x \geq 0$. *Then, for* $n \geq \vartheta/(e^{\lambda-1} - \lambda)$,

$$P(X \geq n) > \left(1 - \lambda e^{1-\lambda}\left(1 - \frac{1}{n+1}\right)^{2+\vartheta/\lambda}\right)^{-1} f_n,$$

$$P(X \geq n) < \left(1 - \lambda e^{1-\lambda}\left(1 + \frac{\vartheta}{(n+1)\lambda}\right)\right)^{-1} f_n.$$

(f) *Let X be distributed according to a zeta distribution with parameter* $\rho > 0$; *the pmf of X is given by* $f_x = c(\rho)/x^{\rho+1}$, $x \geq 1$, *where* $c(\rho) = (\sum_{k=1}^{\infty} k^{-(\rho+1)})^{-1}$. *Here, for* $n \geq 2$,

$$\frac{n}{\rho} f_n < P(X \geq n) < \left(\frac{n}{n-1}\right)^{\rho} \frac{n}{\rho} f_n.$$

PROOF: Define $c_0, c_1, \ldots$ recursively by $f_{n+1} = c_n f_n$, $n \geq 0$. Then,

$$S_n = f_n + \sum_{j=n}^{\infty} c_j f_j, \qquad n \geq 0. \tag{1}$$

Suppose $(f_n)_{n>M}$ is strictly decreasing and thus $c_n < 1$ for $n \geq M$. If $c_n \underset{(>)}{<} c_{n+1}$ for $n \geq M$, then

$$S_n \underset{(<)}{>} (1 - c_n)^{-1} f_n, \qquad n \geq M. \tag{2}$$

Similarly, if $c_n \underset{(<)}{>} c$ for each $n \geq M$ with $0 \leq c < 1$, then

$$S_n \underset{(<)}{>} (1 - c)^{-1} f_n, \qquad n \geq M. \tag{3}$$

For the Poisson distribution, $f_{n+1} = \vartheta f_n/(n+1)$ and, thus, $c_n > c_{n+1}$ and $c_n < 1$ if $n + 1 > \vartheta$.

In case of the negative binomial distribution,

$$f_{n+1} = \frac{(n+r)q}{n+1} f_n.$$

If $r > 1$, then $(n+r)q/(n+1)$ decreases in n. The upper bound thus follows for $n > (qr - 1)/p$ from (2). Since $c_n > q$ for $n \geq 0$, inequality (3) yields the lower bound. Exchanging $<$ and $>$ gives the corresponding inequalities for $r < 1$.

For the binomial distribution,

$$f_{n+1} = \frac{(m-n)p}{(n+1)q} f_n;$$

hence, the upper bound in (c) follows for $n > (m+1)p - 1$.

In case of the logseries distribution, $f_{n+1} = \vartheta n f_n/(n+1)$ and, hence, $c_n < c_{n+1} < \vartheta$.

For the generalized Poisson distribution, we have

$$f_{n+1} = e^{-\lambda}\left(\lambda + \frac{\vartheta}{n+1}\right)\left(1 + \frac{\lambda}{\vartheta + n\lambda}\right)^{n-1} f_n.$$

From $c_n < e^{1-\lambda}(\lambda + \vartheta/(n+1)) = a_n$ (say) and (1), we obtain $S_n < f_n + \sum_{j=n}^{\infty} a_j f_j$. Since (a_j) is decreasing and $a_n < 1$ for $n \geq \vartheta/(e^{\lambda-1} - \lambda)$, the upper bound in (e) follows. On the other hand,

$$c_n > \lambda e^{1-\lambda}\left(1 - \frac{1}{n+1}\right)^{2+\vartheta/\lambda}.$$

Since the right-hand side increases in n, one obtains the lower bound in a similar way.

The inequalities for the zeta distribution follow from the bounds $\int_n^{\infty} g(t)\,dt < \sum_{k=n}^{\infty} g(k) < \int_{n-1}^{\infty} g(t)\,dt$ for each positive and decreasing function g. ■

Remark 2:

(a) Putting $m = 1$ in Proposition 1(ii) of Glynn [7] or $k = 1$ in inequality (16) in Ross [11] gives the upper bound for the Poisson distribution (see Sections 3 and 4). Johnson et al. [9, p. 164] state the simple bound

$$P(X \geq n) \leq 1 - \exp\left\{-\frac{\vartheta}{n}\right\} \qquad (n \geq \vartheta), \tag{4}$$

which is better than the bound in (a) for some values of n near the mode of the distribution. In the tails of the Poisson distribution, however, this bound is much worse and it does not suffice to verify a tightness condition in [10]. The well-known bound

$$P(X \geq n) \leq 1 - [\Gamma(\vartheta + 1)]^{-1} \int_0^{n-1} t^{\vartheta} e^{-t}\,dt, \tag{5}$$

due to Bohman [3], is more complicated than the bound in (a); the numerical comparisons in Section 5 indicate that it is worse.

(b) The bounds for the negative binomial distribution seem to be new. For the geometric distribution (i.e., for the case $r = 1$ in (b)), $P(X \geq n) = f_n/p$.

(c) Bahadur [2] derived the upper bound for the binomial distribution with the help of a quite complicated series representation of the distribution function; see also [9, p. 121]. A similar but slightly worse upper bound is given in Feller [6, p. 151] (see also [9, p. 111] and Diaconis and Zabell [5, p. 289]). The method used in [5] yields the lower bound nf_n/m; however, this is worse than the trivial lower bound in (c).

(d) The bounds in (d) are given in [9, p. 292] (there, the lower bound has the exponent -1 missing).

(e) The proof of (e) yields the upper bound $\vartheta/(e^{\lambda-1} - \lambda)$ on the modal value M of the generalized Poisson distribution. Note that the condition given in Consul [4, p. 17], although being necessary, is not sufficient for defining an upper bound on M.

Remark 3: The bounds on the survivor function yield bounds on the discrete hazard function $h_n = f_n/S_n$, $n \geq 0$. In particular, h_n converges to 1 for $n \to \infty$ in the case of the Poisson distribution and to p in the case of the negative binomial distribution; the corresponding limits are $1 - \vartheta$ for the logseries distribution and $1 - \lambda e^{1-\lambda}$ for the generalized Poisson distribution. In case of the zeta distribution, $\lim_{n\to\infty} nh_n = \rho$.

Remark 4: Using the bounds given in Proposition 1, it is easy to obtain results regarding the behavior of maxima of discrete random variables. Since $\lim_{n\to\infty} nf_n/S_{n+1} = \rho > 0$ for the zeta distribution, this law is in the domain of attraction of the Weibull distribution (see Anderson [1, Thm. 3]). For the negative binomial distribution, the logseries distribution, and the generalized Poisson distribution, $\lim_{n\to\infty} S_n/S_{n+1} > 1$; hence, similar to the geometric distribution, they do not belong to the domain of attraction of any extreme value distribution ([1, Thm. 4]).

3. REFINEMENTS OF THE ELEMENTARY BOUNDS

Some of the bounds given in Proposition 1 can be improved using similar arguments. Assuming that $c_n < 1$ and $c_n \underset{(>)}{<} c_{n+1}$ for $n \geq M$, where the same notation as in Section 2 is used, we obtain

$$S_n = \sum_{j=n}^{n+k-1} f_j + \sum_{j=n}^{\infty}\left(\prod_{i=0}^{k-1} c_{j+i}\right) f_j \underset{(<)}{>} \sum_{j=n}^{n+k-1} f_j + \left(\prod_{i=0}^{k-1} c_{n+i}\right)\sum_{j=n}^{\infty} f_j$$

for $n \geq M$ and any positive integer k. Hence,

$$S_n \underset{(<)}{>} \left(1 - \prod_{i=0}^{k-1} c_{n+i}\right)^{-1} \sum_{j=n}^{n+k-1} f_j, \qquad n \geq M, k \geq 1.$$

Similarly, if $c_n \underset{(<)}{>} c$ for each $n \geq M$ with $0 \leq c < 1$, then

$$S_n \underset{(<)}{>} (1 - c^k)^{-1} \sum_{j=n}^{n+k-1} f_j$$

for $n \geq M$ and any $k \geq 1$. These inequalities yield the bounds in the following proposition.

PROPOSITION 5:

(a) *Let X have a Poisson distribution with parameter $\vartheta > 0$. Then, for $n > \vartheta - 1$ and any $k \geq 1$,*

$$\sum_{j=n}^{n+k-1} f_j < P(X \geq n) < \left(1 - \frac{\vartheta^k}{\prod_{i=1}^{k} n + i}\right)^{-1} \sum_{j=n}^{n+k-1} f_j.$$

(b) *Suppose X has a negative binomial distribution with parameters r and p, where* $r \underset{(<)}{>} 1$ *and* $0 < p < 1$. *Here, for* $n \geq rq/p$ *and any* $k \geq 1$,

$$(1-q^k)^{-1}\sum_{j=n}^{n+k-1} f_j \underset{(>)}{<} P(X \geq n)$$

$$\underset{(>)}{<} \left(1 - q^k \prod_{i=1}^{k} \frac{n+i+r-1}{n+i}\right)^{-1} \sum_{j=n}^{n+k-1} f_j.$$

(c) *Let X have a logseries distribution with parameter* ϑ, $0 < \vartheta < 1$. *Then, for* $n \geq 1$ *and any* $k \geq 1$,

$$\left(1 - \frac{n\vartheta^k}{n+k}\right)^{-1} \sum_{j=n}^{n+k-1} f_j < P(X \geq n) < (1-\vartheta^k)^{-1} \sum_{j=n}^{n+k-1} f_j.$$

Remark 6: The upper bounds for the tail probabilities of the Poisson distribution given in Proposition 1(ii) of Glynn [7] are similar but slightly worse than the bounds in part (a) of Proposition 5.

4. BOUNDS ON THE TAIL PROBABILITIES USING THE IMPORTANCE SAMPLING IDENTITY

If f and g are probability mass functions (or probability densities), the importance sampling identity states that

$$E_f[h(X)] = E_g\left[\frac{h(X)f(X)}{g(X)}\right], \tag{6}$$

where h is a function satisfying $f(x)h(x) = 0$ if $g(x) = 0$ (the subscript on the expectation operator indicates the distribution of the random variable X). From (6), one obtains the equality (see Ross [11])

$$P_f(X > c) = P_g(X > c)E_g\left[\frac{f(X)}{g(X)}\middle| X > c\right]. \tag{7}$$

Using (7), Ross [11] derived bounds on normal, gamma, and Poisson tail probabilities. In the latter case, he put $g(k) = (1-\pi)\pi^k$ $(k = 0,1,\ldots)$, which yields

$$P_f(X \geq n) = \frac{\pi^n}{1-\pi} E_g\left[\frac{f(X+n)}{\pi^{X+n}}\right]. \tag{8}$$

When $\pi = \vartheta/(n+1)$, where ϑ is the Poisson mean, the bounds on the Poisson tail probabilities resulting from (8) are

$$P(X \geq n) < \frac{f(n)}{1-\vartheta/(n+1)}\left[1 - \left(\frac{\vartheta}{n+1}\right)^k + \vartheta^k \prod_{i=1}^{k} \frac{1}{n+i}\right]$$

and

$$P(X \geq n) > \frac{f(n)}{1 - \vartheta/(n+1)} \left(1 - \left(\frac{\vartheta}{n+1}\right)^k\right) \prod_{i=1}^{k-1} \frac{n+1}{n+i}$$

for $n + 1 > \vartheta$ and any $k \geq 1$. For $k = 1$, the bounds in Section 2 for the Poisson distribution are regained.

Equation (8) can be used in a similar way as in [11] to derive bounds for other discrete distributions. For the logseries distribution,

$$\begin{aligned} E_g\left[\frac{\vartheta^{X+n}}{(X+n)\pi^{X+n}}\right] &= E_g\left[\frac{\vartheta^{X+n}}{(X+n)\pi^{X+n}}\,\middle|\,X < k\right](1 - \pi^k) \\ &\quad + E_g\left[\frac{\vartheta^{X+n}}{(X+n)\pi^{X+n}}\,\middle|\,X \geq k\right]\pi^k \\ &= E_g\left[\frac{\vartheta^{X+n}}{(X+n)\pi^{X+n}}\,\middle|\,X < k\right](1 - \pi^k) \\ &\quad + E_g\left[\frac{\vartheta^{X+n+k}}{(X+n+k)\pi^{X+n+k}}\right]\pi^k \end{aligned} \quad \textbf{(9)}$$

for $k \geq 0$. Putting $\pi = \vartheta$, (8) and (9) yield

$$P(X \geq n) < \frac{f(n)}{1 - \vartheta}\left(1 - \frac{k}{n+k}\,\vartheta^k\right)$$

for $k \geq 0$ and

$$P(X \geq n) > \frac{f(n)}{1 - \vartheta}\,\frac{n}{n+k-1}\,(1 - \vartheta^k)$$

for $k \geq 1$. For $k = 0$, one obtains the upper bound in Section 2 for the logseries distribution.

As a further example, we consider the negative binomial distribution with $r > 1$. Setting

$$w(x) = \frac{(x+n+r-1)!}{(x+n)!}\left(\frac{1-p}{\pi}\right)^{x+n}$$

and using (8), we obtain

$$\begin{aligned} P_f(X \geq n) &= \frac{\pi^n}{1-\pi}\,\frac{p^r}{(r-1)!}\,E_g[w(X)] \\ &= \frac{\pi^n}{1-\pi}\,\frac{p^r}{(r-1)!}\,(E_g[w(X)|X < k](1 - \pi^k) + E_g[w(X+k)]\pi^k). \end{aligned}$$

Table 1. Upper Bounds on the Tail Probabilities of the Poisson Distribution

ϑ	n	S_n	UB1	UB2	UB3	k	UB4	UB5	UB6
10	13	.2084	.2552	.2151	.2334	4	.2172	.5366	.3472
	19	.007187	.007464	.007208	.007349	4	.007195	.4092	.03037
100	110	.1706	.2364	.2071	.2171	7	.1868	.5971	.2093
	130	.002282	.002431	.002348	.002377	7	.002287	.5366	.004714
1000	1031	.1673	.249	.2353	.2268	20	.1895	.6209	—
	1095	.001598	.001731	.001701	.001687	20	.001602	.5988	—

Now, assuming that $n > (qr-1)/p$, set $\pi = (1-p)(n+r)/(n+1)$ and note that, for $r > 1$, $w(x)$ is a decreasing function of x. Therefore, we obtain that for any $k \geq 1$,

$$\begin{aligned} P(X \geq n) &< \frac{\pi^n}{1-\pi} \frac{p^r}{(r-1)!} (w(0)(1-\pi^k) + w(k)\pi^k) \\ &= \frac{f(n)}{1-\pi} \left[1 - \pi^k + (1-p)^k \prod_{i=1}^{k} \frac{n+i+r-1}{n+i} \right] \end{aligned}$$

and

$$\begin{aligned} P(X \geq n) &> \frac{\pi^n}{1-\pi} \frac{p^r}{(r-1)!} w(k-1)(1-\pi^k) \\ &= \frac{f(n)}{1-\pi} (1-\pi^k) \left(\frac{n+1}{n+r} \right)^{k-1} \prod_{i=1}^{k-1} \frac{n+i+r-1}{n+i}. \end{aligned}$$

5. NUMERICAL COMPARISONS

This section gives a detailed comparison of the various bounds for the Poisson, the logseries, and the negative binomial distribution. Tables 1 and 2 show some of the results for the Poisson distribution with parameter $\vartheta = 10, 100, 1000$. n is

Table 2. Lower Bounds on the Tail Probabilities of the Poisson Distribution

ϑ	n	S_n	LB1	LB2	LB3	k	LB4
10	13	.2084	.07291	.1942	.1541	4	.188
	19	.007187	.003732	.007066	.006059	4	.00714
100	110	.1706	.02342	.09466	.1073	7	.1405
	130	.002282	.0005753	.001784	.001843	7	.002257
1000	1031	.1673	.007721	.03622	.09871	20	.1299
	1095	.001598	.0001516	.0006353	.001245	20	.001576

Table 3. Lower and Upper Bounds on the Tail Probabilities of $\mathrm{Nb}(r, p)$

r	p	n	S_n	LB1	UB1	LB2	UB2	LB3	UB3	k
2	0.5	8	.01953	.01758	.01978	.01922	.01957	.01775	.01969	5
2	0.3	16	.01928	.01695	.01965	.01827	.01942	.01706	.01954	10
2	0.1	58	.01509	.01309	.01545	.01349	.01536	.01316	.01535	30
2	0.01	620	.01416	.01222	.01453	.01225	.01452	.01225	.01443	300
10	0.5	23	.01003	.006531	.01045	.009031	.01016	.008434	.01029	6
10	0.3	50	.007206	.004448	.007562	.005623	.007407	.005912	.007446	12
10	0.1	180	.00693	.004056	.007341	.004432	.00728	.005454	.00721	30
10	0.01	1934	.006734	.003863	.007159	.003898	.007153	.005217	.007028	300

Table 4. Lower and Upper Bounds on the Tail Probabilities of the Logseries Distribution

ϑ	n	S_n	LB1	UB1	LB2	UB2	LB3	UB3	k
0.5	4	.0382	.03757	.04508	.03816	.03884	.02705	.04133	2
0.7	7	.02607	.02522	.03257	.02586	.02803	.01733	.02973	4
0.9	19	.02277	.02129	.03088	.0218	.02835	.01365	.02716	10
0.99	145	.02332	.02078	.03487	.02089	.03441	.01314	.02966	100

chosen to be approximately $EX + \sqrt{\mathrm{Var}\,X}$ and $EX + 3\sqrt{\mathrm{Var}\,X}$. In these and the following tables, LB1 and UB1 are the elementary lower and upper bounds, respectively, of Section 2. The bounds in Section 3 become better with increasing values of k; LB2 and UB2 are the lower and upper bounds, respectively, with $k = 5$. LB3 and UB3 are the lower and upper bounds, respectively, based on the importance sampling identity; the next entry is the value of k used in the approximations. The upper bounds in (4) and (5) are denoted by UB5 and UB6, respectively.

Gross and Hosmer [8] presented approximations for tail areas of continuous and discrete distributions. For the binomial and the Poisson distribution, they state that these approximations are lower and upper bounds. In the Poisson case, evaluation of the lower bound yields

$$\mathrm{LB4} = \left(1 - \frac{(n+1-\vartheta)(n+2)}{(n+2-\vartheta)(n+1)^2}\right)^{-1} f_n;$$

the upper bound UB4 is more complicated. For other discrete distributions, it is difficult to verify the condition in [8] to ensure that the approximations are indeed lower and upper bounds.

Tables 1 and 2 show that UB4 and LB4 are very close upper and lower bounds on the Poisson tail probabilities, respectively, and the best bounds in the majority of cases. UB2 and UB3 (LB2 and LB3) are comparable, but it should be noted that values of k are selected that yield good bounds. For other values of k, the bounds based on the importance sampling identity can be bad. The upper bound in (5) is mostly worse than the elementary bound UB1. The values of $\vartheta = 1000$ are missing because of problems with the numerical integration routine. The upper bound UB5 is very bad for the selected values of ϑ and n.

Tables 3 and 4 show results for the negative binomial distribution with parameters p and r, denoted by $\mathrm{Nb}(r, p)$, and the logseries distribution with parameters ϑ, respectively. n is chosen to be approximately $EX + 3\sqrt{\mathrm{Var}\,X}$. UB2 and UB3 are comparable for the selected values of k; UB1 does slightly worse. LB2 is the best lower bound in the majority of cases. Since the optimal value of k in Section 4 depends on the distribution, on the parameters, and on the value of n, we prefer the bounds of Section 3.

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