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On the absence of percolation in the lilypond model

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joint work with

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Contents

1. Hard–sphere models

Definition: The space of all *point configurations* in $\mathbb{R}^d$ is defined as

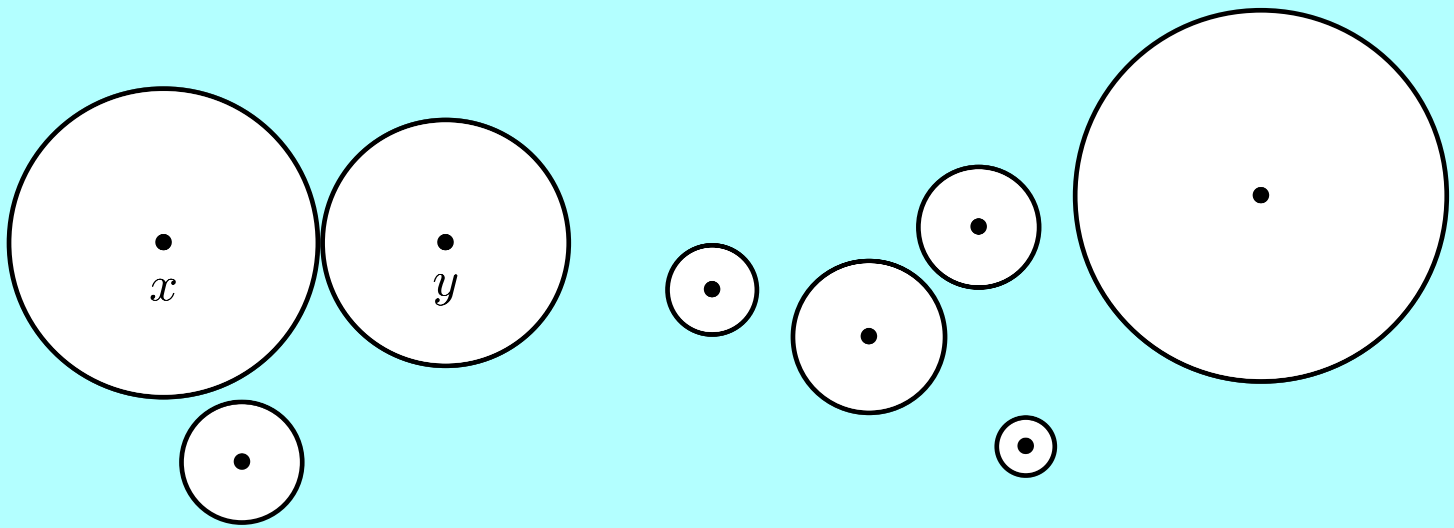
$$\mathbf{N} := \{\varphi \subset \mathbb{R}^d : \varphi \text{ is locally finite}\}.$$

Definition: Consider $\varphi \in \mathbf{N}$ and a mapping $x \mapsto R(\varphi, x) \equiv R(x)$ from φ to $(0, \infty)$ equiping the points of φ with radii of spheres. The marked configuration $\{(x, R(x)) : x \in \varphi\}$ is a *hard-sphere model* if

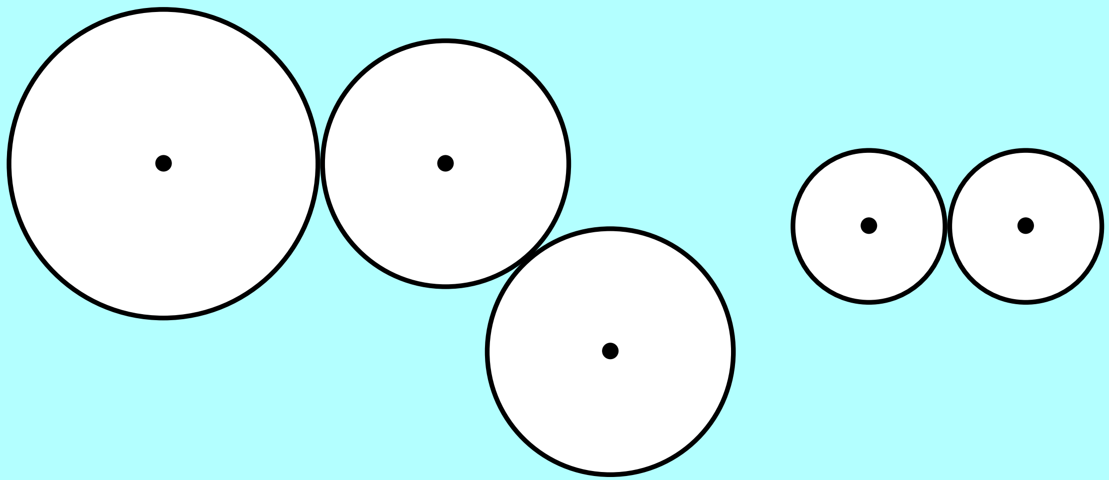
$$R(x) + R(y) \leq \|x - y\|, \quad x, y \in \varphi, x \neq y.$$

Definition: Consider a hard–sphere model $\{(x, R(x)) : x \in \varphi\}$ based on $\varphi \in \mathbf{N}$. Two different points $x, y \in \varphi$ are called *grain neighbours* if

$$R(x) + R(y) = \|x - y\|.$$



Definition: A hard–sphere model $\{(x, R(x)) : x \in \varphi\}$ based on $\varphi \in \mathbf{N}$ is called *lilypond model* if any point $x \in \varphi$ has a smaller grain-neighbour.



2. Descending chains and existence of the lilypond model

Definition: A *descending chain* in $\varphi \in \mathbf{N}$ is a sequence of pairwise different points in φ such that

$$\|x_{i+1} - x_i\| \leq \|x_i - x_{i-1}\|, \quad i \geq 2.$$

Proposition: *Let φ be a locally finite subset of $\mathbb{R}^d$ with* $\operatorname{card} \varphi \geq 2$*. When φ has no descending chain, there is a uniquely determined lilypond model $\{(x, R(x)) : x \in \varphi\}$ based on φ.*

3. Non-lattice type point processes

Definition: A configuration $\varphi \in \mathbf{N}$ is called *non-lattice* if for any $m \geq 2$ and any mutually different points $x_1, \ldots, x_m \in \varphi$ the equality

$$\sum_{1 \leq i < j \leq m} c_{ij} \|x_i - x_j\| = 0$$

for some integers $c_{ij} \in \mathbb{Z}$ $(1 \leq i < j \leq m)$ implies that $c_{ij} = 0$ for all $i < j$.

Lemma: *A point process N whose factorial moment measures are all absolutely continuous is almost surely non-lattice.*

4. Absence of percolation in the lilypond model

Definition: Let $N \neq \emptyset$ be a point process without descending chains. The lilypond model $\{(x, R(N,x)) : x \in N\}$ based on N *percolates* if the random closed set

$$Z := \bigcup_{x \in N} B_{R(x)}(x)$$

contains an unbounded component with positive probability. (Here $B_r(y)$ denotes the closed ball with centre y and radius r.)

Theorem: *If the stationary and non-lattice point process N has almost surely no descending chain then there is no percolation in the lilypond model defined by N.*

5. Absence of descending chain

Definition: Let N be a stationary point process with a positive and finite intensity λ_N. The *Palm probability measure* $\mathbb{P}_0^N$ of N is a probability measure on the underlying probability space satisfying

$$\mathbb{E}\Big[\sum_{x\in N} h(x)\xi(x)\Big] = \lambda_N \mathbb{E}_0[\xi(0)] \int h(x)\,dx,$$

for any non-negative measurable function h on $\mathbb{R}^d$ and any non-negative random field $\{\xi(x) : x \in \mathbb{R}^d\}$ that is jointly stationary with N. Here $\mathbb{E}_0$ denotes expectation with respect to $\mathbb{P}_0^N$.

Assumption: Consider a stationary point process $N \neq \emptyset$ with a finite intensity λ_N and a random field $\xi = \{\xi_x : x \in \mathbb{R}^d\}$ that is *jointly* stationary with N. Assume that

$$\mathbb{P}_0^N((N \setminus \{0\}, \xi) \in \cdot) = \mathbb{E}\left[\mu(N, \xi) \int \mathbf{1}((N \cup \varphi, \xi) \in \cdot)\, \mathbb{Q}_0(d\varphi)\right],$$

where μ is a nonnegative measurable function and $\mathbb{Q}_0$ is a probability measure on $\mathbf{N}$.

Examples: (i) If $\mathbb{Q}_0(\{\emptyset\}) = 1$ and $\mu(N, \xi)$ does not depend on N, then N is a Cox (doubly stochastic Poisson) process.
(ii) If $\mathbb{Q}_0(\{\emptyset\}) = 1$ and $\mu(N, \xi)$ does not depend on ξ, then N is a Gibbs process.
(iii) If μ is constant, then N is a Poisson cluster process.
(iv) If $\mu \equiv 1$ and $\mathbb{Q}_0(\{\emptyset\}) = 1$, then N is a Poisson process.

Theorem: *Assume that the stationary point process N satisfies the assumptions formulated above. Assume also that the function μ and the probability measure $\mathbb{Q}_0$ satisfy certain (exponential) moment conditions. Then N has almost surely no descending chains.*

Poisson case: O. Häggström, R. Meester (1996)

Proof in the Poisson-case:

Take $r > 0$ and $n \in \mathbb{N}$ and define the event

$$C_n := \{\text{there is a } (x_1, \dots, x_n) \in N^{(n)} \text{ such that} \\ r \geq \|x_1\| \geq \|x_2 - x_1\| \geq \dots \geq \|x_n - x_{n-1}\|\}.$$

Then

$$\mathbf{1}_{C_n} \leq \sum_{(x_1,\dots,x_n)\in N^{(n)}} \mathbf{1}(r \geq \|x_1\| \geq \|x_2 - x_1\| \geq \dots \geq \|x_n - x_{n-1}\|).$$

Since N is a Poisson process,

$$\mathbb{P}(C_n) \leq \lambda_N^n \int \cdots \int \mathbf{1}(r \geq \|x_1\| \geq \dots \geq \|x_n - x_{n-1}\|) dx_1 \dots dx_n \\ = \frac{1}{n!} |B_r(0)|^n \to 0$$

as $n \to \infty$. Therefore there is almost surely no descending chain (x_n) in N such that $r \geq \|x_1\|$.

6. Examples

Definition: Let $\xi = \{\xi_x : x \in \mathbb{R}^d\}$ be a stationary non-negative random field. A point process N is a *Cox process* if the conditional distribution of N given ξ is that of a (nonhomogeneous) Poisson process with intensity measure

$$B \mapsto \int_B \xi_x \, dx, \qquad B \subset \mathbb{R}^d.$$

Theorem: *Let N be a Cox process directed by the stationary random intensity $\{\xi_x : x \in \mathbb{R}^d\}$ and assume that*

$$\limsup_{n\to\infty} \frac{\sqrt[n]{\mathbb{E}[\xi_0^n]}}{n} < \infty,$$

Then N has almost surely no descending chains.

Definition: Let N_p be a Poisson process (of *parents*) with finite intensity λ_p and let $\{N^x : x \in N_p\}$ be a family of point processes on $\mathbb{R}^d$ that are conditionally independent given N_p. Assume that the conditional distribution $\mathbb{Q}$ of N^x given N_p is the same for all $x \in N_p$. Then

$$N = \bigcup_{x \in N_p} (N^x + x).$$

is a *Poisson cluster process* with *cluster distribution* $\mathbb{Q}$. It can be assumed that

$$\mathbb{Q} = p_0 \delta_{\emptyset} + \sum_{n=1}^{\infty} p_n \Pi_n,$$

where $\{p_n : n = 0, 1, \ldots\}$ is the *cluster size distribution* and

$$\Pi_n(\cdot) = \mathbb{P}(\{X_{n,1}, \ldots, X_{n,n}\} \in \cdot\}, \quad n \in \mathbb{N},$$

for symmetrically distributed random vectors $X_{n,1}, \ldots, X_{n,n}$.

Theorem: *Let N be a Poisson cluster process as described above and assume that the cluster size distribution $\{p_n : n \in \mathbb{N}\}$ has a finite exponential moment. Assume moreover, that the random vectors $(X_{n,1}, \ldots, X_{n,k})$ have for all $n, k \in \mathbb{N}$ a joint density that is bounded by M^k for some $M > 0$. Then N has almost surely no descending chains.*