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Stationary Random Measures on Homogeneous Spaces

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Abstract This paper discusses stationary random measures on a homogeneous space and their Palm measures. It starts with such fundamental properties as the refined Campbell theorem and then proceeds to consider invariant transports, invariance and transport properties of Palm measures, and stationary partitions. A key tool is a transformation of random measures that permits the extension of recent results for stationary random measures on a group to the more general case of stationary random measures on a homogeneous state space.

Keywords Random measure · Palm measure · Stationarity · Locally compact group · Homogeneous space · Invariant transport-kernel · Stationary partition · Shift-coupling

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1 Introduction

Palm measures and Palm probabilities are fundamental and important concepts in the theory and application of point processes and random measures, see, e.g., Matthes, Kerstan, and Mecke [11], Kallenberg [4], Stoyan, Kendall, and Mecke [18], Daley and Vere-Jones [1], Thorisson [19], Kallenberg [5], and Schneider and Weil [17]. In this paper we consider Palm measures of stationary random measures on a homogeneous space. In his seminal paper [12] Mecke introduced and studied Palm measures of stationary random measures on Abelian groups. Tortrat [20] extended

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some of Mecke's results to an arbitrary locally compact group. Motivated by stochastic geometry and using a different approach, Papangelou [15] introduced and studied Palm measures on homogeneous spaces, and Rother and Zähle [16] generalized some of Mecke's [11] results to the homogeneous case. The most general approach to Palm measures and their invariance properties can be found in Kallenberg [6].

This paper discusses Palm theory for stationary random measures on a homogeneous space S. A simple but crucial tool is a deterministic transformation of a stationary random measure on S into a stationary random measure on the group G. Because this transformation preserves the Palm measure, it allows us to extend recent results for groups to general homogeneous spaces in a short way. Stationary random measures and their Palm measures are introduced in Sect. 3. Sections 4–6 extend some of the results in [8, 9] on invariance and transport properties of Palm measures. In Sect. 7 we establish Mecke's [12] intrinsic characterization of Palm measure in our setting, thereby providing a modest extension of a result in [16]. In the final section we apply the theory to stationary partitions (see [2, 3, 7] for the case $S = G = \mathbb{R}^d$) of (a random subset of) S.

2 Homogeneous Spaces

We consider a topological (multiplicative) group G that is assumed to be a locally compact, second countable Hausdorff space with unit element e and Borel σ-field $\mathcal{G}$. A classical reference concerning such groups is [13], see also Chap. 2 of [5]. A measure ν on G is *locally finite* if it is finite on compact sets. There exists a left-invariant Haar measure λ on G, i.e., a locally finite measure $\lambda \neq 0$ satisfying

$$\int f(hg)\lambda(dg) = \int f(g)\lambda(dg), \quad h \in G, \tag{2.1}$$

for all measurable $f : G \to \mathbb{R}_+$, where $\mathbb{R}_+ := [0, \infty)$. This measure is unique up to a normalization. The *modular function* is a continuous homomorphism $\Delta : G \to (0, \infty)$ satisfying

$$\int f(gh)\lambda(dg) = \Delta\big(h^{-1}\big) \int f(g)\lambda(dg), \quad h \in G, \tag{2.2}$$

for all f as above. It has the property

$$\int f\big(g^{-1}\big)\lambda(dg) = \int \Delta\big(g^{-1}\big) f(g)\lambda(dg). \tag{2.3}$$

The group G is called *unimodular* if $\Delta(g) = 1$ for all $g \in G$.

Next we consider another locally compact second countable Hausdorff space S with Borel σ-field $\mathcal{S}$. We assume that the group G *operates continuously* on S. This means that there is a continuous mapping $(g, x) \mapsto gx$ from $G \times S$ to S such that $ex = x$ and $h(gx) = (hg)x$ for all $h, g \in G$ and $x \in S$. We assume that G operates *transitively* on S, i.e., that the *projection* $\pi_x : G \to S$, $\pi_x(g) := gx$, is surjective for one (and hence for all) $x \in S$. Then S is called a *homogeneous space*. Throughout the paper we also assume that G operates *properly* on S, i.e., that $\pi_x^{-1}K$ is compact in G for any compact $K \subset S$.

A measure μ on S is *invariant* (or G-invariant) if

$$\int f(gx)\mu(dx) = \int f(x)\mu(dx), \quad g \in G, \tag{2.4}$$

for all measurable $f : S \to \mathbb{R}_+$. Up to a constant factor, there is only one such invariant measure which is locally finite, see Theorem 2.29 in [5]. In fact, we can and will choose

$$\mu := \lambda \circ \pi_c^{-1} \tag{2.5}$$

for some fixed $c \in S$. Since G_c is a compact subgroup of G, there is a unique left- and right-invariant probability measure $\kappa(c, \cdot)$ on G_c. It is convenient to extend this measure to G by setting $\kappa(c, G \setminus G_c) := 0$. We have

$$\Delta(g) = 1, \quad g \in G_c. \tag{2.6}$$

Indeed, if $w : S \to \mathbb{R}_+$ is measurable with $\int w\,d\mu = 1$, then (2.5) and (2.2) imply for $g \in G_c$ that

$$1 = \int w(hc)\lambda(dh) = \int w(hgc)\lambda(dh) = \Delta\big(g^{-1}\big) \int w(hc)\lambda(dh) = \Delta\big(g^{-1}\big).$$

For any $x \in S$, we choose some $g_x \in G_{c,x} := \{g \in G : gc = x\}$ and define the probability measure $\kappa(x, \cdot)$ on G by

$$\kappa(x, B) := \int \mathbf{1}\{g_x g \in B\}\kappa(c, dg), \quad B \in \mathcal{G}. \tag{2.7}$$

Because $\kappa(c, \cdot)$ is G_c-invariant, this definition is independent of the choice of g_x. This also implies that

$$\int \mathbf{1}\{g \in \cdot\}\kappa(hx, dg) = \int \mathbf{1}\{hg \in \cdot\}\kappa(x, dg), \quad h \in G,\ x \in S. \tag{2.8}$$

Note that $\kappa(x, \cdot)$ is supported by $G_{c,x}$. Moreover, the proof of Theorem 2.29 in [5] (see also [15]) shows that κ is a kernel disintegrating the Haar measure λ as follows:

$$\int f(g)\lambda(dg) = \iint f(g)\kappa(x, dg)\mu(dx). \tag{2.9}$$

In fact, (2.9) is a straightforward consequence of (2.5), Fubini's theorem, and (2.6).

Common examples of groups (e.g., in stochastic geometry) are the translation group, the rotation group, and the group of rigid motions, all defined as subgroups of the group of bijective affine maps on $\mathbb{R}^d$. The Grassmann manifold of k-dimensional linear subspaces of $\mathbb{R}^d$ is a homogeneous space under the group of rotations. The group of rigid motions operates transitively but not properly on the Grassmann manifold of k-dimensional affine subspaces of $\mathbb{R}^d$. More details on these (and related spaces) can be found in the Appendix of [17].

For future reference, we mention explicitly the important special case of a group acting on itself.

Example 2.1 Let $S := G$ with the group operation $(g, x) \mapsto gx$. Take $c := e$. Then $\mu = \lambda$. Moreover, $G_{c,h} = \{h\}$ and $\kappa(h, \{h\}) = 1$ for all $h \in G$.

3 Stationary Random Measures

We denote by $\mathbf{M}(S)$ the set of all locally finite measures on S, and by $\mathcal{M}(S)$ the cylindrical σ-field on $\mathbf{M}(S)$ which is generated by the evaluation functionals $\nu \mapsto \nu(B)$, $B \in \mathcal{S}$. We often write $\mathbf{M}$ and $\mathcal{M}$ instead of $\mathbf{M}(S)$ and $\mathcal{M}(S)$, respectively. The *support* $\operatorname{supp}\nu$ of a measure $\nu \in \mathbf{M}$ is the smallest closed set $F \subset G$ such that $\nu(G \setminus F) = 0$. By $\mathbf{N} \subset \mathbf{M}$ (resp. $\mathbf{N}_s \subset \mathbf{M}$) we denote the measurable set of all *counting measures* (resp. *simple counting measures*) on S, i.e., the set of all those $\nu \in \mathbf{M}$ with discrete support and $\nu\{x\} := \nu(\{x\}) \in \mathbb{N}_0$ (resp. $\nu\{x\} \in \{0, 1\}$) for all $x \in S$. We can and will identify $\mathbf{N}_s$ with the class of all *locally finite* subsets of S, where a set is called locally finite if its intersection with any compact set is finite.

Although we mostly work on a σ-finite measure space $(\Omega, \mathcal{F}, \mathbb{P})$ ($\mathbb{P}$ need not be a probability measure), we still use probabilistic language. Together with $\mathbb{P}$, we shall consider several other measures on $(\Omega, \mathcal{F})$. A *random measure* on S is a measurable mapping $\xi : \Omega \to \mathbf{M}$. A random measure is a (simple) *point process* on S if $\mathbb{P}(\xi \notin \mathbf{N}) = 0$ (resp. $\mathbb{P}(\xi \notin \mathbf{N}_s) = 0$). A random measure ξ can also be regarded as a *kernel* from Ω to S. Accordingly we write $\xi(\omega, B)$ instead of $\xi(\omega)(B)$. If ξ is a random measure, then the mapping $(\omega, x) \mapsto \mathbf{1}\{x \in \operatorname{supp}\xi(\omega)\}$ is measurable.

We assume that G acts measurably on Ω. This means that there is a family of measurable mappings $\theta_g : \Omega \to \Omega$, $g \in G$, such that $(\omega, g) \mapsto \theta_g\omega$ is measurable, θ_e is the identity on Ω, and

$$\theta_g \circ \theta_h = \theta_{gh}, \quad g, h \in G, \tag{3.1}$$

where $\circ$ denotes composition. In particular, θ_g is a bijection with inverse $\theta_g^{-1} = \theta_{g^{-1}}$. Sometimes $\{\theta_g : g \in G\}$ is called a *measurable flow*. A random measure ξ on S is called *invariant* if

$$\xi(\theta_g\omega, gB) = \xi(\omega, B), \quad \omega \in \Omega,\ g \in G,\ B \in \mathcal{S}, \tag{3.2}$$

where $gB := \{gx : x \in B\}$. This means that

$$\int f(x)\xi(\theta_g\omega, dx) = \int f(gx)\xi(\omega, dx) \tag{3.3}$$

for all measurable $f : S \to \mathbb{R}_+$. We often omit the ω in such relations, i.e., we write (3.3) as $\int f(x)\xi(\theta_g, dx) = \int f(gx)\xi(\theta_e, dx)$ or $\int f(x)\xi(\theta_g, dx) = \int f(gx)\xi(dx)$. Recall that θ_e is the identity on Ω. Still another way of expressing (3.2) is

$$\xi \circ \theta_g = g\xi, \quad g \in G, \tag{3.4}$$

where for $\nu \in \mathbf{M}$ and $g \in G$, the measure $g\nu$ is defined by $g\nu(\cdot) := \int \mathbf{1}\{gx \in \cdot\}\nu(dx)$.

In order to formulate a close relationship between invariant random measures on S and G, we need the following lemma.

Lemma 3.1 *If* $\nu \in \mathbf{M}(S)$, *then* $\nu' := \int \kappa(x, \cdot)\nu(dx) \in \mathbf{M}(G)$.

Proof Let $B \subset G$ be compact. Since π_c is continuous, the set

$$K := \{x \in S : G_{c,x} \cap B \neq \emptyset\} = \pi_c(B)$$

is compact. It follows that

$$\nu'(B) = \int \kappa(x, B)\nu(dx) \leq \int \mathbf{1}\{G_{c,x} \cap B \neq \emptyset\}\nu(dx) = \nu(K) < \infty.$$

Therefore, ν' is locally finite. □

The following lemma provides an explicit representation of Theorem 7.3 in [6] in our more specific situation. It will play a crucial role in the sequel. Invariance of a random measure on G is defined in the setting of Example 2.1.

Lemma 3.2 *If* ξ *is an invariant random measure on* S, *then*

$$\xi' := \int \kappa(x, \cdot)\xi(dx) \tag{3.5}$$

is an invariant random measure on G.

Proof Let $B \in \mathcal{G}$. Since κ is a kernel, $\xi'(B)$ is measurable. If B is compact, then $\xi'(B)$ is finite by Lemma 3.1. Using (2.8) and (3.2), we get that for any $g \in G$ and $B \in \mathcal{S}$,

$$\xi'(\theta_g, gB) = \int \kappa(x, gB)\xi(\theta_g, dx) = \int \kappa(gx, gB)\xi(dx) = \xi'(B).$$

Therefore, ξ' is invariant. □

Remark 3.3 If ξ' is given as in (3.5), then

$$\xi = \int \mathbf{1}\{gc \in \cdot\}\xi'(dg). \tag{3.6}$$

Conversely, if ξ' is an invariant random measure on G, then (3.6) defines an invariant random measure on S.

A measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is called *invariant* if it is invariant under the flow, i.e.,

$$\mathbb{P} \circ \theta_g = \mathbb{P}, \quad g \in G, \tag{3.7}$$

where θ_g is interpreted as a mapping from $\mathcal{F}$ to $\mathcal{F}$ in the usual way:

$$\theta_g A := \{\theta_g \omega : \omega \in A\}, \quad A \in \mathcal{F},\ g \in G.$$

Example 3.4 Consider the measurable space $(\mathbf{M}, \mathcal{M})$ and define, for $\nu \in \mathbf{M}$ and $g \in G$, the measure $\theta_g \nu$ by $\theta_g \nu := g\nu$, i.e., $\theta_g \nu(B) := \nu(g^{-1}B)$, $B \in \mathcal{S}$. Then $\{\theta_g : g \in G\}$ is a measurable flow, and the identity ξ on $\mathbf{M}$ is an invariant random measure. An invariant probability measure on $(\mathbf{M}, \mathcal{M})$ can be interpreted as the distribution of a *stationary random measure*.

Example 3.5 Assume that G is an additive Abelian group and that $S = G$. Consider a flow $\{\tilde{\theta}_g : g \in G\}$ as in [9] (see also [14]). In our current setting this amounts to define $gx := x + g$ and $\theta_g := \tilde{\theta}_g^{-1}$. It is somewhat unfortunate that in the point process literature it is common to define the shift of a measure $\nu \in \mathbf{M}$ by $g \in G$ by $g^{-1}\nu$ and not (as it would be more natural) by $g\nu$. Here we follow the terminology of [6].

Remark 3.6 Our setting accommodates stationary marked point processes (see [1, 11]) and stochastic processes (fields) jointly stationary with a random measure ξ (see [19]). The use of an abstract flow $\{\theta_g : g \in G\}$ acting directly on the underlying sample space makes the notation quite efficient. A more general framework would be to let the group operate on the appropriate state spaces and to replace (3.7) by a distributional invariance, see [6]. However, Theorem 7.2 in [6] shows under quite general assumptions that a stationary random measure can indeed be represented in the invariant way (3.2). Moreover, in applications it is often possible to work with the appropriate canonical spaces, cf. also Example 3.4.

We now fix a σ-finite invariant measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ and an invariant random measure ξ on S. Define the invariant random measure ξ' on G by (3.5). The *Palm measure* $\mathbb{P}_{\xi'}$ of ξ' is the σ-finite measure on Ω defined by

$$\mathbb{P}_{\xi'}(A) = \mathbb{E} \int \mathbf{1}\{\theta_g^{-1} \in A\} w'(g)\xi'(dg), \quad A \in \mathcal{F}, \tag{3.8}$$

where $\mathbb{E}$ denotes integration with respect to $\mathbb{P}$, and $w' : G \to \mathbb{R}_+$ is a measurable function having $\int w'(g)\lambda(dg) = 1$, see [6, 20], and [8]. This definition does not depend on the choice of w'. The measures $\mathbb{P}$ and $\mathbb{P}_{\xi'}$ are connected by the *refined Campbell theorem*

$$\mathbb{E} \int f(\theta_g^{-1}, g)\xi'(dg) = \mathbb{E}_{\mathbb{P}_{\xi'}} \int f(\theta_e, g)\lambda(dg), \tag{3.9}$$

valid for any measurable $f : \Omega \times G \to \mathbb{R}_+$.

Let $w : S \to \mathbb{R}_+$ be a measurable function having $\int w(x)\mu(dx) = 1$. Choosing in (3.8) $w'(g) := w(gc)$, $g \in G$, shows that $\mathbb{P}_{\xi'}$ is given by

$$\mathbb{P}_\xi(A) := \mathbb{E} \iint \mathbf{1}\{\theta_g^{-1} \in A\} w(x)\kappa(x, dg)\xi(dx), \quad A \in \mathcal{F}. \tag{3.10}$$

We call $\mathbb{P}_\xi$ the *Palm measure* of ξ at c. Integration with respect to $\mathbb{P}_\xi$ is denoted by $\mathbb{E}_\xi$. It is easy to see that $\mathbb{P}_\xi$ is concentrated on $\{c \in \operatorname{supp} \xi\}$, that is,

$$\mathbb{P}_\xi(c \notin \operatorname{supp} \xi) = 0. \tag{3.11}$$

The defining equality

$$\mathbb{P}_\xi = \mathbb{P}_{\xi'} \tag{3.12}$$

is a convenient tool to extend results for Palm measures of invariant random measures on groups to the case of a homogeneous state space. We start with rewriting the basic refined Campbell theorem (3.9). In the canonical case of Example 3.4 and for finite intensities, the result was derived in Rother and Zähle [16].

Theorem 3.7 *Let ξ be an invariant random measure on S. Then $\mathbb{P}_\xi$ is σ-finite, and we have for any measurable $f : \Omega \times G \to \mathbb{R}_+$,*

$$\mathbb{E} \iint f\big(\theta_g^{-1}, g\big)\kappa(x, dg)\xi(dx) = \mathbb{E}_\xi \int f(\theta_e, g)\lambda(dg). \tag{3.13}$$

The *intensity* γ_ξ of ξ is defined by

$$\gamma_\xi := \mathbb{P}_\xi(\Omega) = \mathbb{E} \int w(x)\xi(dx). \tag{3.14}$$

We have $\gamma_\xi = \mathbb{E}\xi(B)$ for any $B \in \mathcal{S}$ with $\mu(B) = 1$. The refined Campbell theorem implies the ordinary Campbell theorem

$$\mathbb{E} \int f(x)\xi(dx) = \gamma_\xi \int f(x)\mu(dx) \tag{3.15}$$

for all measurable $f : S \to \mathbb{R}_+$. We just have to use that $f(x) = \int f(gc)\kappa(x, dg)$. In case $0 < \gamma_\xi < \infty$ we can define the *Palm probability measure* of ξ by $\mathbb{P}_\xi^0 := \gamma_\xi^{-1}\mathbb{P}_\xi$.

To derive another corollary of the refined Campbell theorem, we take a measurable function $\tilde{w} : \mathbf{M} \times S \to \mathbb{R}_+$ satisfying

$$\int \tilde{w}(\nu, x)\nu(dx) = 1, \tag{3.16}$$

whenever $\nu \in \mathbf{M}$ is not the null measure. For one example of such a function, we refer to [12]. We then have the *inversion formula*

$$\mathbb{E}\mathbf{1}\{\xi(S) > 0\} f = \mathbb{E}_\xi \int \tilde{w}(\xi \circ \theta_g, gc) f(\theta_g)\lambda(dg) \tag{3.17}$$

for all measurable $f : \Omega \to \mathbb{R}_+$. This is a direct consequence of the refined Campbell theorem (3.13).

The *invariant* σ-field $\mathcal{I} \subset \mathcal{F}$ is the class of all sets $A \in \mathcal{F}$ satisfying $\theta_g A = A$ for all $g \in G$. Let ξ be an invariant random measure with finite intensity and define

$$\hat{\xi} := \mathbb{E}\bigg[\int w(x)\xi(dx) \,\bigg|\, \mathcal{I}\bigg], \tag{3.18}$$

where the conditional expectation is defined as for probability measures. Since $\hat{\xi} \circ \theta_g = \hat{\xi}$, $g \in G$, the refined Campbell theorem (3.13) implies that $\mathbb{E}\mathbf{1}_A \int w(x)\xi(dx) = \mathbb{P}_\xi(A)$ for all $A \in \mathcal{I}$. Therefore definition (3.18) is independent of the choice of w. If $\mathbb{P}$ is a probability measure and $S = G = \mathbb{R}^d$, then $\hat{\xi}$ is called *sample intensity* of ξ, see [11] and [5]. Assuming that $\mathbb{P}(\hat{\xi} = 0) = 0$, we define the *modified Palm measure* $\mathbb{P}^*_\xi$ at c (see [7, 11, 19]) by

$$\mathbb{P}^*_\xi(A) = \mathbb{E}\left[\hat{\xi}^{-1} \iint \mathbf{1}\{\theta_g^{-1} \in A\} w(x)\kappa(x, dg)\xi(dx)\right]. \tag{3.19}$$

Conditioning shows that

$$\mathbb{P}^*_\xi(A) = \mathbb{P}(A), \quad A \in \mathcal{I}. \tag{3.20}$$

Comparing (3.19) and (3.10) yields

$$d\mathbb{P}^*_\xi = \hat{\xi}^{-1} d\mathbb{P}_\xi. \tag{3.21}$$

The refined Campbell theorem (3.13) takes the form

$$\mathbb{E}\left[\hat{\xi}^{-1} \iint f(\theta_g^{-1}, g)\kappa(x, dg)\xi(dx)\right] = \mathbb{E}_{\mathbb{P}^*_\xi} \int f(\theta_e, g)\lambda(dg). \tag{3.22}$$

Remark 3.8 If $\mathbb{P}$ is a probability measure and ξ is a simple point process with a positive and finite intensity, then the Palm probability measure $\mathbb{P}^0_\xi$ can be interpreted as a conditional probability measure given that ξ has a point in c. This could be justified by Theorem 12.8 in [4]. The modified version describes the underlying stochastic experiment as seen from a randomly chosen point of ξ, see [11, 19]. Both measures agree iff $\hat{\xi}$ is $\mathbb{P}$-a.e. constant and in particular if $\mathbb{P}$ is *ergodic*, that is, $\mathbb{P}(A) = 0$ or $\mathbb{P}(\Omega \setminus A) = 0$ for all $A \in \mathcal{I}$.

The above theory can be extended to certain nontransitive situations:

Remark 3.9 Let $(S', \mathcal{S}')$ be some measurable space and ζ a kernel from Ω to $S \times S'$. We call ζ *marked random measure* (on S with mark space S') if $\zeta(\cdot \times S')$ is a random measure on S. Such a marked random measure is *invariant* if $\zeta(\cdot \times B')$ is invariant for all $B' \in \mathcal{S}'$. In this case the *Palm measure* $\mathbb{P}_\zeta$ (at c) of ζ is the measure on $\Omega \times S'$ defined by

$$\mathbb{P}_\zeta(A) = \mathbb{E} \iint \mathbf{1}\{(\theta_g^{-1}, z) \in A\} w(x)\kappa(x, dg)\zeta(d(x, z)), \quad A \in \mathcal{F} \otimes \mathcal{S}'.$$

Note that $\mathbb{P}_\zeta(\cdot \times B')$ is the Palm measure of $\zeta(\cdot \times B')$ at c. The refined Campbell theorem (3.13) takes the form

$$\mathbb{E} \iint f(\theta_g^{-1}, g, z)\kappa(x, dg)\zeta(d(x, z)) = \iint f(\omega, g, z)\lambda(dg)\mathbb{P}_\zeta(d(\omega, z)) \tag{3.23}$$

for all measurable $f : \Omega \times G \times S' \to \mathbb{R}_+$.

Assume that $(\Omega, \mathcal{F})$ is a Borel space. (This is, e.g., the case in Example 3.4.) If $\mathbb{P}_\zeta(\Omega \times \cdot)$ is a σ-finite measure, then we may *disintegrate* $\mathbb{P}_\zeta$ to get another form of (3.23). For simplicity, we even assume that the intensity γ_ξ of $\xi := \zeta(\cdot \times S')$ is finite. Assuming also that $\gamma_\xi > 0$, we can define the *mark distribution* $\mathbb{W}$ of ζ by $\mathbb{W} := \gamma_\xi^{-1}\mathbb{P}_\zeta(\Omega \times \cdot)$. There exists a stochastic kernel $(z, A) \mapsto \mathbb{P}^z_\zeta(A)$ from S' to Ω satisfying

$$\mathbb{P}_\zeta\big(d(\omega, z)\big) = \gamma_\xi \mathbb{P}^z_\zeta(d\omega)\mathbb{W}(dz).$$

Therefore (3.23) can be written as

$$\mathbb{E}\iint f\big(\theta_g^{-1}, g, z\big)\kappa(x, dg)\zeta\big(d(x, z)\big) = \gamma_\xi \iiint f(\omega, g, z)\lambda(dg)\mathbb{P}^z_\zeta(d\omega)\mathbb{W}(dz).$$

The Palm measure at c is invariant under G_c, see [16].

Proposition 3.10 *For all $h \in G_c$, we have $\mathbb{P}_\xi \circ \theta_h = \mathbb{P}_\xi$.*

Proof Let $h \in G_c$. By (2.8) and (2.7) we have that for all $x \in S$,

$$\begin{aligned}\kappa(hx, \cdot) = \int \mathbf{1}\{hg_x g \in \cdot\}\kappa(c, dg) &= \int \mathbf{1}\big\{hg_x gh^{-1} \in \cdot\big\}\kappa(c, dg)\\ &= \int \mathbf{1}\big\{hgh^{-1} \in \cdot\big\}\kappa(x, dg),\end{aligned}$$

where the second equality comes from the right-invariance of $\kappa(c, \cdot)$ on G_c. Using this fact, we get from the definition (3.10) of $\mathbb{P}_\xi$ and invariance of ξ for all $A \in \mathcal{F}$ that

$$\begin{aligned}\mathbb{P}_\xi(\theta_h A) &= \mathbb{E}\iint \mathbf{1}\big\{\theta_g^{-1} \in \theta_h A\big\}w(hx)\kappa(hx, dg)\xi \circ \theta_h^{-1}(dx)\\ &= \mathbb{E}\iint \mathbf{1}\big\{\theta_g^{-1} \circ \theta_h^{-1} \in A\big\}w(hx)\kappa(x, dg)\xi \circ \theta_h^{-1}(dx)\\ &= \mathbb{E}\iint \mathbf{1}\big\{\theta_g^{-1} \in A\big\}w(hx)\kappa(x, dg)\xi(dx),\end{aligned}$$

where we have used the invariance (3.7) for the last equation. By invariance of μ we have $\int w(hy)\mu(dy) = 1$. Since the definition (3.10) is independent of the choice of w, we conclude the assertion. □

4 Transport-Kernels and an Exchange Formula

We first adapt the terminology from [9] to our present more general setting. A *transport-kernel* (on S) is a kernel T from $\Omega \times S$ to S which is Markovian, i.e., which has $T(\omega, x, S) = 1$ for all $(\omega, x) \in \Omega \times S$. We think of $T(\omega, x, B)$ as redistributing a (potential) unit mass at x within S. A *weighted transport-kernel* is a kernel T from $\Omega \times S$ to S such that $T(\omega, x, \cdot)$ is locally finite for all $(\omega, x) \in \Omega \times S$. A weighted transport-kernel T is called *invariant* if

$$T(\theta_g \omega, gx, gB) = T(\omega, x, B), \quad g \in G,\ x \in S,\ \omega \in \Omega,\ B \in \mathcal{S}. \tag{4.1}$$

Quite often we use the short-hand notation $T(x, \cdot) := T(\theta_e, x, \cdot)$. If ξ is an invariant random measure on S and $\eta := \int T(\omega, x, \cdot)\xi(\omega, dx)$ is locally finite for each $\omega \in \Omega$, then η is again an invariant random measure. Our interpretation is that T transports ξ to η in an invariant way.

We now fix a σ-finite invariant measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. We use the function $\Delta^* : S \to (0, \infty)$ defined by

$$\Delta^*(x) := \Delta\bigl(g_x^{-1}\bigr), \quad x \in S, \tag{4.2}$$

where $g_x \in G_{c,x}$. This definition is independent of the choice of g_x. Indeed, if $g, h \in G_{c,x}$, then $g^{-1}h \in G_c$ so that (2.6) implies $1 = \Delta(g^{-1}h)$, i.e. $\Delta(g^{-1}) = \Delta(h^{-1})$. The representation $\Delta^*(x) = \int \Delta(g^{-1})\kappa(x, dg)$ shows that Δ^* is measurable.

The next result is a useful transport property of Palm measures. It generalizes Theorem 4.2 in [9] for Abelian groups and Theorem 2.11 in [8] for general groups.

Theorem 4.1 *Consider two invariant random measures ξ and η on S, and let T and T^* be invariant weighted transport-kernels satisfying*

$$\iint \mathbf{1}\bigl\{(x, y) \in \cdot\bigr\} T(\omega, x, dy)\xi(\omega, dx) = \iint \mathbf{1}\bigl\{(x, y) \in \cdot\bigr\} T^*(\omega, y, dx)\eta(\omega, dy) \tag{4.3}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. Then for any measurable function $f : \Omega \times G \to \mathbb{R}_+$,

$$\mathbb{E}_\xi \iint f\bigl(\theta_g^{-1}, g^{-1}\bigr)\Delta^*(x)\kappa(x, dg)T(c, dx) = \mathbb{E}_\eta \iint f(\theta_e, g)\kappa(x, dg)T^*(c, dx). \tag{4.4}$$

Proof Define a kernel T' from $\Omega \times G$ to G by

$$T'(g, B') := \int \kappa(x, B')T(gc, dx), \quad g \in G,\ B' \in \mathcal{G}. \tag{4.5}$$

Lemma 3.2 implies that T' is a weighted transport-kernel on G. From (2.8) and invariance of T we have that for all $g, h \in G$ and $B' \in \mathcal{G}$,

$$\begin{aligned} T'(\theta_h, hg, hB') &= \int \kappa(x, hB')T(\theta_h, hgc, dx) \\ &= \int \kappa(hx, hB')T(gc, dx) = \int \kappa(x, B')T(gc, dx). \end{aligned}$$

Hence T' is invariant. Define the invariant weighted transport-kernel T'^* on G in terms of T^* as T' in terms of T. Defining ξ' by (3.5), we have

$$\iint \mathbf{1}\{(g,h)\in\cdot\}T'(g,dh)\xi'(dg) = \iiiint \mathbf{1}\{(g,h)\in\cdot\}\kappa(x,dh) \times T(gc,dx)\kappa(y,dg)\xi(dy)$$
$$= \iiiint \mathbf{1}\{(g,h)\in\cdot\}\kappa(y,dg)\kappa(x,dh) \times T(y,dx)\xi(dy),$$

where we have also used Fubini's theorem. A similar equation relates T'^{*} and T^{*}. Applying assumption (4.3), we obtain that

$$\iint \mathbf{1}\{(g,h)\in\cdot\}T'(g,dh)\xi'(dg) = \iint \mathbf{1}\{(g,h)\in\cdot\}T'^{*}(h,dg)\eta'(dh)$$

$\mathbb{P}$-a.e., where $\eta' := \int \kappa(x,\cdot)\eta(dx)$. Hence we can apply the result for groups and (3.12) to obtain the assertion. □

Corollary 4.2 *Let the assumptions of Theorem 4.1 be satisfied. Assume moreover that ξ and η have finite intensities and that $\mathbb{P}(\hat{\xi}=0)=\mathbb{P}(\hat{\eta}=0)=0$. Then for any measurable function $f:\Omega\times G\to\mathbb{R}_+$,*

$$\hat{\xi}\,\mathbb{E}_{\mathbb{P}^*_\xi}\left[\iint f\big(\theta_g^{-1},g^{-1}\big)\Delta^*(x)\kappa(x,dg)T(c,dx)\,\bigg|\,\mathcal{I}\right]$$
$$= \hat{\eta}\,\mathbb{E}_{\mathbb{P}^*_\eta}\left[\iint f(\theta_e,g)\kappa(x,dg)T^*(c,dx)\,\bigg|\,\mathcal{I}\right]$$

$\mathbb{P}$-a.e. for any choice of the conditional expectations.

Proof Define the random variables $X := \iint f(\theta_g^{-1},g^{-1})\Delta^*(x)\kappa(x,dg)T(c,dx)$ and $X' := \iint f(\theta_e,g)\kappa(x,dg)T^*(c,dx)$, and let $A\in\mathcal{I}$. Due to (3.20), we have $\mathbb{P}^*_\xi=\mathbb{P}^*_\eta$ on $\mathcal{I}$. Hence we have to show that

$$\mathbb{E}_{\mathbb{P}^*_\xi}\mathbf{1}_A\hat{\xi}X = \mathbb{E}_{\mathbb{P}^*_\eta}\mathbf{1}_A\hat{\eta}X'.$$

By (3.21) this amounts to $\mathbb{E}_\xi\mathbf{1}_A X = \mathbb{E}_\eta\mathbf{1}_A X'$, i.e., to a consequence of (4.4). □

5 Transport Properties of Palm Measures

In this section we fix an invariant σ-finite measure $\mathbb{P}$ on $(\Omega,\mathcal{F})$. Let ξ and η be two invariant random measures on S. A weighted transport-kernel T on S is called (ξ,η)-*balancing* if

$$\int T(\omega,x,\cdot)\xi(\omega,dx) = \eta(\omega,\cdot) \tag{5.1}$$

for all $\omega\in\Omega$. In the case $\xi=\eta$ we also say that T is ξ-*preserving*. If (5.1) holds for $\mathbb{P}$-a.e. $\omega\in\Omega$, then we say that T is $\mathbb{P}$-a.e. (ξ,η)-balancing.

The following invariance property of Palm measures generalizes results in [9] and [8].

Theorem 5.1 *Consider two invariant random measures* ξ *and* η *on* S *and an invariant weighted transport-kernel* T. *Then* T *is* $\mathbb{P}$*-a.e.* (ξ,η)*-balancing iff*

$$\mathbb{E}_\xi \iint f\big(\theta_g^{-1}\big)\Delta^*(x)\kappa(x,dg)T(c,dx) = \mathbb{E}_\eta f \tag{5.2}$$

for all measurable $f:\Omega\to\mathbb{R}_+$.

Proof Assume first that T is $\mathbb{P}$-a.e. (ξ,η)-balancing. Lemma 5.2 below shows that there exists a invariant transport-kernel T^* satisfying (4.3) for $\mathbb{P}$-a.e. $\omega\in\Omega$. Applying (4.4) to a function not depending on the second argument yields (5.2).

Let us now assume that (5.2) holds. Defining the invariant weighted transport-kernel T' by (4.5), (5.2) can be written as

$$\mathbb{E}_\xi \int f\big(\theta_g^{-1}\big)\Delta\big(g^{-1}\big)T'(e,dg) = \mathbb{E}_\eta f.$$

Hence, from (3.12) and from the result for groups we get that

$$\int T'(g,B')\kappa(x,dg)\xi(dx) = \int \kappa(x,B')\eta(dx)$$

$\mathbb{P}$-a.e. for any $B'\in\mathcal{G}$. Applying this with $B':=\pi_c^{-1}B$ for $B\in\mathcal{S}$ easily yields that (5.1) holds $\mathbb{P}$-a.e. □

The above proof has used the following lemma:

Lemma 5.2 *Assume that* T *is a* $\mathbb{P}$*-a.e.* (ξ,η)*-balancing invariant weighted transport-kernel. Then there exists an invariant transport-kernel* T^* *on* S *such that* (4.3) *holds for* $\mathbb{P}$*-a.e.* $\omega\in\Omega$.

Proof Consider the following measure W on $\Omega\times S\times S$:

$$W := \iiint \mathbf{1}\big\{(\omega,x,y)\in\cdot\big\}T(\omega,x,dy)\xi(\omega,dx)\mathbb{P}(d\omega).$$

Stationarity of $\mathbb{P}$, (3.2), and (4.1) easily imply that

$$\int \mathbf{1}\big\{(\theta_g\omega,gx,gy)\in\cdot\big\}W\big(d(\omega,x,y)\big) = W,\quad g\in G.$$

Moreover, as T is a $\mathbb{P}$-a.e. (ξ,η)-balancing, we have

$$W' := \int \mathbf{1}\big\{(\omega,y)\in\cdot\big\}W\big(d(\omega,x,y)\big) = \iint \mathbf{1}\big\{(\omega,y)\in\cdot\big\}\eta(\omega,dy)\mathbb{P}(d\omega). \tag{5.3}$$

This is a σ-finite measure on $\Omega \times S$. We can now apply Theorem 3.5 in [6] to obtain an invariant transport-kernel T^* satisfying

$$W = \iint \mathbf{1}\big\{(\omega, x, y) \in \cdot\big\} T^*(\omega, y, dx) W'\big(d(\omega, y)\big).$$

(In fact the theorem yields an invariant kernel T' satisfying this equation. But in our specific situation we have $T'(\omega, y, G) = 1$ for W'-a.e. (ω, y), so that T' can be modified in an obvious way to yield the desired T^*.) Recalling the definition of W and the second equation in (5.3) yields

$$\begin{aligned}\iiint & \mathbf{1}\big\{(\omega, x, y) \in \cdot\big\} T(\omega, x, dy)\xi(\omega, dx)\mathbb{P}(d\omega) \\ &= \iiint \mathbf{1}\big\{(\omega, x, y) \in \cdot\big\} T^*(\omega, y, dx)\eta(\omega, dy)\mathbb{P}(d\omega)\end{aligned}$$

and hence the assertion of the lemma. □

6 Existence of Balancing Weighted Transport-Kernels

We fix an invariant σ-finite measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. Our aim is to establish a necessary and sufficient condition for the existence of a balancing invariant weighted transport-kernel T satisfying

$$\int \Delta^*(x) T(c, dx) = 1. \tag{6.1}$$

The following theorem was proved in [9] for Abelian groups and in [8] for general groups.

Theorem 6.1 *Assume that ξ and η are invariant random measures on S with positive and finite intensities. Then there exists a $\mathbb{P}$-a.e. (ξ, η)-balancing invariant weighted transport-kernel satisfying* (6.1) *iff*

$$\mathbb{E}\big[\xi(B) \mid \mathcal{I}\big] = \mathbb{E}\big[\eta(B) \mid \mathcal{I}\big] \quad \mathbb{P}\text{-a.e.} \tag{6.2}$$

for some $B \in \mathcal{S}$ satisfying $0 < \mu(B) < \infty$.

Proof Let $B \in \mathcal{S}$ satisfy $0 < \mu(B) < \infty$ and take $A \in \mathcal{I}$. Applying the refined Campbell theorem (3.13) with $f(\theta_e, g) := \mathbf{1}_A \mathbf{1}\{gc \in B\}$ yields that

$$\mu(B)\mathbb{P}_\xi(A) = \mathbb{E}\big[\mathbf{1}_A \xi(B)\big], \qquad \mu(B)\mathbb{P}_\eta(A) = \mathbb{E}\big[\mathbf{1}_A \eta(B)\big]. \tag{6.3}$$

Assume now that T is a $\mathbb{P}$-a.e. (ξ, η)-balancing invariant weighted transport-kernel satisfying (6.1). Then Theorem 5.1 implies the equality $\mathbb{P}_\xi(A) = \mathbb{P}_\eta(A)$ for all $A \in \mathcal{I}$. Thus (6.3) implies $\mathbb{E}\mathbf{1}_A \xi(B) = \mathbb{E}\mathbf{1}_A \eta(B)$ and hence (6.2).

Let us now assume that (6.2) holds for some $B \in \mathcal{S}$ satisfying $0 < \mu(B) < \infty$. From (6.3) and conditioning we obtain that $\mathbb{P}_\xi = \mathbb{P}_\eta$ on $\mathcal{I}$. By (3.12) and the results for groups, there is an invariant weighted transport-kernel T' on G such that

$$\int \Delta\bigl(g^{-1}\bigr)T'(e, dg) = 1 \tag{6.4}$$

and

$$\iint T'(g, B')\kappa(x, dg)\xi(dx) = \int \kappa(x, B')\eta(dx) \quad \mathbb{P}\text{-a.e.} \tag{6.5}$$

for all $B' \in \mathcal{G}$. Define

$$T(\omega, x, B) := \int T'\bigl(\omega, g, \pi_c^{-1}B\bigr)\kappa(x, dg), \quad x \in S,\ \omega \in \Omega,\ B \in \mathcal{S}. \tag{6.6}$$

Applying (6.5) with $B' := \pi_c^{-1}B$ for $B \in \mathcal{S}$, gives

$$\int T(x, B)\xi(dx) = \eta(B) \quad \mathbb{P}\text{-a.e.} \tag{6.7}$$

It remains to show that T is an invariant weighted transport-kernel satisfying (6.1). By the definition of T' and Δ^*,

$$\begin{aligned}\int \Delta^*(x)T(c, dx) &= \iint \Delta^*(hc)T'(g, dh)\kappa(c, dg)\\ &= \iint \Delta\bigl(h^{-1}\bigr)T'(g, dh)\kappa(c, dg).\end{aligned}$$

Using invariance of T' gives

$$\begin{aligned}\int \Delta^*(x)T(c, dx) &= \iint \Delta\bigl((gh)^{-1}\bigr)T'(e, dh)\kappa(c, dg)\\ &= \iint \Delta\bigl(h^{-1}\bigr)T'(e, dh)\kappa(c, dg) = 1,\end{aligned}$$

where we have used (6.4) and that $\Delta(g^{-1}) = 1$ for $g \in G_c$. Now we take a compact $B \subset S$. Since $g \mapsto \Delta(g^{-1})$ is bounded away from 0 on the compact set $\pi_c^{-1}B = \bigcup_{x\in B} G_{c,x}$, the mapping Δ^* has the same property on B. Therefore (6.1) implies that $T(c, B)$ must be finite. Hence T is a weighted transport-kernel. To show invariance, we take $h \in G$, $x \in S$, and $B \in \mathcal{S}$. Then

$$\begin{aligned}T(\theta_h, hx, hB) &= \int T'\bigl(\theta_h, g, \pi_c^{-1}(hB)\bigr)\kappa(hx, dg)\\ &= \int T'\bigl(\theta_h, hg, h\pi_c^{-1}B\bigr)\kappa(x, dg),\end{aligned}$$

where we have used (2.8) and $\pi_c^{-1}(hB) = h\pi_c^{-1}B$. Therefore invariance of T is implied by the same property of T'. □

7 Mecke's Characterization of Palm Measures

In contrast to the previous sections, we do not fix an invariant measure on $(\Omega, \mathcal{F})$. Instead we consider here a measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ as a candidate for a Palm measure at c of a given invariant random measure w.r.t. some invariant measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. In case of an invariant random measure on an Abelian group (and within a canonical framework) the following fundamental characterization theorem was proved in [12]. The following extension to homogeneous spaces (in the canonical framework of Example 3.4 and for finite intensities) was established in [16]. Again, as in other proofs in this paper, we proceed by reducing the assertion to the group case.

Theorem 7.1 *Let ξ be an invariant random measure on S. The measure $\mathbb{Q}$ is a Palm measure of ξ at c with respect to some σ-finite invariant measure iff $\mathbb{Q}$ is σ-finite and invariant under G_c, $\mathbb{Q}(\xi(S)=0)=0$, and*

$$\mathbb{E}_{\mathbb{Q}} \iint f\big(\theta_g^{-1}, g^{-1}c\big)\Delta^*(x)\kappa(x, dg)\xi(dx) = \mathbb{E}_{\mathbb{Q}} \int f(\theta_e, x)\xi(dx) \tag{7.1}$$

for all measurable $f : \Omega \times S \to \mathbb{R}_+$.

Proof If $\mathbb{Q}$ is a Palm measure of ξ at c, then $\mathbb{Q}$ is σ-finite by Theorem 3.7 and invariant under G_c by Proposition 3.10. The equation $\mathbb{Q}(\xi(S)=0)=0$ holds by (3.11), while the Mecke equation (7.1) is a special case of (4.4).

Let us now conversely assume the stated conditions. We define a random measure ξ' on G by (3.5) and consider a measurable function $f : \Omega \times G \to \mathbb{R}_+$. Applying (7.1) to the function $(\omega, x) \mapsto \int f(\omega, h)\kappa(x, dh)$, we obtain

$$\mathbb{E}_{\mathbb{Q}} \int f(\theta_e, g)\xi'(dg) = \mathbb{E}_{\mathbb{Q}} \iiint f\big(\theta_g^{-1}, h\big)\kappa\big(g^{-1}c, dh\big)\Delta\big(g^{-1}\big)\kappa(x, dg)\xi(dx).$$

Equation (2.8) and Fubini's theorem imply that

$$\mathbb{E}_{\mathbb{Q}} \int f(\theta_e, g)\xi'(dg) = \int \mathbb{E}_{\mathbb{Q}}\bigg[\iint f\big(\theta_g^{-1}, g^{-1}h\big)\Delta\big(g^{-1}\big)\kappa(x, dg)\xi(dx)\bigg]\kappa(c, dh). \tag{7.2}$$

By invariance of $\mathbb{Q}$ under G_c the expectation on the above right-hand side equals

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}} \iint f\big(\theta_g^{-1} \circ \theta_h, g^{-1}h\big)\Delta\big(g^{-1}\big)\kappa(x, dg)\xi \circ \theta_h(dx) \\
&\quad = \mathbb{E}_{\mathbb{Q}} \iint f\big(\theta_{hg}^{-1} \circ \theta_h, (hg)^{-1}h\big)\Delta\big((hg)^{-1}\big)\kappa(x, dg)\xi(dx) \\
&\quad = \mathbb{E}_{\mathbb{Q}} \iint f\big(\theta_g^{-1}, g^{-1}\big)\Delta\big(g^{-1}\big)\kappa(x, dg)\xi(dx),
\end{aligned}$$

where we have used the properties of the modular function and (2.6). Inserting this into (7.2) yields

$$\mathbb{E}_{\mathbb{Q}} \int f(\theta_e, g)\xi'(dg) = \mathbb{E}_{\mathbb{Q}} \int f\big(\theta_g^{-1}, g^{-1}\big)\Delta\big(g^{-1}\big)\xi'(dg).$$

Since $\mathbb{Q}(\xi'(G)=0)=0$, Theorem 2.19 in [8] (Mecke's characterization for general groups) implies that there is a σ-finite invariant measure $\mathbb{P}$ such that $\mathbb{Q}=\mathbb{P}_{\xi'}$. A reference to (3.12) concludes the proof of the theorem. □

Remark 7.2 Let ξ be an invariant random measure on S such that $\mathbb{Q}(\xi(S)=0)=0$. The inversion formula (3.17) shows that there is at most one invariant σ-finite measure $\mathbb{P}$ on $(\Omega,\mathcal{F})$ such that $\mathbb{Q}=\mathbb{P}_{\xi}$.

8 Stationary Partitions

A stationary partition decomposes a random subset Z of S into pairwise disjoint measurable sets. This partition is assumed to be consistent with the flow $\{\theta_g : g\in G\}$. Motivated by recent work in [3] and [2] on *allocations* to a simple point process, stationary partitions were introduced and studied (in the case $S=G=\mathbb{R}^d$) in [7]. Stationary tessellations of classical stochastic geometry (see, e.g., [17, 18]) are a special case.

Let η be an invariant simple point process on S. It is convenient to assume that $\eta(\omega)\in\mathbf{N}_s$ for all $\omega\in\Omega$ and to identify η with its support. A *stationary partition* (based on η) is a pair (Z,π) consisting of a *measurable* set $Z:\Omega\to\mathcal{S}$ and a mapping $\pi:\Omega\times S\to S$ such that both Z and π are *covariant* and such that $\pi(x)\in\eta$ whenever $x\in Z$. We also assume that $\{Z=\emptyset\}=\{\eta=\emptyset\}$. Measurability of Z just means that $(\omega,x)\mapsto\mathbf{1}\{x\in Z(\omega)\}$ is measurable, while covariance of Z means that

$$Z(\theta_g\omega)=gZ(\omega),\quad \omega\in\Omega,\ g\in G. \tag{8.1}$$

The covariance of π is defined by

$$\pi(\theta_g\omega,gx)=g\pi(\omega,x),\quad \omega\in\Omega,\ x\in S,\ g\in G. \tag{8.2}$$

For convenience, we also assume that $\pi(x)=x$, $x\in S$, whenever $\eta=\emptyset$. Define

$$C(\omega,x):=\big\{y\in Z(\omega):\pi(\omega,y)=x\big\},\quad \omega\in\Omega,\ x\in S. \tag{8.3}$$

Note that $C(x)=\emptyset$ whenever $x\notin\eta\neq\emptyset$. The system $\{C(x):x\in\eta\}$ forms a partition of Z into measurable sets, provided that $\eta\neq\emptyset$. Equations (8.1) and (8.2) imply the following covariance property:

$$C(\theta_g\omega,gx)=gC(\omega,x),\quad \omega\in\Omega,\ x\in S,\ g\in G. \tag{8.4}$$

Although we do not make any topological or geometrical assumptions, we refer to $C(x)$ as the *cell* with (generalized) *center* $x\in\eta$. We do not assume that $x\in C(x)$, and some of the cells may be empty.

We now fix a σ-finite invariant measure $\mathbb{P}$ on $(\Omega,\mathcal{F})$. The following theorem generalizes Theorem 7.1 in [7] from the case $S=G=\mathbb{R}^d$ to homogeneous spaces. The former case is also related to Lemma 16 in [2].

Theorem 8.1 *Let* (Z,π) *be a stationary partition. Then for any measurable* $f, \tilde{f}: \Omega \to \mathbb{R}_+$,

$$\mathbb{E}\mathbf{1}\{c \in Z\}\Delta^*\big(\pi(c)\big)\tilde{f}\int f\big(\theta_g^{-1}\big)\kappa\big(\pi(c), dg\big) = \mathbb{E}_\eta f \int \tilde{f}\big(\theta_g^{-1}\big)\mathbf{1}\big\{gc \in C(c)\big\}\lambda(dg). \tag{8.5}$$

Proof Consider the random measure $\xi := \mu(Z \cap \cdot)$. From the covariance (8.1) of Z and invariance of μ we have $\xi(\theta_g, gB) = \int \mathbf{1}\{x \in gB \cap gZ\}\mu(dx) = \xi(B)$ for all $g \in G$ and $B \in \mathcal{S}$. Hence ξ is invariant. An equally simple calculation shows that the Palm measure of ξ at c is given by

$$\mathbb{P}_\xi(A) = \mathbb{P}\big(A \cap \{c \in Z\}\big), \quad A \in \mathcal{F}. \tag{8.6}$$

We define the weighted transport-kernels T and T^* by $T(x,\cdot) := \delta_{\pi(x)}$ and $T^*(x,\cdot) := \mu(C(x) \cap \cdot)$. Since $\pi(x) \in \eta$ whenever $x \in Z$, it is straightforward to check that (4.3) holds even for all $\omega \in \Omega$. By (8.2), T is invariant. Invariance of T^* follows from (8.4) and invariance of μ:

$$T^*(\theta_g, gx, gB) = \mu\big(C(\theta_g, gx) \cap gB\big) = \mu\big(gC(x) \cap gB\big) = T^*(x, B).$$

Theorem 4.1 implies that (4.4) holds. Applying this formula to the measurable function $(\omega, g) \mapsto f(\omega)\tilde{f}(\theta_g^{-1}\omega)$ and taking into account (8.6), the definitions of T and T^* yield the assertion (8.5). □

Putting $\tilde{f} \equiv 1$ in (8.5) yields:

Corollary 8.2 *Let* (Z,π) *be a stationary partition. Then for any measurable* $f: \Omega \to \mathbb{R}_+$,

$$\mathbb{E}\mathbf{1}\{c \in Z\}\int f\big(\theta_g^{-1}\big)\Delta^*\big(\pi(c)\big)\kappa\big(\pi(c), dg\big) = \mathbb{E}_\eta \mu\big(C(c)\big)f. \tag{8.7}$$

Under additional assumptions on f and $\tilde{f}$, Theorem 8.1 can be simplified as follows.

Theorem 8.3 *Let* (Z,π) *be a stationary partition, and let* $f, \tilde{f}: \Omega \times S \to \mathbb{R}_+$ *be invariant and measurable. Then*

$$\mathbb{E}\mathbf{1}\{c \in Z\}\Delta^*\big(\pi(c)\big)\tilde{f}(c)f\big(\pi(c)\big) = \mathbb{E}_\eta f \int_{C(c)} \tilde{f}(x)\mu(dx). \tag{8.8}$$

Proof Apply (8.5) with f (resp. $\tilde{f}$) replaced by $f(\theta_e, c)$ (resp. $\tilde{f}(\theta_e, c)$). □

Corollary 8.4 *Let* (Z,π) *be a stationary partition, and let* $f: \Omega \times S \to \mathbb{R}_+$ *be invariant and measurable. Then*

$$\mathbb{E}\mathbf{1}\{c \in Z\}\Delta^*\big(\pi(c)\big)f\big(\pi(c)\big) = \mathbb{E}_\eta f \mu\big(C(c)\big). \tag{8.9}$$

The special case $f \equiv 1$ of (8.9) gives

$$\mathbb{E}_\eta \mu\big(C(c)\big) = \mathbb{E}\mathbf{1}\{c \in Z\}\Delta^*\big(\pi(c)\big) \tag{8.10}$$

If η has a positive and finite intensity γ_η and G is unimodular, this yields the formula

$$\mathbb{E}_{\mathbb{P}^0_\eta} \mu\big(C(c)\big) = \mathbb{P}(c \in Z)\gamma_\eta^{-1}. \tag{8.11}$$

Define the cell containing $x \in S$ by

$$V(x) := \big\{y \in S : \pi(y) = \pi(x)\big\}.$$

Corollary 8.5 *Let (Z, π) be a stationary partition. Then for any $\beta \geq 0$,*

$$\mathbb{E}\mathbf{1}\{c \in Z\}\Delta^*\big(\pi(c)\big)\mu\big(V(c)\big)^\beta = \mathbb{E}_\eta \mu\big(C(c)\big)^{\beta+1}. \tag{8.12}$$

Proof Define $f(\omega, x) := \mu(C(\omega, x))^\beta$. Equation (8.4) and the invariance of μ imply that f is invariant. Hence we can apply (8.9). It remains to note that $C(\pi(c)) = V(c)$. □

A stationary partition (Z, π) is called *proper* if

$$\mathbb{P}_\eta\big(\mu\big(C(c)\big) = 0\big) = \mathbb{P}_\eta\big(\mu\big(C(c)\big) = \infty\big) = 0. \tag{8.13}$$

For unimodal groups, the second equation is implied by (8.10). The following two results can be proved as in Sect. 5 of [7].

Proposition 8.6 *Let (Z, π) be a stationary and proper partition. Then for any measurable $f : \Omega \to \mathbb{R}_+$,*

$$\mathbb{E}\mathbf{1}\{c \in Z\}\Delta^*\big(\pi(c)\big)\mu\big(V(c)\big)^{-1} f\big(\theta^{-1}_{\pi(c)}\big) = \mathbb{E}_\eta f, \tag{8.14}$$

where $\theta^{-1}_{\pi(c)} : \Omega \to \Omega$ is defined by $\theta^{-1}_{\pi(c)}(\omega) := \theta^{-1}_{\pi(\omega,c)}\omega$.

Corollary 8.7 *Let (Z, π) be a stationary and proper partition. Then* (8.12) *holds for all $\beta \in \mathbb{R}$. In particular, if the intensity γ_η of η is finite, then*

$$\gamma_\eta = \mathbb{E}\mathbf{1}\{c \in Z\}\Delta^*\big(\pi(c)\big)\mu\big(V(c)\big)^{-1}. \tag{8.15}$$

From now on we assume that η has a finite intensity and $\mathbb{P}(\hat{\eta} = 0) = 0$. Just for simplicity, we also assume that $\mathbb{P}$ (and hence also $\mathbb{P}^*_\eta$) is a probability measure. We first note the following consequence of the proof of Theorem 8.1 and Corollary 4.2:

Corollary 8.8 *Under the hypothesis of Theorem* 8.1, *for any measurable $f, \tilde{f} : \Omega \to \mathbb{R}_+$,*

$$\mathbb{E}\left[\mathbf{1}\{c \in Z\}\Delta^*\big(\pi(c)\big)\tilde{f}\int f\big(\theta_g^{-1}\big)\kappa\big(\pi(c), dg\big)\,\middle|\,\mathcal{I}\right]$$
$$= \hat{\eta}\,\mathbb{E}_{\mathbb{P}_\eta^*}\left[f\int \tilde{f}\big(\theta_g^{-1}\big)\mathbf{1}\big\{gc \in C(c)\big\}\lambda(dg)\,\middle|\,\mathcal{I}\right] \tag{8.16}$$

$\mathbb{P}$-a.e. for any choice of the conditional expectations. In particular,

$$\hat{\eta}^{-1}\mathbb{E}\left[\mathbf{1}\{c \in Z\}\Delta^*\big(\pi(c)\big)\int f\big(\theta_g^{-1}\big)\kappa\big(\pi(c), dg\big)\,\middle|\,\mathcal{I}\right] = \mathbb{E}_{\mathbb{P}_\eta^*}\big[f\mu\big(C(c)\big)\,\big|\,\mathcal{I}\big]. \tag{8.17}$$

Let $\alpha > 0$. Essentially following [2] (dealing with the case $S = G = \mathbb{R}^d$), we call a stationary partition (Z, π) (based on η) *α-balanced* if

$$\mathbb{P}\big(\mu\big(C(x)\big) = \alpha\hat{\eta}^{-1} \text{ for all } x \in \eta\big) = 1. \tag{8.18}$$

The significance of α-balanced stationary partitions is due to the following theorem. The result extends Theorem 13 in [3] and Theorem 9.1 in [7] (both dealing with $\alpha = 1$) from $\mathbb{R}^d$ to general homogeneous spaces.

Theorem 8.9 *Let $\alpha > 0$. A stationary partition (Z, π) is α-balanced iff*

$$\mathbb{P}_\eta^* = \alpha^{-1}\mathbb{E}\mathbf{1}\{c \in Z\}\Delta^*\big(\pi(c)\big)\int \mathbf{1}\big\{\theta_g^{-1} \in \cdot\big\}\kappa\big(\pi(c), dg\big). \tag{8.19}$$

Proof If (Z, π) is an α-balanced stationary partition, then (8.19) follows from (8.17).

Assume now that (8.19) holds. Since $\mathbb{P}_\eta^*$ has the invariant density $\hat{\eta}$ with respect to $\mathbb{P}_\eta$, (8.19) implies that

$$\mathbb{P}_{\alpha\eta} = \mathbb{E}\mathbf{1}\{c \in Z\}\Delta^*\big(\pi(c)\big)\hat{\eta}\int \mathbf{1}\big\{\theta_g^{-1} \in \cdot\big\}\kappa\big(\pi(c), dg\big), \tag{8.20}$$

where we have also used the property that $\mathbb{P}_{\alpha\eta} = \alpha\mathbb{P}_\eta$. Using the invariant weighted transport-kernels $T(x, \cdot) := \hat{\eta}\mathbf{1}\{x \in Z\}\delta_{\pi(x)}$, (8.20) reads

$$\mathbb{P}_{\alpha\eta} = \mathbb{E}\iint \mathbf{1}\big\{\theta_g^{-1} \in \cdot\big\}\Delta^*(x)\kappa(x, dg)T(c, dx).$$

Since $\mathbb{P}$ is the Palm measure of μ, we get from Theorem 5.1 that T is $\mathbb{P}$-a.e. $(\mu, \alpha\eta)$-balancing. Therefore,

$$\int \mathbf{1}\big\{x \in Z, \pi(x) \in \cdot\big\}\mu(dx) = \alpha\hat{\eta}^{-1}\eta(\cdot)$$

$\mathbb{P}$-a.e. This just says that (Z, π) is α-balanced. □

Remark 8.10 If an α-balanced stationary partition (Z, π) is given, then (8.19) provides an explicit method for constructing the modified Palm probability measure $\mathbb{P}_\eta^*$

by a *shift-coupling* with the invariant measure $\mathbb{P}$. In the case where $S = G$ is a unimodal group, (8.19) simplifies to

$$\mathbb{P}^*_\eta = \alpha^{-1}\mathbb{E}\mathbf{1}\{e \in Z\}\mathbf{1}\{\theta^{-1}_{\pi(e)} \in \cdot\}. \tag{8.21}$$

Since $\alpha = \mathbb{P}(e \in Z)$, this means that $\mathbb{P}^*_\eta = \mathbb{P}(\theta^{-1}_{\pi(e)} \in \cdot \mid e \in Z)$.

The actual construction of α-balanced partitions is an interesting topic in its own right. Triggered by [10], the case $S = G = \mathbb{R}^d$ was discussed in [2, 3]. Among many other things, it was shown there that α-balanced partitions do actually exist for any $\alpha \le 1$. The occurrence of the sample intensity $\hat{\eta}$ in (8.18) is explained by the spatial ergodic theorem, see Proposition 9.1 in [7]. The paper [3] has also results on discrete groups in the case $\alpha = 1$. In the case of a general homogeneous space with a diffuse invariant measure μ it could be conjectured that α-balanced partitions exist for all $\alpha \le 1$.

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