Günter Last
Institut für Mathematische Stochastik
Universität Karlsruhe (TH)

On Palm measures of stationary spatial point processes

Günter Last

joint work with

Matthias Heveling (Karlsruhe)

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Contents

1. Stationary point processes

Framework:

(i) The space of all *point configurations* in $\mathbb{R}^d$ is defined as

$$\mathbf{N} := \{\varphi \subset \mathbb{R}^d : \varphi \text{ is locally finite}\}.$$

(ii) The *Palm space* is defined by

$$\mathbf{N}_0 := \{\varphi \in \mathbf{N} : 0 \in \varphi\}.$$

(iii) Any $\varphi \in \mathbf{N}$ is identified with a counting measure:

$$\varphi(B) := \text{card}\{x \in \varphi : x \in B\}, \quad B \subset \mathbb{R}^d.$$

(iv) The σ-field $\mathcal{N}$ is the smallest σ-field of subsets of $\mathbf{N}$ making the mappings $\varphi \mapsto \varphi(B)$ for all Borel sets $B \subset \mathbb{R}^d$ measurable.

Definition: A *point process* (on $\mathbb{R}^d$) is a probability measure $\mathbb{P}$ on $(\mathbf{N}, \mathcal{N})$. Given $\mathbb{P}$ we also call the identity map N on $\mathbf{N}$ a point process.

Definition:

(i) The *shifts* $\theta_x : \mathbf{N} \to \mathbf{N}$, $x \in \mathbb{R}^d$, are defined by

$$\theta_x \varphi := \varphi - x, \quad x \in \mathbb{R}^d.$$

(ii) The *shifts* $\theta_x : \mathcal{N} \to \mathcal{N}$, $x \in \mathbb{R}^d$, are defined by

$$\theta_x A := \{\varphi - x : \varphi \in A\}, \quad A \in \mathcal{N}.$$

(iii) A measure $\mathbb{P}$ on $(\mathbf{N}, \mathcal{N})$ is called *stationary* if

$$\mathbb{P} \circ \theta_x = \mathbb{P}, \quad x \in \mathbb{R}^d.$$

Definition: Let $\mathbb{P}$ be a σ-finite and stationary measure on $(\mathbf{N}, \mathcal{N})$. Then

$$\mathbb{P}_0(A) := \iint \mathbf{1}(\theta_x \varphi \in A, x \in [0,1]^d)\, \varphi(dx)\, \mathbb{P}(d\varphi)$$
$$= \int \Big(\sum_{x \in [0,1]^d \cap \varphi} \mathbf{1}(\theta_x \varphi \in A) \Big)\, \mathbb{P}(d\varphi) \quad A \in \mathcal{N},$$

is called the *Palm measure* of $\mathbb{P}$. If the *intensity*

$$\lambda_{\mathbb{P}} := \int \varphi([0,1]^d)\, \mathbb{P}(d\varphi)$$

is positive and finite, then the normalized Palm measure $\lambda_{\mathbb{P}}^{-1}\mathbb{P}_0$ is called *Palm probability measure* of $\mathbb{P}$.

Theorem: *Any σ-finite and stationary measure $\mathbb{P}$ on $(\mathbf{N}, \mathcal{N})$ satisfies the refined Campbell theorem*

$$\mathbb{E}_{\mathbb{P}}\left[\int f(\theta_x N, x)\, N(dx)\right] = \mathbb{E}_{\mathbb{P}_0}\left[\int f(N, x)\, dx\right]$$

for all measurable $f : \mathbf{N} \times \mathbb{R}^d \to [0, \infty)$.

Let

$$V(\varphi) := \{x \in \mathbb{R}^d : \varphi \cap B^0(x, |x|) = \emptyset\},$$

denote the Voronoi cell of $\varphi \in \mathbf{N}_0$ around $0 \in \varphi$, where $B^0(x, r)$ is the open ball with centre x and radius $r > 0$.

Theorem: *Any σ-finite and stationary measure $\mathbb{P}$ on $(\mathbf{N}, \mathcal{N})$ satisfies the inversion formula*

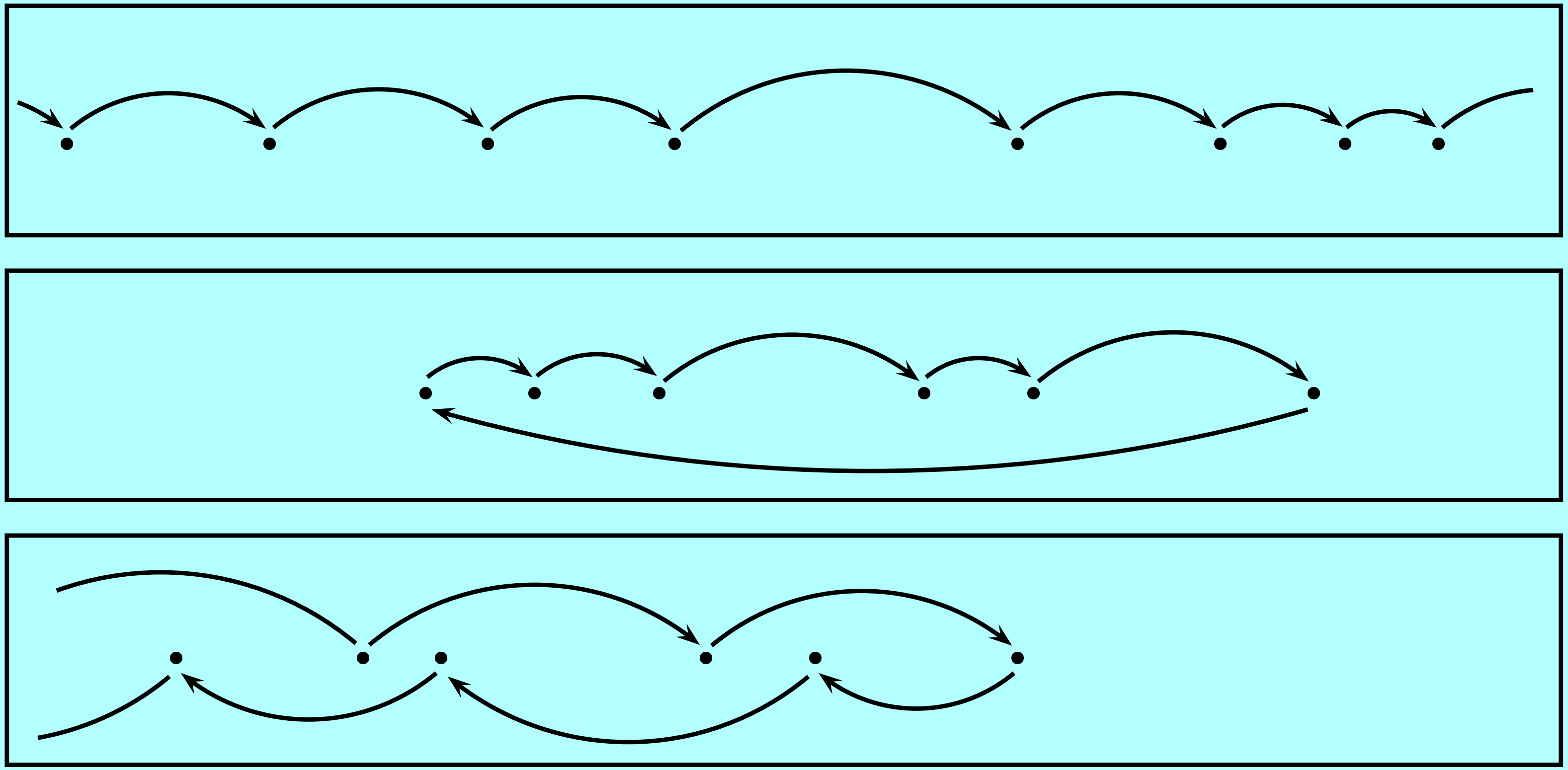
$$\mathbb{E}_{\mathbb{P}}[\mathbf{1}(N \neq \emptyset) f(N)] = \mathbb{E}_{\mathbb{P}_0}\left[\int_{V(N)} f(\theta_x N)\, dx\right],$$

for all measurable $f : \mathbf{N} \to [0, \infty)$.

2. Time- and cycle stationarity in one dimension

Assume that $d = 1$ and consider the following mapping

$$\sigma : \mathbf{N} \times \mathbb{R} \rightarrow \mathbb{R}$$



The mapping $\sigma : \mathbf{N} \times \mathbb{R} \to \mathbb{R}$ creates a point shift $\theta_\sigma : \mathbf{N} \to \mathbf{N}$:

$$\theta_\sigma(\varphi) := \theta_{\sigma(\varphi,0)}(\varphi) = \varphi - \sigma(\varphi, 0).$$

Definition: Assume $d = 1$. A measure $\mathbb{Q}$ on $(\mathbf{N}, \mathcal{N})$ is called *cycle-stationary* if $\mathbb{Q}(0 \notin N) = 0$ and

$$\mathbb{Q}(\theta_\sigma N \in \cdot) = \mathbb{Q}.$$

Theorem: *The Palm measure of a σ-finite and stationary measure on $(\mathbf{N}, \mathcal{N})$ is cycle-stationary. If, conversely, $\mathbb{Q}$ is a σ-finite and cycle-stationary measure, then there is a unique σ-finite stationary measure $\mathbb{P}$ such that $\mathbb{P}(\{\emptyset\}) = 0$ and $\mathbb{Q} = \mathbb{P}_0$.*

References:

E.L. Kaplan (1955) Transformations of stationary random sequences. *Math. Scand.* **3**, 127–149.

C. Ryll–Nardzewski (1961) Remarks on processes of calls. *Proc. 4th Berkeley Symp. Math. Statist. Probab.* **2**, 455-465.

I.M. Slivnyak (1962) Some properties of stationary flows of homogeneous random events. *Th. Probab. Appl.* **7**, 336–341.

3. Bijective point maps

Definition:

(i) A measurable mapping $\pi : \mathbf{N} \times \mathbb{R}^d \to \mathbb{R}^d$ is called *covariant* if

$$\pi(\varphi - y, x - y) = \pi(\varphi, x) - y$$

for all $\varphi \in \mathbf{N}$ and $x, y \in \mathbb{R}^d$. Given a covariant mapping π we write $\pi(\varphi) := \pi(\varphi, 0)$.

(ii) A *point map* is a covariant mapping $\pi : \mathbf{N} \times \mathbb{R}^d \to \mathbb{R}^d$ such that $\pi(\varphi, x) \in \varphi$ for any $\varphi \in \mathbf{N}$ and $x \in \varphi$. (One can always assume that $\pi(\varphi, x) = x$ for all $x \notin \varphi$.)

(iii) A point map π is *bijective* if $\pi(\varphi, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is a bijection for any $\varphi \in \mathbf{N}$.

Example: (Olle Häggström) Define a point map π by

$$\pi(\varphi, x) := \begin{cases} y, & \text{if } x \text{ and } y \in \varphi \setminus \{x\} \text{ are} \\ & \text{mutual nearest neighbors in } \varphi, \\ x, & \text{otherwise.} \end{cases}$$

This map is called *mutual nearest neighbor matching*.

Definition: The *composition* of two point maps $\pi, \tau : \mathbf{N} \times \mathbb{R}^d \to \mathbb{R}^d$ is defined by

$$\pi \circ \tau(\varphi, x) := \pi(\varphi, \tau(\varphi, x)).$$

Lemma:

(i) *The composition of two point maps π and τ is again a poin map. If π and τ are bijective, then so is $\pi \circ \tau$.*

(ii) *If a point map π is bijective then there is a inverse point map π^{-1} such that*

$$\pi \circ \pi^{-1}(\varphi, x) = \pi^{-1} \circ \pi(\varphi, x) = x$$

for all $\varphi \in \mathbf{N}$ and all $x \in \varphi$. This point map is bijective.

Definition: If π is a point map then the associated *point shift* $\theta_\pi : \mathbf{N} \to \mathbf{N}$ is defined by

$$\theta_\pi(\varphi) := \theta_{\pi(\varphi,0)}(\varphi) = \varphi - \pi(\varphi, 0).$$

Lemma: *For any point maps π and τ,*

$$\theta_\pi \circ \theta_\tau = \theta_{\pi\circ\tau}.$$

Lemma: *Let π be a point map. Then π is bijective if and only if $\theta_\pi : \mathbf{N} \to \mathbf{N}$ is a bijection. In this case*

$$(\theta_\pi)^{-1} = \theta_{\pi^{-1}}.$$

Theorem: (Mecke 1975) *Let $\mathbb{P}_0$ be the Palm measure of a stationary σ-finite measure $\mathbb{P}$ and $\pi : \mathbf{N} \times \mathbb{R}^d \to \mathbb{R}^d$ a bijective point map. Then*

$$\mathbb{P}_0(\theta_\pi N \in \cdot) = \mathbb{P}_0.$$

References:

J. Mecke (1975) Invarianzeigenschaften allgemeiner Palmscher Maße. *Math. Nachr.* **65**, 335–344.

H. Thorisson (2000) *Coupling, Stationarity, and Regeneration.* Springer, New York.

4. Point-stationarity

Definition: A measure $\mathbb{Q}$ on $(\mathbf{N}, \mathcal{N})$ is called *point-stationary*, if $\mathbb{Q}(0 \notin N) = 0$ and

$$\mathbb{Q}(\cdot) = \mathbb{Q}(\theta_\pi N \in \cdot)$$

for all bijective point-shifts π.

Theorem: (Heveling/L. 2005) *A measure $\mathbb{Q}$ on $(\mathbf{N}, \mathcal{N})$ is the Palm measure of some stationary σ-finite measure $\mathbb{P}$ iff $\mathbb{Q}$ is σ-finite and point-stationary.*

Proof of the characterization:

Theorem: (Mecke 1967) *A measure* $\mathbb{Q}$ *on* $(\mathbf{N}, \mathcal{N})$ *is the Palm measure of some stationary* σ*-finite measure* $\mathbb{P}$ *iff* $\mathbb{Q}(0 \notin N) = 0$ *and*

$$\mathbb{E}_{\mathbb{Q}}\left[\int f(\theta_x N, -x)\, N(dx)\right] = \mathbb{E}_{\mathbb{Q}}\left[\int f(N, x)\, N(dx)\right]$$

for any measurable $f : \mathbf{N} \times \mathbb{R}^d \to [0, \infty)$.

Proposition: *There exists a countable family* $\pi_n : \mathbf{N} \times \mathbb{R}^d \to \mathbb{R}^d$, $n \in \mathbb{N}$, *of bijective point maps, such that (in particular)*

$$\{\pi_n(\varphi, 0) : n \in \mathbb{N}\} = \varphi, \quad \varphi \in \mathbf{N}_0.$$

5. Matchings

Definition: A point map π is called *matching*, if $\pi \circ \pi = \mathrm{id}_{\mathbb{R}^d}$, i.e.

$$\pi(\varphi, \pi(\varphi, x)) = x, \quad \varphi \in \mathbf{N}, x \in \varphi.$$

Lemma: *A matching π is bijective and*

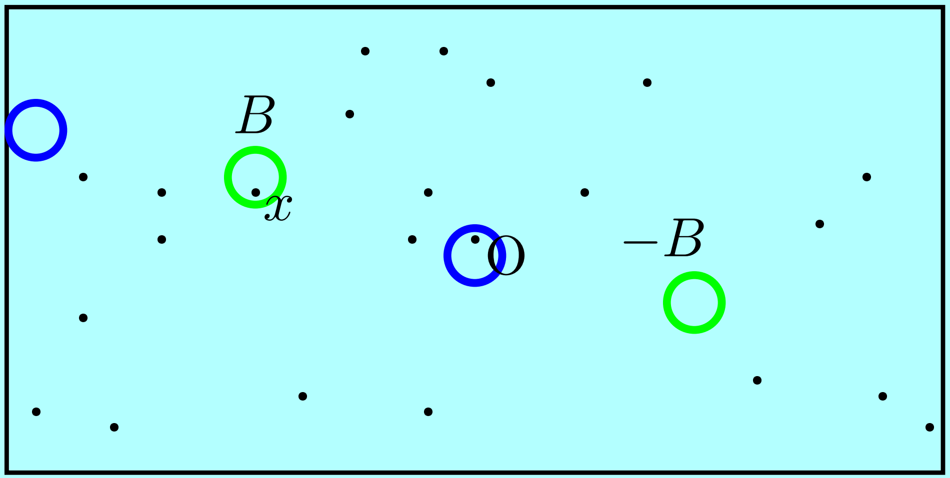
$$\pi^{-1} = \pi.$$

Symmetric area search based on a (small) set $B \subset \mathbb{R}^d \setminus \{0\}$:

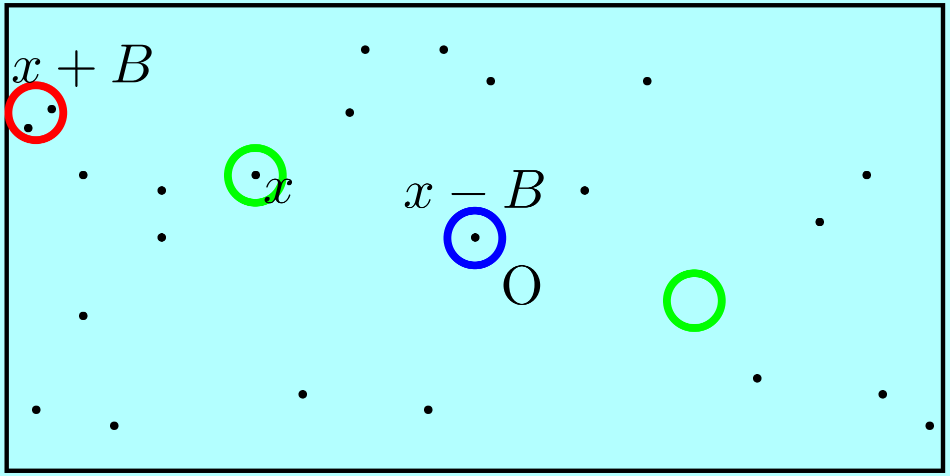
Take $\varphi \in \mathbf{N}_0$. If

$$\varphi \cap (B \cup -B) = \{x\},$$
$$\varphi \cap ((x + B) \cup (x - B)) = \{0\},$$

then we define $\pi(\varphi) := x$. Otherwise we let $\pi(\varphi) := 0$.



Symmetric area search that fails, i.e. $\pi(\varphi) = 0$:



Conclusion: Symmetric area search detects all points in $\varphi \setminus \{0\}$, provided that there is no line passing through 0 that contains 3 or more points of φ.

Proposition: *There exist matchings* $\pi_n : \mathbf{N} \times \mathbb{R}^d \to \mathbb{R}^d$, $n \in \mathbb{N}$, *such that the following two properties hold for any* $\varphi \in \mathbf{N}_0$:

$$\text{(i)} \quad \{x \in \varphi : \theta_x \varphi \neq \varphi\} = \{\pi_n(\varphi) : n \in \mathbb{N}, \pi_n \neq 0\}.$$

$$\text{(ii)} \quad \pi_n(\varphi) = \pi_m(\varphi) \neq 0 \quad \textit{implies} \quad m = n.$$

Proposition: *Let* $\mathbb{Q}$ *be a* σ*-finite measure on* $(\mathbf{N}, \mathcal{N})$ *with the property* $\mathbb{Q}(0 \notin N) = 0$. *Then* $\mathbb{Q}$ *is point-stationary if and only if*

$$\mathbb{Q}(\cdot) = \mathbb{Q}(\theta_\pi N \in \cdot)$$

for all matchings π.

6. The periodicity lattice

Definition: Let $\varphi \in \mathbf{N}$. Then

$$L(\varphi) := \{x \in \mathbb{R}^d : \varphi - x = \varphi\}$$

is called *periodicity lattice* of φ. If $L(\varphi) = \{0\}$, then φ is called *aperiodic*.

Lemma: *The periodicity lattice of* $\varphi \in \mathbf{N} \setminus \{0\}$ *is a discrete subgroup of* $\mathbb{R}^d$.

7. Translation invariant graphs

Definition: Let $\pi : \mathbf{N} \times \mathbb{R}^d \to \mathbb{R}^d$ be a bijective point map. Drawing a directed edge from any $x \in \varphi$ to $\pi(\varphi, x)$ equips $\varphi \in \mathbf{N}$ with the structure of a *directed graph* $G_\pi(\varphi)$ with vertex set φ. In case $\pi(\varphi, x) = x$ the point x is *isolated* in $G_\pi(\varphi)$.

Remark: The graph $G_\pi(\varphi)$ is constructed in a translation invariant way. If there is a directed edge from x to x' in $G_\pi(\varphi)$ then there is a directed edge from $x - y$ to $x' - y$ in $G_\pi(\varphi - y)$.

Remark: Let π be a bijective point map and let $\varphi \in \mathbf{N}$. If the iterates π^n, $n \in \mathbb{Z}$, of π satisfy

$$\{\pi^n(\varphi, x) : n \in \mathbb{Z}\} = \varphi$$

for some $x \in \varphi$ then $G_\pi(\varphi)$ is a (directed) *path*. This path is *doubly-infinite* if φ contains infinitely many points and *cyclic*, otherwise.

Theorem: (Ferrari/Landim/Thorisson 2004, Holroyd/Peres 2003)
Let $\mathbb{P}$ *be a stationary Poisson process on* $(\mathbf{N}, \mathcal{N})$. *Then there is a bijective point shift* $\pi : \mathbf{N} \times \mathbb{R}^d \to \mathbb{R}^d$ *such that*

$$N = \{\pi^n(N,0) : n \in \mathbb{Z}\} \quad \mathbb{P}_0 - a.s.$$

Alternative formulation:

It is possible to give the points of N an ordering isomorphic to the usual ordering of the integers in a deterministic translation-invariant way.

Theorem: (Timar 2004) *Let $\mathbb{P}$ be a stationary and ergodic probability measure on $(\mathbf{N}, \mathcal{N})$ such that N has a positive and finite intensity and is $\mathbb{P}$-a.s. aperiodic. Then there is a bijective point shift $\pi : \mathbf{N} \times \mathbb{R}^d \to \mathbb{R}^d$ such that*

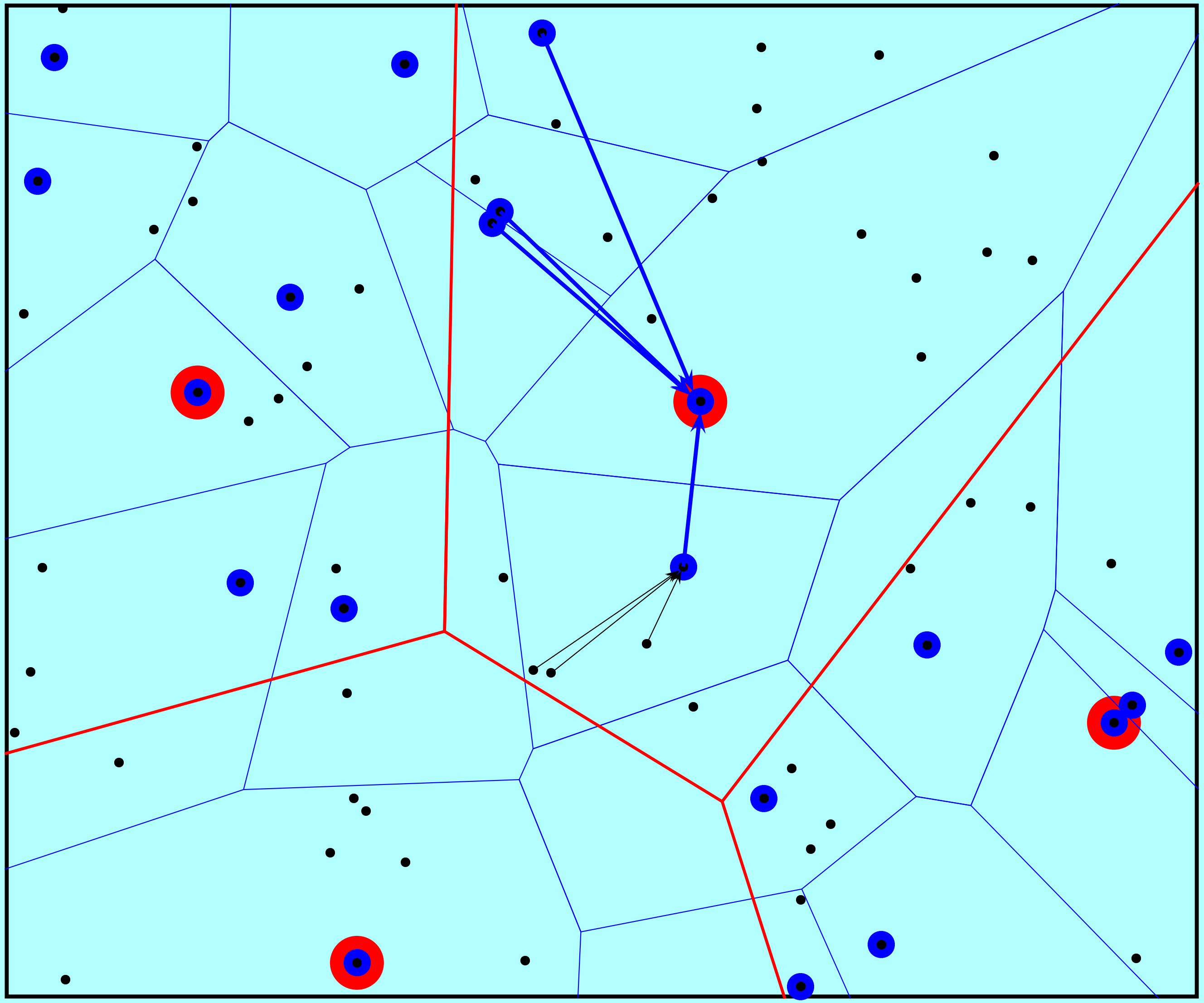
$$N = \{\pi^n(N, 0) : n \in \mathbb{Z}\} \quad \mathbb{P}_0 - a.s.$$

Problem: Is there a bijective point map π such that

$$\varphi = \{\pi^n(\varphi, 0) : n \in \mathbb{Z}\}$$

holds for any aperiodic $\varphi \in \mathbf{N}_0$?

Construction of a covariant tree:



Theorem: *There is a bijective point map* $\pi : \mathbf{N} \times \mathbb{R}^d \to \mathbb{R}^d$ *having the following properties:*

(i) *There is a constant* $c \in \mathbb{N}$ *(depending only on* π *and the dimension* d*) such that the graph* $G_\pi(\varphi)$ *has at most* c *connected components for any aperiodic* $\varphi \in \mathbf{N}$*.*

(ii) *Each component of* $G_\pi(\varphi)$ *is a tree.*

(iii) *For any stationary and aperiodic probability measure* $\mathbb{P}$ *on* $(\mathbf{N}, \mathcal{N})$ *satisfying* $\mathbb{P}(N = \emptyset) = 0$ *the graph* $G_\pi(N)$ *has* $\mathbb{P}$*-a.s. exactly one component.*