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# Stationary partitions and shift-coupling

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Frankfurter Stochastik-Tage 2006

15.03.2006

# 1. Stationary point processes

**Canonical Framework:**

(i) The space of all *point configurations* in $\mathbb{R}^d$ is defined as

$$\mathbf{N} := \{\varphi \subset \mathbb{R}^d : \varphi \text{ is locally finite}\}.$$

(ii) Any $\varphi \in \mathbf{N}$ is identified with a counting measure:

$$\varphi(B) := \text{card}\{x \in \varphi : x \in B\}, \quad B \subset \mathbb{R}^d.$$

(iii) The $\sigma$-field $\mathcal{N}$ is the smallest $\sigma$-field of subsets of $\mathbf{N}$ making the mappings $\varphi \mapsto \varphi(B)$ for all Borel sets $B \subset \mathbb{R}^d$ measurable.

## Flows:

(i) The underlying sample space is $(\Omega, \mathcal{A})$.

(ii) We consider measurable mappings $\theta_x : \Omega \to \Omega$, $x \in \mathbb{R}^d$, satisfying $\theta_0 = \mathrm{id}_\Omega$ and the flow property

$$\theta_x \circ \theta_y = \theta_{x+y}, \quad x, y \in \mathbb{R}^d.$$

(iii) A (simple) *point process* $N$ on $\mathbb{R}^d$ is a measurable mapping $N : \Omega \to \mathbf{N}$.

(iv) A point process $N$ is *adapted* (or *stationary*) if

$$N(\omega, B + x) = N(\theta_x \omega, B), \quad \omega \in \Omega, x \in \mathbb{R}^d, B \in \mathcal{B}^d.$$

## 2. Palm measures

Assumptions: $N$ is an adapted point process and $\mathbb{P}$ a $\sigma$-finite measure on $(\Omega, \mathcal{A})$.

Definition: (i) $\mathbb{P}$ is *stationary* if

$$\mathbb{P} \circ \theta_x = \mathbb{P}, \quad x \in \mathbb{R}^d.$$

(ii) If $\mathbb{P}$ is stationary, then the measure

$$\mathbb{P}_N(A) := \iint \mathbf{1}\{\theta_x \omega \in A, x \in [0,1]^d\} \, N(\omega)(dx) \, \mathbb{P}(d\omega), \quad A \in \mathcal{A},$$

is called the *Palm measure* of $\mathbb{P}$ (with respect to $N$).

(iii) If the *intensity* $\lambda_N := \mathbb{E}_{\mathbb{P}}[N([0,1]^d)]$ of $N$ is positive and finite, then the normalized Palm measure $\lambda_N^{-1}\mathbb{P}_N$ is called *Palm probability measure* of $\mathbb{P}$.

**Theorem:**(refined Campbell's theorem) *Let $N$ be an adapted point process and $\mathbb{P}$ a $\sigma$-finite, stationary measure on $(\mathbf{N}, \mathcal{N})$. Then*

$$\mathbb{E}_{\mathbb{P}}\left[\int f(\theta_x, x)\, N(dx)\right] = \mathbb{E}_{\mathbb{P}_N}\left[\int f(\theta_0, x)\, dx\right]$$

*for all measurable* $f : \Omega \times \mathbb{R}^d \rightarrow [0, \infty)$.

# 3. The modified Palm probability measure

**Definition:** Let $N$ be an adapted point process.

(i) The *invariant $\sigma$-field* $\mathcal{I} \subset \mathcal{A}$ contains all sets $A \in \mathcal{A}$ with the property

$$\theta_x A = A, \quad x \in \mathbb{R}^d.$$

(ii) Let $\mathcal{I}'$ the invariant $\sigma$-field on $\mathbf{N}$, i.e. the system of all sets $C \in \mathcal{N}$ satisfying $C + x = C$ for all $x \in \mathbb{R}^d$. The *invariant $\sigma$-field* generated by $N$ is defined by

$$\mathcal{I}_N = \{\{N \in C\} : C \in \mathcal{I}'\}$$

**Remark:** If $N$ is an adapted point process, then

$$\mathcal{I}_N \subset \mathcal{I}.$$

Assumptions: $\mathbb{P}$ is a stationary probability measure on $(\Omega, \mathcal{A})$ and $N$ is an adapted point process.

Definition: The random variable

$$\hat{N} := \mathbb{E}[N([0,1]^d)|\mathcal{I}].$$

is called *sample intensity* of $N$.

Proposition: *The sample intensity can be chosen $\mathcal{I}_N$-measurable and satisfies*

$$\mathbb{E}_{\mathbb{P}}[N(B)|\mathcal{I}] = |B|_d \cdot \hat{N} \quad \mathbb{P}-a.s.,\ B \in \mathcal{B}^d.$$

*Moreover,*

$$\{\hat{N} = 0\} = \{N = \emptyset\} \quad \mathbb{P}-a.s.$$

**Proposition:** *Consider an increasing sequence $B_n$, $n \in \mathbb{N}$, of convex sets whose inradius converges towards $\infty$ and assume that $N$ has a finite intensity. Then*

$$\hat{N} = \lim_{n\to\infty} \frac{N(B_n)}{|B_n|_d}$$

*holds $\mathbb{P}$-almost surely.*

**Definition:** Let $N$ be an adapted point process and $\mathbb{P}$ a stationary probability measure on $(\Omega, \mathcal{A})$.

(i) $N$ has a *positive and finite sample intensity*, if

$$\mathbb{P}(0 < \hat{N} < \infty) = 1.$$

(ii) If $N$ has a *positive and finite sample intensity* then

$$\mathbb{P}^*_N(A) := \mathbb{E}\left[\hat{N}^{-1} \int \mathbf{1}\{\theta_x \in A, x \in [0,1]^d\}\, N(dx)\right], \quad A \in \mathcal{A},$$

is called *modified Palm probability measure* of $N$.

**Proposition:** *If $N$ has a positive and finite sample intensity then*

$$\mathbb{P}^*_N(A) = \mathbb{P}(A), \quad A \in \mathcal{I}.$$

**Proposition:** *Assume that $\mathbb{P}$ is a stationary probability measure on $(\Omega, \mathcal{A})$ such that $\mathbb{P}(\hat{N} = 0) = 0$ and $\mathbb{E}\hat{N} < \infty$. Then $\mathbb{P}^*_N = \mathbb{P}^0_N$ if and only if*

$$\mathbb{P}(\hat{N} = \mathbb{E}\hat{N}) = 1.$$

**Definition:** An adapted point process satisfying

$$\mathbb{P}(\hat{N} = \mathbb{E}\hat{N}) = 1.$$

is called *pseudo-ergodic*.

Example: Assume that $(\Omega, \mathcal{A}) = (\mathbf{N}, \mathcal{N})$. Consider for some $p \in (0,1)$ the mixture

$$\mathbb{P} := p\mathbb{P}_2 + (1-p)\mathbb{Q}$$

of the distribution of a Poisson process $\mathbb{P}_2$ with intensity 2 and the distribution $\mathbb{Q}$ of the union of $N \cup (N+y)$ $(y \neq 0)$ under the distribution of a Poisson process with intensity 1. Then $\mathbb{P}$ is pseudo-ergodic but not ergodic.

# 4. Stationary partitions

Assumption: $N$ is an adapted point process.

Definition: A *stationary partition* (based on $N$) is a measurable mapping $\pi : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\pi(\omega, x) \in N(\omega), \quad \omega \in \Omega,\ N(\omega) \neq \emptyset,$$

and such that it is *covariant*, i.e.

$$\pi(\theta_y \omega, x - y) = \pi(\omega, x) - y, \quad \omega \in \Omega,\ x, y \in \mathbb{R}^d.$$

For convenience we also assume that $\pi(x) = x$, $x \in \mathbb{R}^d$, whenever $N = \emptyset$.

**Definition:** Let $\pi$ be a stationary partition based on $N$.

(i) The *cell* with *centre* $x \in N$ is the Borel set

$$C(x) = \{y \in \mathbb{R}^d : \pi(y) = x\}.$$

The same definition yields $C(x) = \emptyset$ whenever $x \notin N(\omega)$.

(ii) The cell containing the site $y \in \mathbb{R}^d$ is the Borel set

$$V(y) := \{z \in \mathbb{R}^d : \pi(z) = \pi(y)\}.$$

**Remark:** The system $\{C(x) : x \in N\}$ forms a partition of $\mathbb{R}^d$ into measurable sets provided that $N \neq \emptyset$.

Definition: The *shift*

$$\theta_{\pi(0)} : \Omega \to \Omega$$

associated with a stationary partition $\pi$ is defined by

$$(\theta_{\pi(0)})(\omega) := \theta_{\pi(\omega,0)}(\omega).$$

The shifted point process $N \circ \theta_{\pi(0)} = N - \pi(0)$ has a point at the origin whenever $N \neq \emptyset$

# 5. Basic properties of stationary partitions

**Assumptions:** $\mathbb{P}$ is a stationary probability measure and $N$ is an adapted point process with positive and finite sample intensity.

**Theorem:** *We have for all measurable $f, g : \Omega \to [0, \infty)$ that*

$$\mathbb{E}[f \cdot g(\theta_{\pi(0)})] = \mathbb{E}_{\mathbb{P}_N}\left[g \cdot \int_{C(0)} f \circ \theta_x \, dx\right],$$

*and*

$$\mathbb{E}[f \cdot g(\theta_{\pi(0)})|\mathcal{I}] = \hat{N} \cdot \mathbb{E}_{\mathbb{P}^*_N}\left[g \cdot \int_{C(0)} f \circ \theta_x \, dx \Big| \mathcal{I}\right] \quad \mathbb{P} - a.s.,$$

*for any choice of the conditional expectations.*

**Proposition:** *We have for all measurable* $g : \Omega \to [0, \infty)$ *that*

$$\mathbb{E}_{\mathbb{P}_N}\left[|C(0)|_d \cdot g\right] = \mathbb{E}[g(\theta_{\pi(0)})].$$

**Definition:** We call the stationary partition $\pi$ $\mathbb{P}$-*proper* if

$$0 < |C(\omega, x)|_d < \infty, \quad x \in N(\omega),$$

holds for all $\omega$ outside an event of $\mathbb{P}$-measure 0.

**Proposition:** *If $\pi$ is $\mathbb{P}$-proper, then we have for all measurable $f : \Omega \to [0, \infty)$ that*

$$\mathbb{E}_{\mathbb{P}_N} f = \mathbb{E}[|V(0)|_d^{-1} \cdot f(\theta_{\pi(0)})].$$

**Proposition:** *If $\pi$ is $\mathbb{P}$-proper then the conditional distribution of $-\pi(0)$ given $\theta_{\pi(0)}$ is the uniform distribution on $V(0) - \pi(0)$.*

# 6. Point stationarity

**Assumption:** $N$ is an adapted point process.

**Definition:**

(i) A covariant mapping $\pi : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is called *bijective point map* if $\pi(\omega, x) \in N(\omega)$ whenever $x \in N(\omega)$ and the restriction of $\pi(\omega, \cdot)$ on $N(\omega)$ is bijective.

(ii) A measure $\mathbb{Q}$ on $(\Omega, \mathcal{A})$ is *point-stationary* (with respect to $N$) if $\mathbb{Q}(0 \notin N) = 0$ and

$$\mathbb{Q}(\cdot) = \mathbb{Q}(\theta_{\pi(0)} \in \cdot)$$

holds for all bijective point maps $\pi : \Omega \to \mathbb{R}^d$ such that $\pi(0)$ is $\sigma(N)$-measurable.

**Theorem:** *Let $\mathbb{Q}$ be a $\sigma$-finite measure on $(\Omega, \mathcal{A})$ and let $\pi$ be a stationary partition based on $N$ such that $\pi$ is $\mathbb{Q}$-proper. Let $\mathbb{Q}^0$ be the measure with density $|V(0)|_d^{-1}$ with respect to $\mathbb{Q}$ and assume that $\mathbb{Q}^0(\theta_{\pi(0)} \in \cdot)$ is point-stationary. If the conditional distribution $\mathbb{Q}(-\pi(0) \in \cdot | \theta_{\pi(0)})$ can be chosen as the uniform distribution on $V(0) - \pi(0)$ then $\mathbb{Q}$ is stationary.*

References:

Thorisson (2000) *Coupling, Stationarity, and Regeneration.* Springer, New York. (Voronoi-tessellations)

L. (2005) Stationary partitions and Palm probabilities. Preprint. (General stationary partitions)

# 7. Balanced stationary partitions and shift-coupling

Assumptions: $\mathbb{P}$ is a stationary probability measure and $N$ is an adapted point process with positive and finite sample intensity.

Definition: A stationary partition $\pi$ (based on $N$) is *balanced* (w.r.t. $\mathbb{P}$), if

$$\mathbb{P}(|C(x)|_d = \hat{N}^{-1} \text{ for all } x \in N) = 1,$$

or, equivalently,

$$\mathbb{P}^*_N(|C(0)|_d = \hat{N}^{-1}) = 1.$$

**Proposition:** *Let $\xi$ be a random variable such that $\mathbb{P}(0 < \xi < \infty) = 1$ and $\pi$ a stationary partition based on $N$. Assume that*

$$\mathbb{P}(|C(x)|_d = \xi \text{ for all } x \in N) = 1.$$

*Then $\mathbb{P}(\xi = \hat{N}^{-1}) = 1$.*

**Theorem:** *Let $\pi$ be a stationary partition. Then $\pi$ is balanced if and only if*

$$\mathbb{P}(\theta_{\pi(0)} \in \cdot) = \mathbb{P}^*_N.$$

**Theorem:** *There is a stationary partition that is balanced with respect to $\mathbb{P}$. This partition can be chosen $N$*-measurable, *in the sense that $\pi(0)$ is $\sigma(N)$-measurable.*

Remark: Let $\pi$ be a stationary and $N$-measurable partition and define $\tau := \pi(0)$. Since

$$\mathbb{P}(N - \tau \in \cdot) = \mathbb{P}_N^*(N \in \cdot),$$

the pair $(N, N - \tau)$ is called *shift-coupling* of the stationary and the modified Palm distribution of $N$.

References:

Thorisson (1996). Transforming random elements and shifting random fields. *Annals of Probability* **24**, 2057–2064.

Holroyd and Peres (2005). Extra heads and invariant allocations. *Annals of Probability* **33** 31–52.

L. (2005). Stationary partitions and Palm probabilities. Preprint. (non-ergodic point process case)