The Annals of Probability
2005, Vol. 33, No. 5, 1698–1715
DOI 10.1214/009117905000000224

CHARACTERIZATION OF PALM MEASURES VIA BIJECTIVE POINT-SHIFTS

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The paper considers a stationary point process N in $\mathbb{R}^d$. A point-map picks a point of N in a measurable way. It is called bijective [Thorisson, H. (2000). *Coupling*, *Stationarity*, *and Regeneration*. Springer, New York] if it is generating (by suitable shifts) a bijective mapping on N. Mecke [*Math. Nachr.* **65** (1975) 335–344] proved that the Palm measure of N is point-stationary in the sense that it is invariant under bijective point-shifts. Our main result identifies this property as being characteristic for Palm measures. This generalizes a fundamental classical result for point processes on the line (see, e.g., Theorem 11.4 in [Kallenberg, O. (2002). *Foundations of Modern Probability*, 2nd ed. Springer, New York]) and solves a problem posed in [Thorisson, H. (2000). *Coupling*, *Stationarity*, *and Regeneration*. Springer, New York] and [Ferrari, P. A., Landim, C. and Thorisson, H. (2004). *Ann. Inst. H. Poincaré Probab. Statist.* **40** 141–152]. Our second result guarantees the existence of bijective point-maps that have (almost surely with respect to the Palm measure of N) no fixed points. This answers another question asked by Thorisson. Our final result shows that there is a directed graph with vertex set N that is defined in a translation-invariant way and whose components are almost surely doubly infinite paths. This generalizes and complements one of the main results in [Holroyd, A. E. and Peres, Y. (2003). *Electron. Comm. Probab.* **8** 17–27]. No additional assumptions (as ergodicity, nonlattice type conditions, or a finite intensity) are made in this paper.

1. Introduction. We consider a stationary (simple) point process N in $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Stationarity means distributional invariance under all translations. We let $\mathbb{P}_N$ denote the Palm measure of N (see, e.g., [1, 7, 9, 11]). If N has a positive and finite intensity, then the normalization $\mathbb{P}_N^0$ of $\mathbb{P}_N$ (the Palm probability measure of N) describes the statistical properties of the underlying stochastic experiment as seen from *a typical point* of N located at the origin 0. Note that $\mathbb{P}_N(0 \notin N) = 0$. Thorisson [14] (using a canonical framework) calls a measurable mapping $\pi : \Omega \to \mathbb{R}^d$ *a point-map* if $\pi \in N$ on the event $\{0 \in N\}$. Such a π is called *a bijective point-map* if $x \mapsto \pi \circ \theta_x + x$ is a bijection on N, where $\theta_x : \Omega \to \Omega$, $x \in \mathbb{R}^d$, is the underlying family of shift operators (see Section 2). Mecke ([8], Satz 4.3) proved that bijective point-maps

Received August 2004; revised February 2005.
AMS 2000 subject classifications. Primary 60G55; secondary 60G10.
Key words and phrases. Point process, stationarity, Palm measure, point-map, random graph, tree, point-stationarity.

can be used to shift the point 0 to a (possibly) different point of N without biasing the Palm measure of N (see also Theorem 9.4.1 of [14]). A short proof of this fact is given in Section 3.

Following [14] we call a measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ *point-stationary* if $\mathbb{Q}(0 \notin N) = 0$ and $\mathbb{Q}$ is invariant under all $\sigma(N)$-measurable bijective point-maps. However, unlike [14] we do not allow additional randomization. The main result of our paper (Theorem 4.1) shows that point-stationarity is the characteristic property of Palm measures. This characterization generalizes a fundamental classical result for one-dimensional processes (see, e.g., Theorem 11.4 in [5]) and solves a problem posed in [14] and [2] in complete generality. Our proof is based on a construction of a nested family of bijective point-maps exhausting all points of N and on Mecke's [7] intrinsic characterization of Palm measures.

Thorisson [2, 14] has asked for the existence of a bijective point-map having the property $\mathbb{P}_N^0(\pi \neq 0) = 1$. Then it is possible to shift the typical point of N (in a nonrandomized way) to an almost surely different point of N without biasing the Palm probability measure. In dimension $d = 1$ there are of course many such point-maps, as one can shift to the nearest neighbor to the right, for instance. In dimension $d \geq 2$ the situation is less clear. Shifting to the nearest neighbor of the origin, for instance, does not solve the problem. As the nearest neighbor of the typical point might also be the nearest neighbor of other points, it is certainly not a typical point anymore. In this paper we will prove (see Theorem 5.1) the existence of a $\sigma(N)$-measurable bijective point-map satisfying

$$\mathbb{P}_N(\pi = 0) = 0. \tag{1.1}$$

The third main result in this paper (Theorem 6.1) shows that there is a graph with vertex set N that is constructed from N in a translation-invariant way and whose components are almost surely directed doubly infinite paths. This complements Theorem 2 in [4] which provides (under some additional assumptions on N) an isometry-invariant graph with vertex set N having the same property.

2. Palm measures. All random elements are defined on a measurable space $(\Omega, \mathcal{F})$ equipped with *a measurable flow* $\theta_x : \Omega \to \Omega$, $x \in \mathbb{R}^d$. This is a family of measurable mappings such that $(\omega, x) \mapsto \theta_x\omega$ is measurable, θ_0 is the identity on Ω and

$$\theta_x \circ \theta_y = \theta_{x+y}, \qquad x, y \in \mathbb{R}^d.$$

A random measure N on $\mathbb{R}^d$ (see, e.g., [5]) is called *adapted* to the flow if

$$N(\omega, B + x) = N(\theta_x\omega, B), \qquad \omega \in \Omega, x \in \mathbb{R}^d, B \in \mathcal{B}^d,$$

where $\mathcal{B}^d$ denotes the Borel σ-field on $\mathbb{R}^d$.

Let $\mathbb{P}$ be a σ-finite measure on $(\Omega, \mathcal{F})$ being *stationary* in the sense that

$$\mathbb{P} \circ \theta_x = \mathbb{P}, \qquad x \in \mathbb{R}^d.$$

We are hence working within the $\{\theta_x\}$-framework as introduced in [9] for point processes (see also [11]). Although a canonical setting as used in [14] and Chapter 11 of [5] might be more appealing to the reader, the present more general framework is notationally less cumbersome. In Sections 3 and 4 we will not assume that $\mathbb{P}$ is a probability measure. This allows us to formulate our results in greater generality that is useful even in a probabilistic framework (see, e.g., [16] or the proof of Proposition 4.3). Moreover, our general setting leads to a completely symmetric form of the main result (Theorem 4.1).

If N is a flow-adapted random measure and $\mathbb{P}$ is a σ-finite stationary measure on $(\Omega, \mathcal{F})$, then the "distribution" $\mathbb{P}(N + x \in \cdot)$ of the shifted process $N + x$ is the same for any $x \in \mathbb{R}^d$. Therefore we call N just *stationary*. The *intensity measure* $\Lambda(B) := \mathbb{E}_{\mathbb{P}}[N(B)]$, $B \in \mathcal{B}^d$, of a stationary random measure is given by $\Lambda(dx) = \lambda_{\mathbb{P}}\, dx$, where $\lambda_{\mathbb{P}} := \mathbb{E}_{\mathbb{P}}[N([0,1]^d)]$ is the *intensity* of N and dx refers to integration with respect to Lebesgue measure on $\mathbb{R}^d$. Here $\mathbb{E}_{\mathbb{P}}$ denotes integration with respect to $\mathbb{P}$. The measure

$$\mathbb{P}_N(A) := \iint \mathbf{1}(\theta_x\omega \in A, x \in [0,1]^d) N(\omega)(dx)\mathbb{P}(d\omega), \qquad A \in \mathcal{F}, \tag{2.1}$$

is called the *Palm measure* of $\mathbb{P}$ (with respect to N). It is σ-finite and satisfies the *refined Campbell theorem*

$$\mathbb{E}_{\mathbb{P}}\left[\int f(\theta_x, x) N(dx)\right] = \mathbb{E}_{\mathbb{P}_N}\left[\int f(\theta_0, x)\, dx\right] \tag{2.2}$$

for all measurable $f : \Omega \times \mathbb{R}^d \to [0, \infty)$. If $0 < \lambda_{\mathbb{P}} < \infty$, then we can define the *Palm probability measure* $\mathbb{P}_N^0 := \lambda_{\mathbb{P}}^{-1}\mathbb{P}_N$ of N.

An example of a random measure is $N(\cdot) := \operatorname{card}\{i : \xi_i \in \cdot\}$, where $\{\xi_i : i \in \mathbb{N}\}$ is a (*simple*) *point process* of pairwise distinct points in $\mathbb{R}^d$ (see [5]) that are not allowed to accumulate in bounded sets. Formally we may introduce N as a measurable mapping from Ω to the space $\mathbf{N}$ of all locally finite subsets $\varphi \subset \mathbb{R}^d$ equipped with the σ-field generated by the mappings $\varphi \mapsto \varphi(B) := \operatorname{card}(\varphi \cap B)$, $B \in \mathcal{B}^d$. We will make no distinction between a point process and its associated random measure.

3. Invariance properties of Palm measures. Let N be a point process in $\mathbb{R}^d$ adapted to the flow $\{\theta_x : x \in \mathbb{R}^d\}$. We call a measurable mapping $\pi : \Omega \to \mathbb{R}^d$ *a point-map* if $\pi \in N$ on the event $\{0 \in N\}$. Such a point-map creates *a point-shift* $\theta_\pi : \Omega \to \Omega$, given as the composed mapping $\omega \mapsto \theta_{\pi(\omega)}\omega$. We call both a point-map and the associated point-shift *bijective* if

$$x \mapsto \pi(x) := \pi \circ \theta_x + x \tag{3.1}$$

is a bijection on N whenever $N \neq \varnothing$. In this case it is easy to see that the inverse $x \mapsto \pi^{-1}(x)$ of this mapping is of the form

$$\pi^{-1}(x) = \pi^{-1}(0) \circ \theta_x + x.$$

Hence $\pi^{-1} := \pi^{-1}(0)$ is again a bijective point-map. We call it the *inverse* point-map of π. If π_1 and π_2 are two bijective point maps, then $x \mapsto \pi_2(\pi_1(x))$ is a bijection on N whenever $N \neq \varnothing$. It is easy to see that this bijection is associated with the point map $\pi_2 \circ \pi_1 := \pi_2 \circ \theta_{\pi_1} + \pi_1$. In particular we may define for any $n \in \mathbb{N}$ the nth iterate π^n of a bijective point-shift. Further we define $\pi^0 := 0$, and $\pi^{-n} := (\pi^n)^{-1}$ for $n \in \mathbb{N}$.

The following result can be derived as a special case of Satz 4.3 in [8] (see also Theorem 9.4.1 of [14]).

THEOREM 3.1. *Let $\mathbb{P}_N$ be the Palm measure of a stationary σ-finite measure $\mathbb{P}$ and let $\pi : \Omega \to \mathbb{R}^d$ a bijective point-map. Then*

$$\mathbb{P}_N(\theta_\pi \in \cdot) = \mathbb{P}_N. \tag{3.2}$$

PROOF. Take a measurable function $g : \Omega \to [0, \infty)$ and a Borel set $B \subset \mathbb{R}^d$ of volume 1. By the refined Campbell theorem,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_N}[g \circ \theta_\pi] &= \mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}(x \in B) g(\theta_{\pi \circ \theta_x}) N(dx)\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}(x \in B) g(\theta_{\pi(x)}) N(dx)\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}(\pi^{-1} \circ \theta_y + y \in B) g \circ \theta_y N(dy)\right] \\
&= \mathbb{E}_{\mathbb{P}_N}\left[\int \mathbf{1}(\pi^{-1} + y \in B) g\, dy\right] = \mathbb{E}_{\mathbb{P}_N}[g],
\end{aligned}$$

where we have used the inverse point-map π^{-1} of π to get the third equality. □

The above proof of Theorem 3.1 (being close to the proof in [8]) is of interest even in the case $d = 1$. It seems to be more elegant than the standard textbook proof using a limit argument.

4. Characterization of Palm measures. Thorisson [14] introduced the concept of point-stationarity, which formalizes the intuitive idea of a point process which looks distributionally the same seen from each of its points. Our construction of bijective point-shifts in Proposition 4.2 and our main result Theorem 4.1 permit us to remove the random background used in the original definition and to reduce it to its natural form: distributional invariance under bijective point-shifts.

We call a measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ *point-stationary* (with respect to N) if $\mathbb{Q}(0 \notin N) = 0$ and $\mathbb{Q}$ is invariant under all $\sigma(N)$-measurable bijective point-shifts. The latter means that

$$\mathbb{Q}(\cdot) = \mathbb{Q}(\theta_\pi \in \cdot) \tag{4.1}$$

holds for any bijective point-map $\pi : \Omega \to \mathbb{R}^d$ that is measurable with respect to the σ-field generated by N.

Our aim here is to prove that point-stationarity is the characteristic property of Palm measures. This generalizes a fundamental result for stationary point processes on the real line due to Kaplan [6], Ryll-Nardzewski [10] and Slivnyak [12] (see also Theorem 12.3.II in [1] and Theorem 11.4 in [5]). Our result solves a problem posed in [2] and [14] in complete and striking generality.

THEOREM 4.1. *A measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is the Palm measure of some stationary σ-finite measure $\mathbb{P}$ iff $\mathbb{Q}$ is σ-finite and point-stationary.*

Our proof of Theorem 4.1 is based on Proposition 4.2 below that is of interest in its own right. This purely deterministic result provides some nested and monotone way to exhaust all points of a configuration $\varphi \in \mathbf{N}$ by measurable bijective point-maps.

We will begin with an informal description of our construction of point-maps and assume that $0 \in N$. A first approach might be to fix a Borel set $B \in \mathcal{B}^d$ and then define a point-map that maps 0 to the point in $B \cap N$ if it is unique and to 0 otherwise. However, this point-map π is not bijective, unless we symmetrize using the reflected set $-B := \{-x : x \in B\}$ as follows. If there is a unique point x in $B \cup (-B)$, and if the origin is the unique point in $(B + x) \cup (-B + x)$, then we define $\pi = \pi(0) := x$. Otherwise we define $\pi := 0$. We call this procedure *symmetric area search*. Note that $\pi(0) = x$ if and only if $\pi(x) = 0$, that is, the mapping $x \mapsto \pi(x)$ is self-inverse.

In Figure 1, we illustrate on the left the case where the point 0 is mapped to x. This is not so on the right; here 0 is invariant under the point-map, because there are two points in $(-B + x) \cup (B + x)$. Taking $B \in \mathcal{B}^d$ from finer and finer

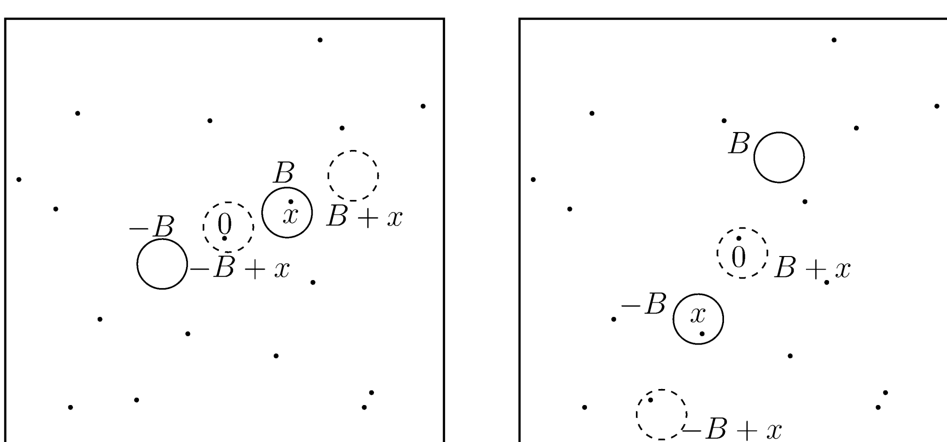

FIG. 1. *Matching by symmetric area search.*

partitions of $\mathbb{R}^d$, we define a family of point-maps such that for any point $x \in N$ such that $-x \notin N$ and $2x \notin N$, there exists a point-map in the family that maps 0 to x. If k $(2 < k < \infty)$ points are equidistantly aligned, we speak of a *k-chain* and generalize the above approach by ordering the points with respect to some translation-invariant order, and then define different point-maps for each specified position in chains of fixed length. Finally, the double-sided infinite case is treated in close analogy to the classical one-dimensional case, and the one-sided infinite case similarly, using some fixed bijection $f:\mathbb{N} \to \mathbb{Z}$.

Let us now formalize our approach. For the sake of clarity it is reasonable to redefine point-maps in a canonical setting. Let $\mathbf{N}_0$ denote the measurable set of all $\varphi \in \mathbf{N}$ such that $0 \in \varphi$. Consider a measurable mapping $\sigma : \mathbf{N}_0 \to \mathbb{R}^d$ such that $\sigma(\varphi) \in \varphi$ for any $\varphi \in \mathbf{N}_0$ and define

$$\sigma(\varphi, x) := \sigma(\varphi - x) + x, \qquad \varphi \in \mathbf{N}, x \in \varphi. \tag{4.2}$$

If $\sigma(\varphi, \cdot)$ is a bijection on φ for any $\varphi \in \mathbf{N}$, then we call σ a *bijective point-map*. Formally we will make no distinction between σ as defined on $\mathbf{N}_0$ and the mapping $(\varphi, x) \mapsto \sigma(\varphi, x)$ as defined on the measurable set $\{(\varphi, x) \in \mathbf{N} \times \mathbb{R}^d : x \in \varphi\}$. The iterates $\sigma^n(\varphi, \cdot)$, $n \in \mathbb{Z}$, of a bijective point-map are defined as in Section 3.

PROPOSITION 4.2. *There exist bijective point-maps $\sigma_{m,n} : \mathbf{N}_0 \to \mathbb{R}^d$, $m, n \in \mathbb{N}$, such that for any $\varphi \in \mathbf{N}$ and $x \in \varphi$*:

(a) *For all $m \in \mathbb{N}$ and all $n, n' \in \mathbb{N}$ with $n \neq n'$*,

$$\{\sigma^i_{m,n}(\varphi, x) : i \in \mathbb{Z}\} \cap \{\sigma^i_{m,n'}(\varphi, x) : i \in \mathbb{Z}\} = \{x\}.$$

(b) *As $m \to \infty$*,

$$\{\sigma^i_{m,n}(\varphi, x) : n \in \mathbb{N}, i \in \mathbb{Z}\} \uparrow \varphi.$$

PROOF. A bijective point-map $\sigma : \mathbf{N}_0 \to \mathbb{R}^d$ is called *a translation-invariant matching* if $\sigma^2(\varphi, 0) = 0$ for any $\varphi \in \mathbf{N}_0$. As explained above, the idea is now to construct translation-invariant matchings that exhaust all points of $\varphi \in \mathbf{N}$ with the possible exception of points that form lattices on lines. We prepare this construction with an analysis of the one-dimensional lattice structures in φ that we will call *chains in φ*. Fix some translation-invariant order $<$ on $\mathbb{R}^d$. A k-tuple $(x_1, \dots, x_k)$ of points in φ, $k \geq 2$, is called a k-chain if $x_{i+1} - x_i = x_j - x_{j-1}$ and $x_i < x_j$ for all $1 \leq i < j \leq k$, $x_1 - (x_2 - x_1) \notin \varphi$ and $x_k + (x_2 - x_1) \notin \varphi$. A sequence $(x_i : i \in \mathbb{N})$ of elements of φ is called a $\propto$-chain if $x_{i+1} - x_i = x_j - x_{j-1}$ for all $1 \leq i < j$ and $x_1 - (x_2 - x_1) \notin \varphi$. Finally we call a sequence $(x_i : i \in \mathbb{Z})$ of elements of φ an ∞-chain if $x_{i+1} - x_i = x_j - x_{j-1}$ and $x_i < x_j$ whenever $i < j$.

Two distinct points $x, y \in \varphi$ generate a chain $M_{x,y}$ in φ in the following way. For $k \in \mathbb{Z}$ let $y_k := x + k(y - x)$, in particular $y_0 = x$ and $y_1 = y$. Then let $m := \inf\{n \in N : y_n \notin \varphi\}$ and $l := \inf\{n \in \mathbb{N} : y_{-n} \notin \varphi\}$, where $\inf \varnothing = \infty$.

After suitable enumeration the (finite or infinite) sequence $(y_i : -l < i < m)$ is an $(m + l - 1)$-chain if both m and l are finite, a $\propto$-chain if one but not both of m and l is finite and an ∞-chain if m and l are both infinite.

Let $x \in \varphi$ and $B \in \mathcal{B}^d$. We call a chain $K = (x_i : i \in I)$ *B-adapted* if $x_2 - x_1 \in B \cup (-B)$. The point x is called *B-uncritical* (in φ) if there exists a unique B-adapted chain through x. This chain is then denoted by K_x^B. Let $k \in \{\propto, 2, 3, \ldots\}$ and $i \in \mathbb{N}$ such that $i < k$. Here we use the convention $i < \propto$ for any $i \in \mathbb{N}$. For any $x \in \varphi$ we then define

$$\sigma_{k,i}^B(\varphi, x) := \begin{cases} y, & \text{if } x \text{ is } B\text{-uncritical and } y \in \varphi \setminus \{x\} \text{ is} \\ & \text{a } B\text{-uncritical point such that} \\ & M_{x,y} = K_x^B = K_y^B = (x_j : j \in I) \text{ is} \\ & \text{a } k\text{-chain with } \{x_i, x_{i+1}\} = \{x, y\}, \\ x, & \text{otherwise.} \end{cases} \tag{4.3}$$

In the following we will derive some basic properties of the mappings $\sigma_{k,i}^B(\varphi, \cdot)$ that are defined this way. For convenience, we will skip indices as well as the argument φ whenever possible. First we note that σ is well defined. Indeed, given a B-uncritical point $x \in \varphi$ there is at most one B-uncritical $y \in \varphi \setminus \{x\}$ satisfying the condition in (4.3). The mapping σ is translation covariant, that is, for $x \in \varphi$ and $z \in \mathbb{R}^d$ we have $\sigma(\varphi - z, x - z) = \sigma(\varphi, x) - z$, because $x, y \in \varphi$ satisfy the condition in (4.3) if and only if $x - z, y - z \in \varphi - z$ do. Also, by the symmetry of x and y in (4.3), we have $\sigma(\varphi, x) = y$ if and only if $\sigma(\varphi, y) = x$ for $x, y \in \varphi$; thus $\sigma^2(\varphi, 0) = 0$, so that σ is a translation-invariant matching.

For any $B, B' \in \mathcal{B}^d$ such that $(B \cup (-B)) \cap (B' \cup (-B')) = \varnothing$, x is both B-uncritical and B'-uncritical, and fos $\sigma_{k,i}^B(x) \neq x$ we have $\sigma_{i,k}^B(x) \neq \sigma_{i',k'}^{B'}(x)$ for all pairs (i, k), (i', k'). This is due to the fact that our matchings always swap neighbors in a chain. But the difference of those points cannot be in both $B \cup (-B)$ and $B' \cup (-B')$. Also, for $(k, i) \neq (k', i')$ we have $\sigma_{k,i}^B(x) \neq \sigma_{k',i'}^B(x)$, whenever $\sigma_{k,i}^B(x) \neq x$. Suppose that $\sigma_{k,i}^B(x) = \sigma_{k',i'}^B(x) = y \neq x$. Then $k = k'$ must be the length of K_x^B, and we have $\{x, y\} = \{x_i, x_{i+1}\} = \{x_{i'}, x_{i'+1}\}$, hence $i = i'$.

Next we choose bounded Borel subsets $B_{m,n} \subset \mathbb{R}^d$, $m, n \in \mathbb{N}$, such that $B_{m,n}$ is for all $m, n \in \mathbb{N}$ the finite union of sets from $\{B_{m+1,i} : i \in \mathbb{N}\}$, and such that

$$\{B_{m,n} : n \in \mathbb{N}\} \cup \{-B_{m,n} : n \in \mathbb{N}\}, \qquad m \in \mathbb{N},$$

is a nested sequence of partitions of $\mathbb{R}^d \setminus \{0\}$ satisfying

$$\lim_{m \to \infty} \sup\{D(B_{m,n}) : n \in \mathbb{N}\} = 0, \tag{4.4}$$

where $D(B)$ denotes the diameter of a set $B \subset \mathbb{R}^d$.

Take $m, n \in \mathbb{N}$, $k \in \{\propto, 2, 3, \dots\}$, $i < k$, $x \in \varphi$, and let $y := \sigma_{k,i}^{B_{m,n}}(\varphi, x)$. Slightly modifying the matchings given by (4.3), we define

$$\sigma_{k,i}^{(m,n)}(\varphi, x) := \begin{cases} x, & \text{if there is an } \infty\text{-chain } K \text{ through } x \\ & \text{such that } y \in \operatorname{aff} K \\ & \text{and an } \infty\text{-chain } K' \text{ through } y \\ & \text{such that } x \in \operatorname{aff} K, \\ \sigma_{k,i}^{B_{m,n}}(\varphi, x), & \text{otherwise,} \end{cases} \tag{4.5}$$

where $\operatorname{aff}(B)$ denotes the affine hull of a set $B \subset \mathbb{R}^d$. Due to the translation invariance of the condition in (4.5) all the properties derived above are shared by these mappings.

Next we define the set

$$C_m(\varphi, x) := \{\sigma_{k,i}^{(m,n)}(\varphi, x) : n \in \mathbb{N}, k \in \{\propto, 2, 3, \dots\}, 1 \le i < k\}$$

of all points in φ that are attained from x by the $\sigma_{k,i}^{(m,n)}$. We will prove that

$$C_m(\varphi, x) \subset C_{m+1}(\varphi, x).$$

If $x, y \in C_m(\varphi, x)$ and $x \neq y$, then x and y are not mutually contained in the affine hull of an ∞-chain through the other point and there exist $m, n \in \mathbb{N}$ and (k, i) such that $\sigma_{k,i}^{(m,n)}(x) = y$. Then there exists a unique $n' \in \mathbb{N}$ such that $y - x \in (B_{m+1,n'} \cup -B_{m+1,n'})$. Therefore x and y are $B_{m+1,n'}$-uncritical and $M_{x,y} = K_x^{B_{m+1,n'}} = K_y^{B_{m+1,n'}}$. Hence $\sigma_{k,i}^{B_{m+1,n'}}(\varphi, x) = y \in C_{m+1}(x)$.

Next let us define the set $L(\varphi, x)$ of all points $y \in \varphi \setminus \{x\}$ such that there is an ∞-chain K through x and $y \in \operatorname{aff} K$ and an ∞-chain K' through y and $x \in \operatorname{aff} K'$. Again, this definition is symmetric; we have $y \in L(x)$ if and only if $x \in L(y)$. In particular, all points $y \in \varphi \setminus \{x\}$ such that $M_{x,y}$ is an ∞-chain are elements of $L(\varphi, x)$. We will show that

$$\bigcup_{m \in \mathbb{N}} C_m(\varphi, x) = \varphi \setminus L(\varphi, x). \tag{4.6}$$

Suppose first that $y \in L(\varphi, x)$. By definition, none of the $\sigma_{k,i}^{(m,n)}$ maps x to y, hence $y \notin C_m(\varphi, x)$ for all $m \in \mathbb{N}$.

Take $y \notin L(x)$ such that $y \neq x$. Then $M_{x,y} := (x_j : j \in I)$ is a k-chain, $k \geq 2$, or a $\propto$-chain and $\{x, y\} = \{x_i, x_{i+1}\}$ for some $i \in I$. For $m \in \mathbb{N}$ sufficiently large there exists an $n \in \mathbb{N}$ such that $x - y \in (B_{m,n} \cup -B_{m,n})$ and x, y are $B_{m,n}$-uncritical. Then $M_{x,y} = K_x^{B_{m,n}} = K_y^{B_{m,n}}$ and $\sigma_{k,i}^{(m,n)}(x) = y \in C_m(x)$.

We will now treat the case of points on ∞-chains. The basic idea is the same as in the case of the real line. One shifts a point to its nearest neighbor in a specified direction. We need to introduce some notation to make this definition formal. Let $S^+ \subset S^{d-1}$ such that $S^{d-1} = S^+ \cup -S^+$ and $S^+ \cap -S^+ = \varnothing$. Then every doubly infinite chain $K = (x_i : i \in \mathbb{Z})$ will be attributed the unique direction $u \in S^+$ such that $x_2 - x_1/\|x_2 - x_1\| \in \{u, -u\}$ and we will write $\vec{K} = u$. Also, we write $|K| := |x_2 - x_1|$ for the "size" of K.

Let $\varphi \in \mathbf{N}$, $x \in \varphi$, $v \in S^+$ and define

$$\delta_v(x) := \inf\{|K| : K \text{ is an } \infty\text{-chain such that } \vec{K} = v \text{ and } x \in \operatorname{aff} K\}.$$

Since φ is locally finite we have $\delta_v(x) > 0$. For $x \in \varphi$ and $t > 0$ we have

$$\operatorname{card}\{v \in S^+ : \delta_v(x) < t\} < \infty. \tag{4.7}$$

In particular, the set $\{\delta_v(x)v : v \in S^+, \delta_v(x) < \infty\}$ is locally finite. If $x, y \in \varphi$ and K, K' are ∞-chains such that $x \in K$, $y \in K'$, $\operatorname{aff} K = \operatorname{aff} K'$ and $\vec{K} = v$, we have $\delta_v(x) = \delta_v(y)$.

A point $x \in \varphi$ is called B*-directed* for some $B \in \mathcal{B}^d$ if there exists a unique direction $u \in S^+$ such that $\delta_u(x)u \in B$. In this case we write $u(x, B) := u$ and otherwise we say that the point x is B*-undirected*. If $x \in \varphi$ is B-directed, then we define the distance to the nearest B-undirected neighbors in $L(x)$ in direction $u(x, B)$ and $-u(x, B)$ by

$$s(x, B) := \inf\{t > 0 : x + tu(x, B) \in L(x) \text{ and } x + tu(x, B) \text{ is } B\text{-undirected}\},$$

$$p(x, B) := \inf\{t > 0 : x - tu(x, B) \in L(x) \text{ and } x - tu(x, B) \text{ is } B\text{-undirected}\},$$

and the B*-directed component* of x in $L(x)$ by

$$N(x, B) := \{y \in L(x) : y = x + tu \text{ for } t \in (-p(B, x, u), s(B, x, u))\}.$$

If $N(x, B)$ has finitely many elements, that is, $p(x, B) < \infty$, $s(x, B) < \infty$, then there exists exactly one point $y \in N(x, B)$ such that $N(x, B) \subset \{y + tu(x, B) : t \geq 0\}$. We denote this point by $F(x, B)$. The next point from x in direction u (if there is one) will be called the successor $S(x, B)$ of x, and we will also use this definition in the doubly infinite case, that is, when $s(x, B) = p(x, B) = \infty$. In the simply infinite case, when either $s(x, B) = \infty$ or $p(x, B) = \infty$, we enumerate the elements of the directed component in the natural way by a function $\eta : N(x, B) \to \mathbb{N}$, such that the end point z satisfies $\eta(z) = 1$.

Note that all of these definitions are again covariant with respect to translations of φ. As a last preparation we introduce a bijective mapping $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) := \begin{cases} \dfrac{n-1}{2}, & \text{if } n \text{ is odd,} \\ -\dfrac{n}{2}, & \text{if } n \text{ is even,} \end{cases}$$

and define $g : \mathbb{N} \to \mathbb{N}$ by

$$g(n) := f^{-1}(f(n) + 1).$$

Then g is a bijection on $\mathbb{N}$ such that for a fixed $n \in \mathbb{N}$ we have

$$\{g^k(n) : k \in \mathbb{Z}\} = \mathbb{N}.$$

Now define for a B-directed point $x \in \varphi$

$$\sigma_\infty^B(\varphi, x) := \begin{cases} S(x, B), & \text{if } p(x, B) + s(x, B) < \infty \\ & \text{and } L(x) \cap \{x + tu : 0 < t < s(x, B)\} \neq \varnothing, \\ F(x, B), & \text{if } p(x, B) + s(x, B) < \infty \\ & \text{and } L(x) \cap \{x + tu : 0 < t < s(x, B)\} = \varnothing, \\ \eta^{-1}(g(\eta(x))), & \text{if } p(x, B) < \infty, s(x, B) = \infty, \\ & \text{or } p(x, B) = \infty, s(x, B) < \infty, \\ S(x, B), & \text{if } p(x, B) = s(x, B) = \infty. \end{cases} \tag{4.8}$$

If x is not B-directed, then we define $\sigma_\infty^B(\varphi, x) := x$.

From the definition it is clear that for any $B, B' \in \mathcal{B}^d$ satisfying $(B \cup -B) \cap (B' \cup -B') = \varnothing$ and any $x \in \varphi$ we have

$$\{(\sigma_\infty^B)^k(x) : k \in \mathbb{Z}\} \cap \{(\sigma_\infty^{B'})^k(x) : k \in \mathbb{Z}\} = \{x\}.$$

Using the sets $B_{m,n}$ we again define a family of mappings by

$$\sigma_\infty^{(m,n)} := \sigma_\infty^{B_{m,n}} \tag{4.9}$$

and denote by

$$D_m(x) := \{y \in \varphi : \text{there exist } n \in \mathbb{N} \text{ and } k \in \mathbb{Z} \text{ such that } (\sigma_\infty^{(m,n)})^k(x) = y\}$$

the set of all points reached from x by $(\sigma_\infty^{(m,n)})^k$ for some $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. We will show that

$$D_m(x) \uparrow L(x) \qquad \text{as } m \to \infty. \tag{4.10}$$

Let $y \in D_m(x)$. Then $y \in L(x)$ and there exist $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $(\sigma_\infty^{(m,n)})^k(x) = y$; in particular y is $B_{m,n}$-directed, that is, there exists a unique $u \in S^+$ such that $\delta_u(x)u \in B_{m,n}$. By the definition of the $B_{m,n}$ there exists a unique n' such that $\delta_u(x)u \in B_{m+1,n'}$. Then y is $B_{m+1,n'}$-directed and $(\sigma_\infty^{(m+1,n')})^k(x) = y$. Hence, $D_m(x) \subset D_{m+1}(x)$.

Now let $z \in L(x)$ and define $v := (z - x)/\|z - x\|$ if $(z - x)/\|z - x\| \in S^+$ and $v := (x - z)/\|x - z\|$ otherwise. Then $\delta_v(z) < \infty$ and by (4.4) and (4.7) there exist $m, n \in \mathbb{N}$ such that z is $B_{m,n}$-directed and $u(z, B_{m,n}) = v$. Then $(\sigma_\infty^{(m,n)})^k(x) = z$ for some $k \in \mathbb{Z}$. Hence, $z \in D_m(x)$.

To finish the proof of the proposition it now suffices to define the point-maps $\sigma_{m,n}$ in such a way that

$$\{\sigma_{m,n} : n \in \mathbb{N}\} = \{\sigma_{k,i}^{(m,n')} : n' \in \mathbb{N}, k \in \{\propto, 2, \dots\}, i < k\} \cup \{\sigma_\infty^{(m,n')} : n' \in \mathbb{N}\}$$

for any $m \in \mathbb{N}$. $\square$

PROOF OF THEOREM 4.1. By Theorem 3.1 it remains to prove the "if" part of the theorem. So we assume that $\mathbb{Q}$ is σ-finite and point-stationary. We show that

$$\mathbb{E}_{\mathbb{Q}}\left[\int f(\theta_x, -x)N(dx)\right] = \mathbb{E}_{\mathbb{Q}}\left[\int f(\theta_0, x)N(dx)\right] \tag{4.11}$$

for any measurable $f : \Omega \times \mathbb{R}^d \to [0, \infty)$, and can then apply Satz 2.5 in [7] to conclude the assertion. Although Mecke proved his result only in a canonical framework, his proof transfers to our more general case with obvious changes (see Proposition II.11 in [9]).

Let π be a bijective point-map. It is easy to check that

$$\pi \circ \theta_{\pi^{-1}} = -\pi^{-1}. \tag{4.12}$$

Define

$$\kappa_\pi := \inf\{n \in \mathbb{N} : \pi^n = 0\}, \tag{4.13}$$

where $\inf \varnothing := \infty$. Since $\pi^n = \pi^{n-1} \circ \theta_\pi + \pi$ we get from (4.12) that

$$\begin{aligned}\kappa_\pi \circ \theta_{\pi^{-1}} &= \inf\{n \in \mathbb{N} : \pi^n \circ \theta_{\pi^{-1}} = 0\}\\ &= \inf\{n \in \mathbb{N} : \pi^{n-1} = \pi^{-1}\} = \inf\{n \in \mathbb{N} : \pi^n = 0\} = \kappa_\pi.\end{aligned}$$

Equation (4.12) does also imply that

$$\theta_\pi \circ \theta_{\pi^{-1}} = \theta_{\pi(\theta_{\pi^{-1}})}(\theta_{\pi^{-1}}) = \theta_{-\pi^{-1}}(\theta_{\pi^{-1}}) = \theta_0.$$

Hence the assumed invariance of $\mathbb{Q}$ under $\theta_{\pi^{-1}}$ yields for any $k \in \mathbb{N} \cup \{\infty\}$ that

$$\mathbb{E}_{\mathbb{Q}}[\mathbf{1}(\kappa_\pi = k) f(\theta_\pi, -\pi)] = \mathbb{E}_{\mathbb{Q}}[\mathbf{1}(\kappa_\pi = k) f(\theta_0, \pi^{-1})]. \tag{4.14}$$

If $\kappa_\pi = 1$, then $\pi = \pi^{-1} = 0$. If $\kappa_\pi = k$ for some finite $k \geq 2$ and $i \in \{1, \dots, k-1\}$, then $(\pi^i)^{-1} = \pi^{k-i}$ and $\kappa_{\pi^i} = k$. Moreover, the points $\pi^1, \dots, \pi^{k-1}$ are all different in this case. Applying (4.14) with π replaced by π^i and summing over $i \in \{0, \dots, k-1\}$, we obtain that

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}(\kappa_\pi = k)\int f(\theta_x, -x)N_\pi(dx)\right]\\ &\qquad = \mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}(\kappa_\pi = k)\int f(\theta_0, x)N_\pi(dx)\right],\end{aligned} \tag{4.15}$$

where N_π is the random counting measure supported by the set $\{\pi^i : i \in \mathbb{Z}\}$. If $\kappa_\pi = \infty$, then $(\pi^i)^{-1} = \pi^{-i}$ for all $i \in \mathbb{Z}$ and the points π^i, $i \in \mathbb{Z}$, are all different. Summing over $i \in \mathbb{Z}$, we obtain (4.15) also in this case, where N_π is defined as before. Hence

$$\mathbb{E}_{\mathbb{Q}}\left[\int f(\theta_x, -x)(N_\pi - \delta_0)(dx)\right] = \mathbb{E}_{\mathbb{Q}}\left[\int f(\theta_0, x)(N_\pi - \delta_0)(dx)\right]. \tag{4.16}$$

We now apply (4.16) to the bijective point-maps $\pi_{m,n} := \sigma_{m,n}(N, 0)$, where the $\sigma_{m,n}$ are as in Proposition 4.2. Summing over n and using property (a) yields that

$$\mathbb{E}_{\mathbb{Q}}\left[\int f(\theta_x, -x) N_m(dx)\right] = \mathbb{E}_{\mathbb{Q}}\left[\int f(\theta_0, x) N_m(dx)\right],$$

where

$$N_m := \delta_0 + \sum_{n\in\mathbb{N}} (N_{\pi_{m,n}} - \delta_0).$$

Taking the limit as $m \to \infty$ and using property (b), we obtain (4.11). $\square$

The *inversion formula* for Palm measures (see Satz 2.4 in [7]) implies that the measure $\mathbb{Q}$ in Theorem 4.1 determines $\mathbb{P}$ on $\{N \neq \varnothing\}$. One useful version of this formula is based on the (random and convex) *Voronoi cell*

$$V := \{x \in \mathbb{R}^d : N \cap S(x, |x|) = \varnothing\}, \tag{4.17}$$

where $|x|$ denotes the length of $x \in \mathbb{R}^d$ and $S(x,r)$ is the open ball with center x and radius $r > 0$. Since N is locally finite we have that the volume $|V|_d$ of V is positive. (If $\mathbb{P}$ is a finite measure, then V is compact almost everywhere on $\{N \neq \varnothing\}$ with respect to both $\mathbb{P}$ and $\mathbb{P}_N$.) We then have for all measurable $f : \Omega \to [0,\infty)$ that

$$\mathbb{E}_{\mathbb{P}}[\mathbf{1}(N \neq \varnothing) f] = \mathbb{E}_{\mathbb{P}_N}\left[\int_V f \circ \theta_x \, dx\right]; \tag{4.18}$$

see also Proposition 10.1 in [13] or Proposition 11.3 in [5].

The following consequence of Theorem 4.1 generalizes Theorem 9.4.1 in [14].

PROPOSITION 4.3. *Let $\mathbb{Q}$ be a σ-finite measure on $(\Omega, \mathcal{F})$ satisfying $\mathbb{Q}(0 \notin N) = 0$ and $\mathbb{Q}(|V|_d = \infty) = 0$. Then $\mathbb{Q}$ is point-stationary iff*

$$\mathbb{E}_{\mathbb{Q}}\left[\int_{V+y} h \circ \theta_x \, dx\right] = \mathbb{E}_{\mathbb{Q}}\left[\int_V h \circ \theta_x \, dx\right] \tag{4.19}$$

holds for all measurable $h : \Omega \to [0,\infty)$ and all $y \in \mathbb{R}^d$.

PROOF. If $\mathbb{Q}$ is point-stationary, then Theorem 4.1 implies that $\mathbb{Q} = \mathbb{P}_N$ for some σ-finite stationary $\mathbb{P}$. Hence (4.19) follows from (4.18).

Let us now assume that (4.19) holds. Then

$$\mathbb{P}(A) := \mathbb{E}_{\mathbb{Q}}\left[\int_V \mathbf{1}_A \circ \theta_x \, dx\right], \qquad A \in \mathcal{F}, \tag{4.20}$$

defines a stationary measure $\mathbb{P}$. Since $\mathbb{Q}$ is σ-finite and $\mathbb{Q}(|V|_d = \infty) = 0$, there is some measurable function $f : \Omega \to (0,\infty)$ such that $\mathbb{E}_{\mathbb{Q}}[|V|_d f] < \infty$. Let τ

denote the lexicographically smallest among the points of N that are closest to the origin and note that

$$\tau \circ \theta_x = -x \qquad \text{on } \{0 \in N\}$$

whenever x is in the interior of V. Then

$$\mathbb{E}_{\mathbb{P}}[f \circ \theta_\tau] = \mathbb{E}_{\mathbb{Q}}\left[\int_V (f \circ \theta_\tau) \circ \theta_x \, dx\right] = \mathbb{E}_{\mathbb{Q}}[|V|_d f] < \infty.$$

Since $f \circ \theta_\tau > 0$, we obtain that $\mathbb{P}$ is σ-finite. Since $\mathbb{Q}(0 \notin N) = 0$, we further have

$$\mathbb{P}(N = \varnothing) = \mathbb{E}_{\mathbb{Q}}[\mathbf{1}(N = \varnothing)|V|_d] = 0. \tag{4.21}$$

Therefore we get from the inversion formula (4.18) and definition (4.20) for all measurable $h : \Omega \to [0, \infty)$

$$\mathbb{E}_{\mathbb{P}_N}\left[\int_V h \circ \theta_x \, dx\right] = \mathbb{E}_{\mathbb{Q}}\left[\int_V h \circ \theta_x \, dx\right]. \tag{4.22}$$

Applying (similarly as above) (4.22) with $h = f \circ \theta_\tau$ for some (arbitrary) measurable $f : \Omega \to [0, \infty)$ and using $\mathbb{Q}(0 \notin N) = \mathbb{P}_N(0 \notin N) = 0$, we obtain

$$\mathbb{E}_{\mathbb{P}_N}[|V|_d f] = \mathbb{E}_{\mathbb{Q}}[|V|_d f].$$

Since $|V|_d < \infty$ almost everywhere with respect to $\mathbb{Q}$, we get the same property with respect to $\mathbb{P}$. This implies $\mathbb{P}_N = \mathbb{Q}$, so that point-stationarity of $\mathbb{Q}$ follows from Theorem 3.1. $\square$

5. Bijective point-maps without fixed points. In this section we will prove the following result announced in the Introduction.

THEOREM 5.1. *There exists a $\sigma(N)$-measurable bijective point-map π satisfying* (1.1) *for any stationary probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$.*

The theorem will be a consequence of a more detailed result. A *lattice* (see, e.g., [3]) is a locally finite set $L \subset \mathbb{R}^d$ such that $x - y \in L$ whenever $x, y \in L$. A locally finite set $L \subset \mathbb{R}^d$ is called *a translated lattice* if $L - x$ is a lattice for some (and then for all) $x \in L$.

THEOREM 5.2. *There exists a translation-invariant matching $\sigma : \mathbf{N}_0 \to \mathbb{R}^d$ such that $\{x \in \varphi : \sigma(\varphi, x) = x\}$ is a translated lattice for any $\varphi \in \mathbf{N}$.*

PROOF. Let σ_l, $l \in \mathbb{N}$, be an arbitrary enumeration of the translation-invariant matchings $\sigma_{i,k}^{B_{m,n}}$ ($m, n \in \mathbb{N}$, $k \in \{\propto, 2, 3, \dots\}$, $i \in \mathbb{N}$, $i < k$) defined by (4.3) in the proof of Proposition 4.2, where the sets $B_{m,n}$ are as in (4.4).

Fix some $\varphi \in \mathbf{N}$ and define

$$\varphi_1 := \{x \in \varphi : \sigma_1(\varphi, x) = x\}.$$

For any $x \notin \varphi_1$ there is a unique $y \in \varphi \setminus \{x\}$ satisfying $\sigma_1(\varphi, x) = y$ and we define $\sigma(\varphi, x) := y$. Assuming that φ_n is given and that σ is defined on $\varphi \setminus \varphi_n$ we define

$$\varphi_{n+1} := \{x \in \varphi_n : \sigma_{n+1}(\varphi_n, x) = x\}$$

and $\sigma(\varphi, x) = y$ whenever $x \in \varphi_n \setminus \varphi_{n+1}$ and $\sigma_{n+1}(\varphi, x) = y$. This defines $\sigma(\varphi, x)$ for any $x \in \varphi \setminus \varphi_\infty$, where

$$\varphi_\infty := \bigcap_{n=1}^{\infty} \varphi_n.$$

For $x \in \varphi_\infty$ we let $\sigma(\varphi, x) := x$. Since σ_n is a translation-invariant matching for any $n \in \mathbb{N}$, we obtain the same property for σ.

It remains to show that φ_∞ is a translated lattice. If $\operatorname{card} \varphi_\infty \leq 1$, then φ_∞ is trivially a translated lattice, so let us assume that $\operatorname{card} \varphi_\infty \geq 2$. We claim that $x + k(y - x) \in \varphi_\infty$ for all $k \in \mathbb{Z}$, whenever $x, y \in \varphi$. Since this property transfers from φ_∞ to $\varphi_\infty - z$ for any $z \in \varphi_\infty$ we may then conclude the assertion.

To prove the claim we consider two distinct points x and y of φ_∞ and suppose that the chain $M_\infty = (x_k : k \in I)$ generated by x and y in φ_∞ is not an ∞-chain. Then x and y have well-defined positions in the chain, say $x = x_i$ and $y = x_j$. If M_∞ is a k-chain, $k \in \mathbb{N}$, then $2x_1 - x_2 \notin \varphi_\infty$ and $2x_k - x_{k-1} \notin \varphi_\infty$ and there exists $l_0 \in \mathbb{N}$ such that $2x_1 - x_2 \notin \varphi_{l_0}$ and $2x_k - x_{k-1} \notin \varphi_{l_0}$. If M_∞ is a $\propto$-chain, then $2x_1 - x_2 \notin \varphi_\infty$ and there exists $l_0 \in \mathbb{N}$ such that $2x_1 - x_2 \notin \varphi_{l_0}$. In both cases we have that $M_{l_0 = M_\infty}$, where M_l is the chain generated by x and y in φ_l. But then even $M_l = M_\infty$ for all $l \geq l_0$.

By (4.4) there exist $m_0, n_0 \in \mathbb{N}$ such that x, y are B_{m_0,n_0}-uncritical in φ. Then for all $m \geq m_0$ there exists (a unique) $n_m \in \mathbb{N}$ such that x and y are B_{m,n_m}-uncritical in φ_l for all $l \geq l_0$. Hence, for some $l \geq l_0$ and $m \geq m_0$ we have $\sigma_l = \sigma_{i,j}^{B_{m,n_m}}$, so $\sigma_l(x) = y$ in contradiction with the choice of x and y.

Hence M_∞ is an ∞-chain in φ_∞ and our claim is proved. $\square$

PROOF OF THEOREM 5.1. We define the flow-adapted point process

$$N_\infty := \{x \in N : \sigma(N, x) = x\},$$

where $\sigma : \mathbf{N}_0 \to \mathbb{R}^d$ is the translation-invariant matching in Theorem 5.2. This theorem implies that $N_\infty(\omega)$ is a translated lattice for any $\omega \in \Omega$. By Theorem 1.3.2 in [3] there exist linearly independent random vectors $U_1, \dots, U_d$ in $\mathbb{R}^d$, that we may assume to be lexicographically ordered and such that

$$N_\infty = \left\{ \sum_{i=1}^{d} \lambda_i U_i : \lambda_i \in \mathbb{Z} \right\}$$

whenever $0 \in N_\infty$ and $\mathrm{conv}(N_\infty) = \mathbb{R}^d$. Here $\mathrm{conv}(A)$ denotes the convex hull of a set $A \subset \mathbb{R}^d$. We may then define a bijective point-map $\pi : \Omega \to \mathbb{R}^d$ by

$$\pi := \begin{cases} \sigma(N,0), & \text{if } 0 \notin N_\infty, \\ U_1, & \text{if } 0 \in N_\infty \text{ and } \mathrm{conv}(N_\infty) = \mathbb{R}^d, \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

Let us now take a stationary probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. Then $\mathbb{P}_N(0 \notin N) = 0$,

$$\mathbb{P}(\{N_\infty = \varnothing\} \cup \{\mathrm{conv}(N_\infty) = \mathbb{R}^d\}) = 1$$

and $\mathbb{P}_N(\mathrm{conv}(N_\infty) \neq \mathbb{R}^d) = 0$. Hence we have $\mathbb{P}_N(\pi = 0) = 0$, as desired. $\square$

6. Translation-invariant graphs. Consider a bijective point-map $\delta : \mathbf{N}_0 \to \mathbb{R}^d$ and let $\varphi \in \mathbf{N}$. Drawing a directed line from any $x \in \varphi$ to $\delta(\varphi, x)$ equips $\varphi \in \mathbf{N}$ with the structure of a *directed graph* $G_\delta(\varphi)$ with vertex set φ. In case $\delta(\varphi, x) = x$ we interpret x as an *isolated* point in $G_\delta(\varphi)$.

A *directed doubly infinite path* as addressed in our next result is formally defined as a directed graph that is isomorphic to the directed graph $\mathbb{Z}$ with $n \in \mathbb{Z}$ pointing to $n+1$ for each $n \in \mathbb{Z}$. Theorem 4.1 complements Theorem 2 in [4]. Regarding the point process N our theorem is more general. We need only stationarity with respect to translations but not ergodicity with respect to all isometries. Also a finite intensity or a nonequidistant property is not required. Note, however, that we have constructed our graphs from N only in a shift-invariant and not in an isometry-invariant way. So given the restrictions on N, the result in [4] is more general.

THEOREM 6.1. *There is a bijective point-map $\delta : \mathbf{N}_0 \to \mathbb{R}^d$ having the following property. For any stationary probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ the components of the directed graph $G_\delta(N)$ are almost everywhere with respect to both $\mathbb{P}$ and $\mathbb{P}_N$ doubly infinite paths.*

PROOF. We use the translation-invariant matching $\sigma : \mathbf{N}_0 \to \mathbb{R}^d$ in Theorem 5.2. Starting with $\varphi_0 := \varphi$ we define two sequences (φ_n) and (ψ_n) of subsets of φ inductively by

$$\varphi_{n+1} := \{\max\{x, \sigma(\varphi_n, x)\} : x \in \varphi_n, \sigma(\varphi_n, x) \neq x\}, \qquad n \in \mathbb{N}_0,$$

$$\psi_{n+1} := \{x \in \varphi_n : \sigma(\varphi_n, x) = x\}, \quad n \in \mathbb{N}_0,$$

where the max is taken with respect to some fixed translation-invariant order $<$ on $\mathbb{R}^d$. Further we define a sequence $\{C_n(x) : x \in \varphi_n\}$, $n \in \mathbb{N}_0$, of systems of pairwise disjoint finite subsets of φ inductively by

$$C_{n+1}(x) := C_n(x) \cup C_n(\sigma(\varphi_n, x)), \qquad x \in \varphi_{n+1},$$

where $C_0(y) := \{y\}$, $y \in \varphi$. Note that the sets $C_n(x)$, $x \in \varphi_n$, contain 2^n elements.

Let

$$\chi_n := \bigcup_{x \in \psi_n} C_{n-1}(x), \qquad \chi_\infty := \bigcup_{n \in \mathbb{N}} \chi_n.$$

We are now following some ideas from [2] and [4]. First we define a directed graph $G = G(\varphi)$ with vertex set $\varphi_\infty := \varphi \setminus \chi_\infty$. We draw a directed edge from $y \in \varphi_\infty$ to $x \in \varphi_\infty$ whenever $x < y$ and there is some $n \in \mathbb{N}$ such that $y = \sigma(\varphi_n, x)$. In this case we call y a *mother* of x and x a *daughter* of y. A point $y \in \varphi_n$ $(n \geq 1)$ has daughters in φ_i for each $i \in \{0, \dots, n-1\}$. This defines a natural ordering of daughters: the larger i the *older* the daughter. By construction the components of G are all *trees*, that is, without cycles. Any point can have at most one a mother. Moreover, any component $C \subset \varphi_\infty$ can contain at most one point without mother. If this occurs, then we call C a *directed* component. We are defining the point-map δ first on the components of G that are not directed, then on the directed components of G and then finally on χ_∞.

Let $C \subset \varphi_\infty$ be a component of G that is not directed. If $x \in C$ has a daughter (i.e., if $x \in \varphi_1$), then we let $\delta(x) \equiv \delta(\varphi, x)$ be the oldest one. If x has no daughter, then we let $\delta(x)$ be the oldest among the younger sisters of x. If x has no younger sister, then we check whether the mother has younger sisters. In this case we define $\delta(x)$ as the oldest among these younger sisters. Since the component C is not directed, it is clear that this *royal succession rule* defines $\delta(x)$ in any case! It is not difficult to check that δ defines a graph on φ_∞ whose components are all directed doubly infinite paths.

Let $C \subset \varphi_\infty$ be a directed component of G. Then there exists a unique point x in C that has no mother. We enumerate the points of the component starting with $x_1 = x$, then taking $x_2 = \sigma(\varphi_1, x)$. We continue with the two points in $C_2(x) \setminus C_1(x)$ and enumerate them such that $x_3 > x_4$. Subsequently we enumerate the 2^{n-1} points of the set $C_n(x) \setminus C_{n-1}(x), n \geq 2$ in descending order. This procedure gives us an enumeration of all points of C. Using the function g from the proof of Proposition 4.2 we define $\delta(x_n) = x_{g(n)}$ for all $n \in \mathbb{N}$. Then δ induces a doubly infinite directed path with vertex set C.

It remains to define δ on χ_∞. We treat each χ_n, $n \in \mathbb{N}$, separately. By Theorem 5.2, ψ_n is a translated lattice. If ψ_n is a singleton $\{x\}$, then we let $\delta(x) := x$. Otherwise ψ_n is empty or contains infinitely many points. In the latter case we find an $m \in \{1, \dots, d\}$ and linearly independent vectors $u_1 < \cdots < u_m$ such that

$$\psi_n = \left\{ x + \sum_{i=1}^{m} \lambda_i u_i : \lambda_i \in \mathbb{Z} \right\}$$

for some (and then for all) $x \in \psi_n$. We then define δ on χ_n in such a way that, for any $x \in \psi_n$, the iterates of δ applied to x satisfy the relationships

$$\{\delta^{k2^{n-1}}(x) : k \in \mathbb{Z}\} = \{x + ku_1 : k \in \mathbb{Z}\}, \qquad \{\delta^k(x) : k \in \mathbb{Z}\} = \bigcup_{k \in \mathbb{Z}} C_{n-1}(x + ku_1).$$

We may omit the details of the pretty obvious construction.

To finish the proof we take some stationary probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ and note that the ψ_n are actually measurable and translation-covariant functions $\varphi \mapsto \psi_n(\varphi)$. In particular, $\psi_n(N)$ is then a stationary point process under $\mathbb{P}$. Hence almost everywhere with respect to both $\mathbb{P}$ and $\mathbb{P}_N$ it is either empty or contains infinitely many points. $\square$

7. Concluding remarks. It is clearly desirable to define point-stationarity as invariance under a preferably small family of bijective point-shifts. In the first version of this paper we asked which Palm measures can be characterized by invariance under just one bijective point-shift θ_π. As shown by the proof of Theorem 4.1 this essentially requires the existence of a $\sigma(N)$-measurable bijective point-map π whose iterates π^n, $n \in \mathbb{Z}$, cover all points of N almost everywhere with respect to the underlying measure $\mathbb{Q}$. In this case we say that $\mathbb{Q}$ has property (P). As shown by complete lattice-structures we can generally not expect Palm measures to have property (P) unless $d = 1$. However, in the meantime it has been established in [15] that the Palm distribution $\mathbb{P}_N^0$ of an ergodic stationary probability measure $\mathbb{P}$ has property (P) if the group of isometries of N is trivial almost everywhere with respect to $\mathbb{P}$. Hence the class of all Palm measures having property (P) is amazingly large. The paper [15] extends recent results for Poisson processes ([2, 4]) and solves a problem posed in [4] in great generality.

Let us call a $\varphi \in \mathbf{N}$ *periodic* if there is some $x \in \mathbb{R}^d \setminus \{0\}$ such that $\varphi + x = \varphi$. Given the results and techniques in [15] we conjecture for $d \geq 2$ that a Palm distribution $\mathbb{Q}$ has property (P) if and only if it is *nonperiodic* in the sense that N is not periodic $\mathbb{Q}$-almost everywhere. Moreover, it should be possible to establish an alternative and somewhat simpler version of Proposition 4.2, by constructing a family of point-shifts that covers the periodic part of N, and then to use a version of the point-shift in [15] to exhaust the remaining points of N.

Acknowledgments. The authors would like to thank Hermann Thorisson for some interesting discussions related to the subjects of the paper. This paper was started during a conference on Applied Probability in Oberwolfach. The authors are grateful to the Mathematisches Forschungsinstitut for providing such an excellent and stimulating frame for the mathematical community.

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