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Unbiased shifts of Brownian motion and balancing random measures

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1. Definition of an unbiased shift

Setting

$B = (B_t)_{t\in\mathbb{R}}$ is a two-sided standard Brownian motion starting in 0 ($B_0 = 0$) defined on its canonical probability space $(\Omega, \mathcal{A}, \mathbb{P}_0)$.

Definition

An unbiased shift (of B) is a random time T (negative values are allowed) such that:

- $B^{(T)} := (B_{T+t} - B_T)_{t\in\mathbb{R}}$ is a Brownian motion,
- $B^{(T)}$ is independent of B_T.

Example

If $T \geq 0$ is a stopping time, then $(B_{T+t} - B_T)_{t\geq 0}$ is a one-sided Brownian motion independent of B_T. However, the example

$$T := \inf\{t \geq 0 \colon B_t = a\}$$

shows that $(B_{T+t} - B_T)_{t\in\mathbb{R}}$ need not be a two-sided Brownian motion.

Example

Consider a deterministic $T \equiv t_0$. Then $B^{(T)} = (B_{t_0+t} - B_{t_0})_{t\in\mathbb{R}}$ is a two-sided Brownian. However, since $B^{(T)}_{-t_0} = -B_{t_0}$, this two-sided motion is not independent of $B_T = B_{t_0}$.

Remark

An unbiased shift with $B_T = 0$ is characterized by

$$(B_{T+t})_{t\in\mathbb{R}} \overset{d}{=} B.$$

According to Mandelbrot (The Fractal Geometry of Nature) „...the process of Brownian zeros is stationary in a weakened form.“ He is using the (non-rigorous) concept of conditional stationarity.
However, the stopping time

$$T := \inf\{t \geq 1 : B_t = 0\}$$

has the property $B_T = 0$. But clearly $B^{(T)}$ is not a Brownian motion. Something is missing!

Definition

Let ℓ^0 be the local time (random measure) at zero. Its right-continuous (generalised) inverse is defined as

$$T_r := \begin{cases} \sup\{t \geq 0 : \ell^0[0,t] = r\}, & r \geq 0, \\ \sup\{t < 0 : \ell^0[t,0] = -r\}, & r < 0. \end{cases}$$

Theorem

Let $r \in \mathbb{R}$. Then T_r is an unbiased shift.

Idea of the proof: The intervals $[T_n, T_{n+1}]$, $n \in \mathbb{Z}$, split B into iid-cycles.

2. Local time

Definition

The local time measure ℓ^x at $x \in \mathbb{R}$ can be defined by

$$\ell^x(C) := \lim_{h \to 0} \frac{1}{h} \int \mathbf{1}\{s \in C, x \le B_s \le x + h\} ds.$$

Hence

$$\int f(B_s, s) ds = \iint f(x, s) \ell^x(ds) dx \quad \mathbb{P}_0\text{-a.s.}$$

for all measurable $f : \mathbb{R}^2 \to [0, \infty)$.

Definition

For $t \in \mathbb{R}$ the shift $\theta_t \colon \Omega \to \Omega$ is given by

$$(\theta_t \omega)_s := \omega_{t+s}, \quad s \in \mathbb{R}.$$

For $x \in \mathbb{R}$ let

$$\mathbb{P}_x := \mathbb{P}_0(B + x \in \cdot), \quad x \in \mathbb{R},$$

where B is the identity on Ω.

Remark

It is a possible to choose a perfect version of local times, that is a (measurable) kernel satisfying for all $x \in \mathbb{R}$ and $\mathbb{P}_x$-a.e. that ℓ^x is diffuse and

$$\ell^x(\theta_t \omega, C - t) = \ell^x(\omega, C), \quad C \in \mathcal{B}, t \in \mathbb{R},$$
$$\ell^x(B, \cdot) = \ell^0(B - x, \cdot).$$

3. Palm measures and allocation rules

Definition

Let

$$\mathbb{P} := \int \mathbb{P}_x dx$$

be the distribution of a Brownian motion with a „uniformly distributed“ starting value.

Remark

A consequence of the stationary increments of B is the invariance property

$$\mathbb{P} = \mathbb{P} \circ \theta_s, \quad s \in \mathbb{R}.$$

Definition

A random measure ξ on $\mathbb{R}$ is a kernel from Ω to $\mathbb{R}$ such that $\xi(\omega, B) < \infty$ for $\mathbb{P}$-a.e. ω and all compact $B \subset \mathbb{R}$. It is called invariant if $\mathbb{P}$-a.e.:

$$\xi(\theta_t\omega, C - t) = \xi(\omega, C), \quad C \in \mathcal{B}, t \in \mathbb{R}.$$

In this case its Palm measure $\mathbb{Q}_\xi$ is defined by

$$\mathbb{Q}_\xi(A) := \mathbb{E} \int \mathbf{1}_{[0,1]}(s)\mathbf{1}_A(\theta_s B)\xi(ds), \quad A \in \mathcal{A}.$$

If the intensity $\lambda_\xi := \mathbb{E}_\mathbb{P}\xi[0,1] = \mathbb{Q}_\xi(\Omega)$ is positive and finite, then the Palm probability measure of ξ is defined by $\mathbb{Q}^0_\xi := \lambda_\xi^{-1}\mathbb{Q}_\xi$.

Theorem (Geman and and Horowitz ’73)

The Palm (probability) measure of the local time ℓ^x is $\mathbb{P}_x$.

Definition

Let ν be a probability measure on $\mathbb{R}$. Define

$$\mathbb{P}_\nu := \int \mathbb{P}_x \nu(dx), \qquad \ell^\nu := \int \ell^x \nu(dx).$$

Corollary

$\mathbb{P}_\nu$ is the Palm probability measure of ℓ^ν.

Remark

In the language of stochastic analysis ℓ^ν is a continuous additive functional with Revuz measure ν.

Definition

An allocation rule is a measurable mapping $\tau : \Omega \times \mathbb{R} \to \mathbb{R}$ that is equivariant in the sense that

$$\tau(\theta_t \omega, s - t) = \tau(\omega, s) - t, \quad s \in \mathbb{R}, t \in \mathbb{R}, \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Theorem (L. and Thorisson '09)

Let ξ and η be two invariant random measures with positive and finite intensities. Let τ be an allocation rule and define $T := \tau(\cdot, 0)$. Then

$$\mathbb{Q}_\xi(\theta_T B \in \cdot) = \mathbb{Q}_\eta$$

iff τ is balancing ξ and η, that is

$$\int \mathbf{1}\{\tau(s) \in \cdot\}\xi(ds) = \eta \quad \mathbb{P}\text{-a.e.}$$

4. Unbiased shifts and balancing allocation rules

Definition (Skorokhod embedding problem)

Let ν be a probability measure on $\mathbb{R}$. A random time T embeds ν if B_T has distribution ν.

Theorem

Let T be a random time and ν be a probability measure on $\mathbb{R}$. Then T is an unbiased shift embedding ν if and only if the allocation rule τ defined by $\tau_T(s) := T \circ \theta_s + s$ is balancing ℓ^0 and ℓ^ν.

Example

Let $r > 0$. Then

$$\tau(s) := \inf\{t > s \colon \ell^0([s,t]) = r\}, \quad s \in \mathbb{R}.$$

Then τ is an allocation rule balancing ℓ^0 with itself. Hence $T_r = \tau(\cdot, 0)$ is an unbiased shift (embedding δ_0).

5. Existence of unbiased shifts

Theorem

Let ν be a probability measure on $\mathbb{R}$ with $\nu\{0\} = 0$. Then the stopping time

$$T := \inf\{t > 0 \colon \ell^0[0,t] = \ell^\nu[0,t]\}$$

embeds ν and is an unbiased shift.

Remark

The above stopping time above was introduced in Bertoin and Le Jan (1992) as a solution of the Skorokhod embedding problem.

Theorem

Let ν be a probability measure on $\mathbb{R}$. Then there exists a nonnegative stopping time that is an unbiased shift embedding ν.

Theorem

Assume that $\nu\{0\} < 1$. Then any unbiased shift T embedding ν satisfies

$$\mathbb{P}_0\{T = 0\} = 0.$$

If, however, $\nu = \delta_0$, then for any $p \in [0, 1]$ there is an unbiased shift $T \geq 0$ embedding ν and such that $\mathbb{P}_0\{T = 0\} = p$.

Theorem

Let ξ and η be jointly stationary orthogonal diffuse random measures on $\mathbb{R}$ with finite and equal intensities. Then the mapping $\tau\colon \Omega \times \mathbb{R} \to \mathbb{R}$, defined by

$$\tau(s) := \inf\{t > s \colon \xi[s,t] = \eta[s,t]\}, \quad s \in \mathbb{R},$$

is an allocation rule balancing ξ and η.

Remark

The previous theorem holds in a more general stationary setting. The assumption of equal intensities has to be replaced by

$$\mathbb{E}\big[\xi[0,1]\big|\mathcal{I}\big] = \mathbb{E}\big[\eta[0,1]\big|\mathcal{I}\big] \quad \mathbb{P}\text{-a.e.},$$

where $\mathcal{I}$ is the invariant σ-field. In the Brownian setting, $\mathbb{P}$ is trivial on $\mathcal{I}$. (If $A \in \mathcal{I}$ then either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A^c) = 0$.)

6. Moment properties of unbiased shifts

Theorem

If T is an unbiased shift embedding a probability measure $\nu \neq \delta_0$, then

$$\mathbb{E}_0 \sqrt{|T|} = \infty.$$

Idea of the proof:

- Take an $x > 0$ such that $\nu[x, \infty) = \mathbb{P}(B_T > x) > 0$.
- On the event $\{B_T > x\}$, T can be bounded from below by the minimum of two independent hitting times for $-x$, independent of B_T.
- Use the moment properties of hitting times.

Theorem

Suppose ν is a distribution with $\nu\{0\} = 0$, and finite absolute mean. If the stopping time $T \geq 0$ is an unbiased shift embedding ν, then

$$\mathbb{E}_0 T^{1/4} = \infty.$$

Theorem

Let ν be as above and let T be the Bertoin/Le Jan stopping time. Then, for all $\beta \in [0, 1/4)$,

$$\mathbb{E}_0 T^{\beta} < \infty.$$

Theorem

Let $T \geq 0$ *be an unbiased shift embedding* δ_0*. If* T *is non-trivial, that is* $\mathbb{P}_0\{T = 0\} < 1$*, then*

$$\mathbb{E}_0 T = \infty.$$

Example

There is an unbiased shift $T \neq 0$ embedding δ_0, such that $\mathbb{E} e^{\lambda |T|} < \infty$ for some $\lambda > 0$.

7. Minimality

Definition

A random time $T \geq 0$ is called minimal unbiased shift if it is an unbiased shift and if for any other unbiased shift S embedding $\mathbb{P}(B_T \in \cdot)$ and such such that $\mathbb{P}_0(0 \leq S \leq T) = 1$ we have that $\mathbb{P}_0(S = T) = 1$.

Theorem

The Bertoin-Le Jan stopping time T is a minimal unbiased shift.

8. References

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