Polynomial parallel volume, convexity and contact distributions of random sets

Daniel Hug, Günter Last and Wolfgang Weil

November 20, 2004

Abstract

We characterize convexity of a random compact set X in $\mathbb{R}^d$ via polynomial expected parallel volume of X. The parallel volume of a compact set A is a function of $r \geq 0$ and is defined here in two steps. First we form the parallel set at distance r with respect to a one- or two-dimensional gauge body B. Then we integrate the volume of this (relative) parallel set with respect to all rotations of B. We apply our results to characterize convexity of the typical grain of a Boolean model via first contact distributions.

1 Introduction

Modern data frequently arise as images of (random) structures in $\mathbb{R}^2$ or $\mathbb{R}^3$. It is one of the main purposes of Stochastic Geometry to provide models for such random spatial data. The basic, most flexible and frequently used model is still the *Boolean model* (see e.g. [6], [13], [19]). A (stationary) Boolean model Z in $\mathbb{R}^d$ is a random closed set

$$Z = \bigcup_{n \in \mathbb{N}} (Z_n + \xi_n),$$

where the ξ_n, $n \in \mathbb{N}$, form a stationary Poisson process Ξ in $\mathbb{R}^d$ (with intensity $\gamma > 0$, say) and where the *grains* $Z_1, Z_2, \dots$ are independent, identically distributed non-empty random compact sets, which are also independent of Ξ. Throughout this paper, we assume that there is an underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$ carrying all random elements. Then a random closed set in the sense of Matheron (see [12]) is a measurable map into the space $\mathcal{F}^d$ of closed subsets of $\mathbb{R}^d$ endowed with the Borel σ-field generated by the Fell-Matheron "hit-or-miss" topology. In particular, a random compact set is a random closed set which is almost surely compact.

AMS 1991 *subject classifications.* Primary 60D05, 52A38, 28A75; secondary 52A22, 52A20, 53C65.
Key words and phrases. Stochastic geometry, Boolean model, contact distribution, random compact set, polynomial parallel volume, convexity, linear section, planar section.

The distribution of the Boolean model Z is determined by γ and the distribution of the *typical grain* X, a random compact set having the distribution of the Z_i. In order to fit a Boolean model to given data, the statistical problem consists in finding appropriate estimates for these two parameters. A simple yet powerful set of tools which is available with most image analysing equipment is given by the contact distribution functions; see [10] for a recent survey. For a compact convex set $B \subset \mathbb{R}^d$ containing the origin 0, the *contact distribution function* H_B of Z (with structuring element B) is defined as the distribution function of the 'B-distance' $d_B(0, Z)$ from 0 to Z, given that 0 is not covered by Z, that is,

$$H_B(r) := \mathbb{P}(d_B(0, Z) \leq r \,|\, 0 \notin Z), \quad r \geq 0, \tag{1.1}$$

with

$$d_B(x, Z) := \inf\{t \geq 0 : (x + tB) \cap Z \neq \emptyset\}, \quad x \in \mathbb{R}^d.$$

As a consequence of the Poisson properties of the Boolean model Z, one easily gets

$$H_B(r) = 1 - \exp\left\{-\gamma \mathbb{E}\left[V_d(X + rB^*) - V_d(X)\right]\right\}, \tag{1.2}$$

where $\mathbb{E}$ denotes mathematical expectation, V_d is the volume (Lebesgue measure) in $\mathbb{R}^d$, and $X + rB^*$ is the Minkowski sum (vector sum) of the random compact set X and the reflection of rB in the origin. At this stage, a commonly made assumption is that the grains are (almost surely) convex. The reason for this is that if $A \subset \mathbb{R}^d$ is a compact convex set, then classical formulas from Convex Geometry (the Steiner formula, respectively its generalization by Minkowski) can be used to obtain the polynomial expansion

$$V_d(A + rB^*) = \sum_{k=0}^{d} r^k \binom{d}{k} V(A[d-k], B^*[k]).$$

The coefficient $V(A[d-k], B^*[k])$ on the right-hand side is a special *mixed volume* of $d - k$ copies of A and k copies of B^*, that is

$$V(A\,[d-k], B^*\,[k]) := V(\underbrace{A, \ldots, A}_{d-k}, \underbrace{B^*, \ldots, B^*}_{k});$$

see [16] for an introduction to mixed volumes and all notions related to convexity which are used throughout the following. Thus, for a Boolean model with convex grains the contact distribution has the following simple form

$$\begin{aligned} H_B(r) &= 1 - \exp\left\{-\sum_{k=1}^{d} r^k \binom{d}{k} \gamma \mathbb{E} V(X[d-k], B^*[k])\right\} \\ &= 1 - \exp\left\{-\sum_{k=1}^{d} r^k \binom{d}{k} \gamma \overline{V}_{d-k,B}\right\} \end{aligned} \tag{1.3}$$

with mean values (*densities*) $\overline{V}_{d-k,B} := \mathbb{E} V(X[d-k], B^*[k])$ of the mixed volumes of the grains. Fitting a polynomial to an empirical function $-\ln(1 - \hat{H}_B)$ that arises from given spatial data then yields estimators for $\gamma \overline{V}_{d-k,B}$, $k = 1, \ldots, d$.

Popular choices for B are the unit ball B^d (then one obtains the *spherical contact distribution*) or a unit segment $[0, u]$ with fixed or varying direction u (then one obtains a *linear contact* *distribution*). While $-\ln(1 - H_{B^d})$ is a polynomial of order d (if X is almost surely convex) with $\gamma V_d(B^d)$ as the leading coefficient and with the quermass densities of the grains as the other coefficients (see e.g. [18]), $-\ln(1 - H_{[0,u]})$ is a linear function and the slope is given by γ times the mean grain projection orthogonal to u (if the latter is averaged over all directions or if Z is isotropic, we obtain the mean surface area of the grains).

The assumption of convex grains is often connected automatically with a Boolean model. The polynomial behaviour of the function $-\ln(1 - H_B)$ has even been suggested as a test for the Boolean model against other model alternatives (see, e.g., the discussion in Section 3.3 of [19]). Here one has to check whether, for various shapes of B, the logarithmic empirical contact distribution function $-\ln(1 - \hat{H}_B)$ is well approximated by a polynomial of degree d.

Our aim in this paper is to explore and clarify the connection between the polynomial behaviour of logarithmic contact distribution functions and the convexity of the grains. As we shall show, for a Boolean model Z, the polynomial behaviour (or, more precisely, the linearity) of $-\ln(1 - H_{[0,u]})$ does in fact imply that the grains are convex, if either Z is isotropic or if we average over all directions u. A similar result holds for the *disc contact distributions* H_B with certain two-dimensional convex bodies (disc bodies) B. But a corresponding result is not valid, for example, if B is a ball of dimension at least three. These two cases of (linear and disc) contact distributions lead to the following definitions. The *average logarithmic linear contact distribution function* (*ALLC-function*) L of a stationary Boolean model Z is given by

$$L(r) := -\int_{\mathbb{S}^{d-1}} \ln(1 - H_{[0,u]}(r))\sigma(du), \quad r \geq 0, \tag{1.4}$$

(σ is the invariant probability measure on the unit sphere $\mathbb{S}^{d-1}$). Furthermore, a *disc body* is defined as a two-dimensional convex body $B \subset \mathbb{R}^d$ which contains the origin in its relative interior and has a smooth (of class C^1) and strictly convex relative boundary. The *average logarithmic disc contact distribution function* (*ALDC-function*) D_B of Z (with respect to B) is then defined as

$$D_B(r) := -\int_{SO_d} \ln(1 - H_{\vartheta B}(r))\nu(d\vartheta), \quad r \geq 0, \tag{1.5}$$

where ν is the Haar probability measure on the rotation group SO_d. Note that $L(r)$ and $D_B(r)$ both can be interpreted as mean logarithmic contact distribution functions with a random structuring element. If B is clear from the context, we omit the subscript of D_B. Finally, a compact subset of $\mathbb{R}^d$ is called *regular* if it is the closure of its interior.

Theorem 1.1. *Assume that the typical grain X of the stationary Boolean model Z in $\mathbb{R}^d$ is almost surely a regular compact set which satisfies the integrability assumption* (5.1). *If the ALLC-function L of Z is linear, then X is almost surely convex.*

Theorem 1.2. *Assume that the typical grain X of the stationary Boolean model Z in $\mathbb{R}^d$, $d \geq 3$, is almost surely a regular compact set which has a deterministically bounded diameter. Let B be a disc body. If the ALDC-function D_B of Z is a polynomial, then X is almost surely convex.*

It turns out that in the plane the assumptions of Theorem 1.2 can be relaxed.

Theorem 1.3. *Let $d = 2$, and let $B \subset \mathbb{R}^2$ be a disc body. Assume that the typical grain X of the stationary Boolean model Z in $\mathbb{R}^2$ has a deterministically bounded diameter. If the function $\ln(1 - H_B)$ or the ALDC-function D_B of Z is a polynomial, then X is almost surely convex.*

Theorems 1.1 and 1.2 are consequences of more general results which will be established in Section 5. There we also discuss connections to queueing theory and some applications. Although our results are mainly motivated by the analysis of Boolean models, in view of the right-hand side of equation (1.2) we first establish general results concerning the mean volume of random dilatations of random compact sets. In Section 3, we study dilatations by random segments, Section 4 is devoted to the investigation of dilatations by random disc bodies. The results obtained here are new even in the special case of deterministic compact sets. For instance, a consequence of Corollary 4.6 yields that if $A \subset \mathbb{R}^d$ is a regular compact set and $B^2 \subset \mathbb{R}^d$ is a two-dimensional unit disc such that

$$t \mapsto \int_{SO(d)} V_d(A + t\vartheta B^2)\nu(d\vartheta), \quad r \geq 0, \tag{1.6}$$

is a polynomial, then A is convex. Note that in the two-dimensional special case of this result, which was first established in [8], the integration over the rotation group has no effect. Finally, Section 2 contains some geometric preparations which are needed for the proofs of our main results.

2 Tools from geometry

We are working in the d-dimensional space $\mathbb{R}^d$ with scalar product $\langle \cdot, \cdot \rangle$ and Euclidean norm $|\cdot|$. For a set $A \subset \mathbb{R}^d$, we denote by $\mathrm{int}(A)$ the interior, by $\mathrm{cl}(A)$ the closure, and by ∂A the boundary of A. The i-dimensional Hausdorff measure on $\mathbb{R}^d$ is denoted by $\mathcal{H}^i$. If the i-dimensional Hausdorff measure is applied to subsets of an i-dimensional subspace, then we also write λ_i instead of $\mathcal{H}^i$. For $z \in \mathbb{R}^d$ and $r \geq 0$, $B^d(z, r) := \{y \in \mathbb{R}^d : |y - z| \leq r\}$ is the ball with centre z and radius r. The unit ball $B^d := B^d(0, 1)$ has volume κ_d and its boundary $\mathbb{S}^{d-1}$ (the unit sphere) has surface content $d\kappa_d$. We denote by $\mathcal{F}^d$, $\mathcal{C}^d$, and $\mathcal{K}^d$ the system of all non-empty subsets of $\mathbb{R}^d$, which are closed, compact, and compact and convex, respectively. The elements of $\mathcal{K}^d$ are called *convex bodies*. We write $\mathrm{conv}(A)$ for the convex hull of a set $A \subset \mathbb{R}^d$. The set $\mathcal{F}^d$ and its subsets are endowed with the usual Fell-Matheron "hit-or-miss" topology (see [12]). Measurability on any of these spaces always refers to the Borel σ-field generated by the Fell-Matheron topology.

2.1 Distances and exoskeleton

Given a convex body $B \in \mathcal{K}^d$ with $0 \in B$, we define the B-distance from a closed set $A \subset \mathbb{R}^d$ to a point $x \in \mathbb{R}^d$ by

$$d_B(A, x) := \inf\{r \geq 0 : x \in A + rB\},$$

where $C + D := \{c + d : c \in C, d \in D\}$ denotes the Minkowski sum of subsets $C, D \subset \mathbb{R}^d$ and $rB := \{rb : b \in B\}$ (cf. [16]). If B^* denotes the reflection of B in the origin, then clearly

$$d_B(A, x) = \inf\{r \geq 0 : (x + rB^*) \cap A \neq \emptyset\}.$$

Thus, $d_{B^*}(A, x)$ coincides with the distance $d_B(x, A)$ used implicitly in the introduction. For a closed set $A \subset \mathbb{R}^d$, the exoskeleton $\mathrm{exo}_B(A)$ of A with respect to B is defined as the set of all points $x \in \mathbb{R}^d \setminus A$ for which $d_B(A, x) < \infty$ and $\mathrm{card}((x + d_B(A, x)B^*) \cap A) \geq 2$. It is easy to check that $\mathrm{exo}_B(A)$ is a countable union of closed sets and hence a Borel set.

We will need the following extension of Theorem 3.2 in [9]. The assumption of strict convexity cannot be omitted in Lemma 2.1, even if the set A is convex. This can be seen by choosing $A = B = [-1, 1]^d$.

Lemma 2.1. *Let $A \subset \mathbb{R}^d$ be a closed set, and let $B \in \mathcal{K}^d$ be strictly convex with $0 \in \operatorname{int}(B)$. Then $V_d(\operatorname{exo}_B(A)) = 0$.*

PROOF. We put $\rho := d_B(A, \cdot)$. Since ρ is Lipschitz (see [7, Lemma 1]), and hence differentiable for $\mathcal{H}^d$-a.e. $x \in \mathbb{R}^d \setminus A$, it is sufficient to show that a point of differentiability of ρ cannot belong to $\operatorname{exo}_B(A)$. Hence suppose that ρ is differentiable at $x \in \mathbb{R}^d \setminus A$. We put $t := \rho(x)$. Let $b_i \in \partial B$ be such that $x - tb_i \in A$, $i = 1, 2$. Then

$$\rho(x - \epsilon b_i) = t - \epsilon, \quad \epsilon \in [0, t]. \tag{2.1}$$

To check this, we first assume that $\rho(x-\varepsilon b_i) = s < t-\varepsilon$. Then it follows that $x-\varepsilon b_i \in A+sB$, and hence $x \in A + sB + \varepsilon B = A + (s + \varepsilon)B$ with $s + \varepsilon < t$, a contradiction. This implies that $\rho(x - \varepsilon b_i) \geq t - \varepsilon$. On the other hand,

$$\rho(x - \varepsilon b_i) = d_B(A, x - \varepsilon b_i) \leq d_B(x - tb_i, x - \varepsilon b_i) = d_B(0, (t - \varepsilon)b_i) = t - \varepsilon,$$

which yields the assertion.

Using (2.1) and the differentiability of ρ at x, the differential $D\rho_x(b_i)$ of ρ at x evaluated at b_i satisfies

$$D\rho_x(b_i) = \lim_{\epsilon \to 0+} \frac{\rho(x - \epsilon b_i) - \rho(x)}{-\epsilon} = \lim_{\epsilon \to 0+} \frac{t - \epsilon - t}{-\epsilon} = 1, \tag{2.2}$$

$i = 1, 2$. For any $v \in \mathbb{R}^d$, we have

$$\rho(x + v) - \rho(x) \leq \min\{r \geq 0 : v \in rB\}. \tag{2.3}$$

Using (2.2), (2.3), the differentiability of ρ at x and the fact that $b := (b_1 + b_2)/2 \in B$, we obtain

$$1 = \frac{1}{2} D\rho_x(b_1) + \frac{1}{2} D\rho_x(b_2) = D\rho_x(b) = \lim_{\epsilon \to 0+} \frac{\rho(x + \epsilon b) - \rho(x)}{\epsilon} \leq \min\{r \geq 0 : b \in rB\} \leq 1,$$

hence $(b_1 + b_2)/2 \in \partial B$. Since B is strictly convex, it follows that $b_1 = b_2$. This shows that $x \notin \operatorname{exo}_B(A)$. $\square$

2.2 L-convex hulls and convexification

For $k \in \{0, \ldots, d\}$, we write $\mathcal{L}_k^d$ for the set of k-dimensional linear subspaces of $\mathbb{R}^d$. Then $\mathcal{L}_k^d$ is a compact subset of $\mathcal{F}^d$ and will be endowed with the subspace topology. This subspace topology coincides with the coarsest topology on $\mathcal{L}_k^d$ for which the map $SO_d \to \mathcal{L}_k^d$, $\vartheta \mapsto \vartheta L_0$ is continuous, where $L_0 \in \mathcal{L}_k^d$ is arbitrary but fixed (see [18, p. 18-19]). The subspace orthogonal to $L \in \mathcal{L}_k^d$ is denoted by $L^\perp \in \mathcal{L}_{d-k}^d$. Let $A \subset \mathbb{R}^d$ be a compact set. Then, for $L \in \mathcal{L}_k^d$ and $k \in \{0, \ldots, d\}$, we define the *L-convex hull* of A by

$$\operatorname{conv}_L(A) := \bigcup_{x \in L^\perp} \operatorname{conv}(A \cap (x + L)).$$

If $\operatorname{conv}_L(A) = A$, then A is said to be *L-convex*. The following lemma shows that $\operatorname{conv}_L(A)$ is always a compact set (that is, the map F in the lemma is well-defined) and provides a required measurability property.

Lemma 2.2. *Let $k \in \{0, \dots, d\}$. Then the map $F : \mathcal{C}^d \times \mathcal{L}_k^d \to \mathcal{C}^d$, $(A, L) \mapsto \operatorname{conv}_L(A)$, is well defined and measurable.*

PROOF. Assume that $A_i \to A$ in $\mathcal{C}^d$, where $A_i \subset B^d(0, R)$ for all $i \in \mathbb{N}$ and some $R > 0$, and $L_i \to L$ in $\mathcal{L}_k^d$, as $i \to \infty$. We show that if $y_i \in F(A_i, L_i)$, for $i \in \mathbb{N}$, and $y_i \to y \in \mathbb{R}^d$, as $i \to \infty$, then $y \in F(A, L)$. This will show, in particular, that $\operatorname{conv}_L(A)$ is compact. Moreover, by Satz 1.1.4 and Satz 1.1.5 in [18], F is upper semicontinuous if restricted (in the first component) to compact sets contained in a fixed ball. But this implies that F is measurable.

To obtain the desired conclusion, observe that, for $i \in \mathbb{N}$, there is some $x_i \in L_i^\perp$ such that $y_i \in \operatorname{conv}(A_i \cap (x_i + L_i))$. By Carathéodory's theorem, for $i \in \mathbb{N}$, there are numbers $\lambda_j^i \in [0, 1]$ and points $z_j^i \in A_i \cap (x_i + L_i)$, $j = 1, \dots, d + 1$, such that

$$y_i = \sum_{j=1}^{d+1} \lambda_j^i z_j^i \qquad \text{and} \qquad \sum_{j=1}^{d+1} \lambda_j^i = 1.$$

The sequence $(x_i)_{i \in \mathbb{N}}$ is bounded, since $(y_i)_{i \in \mathbb{N}}$ is a bounded sequence and $y_i = x_i + v_i$ for some $v_i \in L_i$. Since $A_i, A \subset B^d(0, R)$ for $i \in \mathbb{N}$, the sequences $(z_j^i)_{i \in \mathbb{N}}$ are also bounded. Hence, along a subsequence we get

$$x_i \to x \in L^\perp, \quad z_j^i \to z_j \in A \cap (x + L) \quad \text{and} \quad \lambda_j^i \to \lambda_j \in [0, 1],$$

for $j = 1, \dots, d + 1$, where we used that $A_i \to A$ and $L_i \to L$ as $i \to \infty$. Thus we arrive at

$$y = \sum_{j=1}^{d+1} \lambda_j z_j \qquad \text{and} \qquad \sum_{j=1}^{d+1} \lambda_j = 1.$$

This shows that $y \in \operatorname{conv}(A \cap (x + L))$, and therefore $y \in F(A, L)$. $\square$

The concept of L-convex hulls provides a (partial) convexification of a given compact set $A \subset \mathbb{R}^d$ with respect to a subspace. We now discuss a different kind of convexification, mainly for $d = 2$ and for a restricted class of compact sets $A \subset \mathbb{R}^2$, which is based on the top order surface area measure of a convex body. First, we describe the notion of a measure theoretic outer unit normal of a given set and a general version of the Gauss-Green theorem.

We recall a few concepts from analysis adapted to the present needs, for further details and explicit definitions we refer to [2], [5], [22]. Let $A \subset \mathbb{R}^d$ be a set, and let μ be a (outer) measure over $\mathbb{R}^d$ (cf. [5, p. 53]). The *restriction* $\mu \llcorner A$ of μ to A is the outer measure $(\mu \llcorner A)(B) := \mu(A \cap B)$, where $B \subset \mathbb{R}^d$. Moreover, the d-dimensional density of μ at $x \in \mathbb{R}^d$ is defined by

$$\Theta^d(\mu, x) := \lim_{r \to 0+} \frac{\mu(B^d(x, r))}{\kappa(d) r^d}$$

if the limit exists. These densities can be used to introduce a measure theoretic notion of exterior unit normal. First we define, for $x \in \mathbb{R}^d$ and $u \in \mathbb{S}^{d-1}$, the half spaces $H^+(x, u) := \{y \in \mathbb{R}^d : \langle y - x, u \rangle \geq 0\}$ and $H^-(x, u) := \{y \in \mathbb{R}^d : \langle y - x, u \rangle \leq 0\}$ with common boundary hyperplane $H(x, u)$. Let $A \subset \mathbb{R}^d$ be compact and $x \in \mathbb{R}^d$. Then a unit vector $u \in \mathbb{S}^{d-1}$ is said to be a *measure theoretic outer unit normal* of A at x, if

$$\Theta^d(\mathcal{H}^d \llcorner (H^+(x, u) \cap A), x) = 0 \quad \text{and} \quad \Theta^d(\mathcal{H}^d \llcorner (H^-(x, u) \setminus A), x) = 0.$$

If a measure theoretic outer unit normal of A at x exists, then it is unique and $x \in \partial A$. The outer unit normal of A at x will be denoted by $\nu(A,x) \in \mathbb{S}^{d-1}$ if it exists (see [5, Sect. 4.5]); otherwise we define $\nu(A,x) := 0$.

In the following, we consider a compact set $A \subset \mathbb{R}^d$ which satisfies $\mathcal{H}^{d-1}(\partial A) < \infty$ (although a somewhat weaker assumption would be sufficient). This condition implies that A has *finite perimeter* in the sense of the calculus of variations. Hence, a general version of the Gauss-Green theorem holds, i.e.

$$\int_A \operatorname{div} \varphi(z) \mathcal{H}^d(dz) = \int \langle \varphi(x), \nu(A,x) \rangle \mathcal{H}^{d-1}(dx)$$

for all vector fields $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ of class C^1. In particular, the map $\mathbb{R}^d \to \mathbb{R}^d$, $x \mapsto \nu(A,x)$, is $\mathcal{H}^{d-1}$-measurable (cf. [5, Theorem 4.5.6 (2)]). Therefore, we can define a measure μ_A on the Borel subsets of $\mathbb{S}^{d-1}$ by

$$\mu_A := \int \mathbf{1}\{\nu(A,x) \in \cdot\} \mathcal{H}^{d-1}(dx).$$

The Gauss-Green theorem then shows that

$$\int_{\mathbb{S}^{d-1}} u \mu_A(du) = 0, \tag{2.4}$$

i.e., μ_A is *centred*. In the special case of a convex body $L \in \mathcal{K}^d$ with nonempty interior, the (top order) *surface area measure* $S_{d-1}(L,\cdot)$ of L can be defined as $S_{d-1}(L,\cdot) := \mu_L$. However, the surface area measures of convex bodies are usually introduced in a less technical way as coefficients of a *local Steiner formula* (see [16, Chapter 4]). It is known that L is uniquely defined by its (top order) surface area measure up to a translation. We can fix a translation e.g. by requiring L to have its *Steiner point* $s(L)$ at the origin (cf. [16, Equation (1.7.3)]).

Let $A \subset \mathbb{R}^d$ be a general compact set with $\mathcal{H}^{d-1}(\partial A) < \infty$. Assume in addition that μ_A is not concentrated on a great subsphere. Since condition (2.4) is also satisfied, we can apply Minkowski's existence theorem (see [16, Section 7.1]) which yields the existence of a unique convex body $\operatorname{co}(A) \in \mathcal{K}^d$ with nonempty interior and Steiner point at the origin such that $S_{d-1}(\operatorname{co}(A),\cdot) = \mu_A$. We call $\operatorname{co}(A)$ the *convexification* of A. For less general classes of sets, the convexification has been introduced and studied for $d = 2$ in [4] and [20], and for arbitrary dimension in [1], but in a slightly different manner. It is in fact not obvious that the two approaches lead to the same convexification. We shall show this now, but only in the case which we need later, namely for two-dimensional sets with some additional regularity.

In the following, a compact set $A \subset \mathbb{R}^d$ will be called a *star body with respect to the origin*, if there is a positive continuous function $\rho_A : \mathbb{S}^{d-1} \to (0,\infty)$, the *radial function* of A, such that $A = \{\lambda \rho_A(u) u : \lambda \in [0,1], u \in \mathbb{S}^{d-1}\}$. Since ρ_A is continuous, we have $\partial A = \{\rho_A(u) u : u \in \mathbb{S}^{d-1}\}$. Finally, we say that A is a *star body* if a translate of A is a star body with respect to the origin.

We are going to prove the existence and some additional property of the convexification of a planar star body with finite boundary length. The proof requires some preparations. Let $A \subset \mathbb{R}^2$ be a star body with respect to the origin. Put $u(s) := \cos(s)e_1 + \sin(s)e_2$, $s \in [0,2\pi]$, where (e_1,e_2) is the standard basis. Then the map $J_0 : [0,2\pi] \to \mathbb{R}^2$, $s \mapsto \rho_A(u(s))u(s)$, provides a parametrization of ∂A. In addition, we assume that A also has finite boundary length $L := \mathcal{H}^1(\partial A)$, i.e. the curve J_0 is rectifiable (cf. [3, Lemma 3.2]). Then there is a reparametrization by arc-length, denoted by J, of J_0 which is oriented in the same way as J_0.

We define functions $\rho : [0, L] \to (0, \infty)$ and $v : [0, L] \to \mathbb{S}^1$ by $\rho := |J|$ and $v := J/|J|$, hence $J = \rho v$. Since J is Lipschitz and parametrized by arc-length, J is differentiable at s and $|J'(s)| = 1$, for $\mathcal{H}^1$-a.e. $s \in [0, L]$. Here we call J differentiable at $s = 0$ and $s = L$ if the one-sided derivatives at $s = 0$ and $s = L$ exist and coincide. The chosen orientation of J implies that $\det(v(s), v'(s)) \geq 0$ for $\mathcal{H}^1$-a.e. $s \in [0, L]$. Moreover, $\langle v(s), v'(s)\rangle = 0$ for $\mathcal{H}^1$-a.e. $s \in [0, L]$.

By Theorem 3.2.22 (1) in [5] and since $J'(s) \neq 0$ for $\mathcal{H}^1$-a.e. $s \in [0, L]$, the approximate tangent space $\mathrm{Tan}^1(\mathcal{H}^1 \llcorner \partial A, J(s))$ of ∂A at $J(s)$ is a one-dimensional linear subspace spanned by $J'(s)$, for $\mathcal{H}^1$-a.e. $s \in [0, L]$. The coarea formula also yields that, for $\mathcal{H}^1$-a.e. $x \in \partial A$, J is differentiable at $s = J^{-1}(x)$, $\mathrm{Tan}^1(\mathcal{H}^1 \llcorner \partial A, x)$ is the linear subspace spanned by $J'(s)$, and $|J'(s)| = 1$. Any such point $x \in \partial A$ will be called a *smooth* boundary point of A. If x is a smooth boundary point of A and $s = J^{-1}(x)$, we put $t(x) := J'(s)$ and define $\nu(x)$ as the uniquely determined vector such that $(\nu(x), t(x))$ is a positively oriented orthonormal basis of $\mathbb{R}^2$. The following lemma implies that $\nu(A, x) = \nu(x)$ for $\mathcal{H}^1$-almost all $x \in \partial A$.

Lemma 2.3. *Let $A \subset \mathbb{R}^2$ be a star body with respect to the origin. Assume that A has finite boundary length. If x is a smooth boundary point of A, then $\nu(x) = \nu(A, x)$.*

PROOF. We adopt the notation preceding the statement of the lemma. Let x_0 be a smooth boundary point of A, assume that $s_0 := J^{-1}(x_0) \in (0, L)$, and put $\nu_0 := \nu(x_0)$, $w_0 := t(x_0) = J'(s_0)$. For any $\epsilon \in (0, 1)$, we define the cone

$$C(x_0, w_0, \epsilon) := \{x_0 + \lambda w : |\langle w, w_0\rangle| \geq 1 - \epsilon, \lambda \in \mathbb{R}, w \in \mathbb{S}^1\}.$$

Then, for $r \in (0, 1)$, we have

$$\mathcal{H}^2(C(x_0, w_0, \epsilon) \cap B^2(x_0, r)) = \pi h(\epsilon) r^2, \tag{2.5}$$

where $h(\epsilon) := 4 \arccos(1 - \epsilon) \to 0$ as $\epsilon \to 0$.

In view of the equation $J'(s_0) = \rho'(s_0)v(s_0) + \rho(s_0)v'(s_0)$, we distinguish three cases.

(a) $v'(s_0) \neq 0$. Let $\epsilon \in (0, 1)$ be sufficiently small so that $\{\lambda x_0 : \lambda \geq 0\} \cap C(x_0, w_0, \epsilon) = \{x_0\}$. Here we use that $v(s_0)$ and $J'(s_0)$ are linearly independent. Since J is differentiable at s_0, there is a positive number $\delta > 0$ such that $J(s) \in C(x_0, w_0, \epsilon)$ for $s \in [s_0 - \delta, s_0 + \delta]$. Choose $r_0 \in (0, 1)$ such that

$$B^2(x_0, r_0) \subset \text{pos}\{J(s_0 - \delta), J(s_0 + \delta)\},$$

where $\mathrm{pos}(M)$ denotes the positive hull of a set $M \subset \mathbb{R}^d$, i.e. the smallest convex cone containing M. Then, for $r \in (0, r_0)$, the intermediate value theorem shows that if

$$z \in B^2(x_0, r_0) \cap H^-(x_0, \nu_0) \setminus C(x_0, w_0, \epsilon), \tag{2.6}$$

then

$$\{\lambda z : \lambda \geq 1\} \cap J([s_0 - \delta, s_0 + \delta]) \neq \emptyset, \tag{2.7}$$

and therefore $z \in A$. Hence, we get

$$H^-(x_0, \nu_0) \cap B^2(x_0, r) \setminus A \subset B^2(x_0, r) \cap C(x_0, w_0, \epsilon). \tag{2.8}$$

From (2.8) and (2.5) we conclude that

$$\mathcal{H}^2(H^-(x_0,\nu_0)\setminus A\cap B^2(x_0,r))/(\pi r^2)\le h(\epsilon),$$

which implies that

$$\Theta^2(\mathcal{H}^2\llcorner(H^-(x_0,\nu_0)\setminus A),x_0)=0. \tag{2.9}$$

A similar reasoning leads to

$$H^+(x_0,\nu_0)\cap B^2(x_0,r)\cap A\subset B^2(x_0,r)\cap C(x_0,w_0,\epsilon),$$

from which we deduce that

$$\Theta^2(\mathcal{H}^2\llcorner(H^+(x_0,\nu_0)\cap A),x_0)=0.$$

(b1) $v'(s_0)=0$ and $\rho'(s_0)>0$. Let $\epsilon\in(0,1)$ be fixed. Choose $\delta>0$ and $r_0>0$ as in (a). If $r\in(0,r_0)$ and z satisfies (2.6), then (2.7) holds, and hence $z\in A$. Therefore, (2.8) again implies (2.9). The remaining argument is also essentially the same as in case (a).

(b2) $v'(s_0)=0$ and $\rho'(s_0)<0$. The argument is similar to the one for (b1). □

Let $J:[0,L]\to\mathbb{R}^2$ denote a continuous map which is injective on $[0,L)$ and satisfies $J(0)=J(L)$. As usual we call such a map a *Jordan curve*. The image set $J([0,L])$ will be denoted as a *Jordan arc*. As mentioned before, the Jordan curve J is rectifiable if and only if $J([0,L])$ has finite one-dimensional Hausdorff measure.

Proposition 2.4. *Let $A\subset\mathbb{R}^2$ be a star body with finite boundary length. Then the convexification $\mathrm{co}(A)$ of A exists and contains some translate of $\mathrm{conv}(A)$.*

PROOF. The idea of the proof is to approximate the boundary of A by a sequence of inscribed polygonal Jordan arcs which bound star bodies A_n. We show that the surface area measures μ_{A_n} converge weakly to μ_A. To verify this, it is useful to work with tangent vectors rather than with exterior normal vectors. At this part of the argument, Lemma 2.3 is needed. The required assertions of the proposition can easily be established for the sets A_n. A compactness argument and the established weak continuity result will then allow us to deduce the corresponding assertions for the set A itself.

We adopt the notation of Lemma 2.3. Clearly, we can assume that A is a star body with respect to the origin. Let $L=\mathcal{H}^1(\partial A)<\infty$ denote the boundary length of A. For any $n\ge 3$, we consider a partition $s_{n,i}:=iL/n$, $i=0,1,\dots,n$, of $[0,L]$ and define a piecewise affine map $J_n:[0,L]\to\mathbb{R}^2$ by

$$J_n(s)=\frac{s_{n,i+1}-s}{L/n}J(s_{n,i})+\frac{s-s_{n,i}}{L/n}J(s_{n,i+1}), \tag{2.10}$$

for $s\in[s_{n,i},s_{n,i+1}]$. Then, if $n\in\mathbb{N}$ is sufficiently large, J_n is a polygonal Jordan curve, and the enclosed point set A_n converges to A, as $n\to\infty$, in the Hausdorff metric. Since we are considering star bodies with respect to the origin, this follows from the uniform continuity of the chosen parametrization J. In particular, $0\in\mathrm{int}(A_n)$ if $n\in\mathbb{N}$ is sufficiently large, and hence A_n is a star body with respect to the origin. Let these conditions be satisfied for $n\ge n_0$. For $x\in\partial A_n$ such that $s:=J_n^{-1}(x)\notin\{s_{n,i}:n\ge n_0,i\in\{0,\dots,n\}\}$, we put $t_n(x):=J_n'(s)$ and define $\nu_n(x)$ as the uniquely determined unit vector such that $(\nu_n(x),t_n(x))$

is a positively oriented orthonormal basis of $\mathbb{R}^2$. As in the proof of Lemma 2.3 it follows that $\nu_n(x) = \nu(A_n, x)$ (the condition $|J'| = 1$ can be replaced by $J_n' \neq 0$, in the proof of Lemma 2.3). In order to show that the measures μ_{A_n} converge weakly to μ_A, as $n \to \infty$, we take a continuous function $f : \mathbb{S}^1 \to \mathbb{R}$ and prove that

$$\lim_{n\to\infty} \int_{\partial A_n} f(\nu_n(x))\mathcal{H}^1(dx) = \int_{\partial A} f(\nu(x))\mathcal{H}^1(dx). \tag{2.11}$$

To verify (2.11), we define a continuous map $\tilde{f} : \mathbb{S}^1 \to \mathbb{R}$ by $\tilde{f}(u) := f(\sigma_0(u))$, where σ_0 is the rotation by $-\pi/2$. Then, by the coarea formula and since $\partial A_n = J_n([0, L])$ and $\partial A = J([0, L])$,

$$\begin{aligned}\int_{J_n([0,L])} f(\nu_n(x))\mathcal{H}^1(dx) &= \int_0^L f(\nu_n(J_n(s)))|J_n'(s)|ds \\ &= \int_0^L \tilde{f}(t_n(J_n(s)))|J_n'(s)|ds \end{aligned} \tag{2.12}$$

and

$$\int_{J([0,L])} f(\nu(x))\mathcal{H}^1(dx) = \int_0^L \tilde{f}(t(J(s)))ds. \tag{2.13}$$

Here we used $|J'(s)| = 1$, for $\mathcal{H}^1$-a.e. $s \in [0, L]$, and the injectivity of J and J_n. We will apply Lebesgue's dominated convergence theorem to infer that

$$\lim_{n\to\infty} \int_0^L \tilde{f}(t_n(J_n(s)))|J_n'(s)|ds = \int_0^L \tilde{f}(t(J(s)))ds. \tag{2.14}$$

Subsequently, we verify that Lebesgue's theorem can be applied so that the required conclusion is obtained by combining (2.12), (2.13) and (2.14).

Let $s \in [0, L] \setminus \{s_{n,i} : n \geq n_0, i \in \{0, \dots, n\}\}$ be chosen such that J is differentiable at s and $J'(s) = t(J(s))$ is a unit vector. For any $n \geq n_0$, there is some $i \in \{0, \dots, n-1\}$ such that $s \in (s_{n,i}, s_{n,i+1})$. From (2.10) we get

$$\begin{aligned} J_n'(s) &= \frac{n}{L}\left(J(s_{n,i+1}) - J(s_{n,i})\right) \\ &= \frac{n}{L}\left(J'(s)\left(s_{n,i+1} - s\right) + o(1/n)\right) - \frac{n}{L}\left(J'(s)\left(s_{n,i} - s\right) + o(1/n)\right) \\ &= J'(s) + no(1/n), \end{aligned}$$

and thus

$$\lim_{n\to\infty} J_n'(s) = J'(s). \tag{2.15}$$

Moreover,

$$t_n(J_n(s)) = \frac{J(s_{n,i+1}) - J(s_{n,i})}{|J(s_{n,i+1}) - J(s_{n,i})|} = \frac{\frac{L}{n}J'(s) + o(1/n)}{|\frac{L}{n}J'(s) + o(1/n)|},$$

and therefore

$$\lim_{n\to\infty} t_n(J_n(s)) = J'(s) = t(J(s)). \tag{2.16}$$

Thus, (2.15), (2.16) and the continuity of $\tilde{f}$ yield that

$$\lim_{n\to\infty} \tilde{f}(t_n(J_n(s)))|J_n'(s)| = \tilde{f}(t(J(s))),$$

for $\mathcal{H}^1$-a.e. $s \in [0, L]$. Moreover, $s \mapsto \tilde{f}(t_n(s))|J_n'(s)|$ is almost everywhere bounded on $[0, L]$, uniformly in n, since J is Lipschitz. Hence (2.11) is proved.

For $n \geq n_0$, the surface area measure of the set A_n enclosed by J_n is non-degenerate so that the convexification $\mathrm{co}(A_n)$ is well-defined. Since μ_{A_n} has finite support, $\mathrm{co}(A_n)$ is a polytope. As shown in [20, p. 328] we have

$$\mathrm{conv}(A_n) \subset P_n, \tag{2.17}$$

where P_n is a suitable translate of $\mathrm{co}(A_n)$. The construction leading to P_n can be described as follows: If A_n is not convex, let $i \in \{0, \ldots, n-1\}$ be the smallest integer such that $(J(s_{n,i}), J(s_{n,i+2})) \cap A_n = \emptyset$, where $s_{n,n+1} := s_{n,1}$. We then reflect $J([s_{n,i}, s_{n,i+2}])$ in the midpoint $(J(s_{n,i}) + J(s_{n,i+2}))/2$ and thus obtain a new polygonally bounded Jordan arc which bounds a set A_n' that is again a star body with respect to the origin. This new set A_n' has the same surface area measure as A_n, and therefore the same convexification $\mathrm{co}(A_n)$, but fulfills $A_n \subset A_n'$. Repeating this procedure with A_n', we obtain an increasing sequence of sets with the same convexification. Since the reflections used increase the 'clockwise ordering' of the boundary segments, the algorithm ends after finitely many steps and the terminal set is a translate, P_n, of the convexification $\mathrm{co}(A_n)$. Hence $A_n \subset \mathrm{conv}(A_n) \subset P_n$. It is easy to see that the polytopes P_n have uniformly bounded diameter, and therefore we can assume that the polytopes P_n are uniformly bounded. But then P_n converges towards some $A' \in \mathcal{K}^2$ along a subsequence. By (2.11), and the weak continuity of the surface area measures we obtain that $\mu_A = S_1(A', \cdot) = S_1(\mathrm{co}(A), \cdot)$. Moreover, (2.17) and the convergence of A_n towards A show that $\mathrm{conv}(A) \subset A'$. Since $\mathrm{co}(A)$ is a translate of A', we have now proved the proposition. □

2.3 Differentiation of relative parallel volume

In this subsection we consider a compact set $A \subset \mathbb{R}^d$ and a convex body $B \subset \mathbb{R}^d$ and derive some auxiliary results on the relative parallel sets $A + rB$, $r \geq 0$, and their boundaries.

Lemma 2.5. *Let $A \subset \mathbb{R}^d$ be compact, and let $B \in \mathcal{K}^d$ with $0 \in \mathrm{int}(B)$. Let $A \subset rB^*$ for some $r \geq 0$. Then, for any $t > r$, $A + tB$ is a star body with respect to the origin and $\partial(A + tB)$ is homeomorphic to $\mathbb{S}^{d-1}$.*

PROOF. If $x \in A + tB$, then there is some $a \in A$ with $x \in a + tB$. By assumption, we also have $a \in rB^*$, i.e. $0 \in a + rB$. Then, for any $\lambda \in [0, 1)$, we obtain

$$\begin{aligned}
\lambda x &\in \lambda a + \lambda t B + (1-\lambda)0 \\
&\subset \lambda a + \lambda t B + (1-\lambda)a + (1-\lambda)rB \\
&= a + (\lambda t + (1-\lambda)r)B \\
&\subset a + tB,
\end{aligned}$$

where $\lambda t + (1-\lambda)r < \lambda t + (1-\lambda)t = t$ was used. Hence λx is contained in the interior of $a + tB$. Therefore, for any $u \in \mathbb{S}^{d-1}$, there is a unique point $\varphi(u) \in \mathbb{R}^d$ such that

$$\partial(A + tB) \cap \{su : s \geq 0\} = \{\varphi(u)\}.$$

The map $\varphi : \mathbb{S}^{d-1} \to \partial(A+tB)$ is bijective and the inverse map $x \mapsto \varphi^{-1}(x) = x/|x|$ is continuous. Since $\partial(A+tB)$ is compact, φ is a homeomorphism. $\square$

For $t > 0$, we put

$$[d_B(A,\cdot) = t] := \{z \in \mathbb{R}^d : d_B(A,z) = t\}.$$

If A is a compact convex set, one clearly has

$$[d_B(A,\cdot) = t] = \partial(A+tB). \tag{2.18}$$

In the case of a general compact set, this is no longer true as the following very simple counterexample shows.

Example 2.6. *Let* $A := \mathbb{S}^{d-1}$*. Then*

$$\partial(\mathbb{S}^{d-1} + B^d) = 2\mathbb{S}^{d-1} \subset 2\mathbb{S}^{d-1} \cup \{0\} = [d_{B^d}(\mathbb{S}^{d-1},\cdot) = 1].$$

As a weak substitute for (2.18), we have the next lemma, which is sufficient for the proof of the subsequent proposition.

Lemma 2.7. *Let* $A \subset \mathbb{R}^d$ *be compact, and let* $B \in \mathcal{K}^d$ *be strictly convex with* $0 \in \operatorname{int}(B)$*. Then*

(a) $\partial(A+tB) \subset [d_B(A,\cdot) = t]$ *for all* $t > 0$,

(b) $\mathcal{H}^{d-1}([d_B(A,\cdot) = t] \setminus \partial(A+tB)) = 0$ *for* $\mathcal{H}^1$*-a.e.* $t > 0$.

PROOF. (a) Let $x \in \partial(A+tB)$ be given. Then $x \in A+tB$ and there is a sequence of points x_i, $i \in \mathbb{N}$, with $x_i \notin A+tB$ and $x_i \to x$ as $i \to \infty$. Hence $d_B(A,x) \le t$ and $d_B(A,x_i) > t$ for $i \in \mathbb{N}$. The latter implies that $d_B(A,x) \ge t$, and thus $x \in [d_B(A,\cdot) = t]$.

(b) By Lemma 2.1, $\mathcal{H}^d(\text{exo}_B(A)) = 0$. The coarea formula, applied to the Lipschitz distance function $d_B(A,\cdot)$ (cf. [7, Lemma 1]) then shows that

$$\begin{aligned} 0 &= \int_{\text{exo}_B(A)} J_1 d_B(A,\cdot)(x)\mathcal{H}^d(dx) \\ &= \int_0^\infty \int_{[d_B(A,\cdot)=t]} \mathbf{1}\{y \in \text{exo}_B(A)\}\mathcal{H}^{d-1}(dy)dt, \end{aligned}$$

where $J_1 d_B(A,\cdot)$ denotes the one-dimensional approximate Jacobian of $d_B(A,\cdot)$. Hence, for $\mathcal{H}^1$-a.e. $t > 0$,

$$\mathcal{H}^{d-1}(\text{exo}_B(A) \cap [d_B(A,\cdot) = t]) = 0. \tag{2.19}$$

Let $t > 0$ be chosen such that (2.19) is satisfied, and choose any $x \in [d_B(A,\cdot) = t] \setminus \text{exo}_B(A)$. Then $x \in A+tB$, and we have to show that $x \notin \operatorname{int}(A+tB)$. Since $x \notin \text{exo}_B(A)$, we have $(x+tB^*) \cap A = \{y\}$, i.e. $x = y+tb$ for a uniquely determined point $b \in \partial B$. A continuity argument shows (cf. [16, Theorem 1.8.8]) that there is some $\epsilon_0 > 0$ such that, for $s \in (0,\epsilon_0]$ and $a \in B^d(y,\epsilon_0) \setminus \operatorname{int}(x+tB^*)$, we have $a - sb \notin x+tB^*$, and therefore $x+sb \notin a+tB$. Next we choose $\epsilon_1 > 0$ sufficiently small such that $A \cap (x+tB^*+\epsilon_1 B) \subset B^d(y,\epsilon_0)$. Hence, if $a \in A \cap (x+tB^*+\epsilon_1 B)$, then $a \in B^d(y,\epsilon_0) \setminus \operatorname{int}(x+tB^*)$, and thus $x+sb \notin a+tB$ for $s \in (0,\epsilon_0]$. Moreover, if $a \in A \setminus (x+tB^*+\epsilon_1 B)$, then $x+sb \notin a+tB$ for $s \in (0,\epsilon_1]$. Hence, we finally get $x + sb \notin A + tB$ for all $s \in (0, \epsilon_2]$, where $\epsilon_2 := \min\{\epsilon_0, \epsilon_1\}$. This proves the existence of a sequence $x_i \notin A + tB$, $i \in \mathbb{N}$, such that $x_i \to x$ as $i \to \infty$. $\square$

We will now investigate the differentiation of the volume of the relative parallel sets $A+tB$, $t > 0$, where $A \subset \mathbb{R}^d$ is compact and $B \in \mathcal{K}^d$ is strictly convex with $0 \in \operatorname{int}(B)$. The set of all strictly convex $B \in \mathcal{K}^d$ with $0 \in \operatorname{int}(B)$ will be denoted by $\mathcal{K}^d_*$. Let $A \in \mathcal{C}^d$ and let $B \in \mathcal{K}^d_*$. If $x \in \mathbb{R}^d \setminus A$ and $\rho := d_B(A, \cdot)$ is differentiable at x, then we define

$$\nu_B(A, x) := \nabla\rho(x)/|\nabla\rho(x)|;$$

in all other cases, we define $\nu_B(A, x)$ as the zero vector. Then the map $\mathcal{C}^d \times \mathcal{K}^d_* \times \mathbb{R}^d \to \mathbb{R}^d$, $(A, B, x) \mapsto \nu_B(A, x)$, is Borel measurable. The proof of this assertion is based on the fact that $\mathcal{C}^d \times \mathcal{K}^d_* \times \mathbb{R}^d \to \mathbb{R}^d$, $(A, B, x) \mapsto d_B(A, x)$ is continuous and the set of all (A, B, x) such that $d_B(A, \cdot)$ is differentiable at x, can be written as a countable intersection of a countable union of closed sets.

A heuristic argument for the first assertion of the following proposition is given in [7, Remark 3]. As usual, the support function $h(B, \cdot) : \mathbb{R}^d \to \mathbb{R}$ of a compact, convex set $B \subset \mathbb{R}^d$ is defined by $h(B, u) := \max\{\langle x, u\rangle : x \in B\}$ for $u \in \mathbb{R}^d$.

Proposition 2.8. *Let $A \subset \mathbb{R}^d$ be compact, and let $B \in \mathcal{K}^d$ be strictly convex with $0 \in \operatorname{int}(B)$. Then, for any non-negative measurable function $f : \mathbb{R}^d \to \mathbb{R}$,*

$$\int_{\mathbb{R}^d \setminus A} f(z)\mathcal{H}^d(dz) = \int_0^\infty \int_{\partial(A+sB)} f(x)h(B, \nu(A + sB, x))\mathcal{H}^{d-1}(dx)ds.$$

For $\mathcal{H}^1$-a.e. $t > 0$, $\mathcal{H}^{d-1}(\partial(A + tB)) < \infty$ and

$$\frac{d}{dt}V_d(A + tB) = \int_{\partial(A+tB)} h(B, \nu(A + tB, x))\mathcal{H}^{d-1}(dx). \tag{2.20}$$

Moreover, $\nu(A + tB, x) = \nu_B(A, x) \in \mathbb{S}^{d-1}$ is satisfied for $\mathcal{H}^{d-1}$-a.e. $x \in \partial(A + tB)$ and $\mathcal{H}^1$-a.e. $t > 0$.

PROOF. We put $\rho := d_B(A, \cdot)$. We already used that ρ is Lipschitz. The Jacobian of ρ satisfies $J_1\rho(x) > 0$ for $\mathcal{H}^d$-a.e. $x \in \mathbb{R}^d \setminus A$; cf. the proof of Lemma 2.1. The coarea formula and Lemma 2.7 then imply the assertions of the proposition, if we can show that $J_1\rho(x)^{-1} = h(B, \nu_B(A, x))$ and $\nu_B(A, x) = \nu(A + \rho(x)B, x)$ whenever ρ is differentiable at $x \in \mathbb{R}^d \setminus A$.

For the proof, let ρ be differentiable at $x \in \mathbb{R}^d \setminus A$. We put $t := \rho(x)$ and $u := \nu_B(A, x)$. The proof of Lemma 2.1 shows that $x \notin \operatorname{exo}_B(A)$. But then the argument provided for Lemma 2.7 (b) yields that $x \in \partial(A + tB)$. Using again that $x \notin \operatorname{exo}_B(A)$, we obtain the existence of a unique point $a \in A$ such that $x \in \partial(a + tB)$. Let $\lambda_0 > 0$ be fixed and let $v \in H(0, u) \cap \mathbb{S}^{d-1}$. Then, for any $s > 0$ and $\lambda \in [0, \lambda_0]$,

$$\rho(x + s(u + \lambda v)) = t + s|\nabla\rho(x)| + R(s(u + \lambda v))s\sqrt{1 + \lambda^2},$$

where $R(w) \to 0$ as $w \to 0$; hence,

$$\rho(x + s(u + \lambda v)) - t = s\left(|\nabla\rho(x)| + R(s(u + \lambda v))\sqrt{1 + \lambda^2}\right). \tag{2.21}$$

If $s > 0$ is sufficiently small (depending on λ_0), then the right-hand side of (2.21) is positive for all $v \in H(0,u) \cap \mathbb{S}^{d-1}$ and $\lambda \in [0, \lambda_0]$, hence $x + s(u + \lambda v) \notin A + tB$. In particular, $x + s(u + \lambda v) \notin a + tB$, first for $\lambda \in [0, \lambda_0]$, $v \in H(0,u) \cap \mathbb{S}^{d-1}$ and sufficiently small $s > 0$, but then for all $s > 0$, by the convexity of $a + tB$. Since $\lambda_0 > 0$ can be chosen arbitrarily large, it follows that u is an exterior unit normal vector of $a + tB$ at x, hence $h(a + tB, u) = \langle x, u \rangle$. As $x \in \partial(a + tB)$, there is a unique point $b \in \partial B$ with $x = a + tb$, and thus $h(B, u) = \langle b, u \rangle$.

Since $\rho(x - \varepsilon b) = t - \varepsilon$ for $\varepsilon \in (0, t)$ (cf. the proof of Lemma 2.1) and ρ is differentiable at x, we deduce that $D\rho(x)(b) = 1$. Writing b in the form $b = \tilde{b} + \langle b, u \rangle u$ with $\langle \tilde{b}, u \rangle = 0$, we get $D\rho(x)(u) = D\rho(x)(\langle b, u \rangle^{-1} b) = \langle b, u \rangle^{-1} = h(B, u)^{-1}$. This finally shows that $J_1 \rho(x) = |\nabla \rho(x)| = |D\rho(x)(u)| = h(B, u)^{-1}$.

From (2.21) we can further deduce that, for given $\lambda_0 > 0$, suitably chosen $s_0 = s_0(\lambda_0) > 0$ and $r \in (0, s_0)$,

$$H^+(x,u) \cap B^d(x,r) \cap (A + tB) \subset H^+(x,u) \cap B^d(x,r) \setminus \tilde{C}(x,u,\lambda_0,s_0),$$

where

$$\tilde{C}(x,u,\lambda_0,s_0) := \{x + s(u + \lambda v) : v \in H(0,u) \cap \mathbb{S}^{d-1}, \lambda \in [0,\lambda_0], s \in [0,s_0]\}.$$

For $r \in (0, s_0)$, $f(\lambda_0) := \mathcal{H}^d(H^+(x,u) \cap B^d(x,r) \setminus \tilde{C}(x,u,\lambda_0,s_0))/r^d$ is independent of s_0 and r, and $f(\lambda_0) \to 0$ as $\lambda_0 \to \infty$. This implies that $\Theta^d(\mathcal{H}^d \llcorner (H^+(x,u) \cap (A + tB)), x) = 0$. Similarly, we have

$$H^-(x,u) \cap B^d(x,r) \setminus (A + tB) \subset H^-(x,u) \cap B^d(x,r) \setminus \tilde{C}(x,-u,\lambda_0,s_0),$$

where $s_0 = s_0(\lambda_0)$ and $r \in (0, s_0)$. Hence $\Theta^d(\mathcal{H}^d \llcorner (H^-(x,u) \setminus (A + tB)), x) = 0$, which completes the proof. $\square$

3 Dilatation by random segments and convexity

In this section, we consider the dilatation of a random compact set X by a random segment $t[0, U]$ of length $t \geq 0$, where U is a random unit vector. We prove that if the average volume of such a dilatation is a polynomial in the parameter t, then, with probability one, almost all sections of X by lines parallel to U are convex.

For a unit vector $u \in S^{d-1}$, we put $\hat{u} := \text{span}\{u\}$ and define $u^\perp$ as the subspace orthogonal to $\hat{u}$. We denote the $\hat{u}$-convex hull of a set $A \in \mathcal{C}^d$ by $A_u := \text{conv}_{\hat{u}}(A)$. Further, for $C \in \mathcal{C}^d$ and $L \in \mathcal{L}_k^d$, we write $C|L$ for the orthogonal projection of C on L. The measurability of the map $\mathcal{C}^d \times \mathcal{L}_k^d \to \mathcal{C}^d$, $(C, L) \mapsto C|L$ is established in [12, Lemma 3-5-3].

In the following theorem, we consider a random compact set X in $\mathbb{R}^d$ and a random unit vector U. Assuming that $t \mapsto \mathbb{E} V_d(X + t[0, U])$, for $t \geq 0$, is a polynomial, we aim at showing that X must satisfy some convexity property. Clearly, we have $\mathbb{E} V_d(X + t[0, U]) < \infty$. For our proof, however, we need the stronger assumption that

$$\mathbb{E} V_d(\text{conv}(X) + t[0, U]) < \infty \tag{3.1}$$

holds for some (and thus for all) $t > 0$. This follows, for instance, if

$$\mathbb{E} V_d(\text{conv}(X) + B^d) < \infty \tag{3.2}$$

is satisfied.

Theorem 3.1. *Let X be a random compact set in $\mathbb{R}^d$, and let U be a random vector in $\mathbb{S}^{d-1}$ such that* (3.1) *is satisfied. Assume that*

$$t \mapsto \mathbb{E}V_d(X + t[0,U]), \qquad t \geq 0,$$

is a polynomial. Then, with probability one, $X \cap (x + \hat{U})$ is a segment for λ_{d-1}-a.e. $x \in U^\perp$.

PROOF. We consider functions

$$\phi(t) := \mathbb{E}V_d(X + t[0,U]), \qquad t \geq 0,$$

and

$$\psi(t) := \mathbb{E}V_d(X_U + t[0,U]), \qquad t \geq 0.$$

Since $X \subset X_U$, we obtain

$$\phi(t) \leq \psi(t), \qquad t \geq 0. \tag{3.3}$$

Next we prove that ψ is a polynomial of degree one. For this, we apply Fubini's theorem to get

$$\begin{aligned}
V_d(X_U + t[0,U]) &= \int_{U^\perp} V_1((X_U + t[0,U]) \cap (x + \hat{U}))\lambda_{d-1}(dx) \\
&= \int_{U^\perp} V_1(\mathrm{conv}(X \cap (x + \hat{U})) + t[0,U])\lambda_{d-1}(dx) \\
&= \int_{U^\perp} V_1(\mathrm{conv}(X \cap (x + \hat{U})))\lambda_{d-1}(dx) \\
&\qquad + t\int_{U^\perp} \mathbf{1}\{X \cap (x + \hat{U}) \neq \emptyset\}\lambda_{d-1}(dx) \\
&= V_d(X_U) + t\lambda_{d-1}(X|U^\perp).
\end{aligned} \tag{3.4}$$

Using (3.4), we obtain

$$\psi(t) = \mathbb{E}V_d(X_U) + t\mathbb{E}\lambda_{d-1}(X|U^\perp) =: b_0 + tb_1, \qquad t \geq 0, \tag{3.5}$$

where $b_0, b_1 \in \mathbb{R}$ due to the integrability condition (3.1).

By the assumption of polynomial volume growth, we have

$$\phi(t) = \sum_{i=0}^{m} a_i t^i, \qquad t \geq 0, \tag{3.6}$$

where $a_i \in \mathbb{R}$ and $m \in \mathbb{N}$. Since $\phi(t) \geq 0$ for $t \geq 0$, we get $a_m \geq 0$. From (3.3), (3.5) and (3.6), it follows that we can choose $m \leq 1$ and that $0 \leq a_1 \leq b_1$. On the other hand,

$$0 \leq \mathbb{E}\left[V_d(X_U + t[0,U]) - V_d(X + t[0,U])\right] = (b_0 - a_0) + (b_1 - a_1)t,$$

where again we have used the integrability assumption (3.1). For any $t \geq 0$, we define the non-negative random variable

$$\begin{aligned}
f(t) :=& V_d(X_U + t[0,U]) - V_d(X + t[0,U]) \\
=& \int_{U^\perp} V_1\left(\left(\mathrm{conv}(X \cap (x + \hat{U})) + t[0,U]\right) \setminus \left(X \cap (x + \hat{U}) + t[0,U]\right)\right)\lambda_{d-1}(dx).
\end{aligned}$$

If we write $\operatorname{conv}(X\cap(x+\hat{U}))=[x_1,x_2]$ for some $x_1,x_2\in\mathbb{R}^d$, then $X\cap(x+\hat{U})=:I\subset[x_1,x_2]$ and $x_1,x_2\in I$. Therefore,

$$([x_1,x_2]+t[0,U])\setminus(I+t[0,U])=[x_1,x_2]\setminus(I+t[0,U])$$

is non-increasing as t increases and is the empty set for $t>|x_1-x_2|$. This shows that $t\mapsto f(t)$ is non-increasing and $f(t)\to 0$ as $t\to\infty$. From this we conclude that $b_1\le a_1$, and hence $a_1=b_1$. Moreover, since $f(0)$ is integrable, we obtain

$$0=\lim_{t\to\infty}\mathbb{E}\,[f(t)]=\lim_{t\to\infty}\,[(b_0-a_0)+(b_1-a_1)t]=b_0-a_0,$$

whence $a_0=b_0$. But this implies that $\phi(t)=\psi(t)$ for all $t\ge 0$, and thus

$$\begin{aligned}\mathbb{E}\int_{U^\perp}&V_1(\operatorname{conv}(X\cap(x+\hat{U}))+t[0,U])\lambda_{d-1}(dx)\\&=\mathbb{E}\int_{U^\perp}V_1(X\cap(x+\hat{U})+t[0,U])\lambda_{d-1}(dx).\end{aligned}\tag{3.7}$$

The required measurability follows from the auxiliary results provided in [17, p. 192-3]. From

$$V_1(X\cap(x+\hat{U})+t[0,U])\le V_1(\operatorname{conv}(X\cap(x+\hat{U}))+t[0,U])$$

and (3.7), we deduce that, for λ_{d-1}-a.e. $x\in U^\perp$ and $\mathbb{P}$-a.s.

$$V_1(\operatorname{conv}(X\cap(x+\hat{U}))+t[0,U])=V_1(X\cap(x+\hat{U})+t[0,U])$$

for all $t\in(0,\infty)\cap\mathbb{Q}$. Moreover, $\operatorname{conv}(X\cap(x+\hat{U}))+t[0,U]$ is the closure of its relative interior whenever $X\cap(x+\hat{U})\neq\emptyset$. Hence, $\mathbb{P}$-a.s. and for λ_{d-1}-a.e. $x\in U^\perp$, the set $X\cap(x+\hat{U})+t[0,U]$ is convex for all $t\in(0,\infty)\cap\mathbb{Q}$, and hence for all $t>0$. This yields the assertion of the theorem. □

From Theorem 3.1 we can deduce various results as special cases. In our first result, we consider a random unit vector with a special distribution. Clearly, the integrability assumption could be weakened slightly as in the statement of Theorem 3.1.

Theorem 3.2. *Let $X\subset\mathbb{R}^d$ be a random regular compact set for which* (3.2) *is satisfied, and let τ be a finite measure on $\mathbb{S}^{d-1}$ which dominates spherical Lebesgue measure. Assume that*

$$t\mapsto\mathbb{E}\int_{\mathbb{S}^{d-1}}V_d(X+t[0,u])\tau(du),\qquad t\ge 0,$$

is a polynomial. Then X is almost surely convex.

PROOF. We may assume that τ is a probability measure. Let U be a random unit vector with distribution τ and independent of X. Recall that σ denotes normalized spherical Lebesgue measure. Then, by Theorem 3.1, the assumption, and by independence, the random set $X\cap(y+\hat{u})$ is a.s. convex for σ-a.e. $u\in\mathbb{S}^{d-1}$ and λ_{d-1}-a.e. $y\in u^\perp$. Since X is a regular compact set, an approximation argument yields the almost sure convexity of X. □

As further special consequences of Theorem 3.1, we obtain the following corollaries which deal with the case of a deterministic compact set.

Corollary 3.3. *Let $A \subset \mathbb{R}^d$ be a regular compact set. Assume that*

$$t \mapsto \int_{\mathbb{S}^{d-1}} V_d(A + t[0,u])\sigma(du), \qquad t \geq 0,$$

is a polynomial. Then A is convex.

Corollary 3.4. *Let $A \subset \mathbb{R}^d$ be a regular compact set. Assume that $V_d(A + t[0,u])$, $t \geq 0$, is a polynomial in t, for σ-a.e. vector $u \in \mathbb{S}^{d-1}$. Then A is convex.*

Theorem 3.1 in particular holds for a fixed (deterministic) unit vector u. But even if X is also deterministic and regular, we cannot conclude that all linear sections of X in direction u are convex if we merely know that $t \mapsto V_d(X + t[0,u])$, $t \geq 0$, is a polynomial. Consider, for instance, the deterministic set $X := \text{conv}\{0, -e_1, e_2\} \cup \text{conv}\{-e_2, -2e_2, e_1 - e_2\}$ and the direction $u = e_2$. In particular, we have $X \neq X_U$ in this case.

4 Dilatation by random disc bodies and convexity

We recall from the introduction that a two-dimensional convex body containing the origin in its relative interior is a disc body, if its relative boundary is smooth (of class C^1) and strictly convex. In analogy to the previous section, we now investigate the dilatation of a random compact set X by a random disc body tY, where Y is a given random disc body and $t \geq 0$ is a scaling parameter. If the average volume of such a dilatation is a polynomial in the parameter t, then we can show that, with probability one, almost all planar sections of X by two-dimensional planes parallel to Y are convex.

4.1 The two-dimensional deterministic case

The aim of this subsection is to establish the following generalization of Theorem 1 in [8]. The result will be extended to higher dimensions and to random sets subsequently.

Theorem 4.1. *Let $A \subset \mathbb{R}^2$ be compact, and let $B \in \mathcal{K}^2$ be a disc body. Assume that*

$$t \mapsto V_2(A + tB), \qquad t \geq 0,$$

is a polynomial. Then A is convex.

PROOF. The proof is divided into three steps. The general aim is to show that A coincides with its convex hull $C := \text{conv}(A)$. For this we can assume that $A \neq \emptyset$. By translation invariance, we can also assume that $0 \in \text{int}(B)$. Further, let r_0 be the smallest number $r \geq 0$ such that $A \subset z + rB^*$ for some $z \in \mathbb{R}^d$. Then again by translation invariance, we can assume that $A \subset r_0 B^*$.

I. Since C is convex,

$$V_2(C + tB) = V_2(C) + 2tV(C,B) + t^2 V_2(B), \quad t \geq 0, \tag{4.1}$$

where $V(C,B)$ is the *mixed area* of C and B (see e.g. [16]). Taking some $a \in A \subset C$, we deduce that $\{a\} + tB \subset A + tB \subset C + tB$, and hence

$$t^2 V_2(B) \leq V_2(A + tB) \leq V_2(C + tB). \tag{4.2}$$

From (4.1), (4.2) and the assumption, we can conclude that $V_2(A+tB)$ is a polynomial in $t \geq 0$ of degree at most two, i.e. there are constants $c_0, c_1, c_2 \in \mathbb{R}$ such that

$$V_2(A+tB) = c_0 + c_1 t + c_2 t^2, \qquad t \geq 0. \tag{4.3}$$

A comparison of (4.1), (4.2) and (4.3) shows that

$$c_2 = V_2(B) \quad \text{and} \quad c_1 \leq 2V(C, B),$$

and therefore,

$$\frac{d}{dt} V_2(A+tB) = c_1 + 2c_2 t \leq 2V(C, B) + 2V_2(B)t = \frac{d}{dt} V_2(C+tB). \tag{4.4}$$

Combining (4.4) and (2.20), we obtain, for $\mathcal{H}^1$-a.e. $t > 0$,

$$\int_{\partial(A+tB)} h(B, \nu(A+tB, x))\mathcal{H}^1(dx) \leq \int_{\partial(C+tB)} h(B, \nu(C+tB, x))\mathcal{H}^1(dx). \tag{4.5}$$

II. We fix $t > r_0$ such that (4.5) is satisfied and put $A' := A + tB$. Then, in particular, $\mathcal{H}^1(\partial A') < \infty$. Since $t > r_0$, Lemma 2.5 shows that $A' = A + tB$ is a star body with respect to the origin. Using Proposition 2.4 and the notation preceding it, we can rewrite (4.5) in the form

$$\int_{\mathbb{S}^1} h(B, u) S_1(\mathrm{co}(A'), du) \leq \int_{\mathbb{S}^1} h(B, u) S_1(C', du), \tag{4.6}$$

where $C' := C + tB$ is the convex hull of $A' = A + tB$. Proposition 2.4 ensures the existence of a translate K of $\mathrm{co}(A')$ satisfying $C' = \mathrm{conv}(A') \subset K$. By the translation invariance of mixed areas, using a special case of formula (5.1.18) in [16], and by (4.6), we obtain $V(B, K) \leq V(B, C')$. Hence, the symmetry of mixed areas yields

$$\int_{\mathbb{S}^1} \left[h(C', u) - h(K, u)\right] S_1(B, du) \geq 0. \tag{4.7}$$

Since $C' \subset K$, equality must hold in (4.7), and thus $h(C', u) = h(K, u)$ for all $u \in \mathbb{S}^1$ which are in the support of $S_1(B, \cdot)$. As B is smooth, we conclude that $K = C' = \mathrm{conv}(A')$. But then

$$\mathcal{H}^1(\partial A') = S_1(K, \mathbb{S}^1) = S_1(C', \mathbb{S}^1) = \mathcal{H}^1(\partial C'),$$

and we can infer as in [8] that A' is convex.

III. So far we have shown that $A+rB$ is convex for $\mathcal{H}^1$-a.e. $r > r_0$. Hence $A+rB$ is convex for every $r > r_0$, and thus $A + rB = \mathrm{conv}(A + rB) = C + rB$ whenever $r > r_0$. In particular, $V_2(A + rB) = V_2(C + rB)$ for $r > r_0$, and hence by (4.3) and the convexity of C and B,

$$V_2(A) + c_1 r + V_2(B) r^2 = V_2(C) + 2V(C, B) r + V_2(B) r^2, \tag{4.8}$$

first for $r > r_0$, but then also for any $r \in \mathbb{R}$. But this shows that $V_2(A+rB) = V_2(C+rB)$ for all $r \geq 0$. Since $A + rB$ is compact and $C + rB$ has non-empty interior for $r > 0$, we deduce that $A + rB = C + rB$ is convex for any $r > 0$. This implies the convexity of A. □

From the proof of Theorem 4.1 and using Lemma 2.3, we can extract the following result, which will be needed later for establishing an extension in general dimensions.

Lemma 4.2. *Let $A \subset \mathbb{R}^2$ be a star body with finite boundary length, and let $C := \operatorname{conv}(A)$. Let $B \in \mathcal{K}^2$ be a disc body. Then*

$$\int_{\partial C} h(B, \nu(C, z))\mathcal{H}^1(dz) \leq \int_{\partial A} h(B, \nu(A, z))\mathcal{H}^1(dz),$$

with equality if and only if A is convex.

The following example shows that the regularity assumptions on B cannot be completely omitted in the statement of Theorem 4.1.

Example 4.3. Let $T_1 := \operatorname{conv}\{0, e_1 + (\sqrt{3}/3)e_2, e_1 - (\sqrt{3}/3)e_2\}$ denote a unilateral triangle and let $T_2 := -T_1$. Then we define $A := T_1 \cup T_2$. Further, we define a parallelogram by $B := (T_1 - e_1) \cup (T_2 + e_1)$. Then A is not convex, but

$$\begin{aligned} V_2(A + tB) &= V_2(T_1 + tB) + V_2(T_2 + tB) - V_2(tB) \\ &= 2V_2(T_1) + 2tV(T_1 + T_2, B) + t^2 V_2(B) \end{aligned}$$

is a polynomial for $t \geq 0$.

4.2 General dimensions and random compact sets

We now extend the results and arguments of the previous subsection to random compact sets and random disc bodies in general dimensions. For a disc body $B \subset \mathbb{R}^d$, we put $\hat{B} := \operatorname{span}(B)$ and $B^\perp := \hat{B}^\perp$. As in the case of dilatations by random segments, we need some integrability hypothesis. If X is a random compact set in $\mathbb{R}^d$ and Y is a random disc body, then we will have to assume that

$$\mathbb{E}V_d(\operatorname{conv}(X) + tY) < \infty \tag{4.9}$$

is satisfied for some (and hence for all) $t > 0$. If Y is almost surely contained in a ball of fixed radius, then (4.9) follows from (3.2). In particular, (4.9) is satisfied if also X is almost surely contained in a ball of fixed radius. In order to be able to apply in the following proof Lemma 4.2 to the sections $(X - x) \cap \hat{Y}$, we will have to use Lemma 2.5. This forces us to impose an additional assumption concerning the relative size of these sections with respect to Y^*. For instance, if X has a deterministically bounded diameter and Y always contains a two-dimensional ball of fixed radius and centre at the origin, then this assumption is satisfied. It would be nice to be able to remove these additional assumptions, but this would probably require a completely different method of proof.

Theorem 4.4. *Let $X \subset \mathbb{R}^d$ be a random compact set, and let $Y \subset \mathbb{R}^d$ be a random disc body such that* (4.9) *is satisfied. Assume that there is a constant $r_0 > 0$ such that, with probability one, $(X - x) \cap \hat{Y} \subset r_0 Y^*$ for λ_{d-2}-a.e. $x \in Y^\perp$. Further, assume that*

$$t \mapsto \mathbb{E}V_d(X + tY), \qquad t \geq 0,$$

is a polynomial. Then, with probability one, $X \cap (x + \hat{Y})$ is convex for λ_{d-2}-a.e. $x \in Y^\perp$.

PROOF. For a compact set $A \subset \mathbb{R}^d$ and a disc body B, we abbreviate the $\hat{B}$-convex hull of A by A_B, hence

$$A_B = \bigcup_{x \in B^\perp} \operatorname{conv}(A \cap (x + \hat{B})).$$

In the following, we will compare the functions

$$\Phi(t) := \mathbb{E}V_d(X + tY), \qquad t \geq 0,$$

and

$$\Psi(t) := \mathbb{E}V_d(X_Y + tY), \qquad t \geq 0.$$

Since $X \subset X_Y$, we obtain

$$\Phi(t) \leq \Psi(t), \qquad t \geq 0. \tag{4.10}$$

We show that Ψ is a polynomial of degree at most two. By Fubini's theorem, we get

$$\begin{aligned} V_d(X_Y + tY) &= \int_{Y^\perp} V_2((X_Y + tY) \cap (x + \hat{Y}))\lambda_{d-2}(dx) \\ &= \int_{Y^\perp} V_2(\operatorname{conv}(X \cap (x + \hat{Y})) + tY)\lambda_{d-2}(dx). \end{aligned} \tag{4.11}$$

Here we used the definition of X_Y and the relation

$$(A + tY) \cap (x + \hat{Y}) = A \cap (x + \hat{Y}) + tY, \tag{4.12}$$

which holds for any compact set $A \subset \mathbb{R}^d$. Relation (4.12) will be applied repeatedly.

We denote by $V(K, M)$ the mixed area of any two-dimensional convex bodies $K, M \in \mathcal{K}^d$ which lie in parallel affine planes. (Here we use the fact that mixed areas are translation invariant.) For a compact set $A \subset \mathbb{R}^d$ and a two-dimensional convex body $B \in \mathcal{K}^d$, we define

$$\tilde{V}(A, B) := 2\int_{B^\perp} V(\operatorname{conv}(A \cap (x + \hat{B})), B)\lambda_{d-2}(dx).$$

By the multilinearity of mixed areas, we deduce from (4.11)

$$\begin{aligned} V_d(X_Y + tY) &= V_d(X_Y) + t\tilde{V}(X, Y) + t^2 V_2(Y)\int_{Y^\perp} \mathbf{1}\{X \cap (x + \hat{Y}) \neq \emptyset\}\lambda_{d-2}(dx) \\ &= V_d(X_Y) + t\tilde{V}(X, Y) + t^2 V_2(Y)\lambda_{d-2}(X|Y^\perp). \end{aligned} \tag{4.13}$$

By the measurability results established so far and by [17, p. 192-3], each term on the right-hand side of (4.13) is a measurable function. Thus, taking expected values on both sides, we find

$$\begin{aligned} \Psi(t) &= \mathbb{E}V_d(X_Y) + t\mathbb{E}\tilde{V}(X, Y) + t^2\mathbb{E}[V_2(Y)\lambda_{d-2}(X|Y^\perp)] \\ &=: b_0 + b_1 t + b_2 t^2, \qquad t \geq 0, \end{aligned}$$

with finite constants $b_i \in \mathbb{R}$, since (4.9) is satisfied. By assumption,

$$\Phi(t) = \sum_{i=0}^{m} a_i t^i, \qquad t \geq 0,$$

with $a_i \in \mathbb{R}$. From (4.10) it follows that we can choose $m \leq 2$ and that $0 \leq a_2 \leq b_2$. On the other hand,

$$\begin{aligned} \Phi(t) &= \mathbb{E}V_d(X + tY) \\ &= \mathbb{E}\int_{Y^\perp} V_2((X + tY) \cap (x + \hat{Y}))\lambda_{d-2}(dx) \\ &\geq \mathbb{E}\int_{Y^\perp} V_2(tY)\mathbf{1}\{X \cap (x + \hat{Y}) \neq \emptyset\}\lambda_{d-2}(dx) \\ &= t^2\mathbb{E}[V_2(Y)\lambda_{d-2}(X|Y^\perp)] = t^2 b_2. \end{aligned}$$

Therefore, we also have $a_2 \geq b_2$, and thus $a_2 = b_2$. But then (4.10) yields that $a_1 \leq b_1$. Furthermore, for all $t > 0$,

$$\frac{d}{dt}\Phi(t) = a_1 + 2a_2 t \leq b_1 + 2b_2 t = \frac{d}{dt}\Psi(t). \tag{4.14}$$

Our next aim is to derive a converse estimate to (4.14). First, however, we have to discuss some measurability issues. For a closed set $A \in \mathcal{F}^2$ and a Borel measurable function $f : \mathbb{R}^2 \to [0, \infty)$, we define

$$\int_{\partial A} f(z)\mathcal{H}^1(dz) := 0$$

if $\mathcal{H}^1(\partial A) = \infty$. By Theorem 2.1.3 in [21], the set $\{A \in \mathcal{F}^2 : \mathcal{H}^1(\partial A) < \infty\}$ is Borel measurable. With this convention, another application of Theorem 2.1.3 in [21] yields that the map $\mathcal{F}^2 \to [0, \infty)$, $A \mapsto \mathcal{H}^1(\partial A \cap C)$, is Borel measurable for any compact set $C \subset \mathbb{R}^d$. Arguing as in [17, p. 192-3] and by the considerations preceding Proposition 2.8, we find that the map

$$(\omega, x, t) \mapsto \int_{\partial((X(\omega)-x)\cap\hat{Y}(\omega)+tY(\omega))} h(Y(\omega), \nu_{Y(\omega)}((X(\omega) - x) \cap \hat{Y}(\omega), z))\mathcal{H}^1(dz)$$

is Borel measurable on $\Omega \times \mathbb{R}^d \times (0, \infty)$, where Ω is the underlying sample space. For a function $g : [0, \infty) \to [0, \infty)$, we define

$$\frac{d}{dt}^{*} g(t) := \liminf_{n\to\infty} n(g(t + n^{-1}) - g(t)).$$

Note that this definition is related to the lower-right derivative of g at t. Now, if $(\omega, x) \in \Omega \times Y(\omega)^{\perp}$ is fixed, then Proposition 2.8 shows that

$$\frac{d}{dt}^{*} V_2((X - x) \cap \hat{Y} + tY) = \int_{\partial((X-x)\cap\hat{Y}+tY)} h(Y, \nu_Y((X - x) \cap \hat{Y}, z))\mathcal{H}^1(dz) \tag{4.15}$$

for $\mathcal{H}^1$-a.e. $t > 0$. Moreover, both sides of (4.15) are Borel measurable functions of (ω, x, t). Hence, Fubini's theorem implies that (4.15) holds for λ_{d-2}-a.e. $x \in Y^{\perp}$, $\mathbb{P}$-a.s. and for $\mathcal{H}^1$-a.e. $t > 0$. For any such $t > 0$, Fatou's lemma and (4.15) yield that

$$\begin{aligned}
\frac{d}{dt}\Phi(t) &= \lim_{n\to\infty} n\mathbb{E}[V_d(X + (t + n^{-1})Y) - V_d(X + tY)] \\
&= \lim_{n\to\infty} \mathbb{E} \int_{Y^{\perp}} n[V_2((X + (t + n^{-1})Y) \cap (x + \hat{Y})) \\
&\qquad\qquad - V_2((X + tY) \cap (x + \hat{Y}))]\lambda_{d-2}(dx) \\
&= \lim_{n\to\infty} \mathbb{E} \int_{Y^{\perp}} n[V_2((X - x) \cap \hat{Y} + (t + n^{-1})Y) \\
&\qquad\qquad - V_2((X - x) \cap \hat{Y} + tY)]\lambda_{d-2}(dx) \\
&\geq \mathbb{E} \int_{Y^{\perp}} \frac{d}{dt}^{*} V_2((X - x) \cap \hat{Y} + tY)\lambda_{d-2}(dx) \\
&= \mathbb{E} \int_{Y^{\perp}} \int_{\partial((X-x)\cap\hat{Y}+tY)} h(Y, \nu_Y((X - x) \cap \hat{Y}, z))\mathcal{H}^1(dz).
\end{aligned} \tag{4.16}$$

In addition, for any $t > 0$ we obtain from the multilinearity of mixed areas

$$\begin{aligned}
\frac{d}{dt}\Psi(t) &= \lim_{n\to\infty} n\mathbb{E}[V_d(X_Y + (t+n^{-1})Y) - V_d(X_Y + tY)] \\
&= \lim_{n\to\infty} \mathbb{E}\int_{Y^\perp} n\big[V_2((X_Y - x)\cap\hat{Y} + (t+n^{-1})Y) \\
&\qquad\qquad - V_2((X_Y - x)\cap\hat{Y} + tY)\big]\lambda_{d-2}(dx) \\
&= \lim_{n\to\infty} \mathbb{E}\int_{Y^\perp} \big[2V((X_Y - x)\cap\hat{Y} + tY, Y) \\
&\qquad\qquad + n^{-1}V_2(Y)\mathbf{1}\{(X_Y - x)\cap\hat{Y} \neq \emptyset\}\big]\lambda_{d-2}(dx) \\
&= \mathbb{E}\int_{Y^\perp}\int_{\partial(\mathrm{conv}((X-x)\cap\hat{Y})+tY)} h(Y, \nu_Y(\mathrm{conv}((X-x)\cap\hat{Y}), z))\mathcal{H}^1(dz)\lambda_{d-2}(dx).
\end{aligned} \tag{4.17}$$

Let $r_0 > 0$ be such that $(X-x)\cap\hat{Y} \subset r_0 Y^*$ is satisfied for λ_{d-2}-a.e. $x \in Y^\perp$ and $\mathbb{P}$-a.s. Let us choose $t > r_0$. In view of Lemma 2.5 we can apply Lemma 4.2 with $A = (X-x)\cap\hat{Y} + tY$ and $B = Y$ whenever $(X-x)\cap\hat{Y} \neq \emptyset$. Using Fubini's theorem and Proposition 2.8, we hence get, for λ_{d-2}-a.e. $x \in Y^\perp$, $\mathbb{P}$-a.s. and for $\mathcal{H}^1$-a.e. $t > r_0$,

$$\begin{aligned}
&\int_{\partial(\mathrm{conv}((X-x)\cap\hat{Y})+tY)} h(Y, \nu_Y(\mathrm{conv}((X-x)\cap\hat{Y}), z))\mathcal{H}^1(dz) \\
&\qquad \le \int_{\partial((X-x)\cap\hat{Y}+tY)} h(Y, \nu_Y((X-x)\cap\hat{Y}, z))\mathcal{H}^1(dz),
\end{aligned} \tag{4.18}$$

with equality if and only if $(X-x)\cap\hat{Y} + tY$ is convex. Combining (4.16), (4.17) and (4.18), we obtain

$$\frac{d}{dt}\Psi(t) \le \frac{d}{dt}\Phi(t), \tag{4.19}$$

for $\mathcal{H}^1$-a.e. $t > r_0$. From (4.14) and (4.19), we conclude that (4.19) indeed holds with equality, for $\mathcal{H}^1$-a.e. $t > r_0$. Therefore equality must hold in (4.18), for λ_{d-2}-a.e. $x \in Y^\perp$, $\mathbb{P}$-a.s. and for $\mathcal{H}^1$-a.e. $t > r_0$. Hence, $(X-x)\cap\hat{Y} + tY$ is convex, for λ_{d-2}-a.e. $x \in Y^\perp$, $\mathbb{P}$-a.s. and for $\mathcal{H}^1$-a.e. $t > r_0$. The same conclusion is then also available for all $t > r_0$.

Next we prove the convexity of $(X-x)\cap\hat{Y} + tY$, for all $t > 0$, λ_{d-2}-a.e. $x \in Y^\perp$, and $\mathbb{P}$-a.s. This implies the assertion of the theorem. In fact, for $t > r_0$,

$$\begin{aligned}
a_0 + a_1 t + a_2 t^2 = \Phi(t) &= \mathbb{E}\int_{Y^\perp} V_2((X-x)\cap\hat{Y} + tY)\lambda_{d-2}(dx) \\
&= \mathbb{E}\int_{Y^\perp} V_2(\mathrm{conv}((X-x)\cap\hat{Y}) + tY)\lambda_{d-2}(dx) \\
&= \Psi(t) = b_0 + b_1 t + b_2 t^2.
\end{aligned}$$

Since two polynomials, which are equal for $t > r_0$, must be equal for all $t \ge 0$, we infer that

$$V_2((X-x)\cap\hat{Y} + tY) = V_2(\mathrm{conv}((X-x)\cap\hat{Y}) + tY), \tag{4.20}$$

for λ_{d-2}-a.e. $x \in Y^\perp$, $\mathbb{P}$-a.s. and for $t \in (0,\infty)\cap\mathbb{Q}$. The convex set $\mathrm{conv}((X-x)\cap\hat{Y}) + tY$ is the closure of its interior (for $t > 0$), the set $(X-x)\cap\hat{Y} + tY$ is compact, and therefore (4.20) implies the required assertion. □

As a consequence of Theorem 4.4, we obtain the following theorem. Instead of a general random disc body Y, we now consider a random disc body which is generated by randomly rotating (independently of the random compact set X) a fixed disc body B. We also emphasize that for $d = 2$ the regularity assumption on the random set X can be omitted in the following two results. Moreover, for $d = 2$ similar results hold without taking rotational averages.

Theorem 4.5. *Let $d \geq 2$, let $X \subset \mathbb{R}^d$ be a random regular compact set, and let $B \subset \mathbb{R}^d$ be a disc body. Assume that $X \subset r_0 B^d$ almost surely, for some $r_0 > 0$. Further, assume that ν is a finite measure on SO_d dominating the Haar measure such that*

$$t \mapsto \mathbb{E} \int_{SO_d} V_d(X + t\vartheta B)\nu(d\vartheta), \qquad t \geq 0,$$

is a polynomial. Then X is almost surely convex.

PROOF. Let ζ be a random rotation that is independent of X and whose distribution is the normalized ν. It is easy to check that the assumptions of Theorem 4.4 are fulfilled. The required result then follows from Theorem 4.4 applied to $Y := \zeta B$ and a straigthforward approximation argument. □

Again we consider the special case of a deterministic compact set A.

Corollary 4.6. *Let $d \geq 2$, let $A \subset \mathbb{R}^d$ be a regular compact set, and let $B \subset \mathbb{R}^d$ be a disc body. Assume that*

$$t \mapsto \int_{SO_d} V_d(A + t\vartheta B)\nu(d\vartheta), \qquad t \geq 0,$$

is a polynomial. Then A is convex.

5 Characterization of convexity in Boolean models

In this section, we consider a stationary Boolean model

$$Z = \bigcup_{n \in \mathbb{N}} (Z_n + \xi_n),$$

where the ξ_n, $n \in \mathbb{N}$, form a stationary Poisson process in $\mathbb{R}^d$ with positive and finite intensity γ and where the grains $Z_1, Z_2, \dots$ form a sequence of independent, identically distributed random elements in $\mathcal{C}^d$ which is independent of $\{\xi_n : n \in \mathbb{N}\}$. We assume that

$$\mathbb{E}V_d(\mathrm{conv}(X) + B^d) < \infty, \tag{5.1}$$

where X denotes a typical grain. Assumption (5.1) guarantees that each compact set is intersected by only a finite number of the (shifted) grains $Z_n + \xi_n$, $n \in \mathbb{N}$. Hence Z is indeed a random closed set (see [12] and [19] for more details). For the latter result, a weaker condition than (5.1) would be sufficient, but we shall need (5.1) later when we apply Theorem 3.1. The *capacity functional* of the Boolean model is given by

$$\mathbb{P}(Z \cap C \neq \emptyset) = 1 - \exp\left[-\gamma \mathbb{E}V_d(X + C^*)\right], \quad C \in \mathcal{C}^d. \tag{5.2}$$

By (5.1), this function is strictly less than 1. In particular, we obtain for the *volume fraction* $p := \mathbb{P}(0 \in Z)$ that

$$p = 1 - \exp\left[-\gamma \mathbb{E} V_d(X)\right] < 1. \tag{5.3}$$

We consider the contact distribution function H_B as given by (1.1), with a structuring element $B \in \mathcal{K}^d$ containing the origin. Equations (5.2) and (5.3) imply the representation (1.2). If X is almost surely convex, then H_B is given by (1.3).

5.1 Linear sections

We take some $u \in \mathbb{S}^{d-1}$ and consider the linear contact distribution $H_{[0,u]}$ of the Boolean model Z. If the typical grain X is almost surely convex, we obtain from (1.3) that

$$H_{[0,u]}(r) = 1 - \exp\left\{-\frac{\gamma r}{2} \int_{\mathbb{S}^{d-1}} |\langle u, v\rangle| \bar{S}_{d-1}(dv)\right\}, \tag{5.4}$$

where

$$\bar{S}_{d-1} := \mathbb{E} S_{d-1}(X, \cdot)$$

is the mean surface area measure of the typical grain. More generally, in case $X = X_u$ holds almost surely, (3.4) implies that

$$H_{[0,u]}(r) = 1 - \exp\left\{-\gamma r \mathbb{E} \lambda_{d-1}(X | u^\perp)\right\}.$$

This is an exponential distribution whose parameter is determined by the intensity γ and – in the convex case – by the *cosine transform* of $\bar{S}_{d-1}$. If X is *isotropic* (i.e. distributionally invariant under rotations), then $H_l := H_{[0,u]}$ is independent of u. For a convex typical grain X, we then obtain

$$H_l(r) = 1 - \exp\left\{-\gamma r \frac{2\kappa_{d-1}}{d\kappa_d} \mathbb{E} V_{d-1}(X)\right\}.$$

For a general stationary Boolean model, we consider the LLC-function $L(r)$ defined by (1.4). If X is convex, (5.4) implies that

$$L(r) = \frac{\kappa_{d-1}}{d\kappa_d} \lambda_{d-1} r, \quad r \geq 0,$$

where $\lambda_{d-1} := \gamma \bar{S}_{d-1}(\mathbb{S}^{d-1})$ is the *surface area density* of the particle process $\{Z_n + \xi_n : n \in \mathbb{N}\}$,

In view of (1.2) we can apply Theorem 3.2 to obtain Theorem 1.1. Actually, Theorem 3.1 provides the following more detailed information.

Theorem 5.1. *Assume that the typical grain X of the stationary Boolean model Z satisfies* (5.1), *and let $u \in \mathbb{S}^{d-1}$. Assume that the linear contact distribution function $H_{[0,u]}$ is exponential. Then, with probability one, $X \cap (\hat{u} + x)$ is a segment for λ_{d-1}-a.e. $x \in u^\perp$.*

Theorem 5.1 is a purely one-dimensional result making an assertion about the section of Z with a line. For a more detailed discussion of this theorem and related issues, we assume that the Boolean model Z has a regular typical grain and fix some $u \in S^{d-1}$. For any $x \in \partial Z \cap \hat{u}$ we define the length of the *external chord* starting at x by

$$\zeta(x) := \inf\{t > 0 : x + tu \in Z\}.$$

We assume that $N := \{x \in \partial Z \cap \hat{u} : \zeta(x) > 0\}$ is almost surely locally finite. Hence N is a stationary point process in $\hat{u}$ and we assume in addition that N has a finite intensity.

We now consider the *mark distribution* of the stationary marked point process $\{(x, \zeta(x)) : x \in N\}$ and its associated distribution function C_u, describing the statistics of a *typical* external chord in direction u. By a classical point process argument (see e.g. [19]) we have that

$$H_{[0,u]}(r) = \frac{1}{m_u} \int_0^r (1 - C_u(s))ds, \quad r \geq 0, \tag{5.5}$$

where m_u is the mean of C_u. This mean is finite since it is less than the reciprocal of the (positive) intensity of N. In particular $H_{[0,u]}$ is exponential if and only if C_u is exponential. Therefore Theorem 5.1 implies the following result.

Theorem 5.2. *Assume that the typical grain X of the stationary Boolean model Z satisfies* (5.1). *Let $u \in \mathbb{S}^{d-1}$ and assume that $\{x \in \partial Z \cap \hat{u} : \zeta(x) > 0\}$ is almost surely locally finite and of finite intensity. If the external chord length distribution function C_u in direction u is exponential, then, with probability one, $X \cap (\hat{u} + x)$ is a segment for λ_{d-1}-a.e. $x \in u^\perp$.*

It can easily be shown that

$$Z \cap \hat{u} = \bigcup_{n \in \mathbb{N}} (Z'_n + \xi'_n),$$

where the pairs (ξ'_n, Z'_n), $n \in \mathbb{N}$, form an independently marked Poisson process (with points in $\hat{u}$ and compact subsets of $\hat{u}$ as marks) such that $(Z_n + \xi_n) \cap \hat{u} = (Z'_n + \xi'_n)$. Hence $Z \cap \hat{u}$ is a Boolean model in $\hat{u}$. We may interpret this model as an infinite server system with $\hat{u}$ denoting the time axis (cf. [11]). A customer arriving at epoch ξ'_n requires service during the time epochs covered by the random set Z'_n. All customers that are in the system are being served with rate 1. In contrast to the classical case the sets Z'_n need not be intervals. Instead the service of a customer can be *interrupted* several times. The complement of Z (in $\hat{u}$) can be written as countable union of successive *idle times*. Under the assumptions of Theorem 5.2, the length of a typical idle time is exponentially distributed if and only if the service of the individual customers is never interrupted.

While the above relationships between the idle times of an infinite server (with a Poisson input stream and independent and i.i.d. service periods) and a one-dimensional Boolean model are of course well-known, we are not aware of a result characterizing connected service periods via exponential idle times. Apparently, Theorem 1.1 does not seem to allow a queueing interpretation.

5.2 Planar sections

We take some disc body $B \subset \mathbb{R}^d$ and consider the contact distribution H_B of the Boolean model Z. If the typical grain X is almost surely convex, then we can use (4.13) to deduce from (1.2) that

$$H_B(r) = 1 - \exp\left\{-\gamma r \mathbb{E}\tilde{V}(X, B^*) - \gamma r^2 V_2(B) \mathbb{E}\lambda_{d-2}(X|B^\perp)\right\}. \tag{5.6}$$

Actually this formula remains true if we merely assume that $X = X_B$ almost surely.

Returning to the case of a general typical grain, we now consider the ALDC-function of Z as given by (1.5). In view of (1.2), Theorem 4.5 implies Theorem 1.2, while Theorem 4.4 yields the following more detailed result.

Theorem 5.3. *Assume that the typical grain X of the stationary Boolean model Z has a deterministically bounded diameter. Let $B \subset \mathbb{R}^d$ be some disc body such that $\ln(1 - H_B)$ is a polynomial. Then, with probability one, $X \cap (\hat{B} + x)$ is convex for λ_{d-2}-almost all $x \in B^\perp$.*

5.3 Some examples

We consider a stationary point process $M := \{\xi_n : n \in \mathbb{N}\}$ in $\mathbb{R}^d$. Furthermore, let $Z_1, Z_2, \ldots$ be independent, identically distributed non-empty random compact sets which are also independent of M. If the particle process $\{Z_n + \xi_n : n \in \mathbb{N}\}$ is locally finite, then

$$Z := \bigcup_{n \in \mathbb{N}} (Z_n + \xi_n)$$

is a random closed set that is called a *germ-grain model*. The statistical properties of a general germ-grain model are complicated. Explicit analytic formulas are almost never available, even for the most simple characteristics such as volume fraction or mean surface area. An important exception is the Boolean model, where M is a Poisson process. If X (a typical grain with the distribution of Z_1) is convex, then a common tool for checking a Boolean hypothesis is to use the empirical contact distribution functions $\hat{H}_B$ for suitable gauge bodies B (see [14], [19, 3.3]). According to (1.3), plotting $r \mapsto \ln(1 - \hat{H}_B(r))$ should approximately yield a polynomial of degree d with vanishing absolute term.

As we will show by means of examples, one has to be careful when applying this method. Our first example is a germ-grain model on the line with all contact distributions being exponential (as in the Boolean model) but having a lattice process of germs far away from a Poisson process.

Example 5.4. Let $Y \geq 0$ and $U \geq 0$ be independent random variables where Y has density $f(x) = xe^{-x}$, $x \geq 0$, and U is uniformly distributed on $[0, 1]$. Then $M := \{(U + k)Y : k \in \mathbb{Z}\}$ is a stationary point process on the line. A straightforward calculation shows that UY has density $t \mapsto \int_t^\infty f(x)/x dx = e^{-t}$. Hence, for any $a \in \mathbb{R}$ and $r > 0$,

$$\mathbb{P}(M \cap [a, a + r] = \emptyset) = \mathbb{P}(M \cap [0, r] = \emptyset) = \mathbb{P}(UY > r) = e^{-r}.$$

These are the same probabilities as in case of a Poisson process of unit intensity.

We now consider the germ-grain model Z based on M and the deterministic typical grain $X := [0, 1]$. Then we have for any interval $[a, b]$ with $b - a = 1$ and $0 \in [a, b]$ and any $r \geq 0$ that

$$\mathbb{P}(Z \cap [ra, rb] = \emptyset \,|\, 0 \notin Z) = \frac{\mathbb{P}(Z \cap [ra, rb] = \emptyset)}{\mathbb{P}(0 \notin Z)} = \frac{\mathbb{P}(Z \cap [0, r] = \emptyset)}{\mathbb{P}(0 \notin Z)}$$
$$= \frac{\mathbb{P}(M \cap [-1, r] = \emptyset)}{\mathbb{P}(M \cap [-1, 0] = \emptyset)} = e^{-r}.$$

Hence all contact distribution functions of Z are exponential.

There are other examples of stationary point processes M on the line (closer to the Poisson process) so that $M(I)$ is even Poisson distributed for all intervals I. Moran (see [15]) has found a planar process M such that $M(L)$ is Poisson distributed for all compact, convex sets $L \subset \mathbb{R}^2$:

Example 5.5. We consider the process M constructed in [15]. This is a stationary point process in the plane that is not Poisson and such that, for any $L \in \mathcal{K}^2$, $M(L)$ is Poisson distributed with parameter $V_2(L)$. We consider the germ-grain grain model Z based on M and a deterministic typical grain $K \in \mathcal{K}^2$. Then, for any $B \in \mathcal{K}^2$ with $0 \in B$ and any $r \geq 0$,

$$\begin{aligned}\mathbb{P}(0 \notin Z + rB^*) &= \mathbb{P}(0 \in M + (K + rB^*)) = \mathbb{P}(M \cap (K^* + rB) = \emptyset) \\ &= \exp[-V_2(K + rB^*)] = \exp[-V_2(K) - 2rV(K, B^*) - r^2 V_2(B)].\end{aligned}$$

Therefore

$$1 - H_B(r) = \mathbb{P}(0 \notin Z + rB^* \,|\, 0 \notin Z) = \exp[-2rV(K, B^*) - r^2 V_2(B)],$$

which coincides with the corresponding result for a Boolean model with typical grain K. This example can be generalized to any dimension.

Moran's example has still much in common with a Poisson process. Our final example is a stationary lattice germ process M such that a germ-grain model with deterministic spherical grains has a spherical contact distribution function that is of the same form as in a Boolean model with the same typical grain.

Example 5.6. Let Y_1, Y_2, U_1 and U_2 be independent non-negative random variables. Assume that U_1, U_2 are uniformly distributed on $[0, 1]$ and that Y_1, Y_2 have density $f(x) = 4\pi^{-1/2}x^2 e^{-x^2}$, $x \geq 0$. We consider the stationary point process

$$M := \{((U_1 + m)Y_1, (U_2 + n)Y_2) : m, n \in \mathbb{Z}\}.$$

We would like to compute the probability of $\{M \cap rB^2 = \emptyset\}$ for any $r \geq 0$.

On the event $\{U_1 \geq 1/2, U_2 \geq 1/2\}$ we have $M \cap rB^2 = \emptyset$ if and only if

$$(1 - U_1)^2 Y_1^2 + (1 - U_2)^2 Y_2^2 > r^2.$$

Hence

$$\begin{aligned}&\mathbb{P}(M \cap rB^2, U_1 \geq 1/2, U_2 \geq 1/2) \\ &\quad = \mathbb{P}((1 - U_1)^2 Y_1^2 + (1 - U_2)^2 Y_2^2 > r^2, U_1 \geq 1/2, U_2 \geq 1/2) \\ &\quad = \frac{1}{4}\mathbb{P}(U_1^2 W_1^2 + U_1^2 W_1^2 > r^2),\end{aligned}$$

where $W_i := Y_i/2$, $i = 1, 2$. The random variables $U_i W_i$ have density

$$t \mapsto \int_t^\infty 2f(2x)/x dx = \frac{32}{\sqrt{\pi}} \int_t^\infty x \exp[-4x^2] dx = \frac{4}{\sqrt{\pi}} \exp[-4t^2].$$

This is the density of $|V|/\sqrt{8}$, where V is a standard normal random variable. From the well-known convolution property of Gamma distributions, we now obtain $\mathbb{P}(U_1^2 W_1^2 + U_2^2 W_2^2 > t) =$

e^{-4t} for all $t \geq 0$. Treating the other cases (for instance $U_1 \geq 1/2$ and $U_2 \leq 1/2$) in the same way, we finally obtain that

$$\mathbb{P}(M \cap rB^2 = \emptyset) = e^{-4r^2}, \quad r \geq 0,$$

just as in case of a Poisson process of rate $4/\pi$.

We now consider the germ-grain model Z based on M and the deterministic typical grain B^d. Then

$$1 - H_{B^d}(r) = \frac{\mathbb{P}(0 \notin M + (r+1)B^d)}{\mathbb{P}(0 \notin M + B^d)} = e^{-4(r+1)^2+4} = e^{4r^2-8r},$$

as in a Boolean model.

The above examples reveal problems that may arise in using contact distributions to identify a Boolean model within germ-grain models with convex (deterministic) grains. However, the results of the present paper show that contact distribution functions could be used to identify Boolean models with *convex* typical grains within the much larger class of Boolean models with *compact* typical grains. If the empirical logarithmic distribution functions $\ln(1 - \hat{H}_{[0,u]})$, $u \in \mathbb{S}^{d-1}$, are approximately linear for all directions u within a preferably large finite set, then there is no reason to reject a convexity hypothesis. The same can be said if $\ln(1 - \hat{H}_{\vartheta B})$ is approximately quadratic for some disc body B and for all rotations ϑ within a preferably large finite set. Both ideas are in good agreement with the fact that in many applications only linear or planar sections of a three-dimensional set are available.

References

[1] K. Böröczky, I. Bárány, E. Makai Jr., J. Pach, Maximal volume enclosed by plates and proof of the chessboard conjecture, *Discrete Math.* **60** (1986), 101–120.

[2] L.C. Evans, R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, 1992.

[3] K.J. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, Cambridge, 1985.

[4] I. Fáry, E. Makai Jr., Isoperimetry in variable metric, *Stud. Sci. Math. Hungar.* **17** (1982), 143–158.

[5] H. Federer, *Geometric Measure Theory*, Springer, Berlin, 1969.

[6] P. Hall, *Introduction to the Theory of Coverage Processes*, Wiley, New York, 1988.

[7] M.B. Hansen, A.J. Baddeley, R.D. Gill, First contact distributions for spatial patterns: regularity and estimation, *Adv. in Appl. Probab.* **31** (1999), 15–33.

[8] M. Heveling, D. Hug, G. Last, Does polynomial parallel volume imply convexity? *Math. Ann.* **328** (2004), 469–479.

[9] D. Hug, G. Last, On support measures in Minkowski spaces and contact distributions in stochastic geometry, *Ann. Probab.* **28** (2000), 796–850.

[10] D. Hug, G. Last, W. Weil, A survey on contact distributions. 317–357. *Statistical Physics and Spatial Statistics*, Lecture Notes in Physics **600**, Morphology of Condensed Matter, Physics and Geometry of Spatially Complex Systems, ed. by K. Mecke and D. Stoyan, Springer, Berlin, 2002.

[11] J.F.C. Kingman, *Poisson Processes*, Oxford Studies in Probability, Clarendon Press, Oxford, 1993.

[12] G. Matheron, *Random Sets and Integral Geometry*, John Wiley, New York, 1975.

[13] R. Meester, R. Roy, *Continuum Percolation*. Cambridge University Press, Cambridge, 1996.

[14] I. Molchanov, *Statistics of the Boolean model for Practitioners and Mathematicians*, Wiley, Chichester, 1997.

[15] P.A.P. Moran, A quasi-Poisson point process in the plane, *Bull. London Math. Soc.* **8** (1976), 69–70.

[16] R. Schneider, *Convex Bodies: the Brunn-Minkowski Theory*, Cambridge University Press, 1993.

[17] R. Schneider, W. Weil, *Integralgeometrie*, Teubner, Stuttgart, 1992.

[18] R. Schneider, W. Weil, *Stochastische Geometrie*, Teubner, Stuttgart, 2000.

[19] D. Stoyan, W.S. Kendall, and J. Mecke, *Stochastic Geometry and its Applications*, 2nd edn., John Wiley, New York, 1995.

[20] W. Weil, Support functions on the convex ring in the plane and support densities for random sets and point processes, *Rend. Circ. Mat. Palermo (2), Suppl.*, **35** (1994), 323–344.

[21] M. Zähle, Random processes of Hausdorff rectifiable closed sets, *Math. Nachr.* **108** (1982), 49–72.

[22] W.P. Ziemer, *Weakly Differentiable Functions*, Springer, New York, 1989.

Authors' addresses:

Daniel Hug, Mathematisches Institut, Universität Freiburg, Eckerstr. 1, D-79104 Freiburg, Germany, e-mail: daniel.hug@math.uni-freiburg.de

Günter Last, Institut für Mathematische Stochastik, Universität Karlsruhe (TH), Englerstr. 2, D-76128 Karlsruhe, Germany, e-mail: last@math.uni-karlsruhe.de

Wolfgang Weil, Mathematisches Institut II, Universität Karlsruhe (TH), Englerstr. 2, D-76128 Karlsruhe, Germany, e-mail: weil@math.uni-karlsruhe.de