Distributional properties of Poisson Voronoi tessellations

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1. Voronoi tessellations

Definition:

(i) The space of all *point configurations* in $\mathbb{R}^d$ is defined as

$$\mathbf{N} := \{\varphi \subset \mathbb{R}^d : \varphi \text{ is locally finite}\}.$$

(ii) Any $\varphi \in \mathbf{N}$ is identified with a counting measure:

$$\varphi(B) := \mathrm{card}\{x \in \varphi : x \in B\}, \quad B \subset \mathbb{R}^d.$$

(iii) The σ-field $\mathcal{N}$ is the smallest σ-field of subsets of $\mathbf{N}$ making the mappings $\varphi \mapsto \varphi(B)$ for all Borel sets $B \subset \mathbb{R}^d$ measurable.

Definition: The points of $\varphi \in \mathbf{N}$ are in *general quadratic position* if the following two conditions are satisfied.

(i) Any $k \in \{2, \ldots, d+2\}$ points of φ are in general position.

(ii) No $d+2$ points of φ lie on the boundary of some ball.

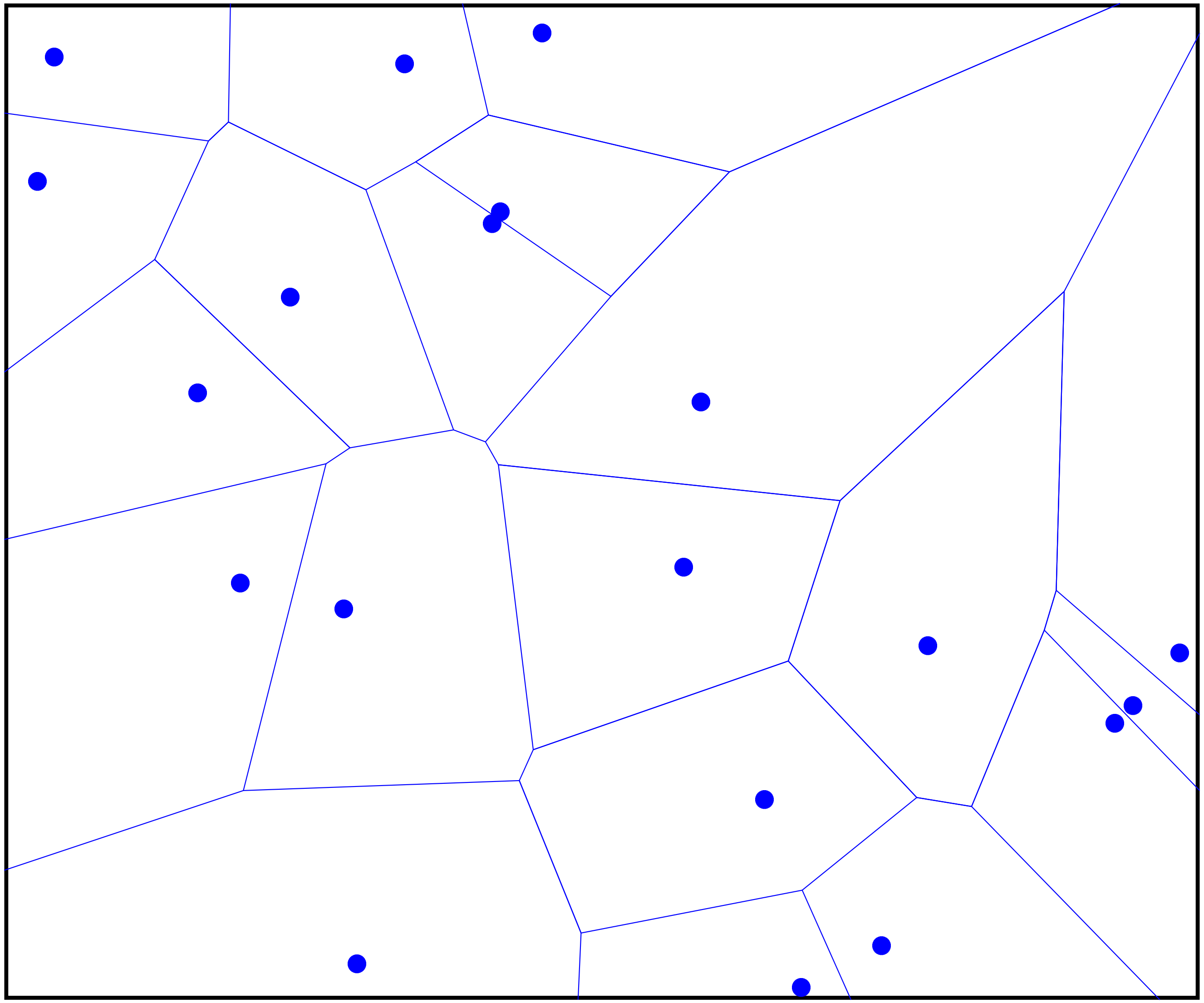
Definition: Let $\varphi \in \mathbf{N}$.

(i) The *Voronoi cell* $C(\varphi, x)$ of $x \in \varphi$ is the set of all sites $y \in \mathbb{R}^d$ whose distance from x is smaller or equal than the distances to all other points of φ.

(ii) The *Voronoi tessellation* based on φ is the system

$$\mathcal{S}_d(\varphi) := \{C(\varphi, x) : x \in \varphi\}.$$

Remark: If the convex hull of φ coincides with $\mathbb{R}^d$, then all Voronoi cells are bounded and $\mathcal{S}_d(\varphi)$ is a *face-to face* tessellation. Moreover, if the point of φ are in general quadratic position, then $\mathcal{S}_d(\varphi)$ is also *normal* in the sense that any k-face is contained in exactly $d-k+1$ cells.



Definition: Let C be a convex polytope. Then

$$C = \bigcup_{k\in\{0,\dots,d\}} \bigcup_{C\in\mathcal{S}_k(C)} \operatorname{relint} F,$$

where $\mathcal{S}_k(C)$ is a finite set of k-dimensional polytopes whose affine hulls are pairwise not equal. A polytope $F \in \mathcal{S}_k(C)$ is called a *k-face* of C.

Definition: Let $\varphi \in \mathbf{N}$ and $k \in \{0, \dots, d\}$. The system of all k-faces of the Voronoi tessellation $\mathcal{S}_d(\varphi)$ is defined by

$$\mathcal{S}_k(\varphi) := \bigcup_{C\in\mathcal{S}_d(\varphi)} \mathcal{S}_k(C).$$

2. Stationary point processes and random measures

Definition:

(i) For any $x \in \mathbb{R}^d$ the shift $\theta_x : \mathbf{N} \to \mathbf{N}$ is defined by

$$\theta_x \varphi = \varphi - x.$$

(ii) A probability measure $\mathbb{P}$ on $(\mathbf{N}, \mathcal{N})$ is *stationary* if

$$\mathbb{P} \circ \theta_x = \mathbb{P}, \quad x \in \mathbb{R}^d.$$

Assumption: $\mathbb{P}$ is a stationary probability measure on $(\mathbf{N}, \mathcal{N})$.

Definition:

(i) $\mathbf{M}$ denotes the space of all locally finite measures on $\mathbb{R}^d$.

(ii) The σ-field $\mathcal{M}$ is the smallest σ-field of subsets of $\mathbf{M}$ making the mappings $\alpha \mapsto \alpha(B)$ for all Borel sets $B \subset \mathbb{R}^d$ measurable.

(iii) A random measure M is a measurable mapping from $\mathbf{N}$ to $\mathbf{M}$.

(iv) A random measure M is *stationary* if

$$M(\varphi, B + x) = M(\theta_x \varphi, B), \quad \varphi \in \mathbf{M}, x \in \mathbb{R}^d, B \in \mathcal{B}^d.$$

Remark: The identity N on $\mathbf{N}$ is a stationary random measure.

Definition: Let M be a stationary random measure.

(i) The *intensity* of M is the number

$$\lambda_M := \mathbb{E}[M([0,1]^d)].$$

(ii) If λ_M is positive and finite, then

$$\mathbb{P}^0_M(A) := \frac{1}{\lambda_M}\mathbb{E}\left[\int \mathbf{1}\{\theta_x N \in A, x \in [0,1]^d\}\, M(dx)\right], \quad A \in \mathcal{N},$$

is called *Palm probability measure* of M.

3. Typical faces

Assumption: $\mathbb{P}$ is a stationary probability measure on $(\mathbf{N}, \mathcal{N})$ such that almost all $\varphi \in \mathbf{N}$ are non-empty and the points of almost all $\varphi \in \mathbf{N}$ are in general quadratic position. We consider the (random) Voronoi tessellation

$$\mathcal{S}_d(N) = \{C(N, x) : x \in N\}$$

generated by N.

Definition: Let $k \in \{0, \dots, d\}$ and $F \in \mathcal{S}_k$. Take some y in the relative interior of F and assume that the points of N are in general position. Then there are exactly $d - k + 1$ different points $X_0, \dots, X_{d-k} \in N$ (the *neighbours of* F) such that the distances $R_y := \|X_i - y\|$ are the same for all i and such that the open ball with centre y and radius R_y does not contain any point of N. Let $\pi_k(F)$ denote the centre of the unique $(d-k)$-dimensional ball in the affine hull of the neighbours containing the neighbours on its boundary. Define the stationary point process of centres of k-faces by

$$N_k := \{\pi_k(F) : F \in \mathcal{S}_k(N)\}.$$

Assumption: For any $k \in \{0, \dots, d\}$ the intensity

$$\lambda_k := \mathbb{E}[N_k([0,1]^d)]$$

is assumed to be finite.

Remark: We have a.s. that $N = N_d$ and hence $\lambda_d = \lambda$.

Definition: Let $k \in \{0, \dots, d\}$. Under the Palm probability measure $\mathbb{P}^0_{N_k}$ we denote by $C_k \in \mathcal{S}_k(N)$ the k-face satisfying $\pi(C_k) = 0$. The distribution

$$\mathbb{P}^0_{N_k}(C_k \in \cdot)$$

is the distribution of the *typical* k-face of the Voronoi tessellation based on N.

Definition: For any $k \in \{0, \dots, d\}$ we define the stationary random measure

$$M_k := \sum_{F \in \mathcal{S}_k(N)} \mathcal{H}^k(F \cap \cdot),$$

where $\mathcal{H}^k$ denotes k-dimensional Hausdorff measure on $\mathbb{R}^d$.

Assumption: For any $k \in \{0, \dots, d\}$ the intensity

$$\mu_k := \mathbb{E}[M_k([0,1]^d)]$$

is assumed to be finite.

Remark: We have $M_0 = N_0$ and hence $\lambda_0 = \mu_0$.

Definition: Let $k \in \{0, \dots, d\}$. Under the Palm probability measure $\mathbb{P}^0_{M_k}$ we denote by $F_k \in \mathcal{S}_k(N)$ the k-face satisfying $0 \in F_k$. The distribution

$$\mathbb{P}^0_{M_k}(F_k \in \cdot)$$

can be interpreted as an *area-biased* version of the distribution of the *typical* k-face.

Proposition: *Consider $k \in \{0, \dots, d\}$ and a measurable and shift-invariant function $g : \mathbf{N} \to [0, \infty)$. Then*

$$\mu_k \mathbb{E}^0_{M_k}[g] = \lambda_k \mathbb{E}^0_{N_k}\left[\mathcal{H}^k(C_k) \cdot g\right],$$

$$\lambda_k \mathbb{E}^0_{N_k}[g] = \mu_k \mathbb{E}^0_{M_k}\left[\mathcal{H}^k(F_k)^{-1} \cdot g\right].$$

4. Mean values for typical faces

Corollary: *For any* $k \in \{0, \dots, d\}$ *we have*

$$\mu_k = \lambda_k \mathbb{E}^0_{N_k}\left[\mathcal{H}^k(C_k)\right],$$
$$\lambda_k = \mu_k \mathbb{E}^0_{M_k}\left[\mathcal{H}^k(F_k)^{-1}\right].$$

In particular

$$\mathbb{E}^0_{N_d}[\mathcal{H}^d(C_d)] = \lambda^{-1},$$
$$\mathbb{E}[\mathcal{H}^d(F_d)^{-1}] = \lambda.$$

Proposition: *We have*

$$\sum_{j=0}^{d}(-1)^j \lambda_j = 0.$$

Definition: Let $\mathcal{P}^d$ denote the system of all convex polytopes in $\mathbb{R}^d$. For $k \in \{0, \dots, d\}$ we define $\nu_k : \mathcal{P}^d \to \mathbb{N}$ by

$$\nu_k(F) := \operatorname{card} \mathcal{S}_k(F).$$

Proposition: *Consider the planar case $d = 2$. Then $\lambda_0 = 2\lambda$ and $\lambda_1 = 3\lambda$. Moreover,*

$$\mathbb{E}^0_{N_2}[\mathcal{H}^2(C_2)] = \frac{1}{\lambda},$$

$$\mathbb{E}^0_{N_2}[\mathcal{H}^1(\partial C_2)] = \frac{2\mu_1}{\lambda},$$

$$\mathbb{E}^0_{N_2}[\nu_0(C_2)] = 6,$$

$$\mathbb{E}^0_{N_1}[\mathcal{H}^1(C_1)] = \frac{\mu_1}{3\lambda}.$$

Theorem: *If N is a stationary Poisson process of intensity λ then the intensities μ_k are explicitly known. In case $d = 2$ we have*

$$\mu_0 = 2\lambda, \quad \mu_1 = 2\sqrt{\lambda}$$

and in case $d = 3$ we have

$$\mu_0 = \frac{24\pi^2}{35}\lambda, \quad \mu_1 = \frac{48\pi^2}{35}\lambda, \quad \mu_2 = \Big(\frac{24\pi^2}{35} + 1\Big)\lambda.$$

Problem: Assume that N is a stationary Poisson process. Determine the distributions

$$\mathbb{P}^0_{N_k}(C_k \in \cdot), \quad k = 0, \ldots, d,$$

and

$$\mathbb{P}^0_{M_k}(F_k \in \cdot), \quad k = 0, \ldots, d.$$

5. The neighbours of a typical vertex

Assumption: N is a stationary Poisson process of intensity $\lambda > 0$.

Definition: Consider the probability measure $\mathbb{P}^0_{N_0}$.

(i) Almost surely there are exactly $d + 1$ different points $X_0, \dots, X_d \in N$ (lexicographically ordered) such that

$$\{0\} = C(N, X_0) \cap \cdots \cap C(N, X_d).$$

The points $X_0, \dots, X_d$ are the *neighbours* of the origin.

(ii) Let $R := |X_0| = \cdots = |X_d|$ denote the distance to the neighbours and define the unit vectors

$$U_0 := \frac{X_0}{R}, \dots, U_d := \frac{X_d}{R}.$$

Theorem: *Under the probability measure* $\mathbb{P}^0_{N_0}$ *the following holds.*

(i) *The random variables* $(\{x \in N : |x| > R\}, R)$ *and* $(U_0, \ldots, U_d)$ *are independent.*

(ii) R^d *is Gamma distributed with shape parameter* d *and scale parameter* $\gamma\kappa_d$.

(iii) *The conditional distribution of* $\{x \in N : |x| > R\}$ *given* $R = r$ *is the distribution of a Poisson process restricted to the complement of the ball* $B(0, r)$.

(iv) $\{U_0, \ldots, U_d\}$ *has distribution*

$$c_0^{-1} \int \cdots \int \mathbf{1}\{\{u_0, \ldots, u_d\} \in \cdot\} \Delta_d(u_0, \ldots, u_{d-k})\, \mathbb{S}(du_0) \ldots \mathbb{S}(du_d)$$

where $\Delta_d(u_0, \ldots, u_d)$ *is the volume of the simplex spanned by the vectors* $u_0, \ldots, u_d$, $\mathbb{S}$ *is the uniform distribution on the unit sphere* S^{d-1} *and* c_0 *is an explicitly known constant.*

6. The length-biased distribution of the neigbours of a typical face

Assumption: N is a stationary Poisson process of intensity $\lambda > 0$.

Definition: Consider the probability measure $\mathbb{P}^0_{M_k}$ for some fixed $k \in \{1, \dots, d-1\}$.

(i) Almost surely there is exactly one k-face $F_k \in \mathcal{S}_k(N)$ such that $0 \in F_k$.

(ii) Almost surely there are exactly $d - k + 1$ different points $X_0, \dots, X_{d-k} \in N$ (lexicographically ordered) such that

$$F_k = C(N, X_0) \cap \cdots \cap C(N, X_{d-k}).$$

The points $X_0, \dots, X_{d-k}$ are the *neighbours* of F_k.

(iii) Let

$$R := |X_0| = \cdots = |X_{d-k}|$$

denote the distance of the origin from the neighbours.

(iv) There is a unique $(d-k)$-dimensional ball in the affine hull of the neighbours containing the neighbours on its boundary. We let Z denote the centre of this ball.

(v) Let

$$R' := |X_0 - Z|, \qquad R'' := |Z|.$$

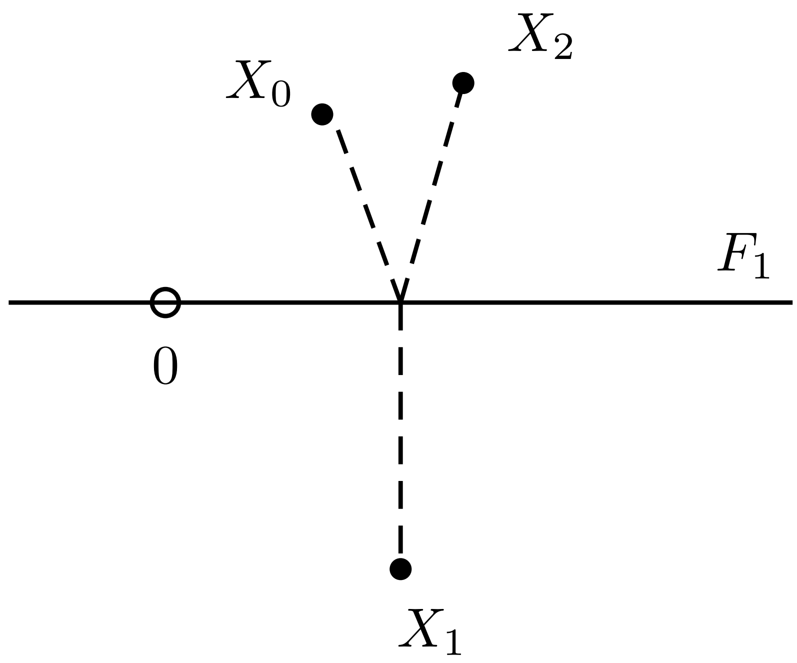
so that

$$R^2 = {R'}^2 + {R''}^2.$$

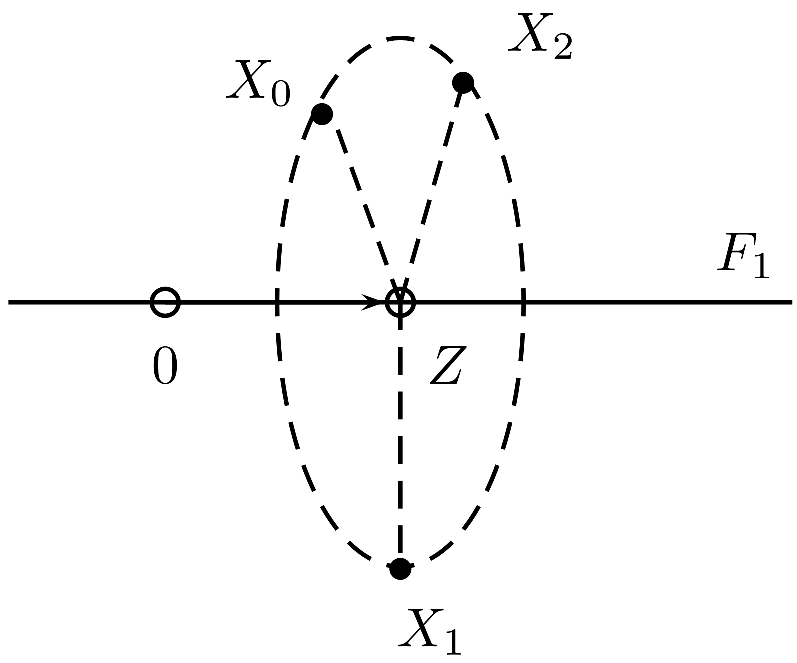
(vi) Define the unit vectors

$$U_0 := \frac{X_0 - Z}{R'}, \dots, U_{d-k} := \frac{X_{d-k} - Z}{R'}, \quad U := \frac{Z}{|Z|}.$$

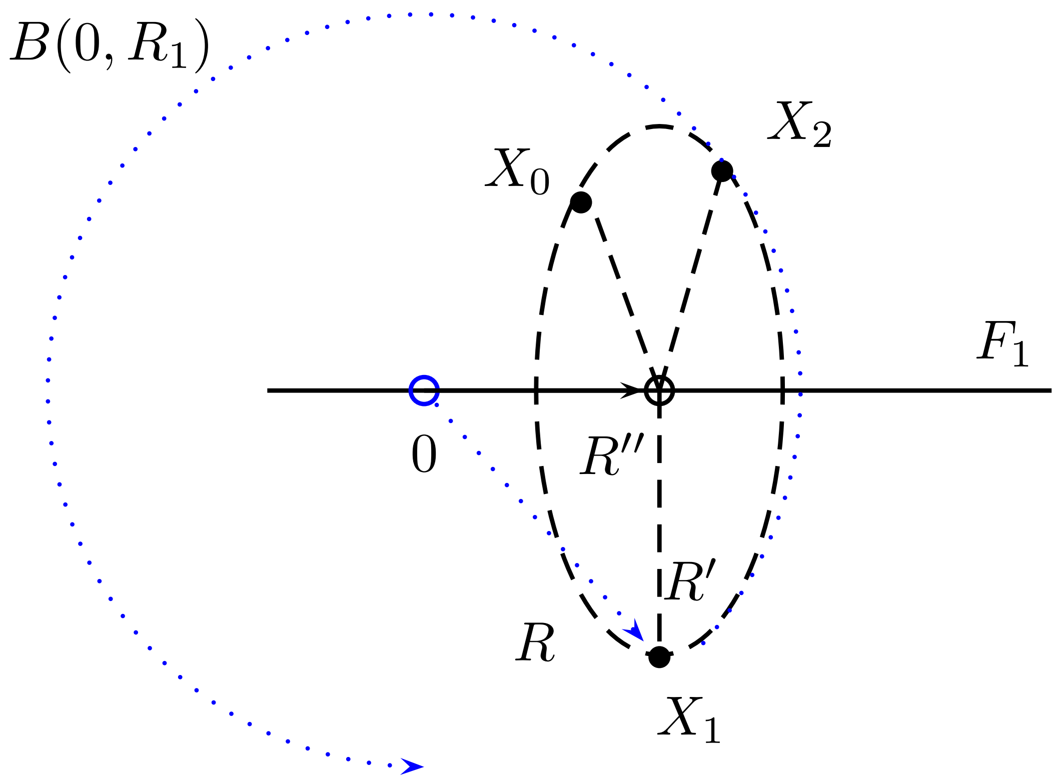
Situation under $\mathbb{P}^0_{M_k}$ for $k = 1$, $d = 3$.



Situation under $\mathbb{P}^0_{M_k}$ for $k = 1$, $d = 3$.



Situation under $\mathbb{P}^0_{M_k}$ for $k = 1$, $d = 3$.



Theorem: *Under the probability measure $\mathbb{P}^0_{M_k}$ the following holds.*

(i) *The random variables* $(\{x \in N : |x| > R\}, R)$, R'^2/R^2, *and* $(U_0, \ldots, U_{d-k}, U)$ *are independent.*

(ii) R^d *is Gamma distributed with shape parameter* $d - k + k/d$ *and scale parameter* $\gamma\kappa_d$.

(iii) *The conditional distribution of* $\{x \in N : |x| > R\}$ *given* $R = r$ *is the distribution of a Poisson process restricted to the complement of the ball* $B(0, r)$.

(iv) R'^2/R^2 *has a Beta distribution with parameters* $d(d-k)/2$ *and* $k/2$.

(v) *Fix a* $d-k$*-dimensional linear subspace* $L \subset \mathbb{R}^d$*. The random pair* $(\{U_{k,0},\dots,U_{k,d-k}\},U_k)$ *has distribution*

$$c_k^{-1}\int\cdots\int \mathbf{1}\{(\{\vartheta u_0,\dots,\vartheta u_{d-k}\},\vartheta u))\in\cdot\}$$
$$\Delta_{d-k}(u_0,\dots,u_{d-k})^{k+1}\mathbb{S}_L(du_0)\dots\mathbb{S}_L(du_{d-k})\,\mathbb{S}_{L^\perp}(du)\,\nu(d\vartheta),$$

where $\Delta_{d-k}(u_0,\dots,u_{d-k})$ *is the* $(d-k)$*-dimensional volume of the simplex spanned by the vectors* $u_0,\dots,u_{d-k}$*,* ν *is the uniform distribution on the rotation group* SO_d*,* c_k *is an explicitly known constant, and* $\mathbb{S}_L$ *and* $\mathbb{S}_{L^\perp}$ *are the uniform distributions (normalized Haar measures) on the unit spheres in* L *and the orthogonal complement* $L^\perp$ *of* L*, respectively.*

7. The distribution of the typical edge and its neighbours

Definition: Assume that N is a stationary Poisson process of intensity $\lambda > 0$ and consider the Palm probability measure $\mathbb{P}^0_{N_1}$.

(i) Let L denote the length of the typical edge C_1. Further let Φ_1 denote the set of the d unit vectors pointing from $\pi_1(C_1)$ to the neighbours of C_1.

(ii) Let α' and α'' denote the angles in $[0, \pi]$ spanned by the edge C_1 and the vectors pointing from the endpoints of C_1 to one of the neighbours of C_1.

(iii) Let ξ denote the volume of the union of the two balls centered at the endpoints of the edge C_1 whose radii are giben by the respective distances from the endpoints to one of the neighbours of C_1.

Theorem: *Under $\mathbb{P}^0_{N_1}$ we have the following assertions:*

(i) *The random variables (α', α''), ξ and Φ_1 are independent.*

(ii) *The random variable ξ has a Gamma distribution with shape parameter $d + 1$ and scale parameter 1.*

(iii) *The distribution of $(\cos\alpha', \cos\alpha'')$ has an explicitly known and integral-free density.*

(iv) *The distribution of Φ_1 is the same as the distribution of the corresponding unit vectors under $\mathbb{P}_{M_1}$. It has been given in Section 6.*

Remark: Under $\mathbb{P}^0_{N_1}$ the random variables α', α'', ξ and Φ determine the edge C_1 and the positions of its neighbours.

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