Curvature measures and fractals

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1. Prof. Dr. Martina Zähle

2. Prof. Dr. Michel Lapidus

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Zusammenfassung

In der vorliegenden Arbeit werden die Begriffe *fraktale Krümmungen* und *fraktale Krümmungsmaße* eingeführt mit der Intention, charakteristische Größen bzw. geometrische Maße für fraktale Mengen bereitzustellen, die die klassischen Krümmungsbegriffe, die für solche Mengen oft nicht erklärt oder nicht sinnvoll sind, ersetzen können.

Die neuen Größen werden aus den klassischen im wesentlichen folgendermaßen abgeleitet: Krümmungen und Krümmungsmaße sind zwar gewöhnlich für eine fraktale Menge $F \subset \mathbb{R}^d$ selbst nicht erklärt, jedoch oft für die ϵ-Parallelmengen von F, d.h. für die Mengen

$$F_\epsilon = \left\{ x \in \mathbb{R}^d : d(x, F) \leq \epsilon \right\} .$$

Läßt man nun ϵ gegen Null konvergieren und reskaliert geeignet, so erhält man die fraktalen Krümmungen als Grenzwert der klassischen. Ist etwa $C_k(F_\epsilon)$ die k-te totale Krümmung von F_ϵ, so ist die k-te fraktale Krümmung von F gegeben durch

$$C_k^f(F) := \lim_{\epsilon \to 0} \epsilon^{s_k} C_k(F_\epsilon) ,$$

falls der Grenzwert existiert, wobei der Skalierungsexponent $s_k = s_k(F)$ zunächst geeignet zu wählen ist. Ähnlich werden fraktale Krümmungsmaße als in derselben Weise reskalierte schwache Grenzwerte der klassischen Krümmungsmaße der Parallelmengen eingeführt. Der bekannte Minkowski-Inhalt und die kürzlich eingeführten fraktalen Eulerzahlen ordnen sich als Spezialfälle in dieses Konzept ein.

Als eine erste Anwendung des Konzeptes werden hier die fraktalen Krümmungen und Krümmungsmaße selbstähnlicher Mengen untersucht. Unter Voraussetzung der offenen-Mengen-Bedingung wird, ausgehend von den klassischen Krümmungsmaßen im Konvexring, sowohl die Existenz fraktaler Krümmungen als auch die Konvergenz der Krümmungsmaße gezeigt. Dabei ist für manche Mengen (d.h. im arithmetischen Fall) eine zusätzliche Mittelung der Grenzwerte erforderlich. Mit dieser Untersuchung hoffen wir, einerseits einen Beitrag zur Diskussion um eine weitere Verallgemeinerung der Krümmungsbegriffe zu leisten, andererseits mit den fraktalen Krümmungen Größen bereitzustellen, die der praktischen Klassifizierung, Beschreibung und Unterscheidung fraktaler Strukturen dienlich sind.

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Introduction

There were two main incentives for working on the subject presented in this thesis. On the one hand there is the long standing quest in classical geometry to extend the notions of curvature as far as possible. Curvatures and Curvature measures have been identified for many classes of sets as in a sense canonical objects for the description of their geometry. On the other hand there is the desire to understand the geometry of fractal sets better, in particular, to find quantitative measures to describe the geometric features of fractals. In this thesis it is intended to make a contribution to both of these goals by introducing the notion of *fractal curvatures*.

A milestone in the development of curvature measures was their introduction for sets with positive reach by Herbert Federer in his famous paper [8], which joined and unified the two existing notions of curvature for differentiable manifolds and convex bodies. Federer's work effected the further development of the formerly independent parts of the theory and led for instance to an axiomatic characterization of curvature measures of convex sets by Rolf Schneider [28] and their additive extension to the convex ring by Helmut Groemer [12]. It was also the starting point for further generalisations and extensions of the notion of curvature measures for instance to unions of sets with positive reach and Lipschitz manifolds by Martina Zähle [32, 33, 34], Jan Rataj and Martina Zähle [24, 25, 26] and Joseph H. G. Fu [10]. In a very recent paper by Daniel Hug, Günther Last und Wolfgang Weil [14], curvature measures were introduced for general closed sets, which marks the tentative culmination of this generalisation process. In contrast to the previously mentioned curvature measures they are non-additive. However, since fractal sets are usually considered as being closed, one could feel tempted to see those curvature measures as the endpoint and solution to the above raised questions. Unfortunately, it seems that, although these measures are defined for fractal sets, they are not capable of sufficiently enlightening the "fractal" features of their structure.

In fractal geometry on the other hand, apart from the different concepts of dimension there have been up to now very few generally accepted quantitative measures, which are able to provide additional information on the geometric structure of fractal sets. One of them is the *Minkowski content* which was proposed by Benoit Mandelbrot in [23] as a *measure of lacunarity*, i.e. as a characteristic for the porosity of fractal sets. In the same paper he also remarked the insufficiency of having just one such measure available:

> "But, in general, one must not expect the search for a measure of lacunarity to converge to a single number, because the task of specifying a general texture numerically is at least as difficult as the task of specifying the shape of a curve."

The Minkowski content is derived from the classical notion of volume or Lebesgue measure by a limiting procedure. Instead of looking at the volume of a compact set $F \subset \mathbb{R}^d$

itself – which, in case F is a fractal, typically is zero and thus not interesting – one studies the volume of its ϵ-parallel sets

$$F_\epsilon = \left\{ x \in \mathbb{R}^d : d(x,F) \le \epsilon \right\},$$

and, in particular, how the volume evolves as ϵ tends to zero. If the volume $\lambda_d(F_\epsilon)$ is rescaled appropriately, namely with the factor ϵ^{m-d}, where $m = m(F)$ denotes the *Minkowski dimension* (or *box dimension*) of F, often some limit exists and is then termed the *Minkowski content* of the set F:

$$M(F) := \lim_{\epsilon \to 0} \epsilon^{m-d} \lambda_d(F_\epsilon).$$

We introduce *fractal curvatures* in the same way as the Minkowski content replacing the volume by some curvature measure. Supposed curvature measures $C_0(F_\epsilon, \cdot), \ldots, C_d(F_\epsilon, \cdot)$ are defined for the parallel sets F_ϵ of F, we consider first for each $k = 0, \ldots, d$ the total curvatures $C_k(F_\epsilon) := C_k(F_\epsilon, \mathbb{R}^d)$. The *k-th fractal curvature* of F is the limit

$$C_k^f(F) := \lim_{\epsilon \to 0} \epsilon^{s_k} C_k(F_\epsilon),$$

provided this limit exists. Here the right choice of the scaling exponent s_k is one of the central questions. Due to the nature of curvature measures, the Minkowski content appears as a special case in this concept. It coincides with the d-th fractal curvature, since $C_d(F_\epsilon, \cdot) = \lambda_d(F_\epsilon \cap \cdot)$. In [22], *fractal Euler numbers* were studied by Marta Llorente and the author, which are derived in a similar manner from the Euler characteristic of the parallel sets. These numbers fit into the presented framework too. They are closely related to the 0-th fractal curvature.

The concept of fractal curvatures comes up very naturally not only because of the direct analogy with the Minkowski content. It is based on parallel sets, which play an important role for the definition of "classical" curvature measures via Steiner type formulas too. Moreover, under certain assumptions on F, for instance if F is convex, the "classical" curvatures $C_k(F_\epsilon)$ (and curvature measures $C_k(F_\epsilon, \cdot)$) are known to converge to the corresponding curvature $C_k(F)$ (or curvature measure $C_k(F, \cdot)$, respectively) of F as ϵ approaches zero. The new feature in the fractal case is the need to rescale the curvatures $C_k(F_\epsilon)$.

In the present exposition we restrict ourselves to curvature measures as defined in the convex ring, i.e. we assume throughout that the parallel sets F_ϵ (but not the fractal sets F themselves, of course) are finite unions of convex sets. We are aware of the loss of generality this restriction implies, but this will keep the presentation simpler and allow concentration on the new ideas. Nevertheless, we are convinced that fractal curvatures can be introduced in a more general setting. This investigation is just a starting point.

We apply the concept to self-similar sets. We will show that for most self-similar sets satisfying the open set condition, fractal curvatures exist, provided that they are not too regular (i.e. provided they are non-arithmetic). Since many popular examples of self-similar sets like the Sierpinski gasket are in fact very regular, it turns out to be useful to study averaged limits of total curvatures as well. *Average fractal curvatures* are shown to exist also for regular self-similar sets. The results on the existence of total curvatures are quite analogical to known results for the Minkowski content (cf. in particular Dimitris Gatzouras [11]) and are derived in a similar way by application of the Renewal Theorem to some appropriate renewal equation. We want to propose these quantities as geometric characteristics for fractal sets, which could play a similar role for fractal sets as the classical total curvatures (or equivalently volume, surface area, Euler characteristic and so on) in classical geometry. Therefore we will also demonstrate in several examples how fractal curvatures are determined and interpreted.

Up to this point the analogies with the Minkowski content were a good guide to lead us. But we will go one step further and study not only the global but also the local limiting behaviour. Fractal curvatures are limits of the total curvatures of the parallel sets like the Minkowski content is a limit involving the "total" volume of the parallel sets. This does not take into account the fact that curvature measures are geometric measures describing the local behaviour of the sets. Naturally the question appears, what happens to the curvature measures $C_k(F_\epsilon, \cdot)$ as ϵ tends to zero. For self-similar sets F, it turns out that, if the curvature measures are rescaled appropriately, limits can be obtained in the sense of weak convergence. Again some averaging does improve the convergence behaviour for very regular sets. The derived limit measures could be regarded as *fractal curvature measures* of the initial set F. They turn out to be self-similar measures. Each fractal curvature measure is some multiple of the Hausdorff measure on F and its total mass is given by the corresponding fractal curvature. The geometric interpretation of this convergence is as follows: If B is a "nice" set, then the rescaled curvature $\epsilon^{s_k} C_k(F_\epsilon, B)$ of F_ϵ in the set B converges to the corresponding "fractal curvature" of F in B.

In the case $k = d$, this weak convergence result extends the known results for the Minkowski content. It provides a local characterization of the limiting behaviour of the volume of the parallel sets. In this statement the assumption on the parallel sets of being finite unions of convex sets can be dropped. It is shown to be valid for arbitrary self-similar sets satisfying the open set condition.

Summing up, we introduce fractal curvatures, which we propose as quantitative measures for the geometry of fractal sets, and fractal curvature measures, which we regard as a replacement of classical curvature measures in the fractal world. Results on their existence and their explicit characterization are obtained for certain classes of self-similar sets in $\mathbb{R}^d$.

The thesis is organized as follows. In Chapter 1 the concept of fractal curvatures is introduced and all the main results are presented. Most of the proofs are postponed to later Chapters. Section 1.1 provides the setting and the main definitions, while in 1.2 self-similar sets are introduced and fractal curvatures are studied for those sets. In Section 1.3, we discuss several examples to illustrate the results of the preceding section and show how fractal curvatures are practically computed and interpreted. Then we turn to the localisation and in Section 1.4 study weak limits of rescaled curvature measures.

Chapter 2 provides a brief introduction to curvature measures in the convex ring. We summarize their most important properties and collect in Section 2.2 a number of statements about their variation measures. In Chapter 3 we make the first steps towards the proofs of the main results by discussing the Renewal Theorem and reformulating it in a way which is most convenient for us. Chapter 4 is devoted to the proofs of the results of Section 1.2 on fractal curvatures and the associated scaling exponents, while in Chapter 5 the results of Section 1.4 are proved. An Appendix recalls some facts about signed measures and especially the notion of weak convergence of signed measures, which has seldom been used in the literature.

I am most grateful to all those who helped me along the way of writing this thesis with their encouragement and support, with stimulating discussions or valuable comments and suggestions. Especially, I wish to thank Professor Martina Zähle for her excellent supervision of my research during three years and for the confidence she has shown in me and my work.

1. Main results and examples

1.1. The concept of fractal curvatures

Convex ring. A set $K \subseteq \mathbb{R}^d$ is said to be *convex* if for any two points $x, y \in K$ the line segment connecting them is a subset of K. Denote by $\mathcal{K}^d$ the family of all *convex bodies*, i.e. of all nonempty compact convex sets in $\mathbb{R}^d$. A set K is called *polyconvex* if it can be represented as a finite union of convex bodies. The class $\mathcal{R}^d$ of all polyconvex sets in $\mathbb{R}^d$ is called the *convex ring*. It is stable with respect to finite unions and intersections.

Curvature measures. For each set $K \in \mathcal{R}^d$ there exist $d+1$ totally finite signed measures $C_0(K, \cdot\,)$, $C_1(K, \cdot\,)$, ..., $C_d(K, \cdot\,)$, called the *curvature measures* of K, which describe the local geometry of the set K. For convex bodies K, they are characterized by the Local Steiner formula and by additive extension they are generalized to arbitrary polyconvex sets K. The total mass $C_k(K) := C_k(K, \mathbb{R}^d)$ of the measure $C_k(K, \cdot\,)$ is called the *k-th total curvature* of K. It is also known as *k-th intrinsic volume* of K. Curvature measures of polyconvex sets are well understood and have many advantageous properties, including covariance with respect to Euclidean motions, homogeneity, locality and additivity. In Chapter 2 we give a more detailed introduction to them and also hints to the literature. For the moment we just add a few words on their geometric meaning. $C_d(K, \cdot\,)$ is nothing but the volume restricted to K, i.e. the d-dimensional Lebesgue measure $\lambda_d(K \cap \cdot\,)$, while $C_{d-1}(K, \cdot\,)$ is half the surface area of K, provided that K is the closure of its interior, $K = \overline{\operatorname{int} K}$. Both measures are always positive in contrast to the situation for $k = 0, \ldots, d-2$, for which $C_k(K, \cdot\,)$ is a signed measure in general. Except for $k = d$, $C_k(K, \cdot\,)$ is concentrated on the boundary ∂K of K. For convex bodies, $C_k(K, \cdot\,)$ has an interpretation in terms of the k-dimensional volumes of the projections of K to k-dimensional linear subspaces. More precisely, $C_k(K)$ is the average of these volumes over all subspaces. In general, the numbers $C_k(K, B)$ describe the different aspects of the "curvedness" of ∂K inside the set B. Finally, by the Gauß Bonnet formula, the 0-th total curvature $C_0(K)$ coincides with the *Euler characteristic* of K, the topological invariant defined in algebraic topology.

The *positive, negative* and *total variation measure* $C_k^+(K, \cdot\,)$, $C_k^-(K, \cdot\,)$ and $C_k^{\mathrm{var}}(K, \cdot\,)$ of the signed measure $C_k(K, \cdot\,)$ are defined respectively, by setting for each Borel set $A \subseteq \mathbb{R}^d$

$$C_k^+(K, A) := \sup_{B \subseteq A} C_k(K, B) \quad \text{and} \quad C_k^-(K, A) := -\inf_{B \subseteq A} C_k(K, B)$$

and

$$C_k^{\mathrm{var}}(K, A) := C_k^+(K, A) + C_k^-(K, A).$$

The variations are positive measures and satisfy the relation

$$C_k(K, \cdot\,) = C_k^+(K, \cdot\,) - C_k^-(K, \cdot\,), \tag{1.1.1}$$

called the *Jordan decomposition* of $C_k(K, \cdot)$. Positive and negative variation measures are useful for localizing "positive" and "negative" curvature or, more figuratively, to distinguish locally convexity from concavity. Some of the properties of curvature measures carry over to their variation measures. For more details we refer to the discussion in Section 2.2.

Central assumption. Fractal sets F, which we are interested in here, are usually not polyconvex. Their curvature measures $C_k(F, \cdot)$ are typically not defined in any classical sense. To investigate their geometric properties, we study the curvature measures of their parallel sets. For a compact set $F \subset \mathbb{R}^d$ and $\epsilon > 0$, the *ϵ-parallel set* of F is the set

$$F_\epsilon := \left\{ x \in \mathbb{R}^d : d(x, F) \leq \epsilon \right\},$$

where $d(x, F) = \inf_{y \in F} d(x, y)$ denotes the Euclidean distance between x and F. In particular, we want to study how the curvature measures of the parallel sets F_ϵ behave as ϵ tends to zero. In general, curvature measures need not be defined for the parallel sets. Therefore, unless indicated otherwise, throughout the paper we make the following assumption:

$$\text{Assume that } F_\epsilon \in \mathcal{R}^d \text{ for all } \epsilon > 0\,.$$

The supposition of F having polyconvex parallel sets ensures the existence of the $d+1$ curvature measures $C_0(F_\epsilon, \cdot)$, $C_1(F_\epsilon, \cdot)$, $\ldots$, $C_d(F_\epsilon, \cdot)$ for each parallel set F_ϵ. In order to introduce the concepts below it is essential to have curvature measures defined for the parallel sets. These need not necessarily be the curvature measures from the convex ring setting. Generalizations (in particular of the Definitions 1.1.2, 1.1.5 and 1.1.6 below) are possible, for instance to sets whose parallel sets are unions of sets with positive reach. For simplicity, we restrict ourselves to the polyconvex setting here. One particular reason is the property of polyconvex sets to have polyconvex parallel sets. If $F_\epsilon \in \mathcal{R}^d$ for some $\epsilon > 0$, then $F_{\epsilon+\delta} \in \mathcal{R}^d$ for all $\delta > 0$. Therefore, the existence of arbitrary small $\epsilon > 0$ such that $F_\epsilon \in \mathcal{R}^d$ is already sufficient to ensure that all parallel sets are polyconvex. Conversely, if there exist some $\epsilon_0 > 0$ such that F_{ϵ_0} is not polyconvex, then the same holds for all smaller parallel sets of F.

Fact 1.1.1. Either $F_\epsilon \in \mathcal{R}^d$ for all $\epsilon > 0$ or there exists $\epsilon_0 > 0$ such that $F_\epsilon \notin \mathcal{R}^d$ for all $0 < \epsilon \leq \epsilon_0$.

For self-similar sets we even have the dichotomy that either all or none of their parallel sets are polyconvex (cf. Proposition 1.2.1).

Scaling exponents. We first concentrate on the total curvatures $C_k(F_\epsilon) = C_k(F_\epsilon, \mathbb{R}^d)$ of the parallel sets F_ϵ. The typical behaviour of $C_k(F_\epsilon)$ as $\epsilon \to 0$ is either to converge to zero or to tend to $\pm\infty$. To obtain more information about the limiting behaviour some rescaling is advisable. We study the expressions $\epsilon^t C_k(F_\epsilon)$ where the exponent $t \in \mathbb{R}$ has to be chosen appropriately for each k (and F). If $t \in \mathbb{R}$ is chosen too small, $\epsilon^t C_k(F_\epsilon)$ will tend to $\pm\infty$, while $\epsilon^t C_k(F_\epsilon)$ tends to 0 if t is too large. So the exponent should be at the borderline between the two extremes. Taking the infimum over all t for which $\epsilon^t C_k(F_\epsilon)$ approaches zero or the supremum over those t for which $\epsilon^t \left|C_k(F_\epsilon)\right|$ is unbounded seems a reasonable choice for the exponent, especially if both numbers happen to coincide. But we have to take the local character of curvature measures into account and the fact that they are, in general, signed measures. The total mass $C_k(F_\epsilon)$ can be zero, while at the same time locally the measure $C_k(F_\epsilon, \cdot)$ is very large. The positive curvature in some part can equal out the negative curvature in some other part of the set to give total mass zero. Therefore it is more reasonable to use the total variation for the definition of the scaling exponent.

Definition 1.1.2. Let $F \subseteq \mathbb{R}^d$ a compact set with polyconvex parallel sets, and let $k \in \{0, 1, \ldots, d\}$. The *k-th curvature scaling exponent* of F is the number

$$s_k = s_k(F) := \inf \left\{ t : \epsilon^t C_k^{\text{var}}(F_\epsilon) \to 0 \text{ as } \epsilon \to 0 \right\} .$$

s_k is well defined at least in the sense of being an element of $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. If $\liminf_{\epsilon\to 0} \epsilon^{s_k} C_k^{\text{var}}(F_\epsilon) > 0$, then, clearly, s_k is the only interesting exponent for this expression, since, for all $t > s_k$, $\epsilon^t C_k^{\text{var}}(F_\epsilon) \to 0$ as $\epsilon \to 0$ and for all $t < s_k$, $\epsilon^t C_k^{\text{var}}(F_\epsilon) \to +\infty$. Then s_k is equivalently characterized by the expression $\underline{s}_k := \sup \left\{ t : \epsilon^t C_k^{\text{var}}(F_\epsilon) \to \infty \text{ as } \epsilon \to 0 \right\}$. In general, $\underline{s}_k$ need not coincide with s_k (but always $\underline{s}_k \leq s_k$) and one should then better speak of lower and upper scaling exponents, respectively. However, here we will only bother about s_k. Observe that $s_k(F)$ is invariant with respect to Euclidean motions and scaling.

Remark 1.1.3. (i) Since $C_d^{\text{var}}(F_\epsilon) = C_d(F_\epsilon) = \lambda_d(F_\epsilon)$, the number $d - s_d(F)$ corresponds to the upper Minkowski dimension (or box dimension) of F, which is defined more generally, namely for arbitrary (compact) sets.

(ii) In [22], Marta Llorente and the author introduced the Euler exponent $\sigma = \sigma(F)$ of F as the infimum $\inf \left\{ t \geq 0 : \epsilon^t \left| \chi(F_\epsilon) \right| \to 0 \text{ as } \epsilon \to 0 \right\}$, where χ denotes the Euler characteristic. $\sigma(F)$ is defined for more general sets F than those discussed here. For sets F with $F_\epsilon \in \mathcal{R}^d$ for all $\epsilon > 0$, however, the Euler exponent is closely related to $s_0(F)$, since, by the Gauss Bonnet formula, $C_0(F_\epsilon) = \chi(F_\epsilon)$. The main difference is that in the definition of $\sigma(F)$ we work with absolute values $|C_0(F_\epsilon)| = |\chi(F_\epsilon)|$, while for $s_0(F)$ we used the total variations $C_0^{\text{var}}(F_\epsilon)$. Often both exponents coincide, but sometimes they differ. This corresponds very well to the different geometric meaning of $\chi(F_\epsilon)$ as a topological invariant and $C_0(F_\epsilon, \cdot)$ as a curvature measure. For the different interpretations of σ and s_0 confer Example 1.3.5. Note that in general $\sigma(F) \leq s_0(F)$.

Remark 1.1.4. By replacing $C_k^{\text{var}}(F_\epsilon)$ in the definition of s_k with $C_k^+(F_\epsilon)$ and $C_k^-(F_\epsilon)$, scaling exponents s_k^+ and s_k^- can be defined. Since $C_k^{\text{var}}(F_\epsilon) = C_k^+(F_\epsilon) + C_k^-(F_\epsilon)$, they satisfy the relation $s_k = \max \left\{ s_k^+, s_k^- \right\}$. Hence one of these two exponents must always coincide with s_k, while the second one can be smaller. This does in fact happen as we will see later. (cf. Remark 1.3.6)

Fractal curvatures. Having defined the scaling exponents for total curvatures, we can now ask for the existence of rescaled limits.

Definition 1.1.5. Let $k \in \{0, 1, \ldots, d\}$. If the limit

$$C_k^f(F) := \lim_{\epsilon\to 0} \epsilon^{s_k} C_k(F_\epsilon)$$

exists, then it is called the *k-th fractal (total) curvature* of the set F.

Unfortunately, the existence of such limits is not always ensured. From the study of local quantities of self-conformal sets like densities or tangent measure distributions it is well-known that, in general, average limits provide better results. It turns out that this is a useful tool for our purposes too.

Definition 1.1.6. Let $k \in \{0, 1, \ldots, d\}$. If the limit

$$\overline{C}_k^f(F) := \lim_{\delta \to 0} \frac{1}{|\log \delta|} \int_\delta^1 \epsilon^{s_k} C_k(F_\epsilon) \frac{d\epsilon}{\epsilon}$$

exists, then it is called the *k-th average fractal (total) curvature* of the set F.

In both definitions one could also work with upper and lower limits. However, here we concentrate on the existence of the (average) limits. Note that whenever the limit in Definition 1.1.5 exists then the corresponding average limit exists as well and coincides with the limit. Moreover, (average) fractal curvatures have the following properties which are due to the corresponding properties of the total curvatures:
Motion invariance: If g is an Euclidean motion, then $C_k^f(gF) = C_k^f(F)$.
Scaling property: If $\lambda > 0$ and $\lambda F := \{\lambda x : x \in F\}$, then $C_k^f(\lambda F) = \lambda^{s_k + k} C_k^f(F)$.
Consistency: If $F \in \mathcal{R}^d$, then $C_k^f(F)$ exists and coincides with $C_k(F)$.
The consistency with the classical total curvatures is easily derived for convex sets from the continuity of curvature measures and in general from the additivity. All properties hold likewise for the average fractal curvatures.

1.2. Fractal curvatures for self-similar sets

We want to apply the above concepts to study self-similar sets. In this section we obtain some results concerning the existence of (average) fractal curvatures of these sets. The main statements are presented without proofs here. The proofs will be given in Chapter 4.

Self-similar sets. Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, \ldots, N$, be contracting similarities. Denote the contraction ratio of S_i by $r_i \in (0, 1)$ and set $r_{\max} := \max_{i=1,\ldots,N} r_i$ and $r_{\min} := \min_{i=1,\ldots,N} r_i$. It is a well known fact in fractal geometry (cf. Hutchinson [15]), that for such a system $\{S_1, \ldots, S_N\}$ of similarities there is a unique, non-empty, compact subset F of $\mathbb{R}^d$ such that $\mathbf{S}(F) = F$, where $\mathbf{S}$ is the set mapping defined by

$$\mathbf{S}(A) = \bigcup_{i=1}^N S_i A, \quad A \subseteq \mathbb{R}^d.$$

F is called the *self-similar set* generated by the system $\{S_1, \ldots, S_N\}$. Moreover, the unique solution s of $\sum_{i=1}^N r_i^s = 1$ is called the *similarity dimension* of F. The system $\{S_1, \ldots, S_N\}$ is said to satisfy the *open set condition* (OSC) if there exists an open, non-empty, bounded subset $O \subset \mathbb{R}^d$ such that $\bigcup_i S_i O \subseteq O$ and $S_i O \cap S_j O = \varnothing$ for all $i \neq j$. In [1], O was called a *feasible open set* of the S_i, or of F, which we adopt here. For

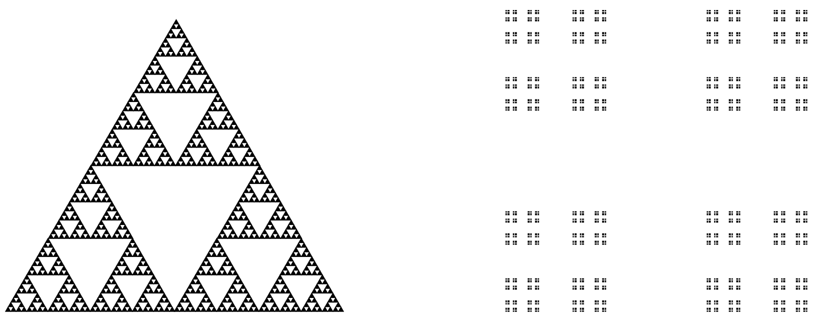

Figure 1.1.: The Sierpinski gasket has polyconvex parallel sets, while the Cantor set on the right does not.

convenience, we also say that F satisfies OSC, always having in mind a particular system of similarities generating F. If additionally $O \cap F \neq \varnothing$ holds for some feasible open set O of F, then $\{S_1, \dots, S_N\}$ (or F) is said to satisfy the *strong open set condition* (SOSC). It was shown by Andreas Schief in [27] that in $\mathbb{R}^d$ SOSC is equivalent to OSC, i.e. if F satisfies OSC, then there exists always some feasible open set O such that $O \cap F \neq \varnothing$. It is easily seen that $F \subseteq \overline{O}$ for each feasible open sets O of F.

Let $h > 0$. A finite set of positive real numbers $\{y_1, ..., y_N\}$ is called *h-arithmetic* if h is the largest number such that $y_i \in h\mathbb{Z}$ for $i = 1, \dots, N$. If no such number h exists for $\{y_1, ..., y_N\}$, the set is called *non-arithmetic*. We attribute these properties to the system $\{S_1, \dots, S_N\}$ or to F if and only if the set $\{-\log r_1, \dots, -\log r_N\}$ has them. In this sense, each set F (generated by some $\{S_1, \dots, S_N\}$) is either h-arithmetic for some $h > 0$ or non-arithmetic.

Parallel sets of self-similar sets. In order that the notions we introduced above are defined for a self-similar set, we have to make sure that its parallel sets are polyconvex. It turns out that not all self-similar sets F have this property. But at least there is the dichotomy that either all or none of the parallel sets of F are polyconvex. This was already shown in [22].

Proposition 1.2.1. *Let F a self-similar set. If $F_\epsilon \in \mathcal{R}^d$ for some $\epsilon > 0$, then $F_\epsilon \in \mathcal{R}^d$ for all $\epsilon > 0$.*

Therefore it suffices to check for an arbitrary parallel set F_ϵ, whether or not it is polyconvex, to know that for all parallel sets of F. For completeness, a simple proof of this statement is included in Section 4.1. Sierpinski gasket and Sierpinski carpet are self-similar sets with polyconvex parallel sets, while for instance the von Koch curve or the Cantor set in Figure 1.1 do not have this property. In $\mathbb{R}$ all self-similar sets have polyconvex parallel sets, since for each $\epsilon > 0$, $F_\epsilon \in \mathbb{R}$ consists of a finite number of intervals. In the sequel assume that $F_\epsilon \in \mathcal{R}^d$ for some (and thus all) $\epsilon > 0$.

Scaling exponents of self-similar sets. The first question that arises is the one for the right scaling exponents $s_k(F)$ of F. The following result provides an upper bound for the k-th scaling exponent $s_k(F)$.

Theorem 1.2.2. *Let F be a self-similar set satisfying OSC and $F_\epsilon \in \mathcal{R}^d$, and let $k \in \{0, \dots, d\}$. The expression $\epsilon^{s-k} C_k^{\mathrm{var}}(F_\epsilon)$ is uniformly bounded in $(0,1]$, i.e. there is a constant M such that for all $\epsilon \in (0,1]$, $\epsilon^{s-k} C_k^{\mathrm{var}}(F_\epsilon) \le M$.*

The proof is postponed to Section 4.5. The stated boundedness of the expression $\epsilon^{s-k} C_k^{\mathrm{var}}(F_\epsilon)$ has the following immediate implications.

Corollary 1.2.3. $s_k \le s - k$

Proof. By Theorem 1.2.2, $\limsup_{\epsilon \to 0} \epsilon^{s-k} C_k^{\mathrm{var}}(F_\epsilon) \le M$ and thus for each $t > 0$, $\lim_{\epsilon \to 0} \epsilon^{s-k+t} C_k^{\mathrm{var}}(F_\epsilon) \le \lim_{\epsilon \to 0} \epsilon^t \limsup_{\epsilon \to 0} \epsilon^{s-k} C_k^{\mathrm{var}}(F_\epsilon) = 0$. □

Corollary 1.2.4. *The expression $\epsilon^{s-k} |C_k(F_\epsilon)|$ is bounded in $(0,1]$ by M.*

Proof. Observe that $|C_k(F_\epsilon)| \le C_k^{\mathrm{var}}(F_\epsilon)$ for each $\epsilon > 0$. □

Corollary 1.2.3 provides the upper bound $s - k$ for the k-th scaling exponent $s_k(F)$ and raises the question whether the equality $s_k = s - k$ holds. It will turn out that for most self-similar sets (and most k) indeed $s_k = s - k$. Unfortunately, this is not always the case as the following example shows.

Example 1.2.5. The unit cube $Q = [0,1]^d \subset \mathbb{R}^d$ is a self-similar set generated for instance by a system of 2^d similarities each with contraction ratio $\frac{1}{2}$, which has similarity dimension $s = d$. For the curvature measures of its parallel sets no rescaling is necessary. Since Q is convex, its parallel sets Q_ϵ are and so the continuity implies that, for $k = 0, \dots, d$, $C_k(Q_\epsilon, \cdot) \to C_k(Q, \cdot)$ as $\epsilon \to 0$. Therefore, $s_k(Q) = 0$, which for $k < d$ is certainly different to $d - k$.

We investigate now the limiting behaviour of the expression $\epsilon^{s-k} C_k(F_\epsilon)$ as $\epsilon \to 0$. Since such degenerate examples like the cubes exist, we can not expect that this will always be the right expression to study.

Scaling functions. For a self-similar set F and $k \in \{0, \dots, d\}$, define the *k-th curvature scaling function* $R_k : (0, \infty) \to \mathbb{R}$ by

$$R_k(\epsilon) = C_k(F_\epsilon) - \sum_{i=1}^{N} \mathbf{1}_{(0, r_i]}(\epsilon) C_k((S_i F)_\epsilon). \tag{1.2.1}$$

The function R_k allows to formulate a renewal equation so that the required information on the limiting behaviour of the expression $\epsilon^{s-k} C_k(F_\epsilon)$ can be obtained from the Renewal theorem. Therefore it is essential to understand its properties. Geometrically the meaning of $R_k(\epsilon)$ is the following: Since $F_\epsilon = \bigcup_{i=1}^N (S_i F)_\epsilon$, the additivity of the total curvatures implies that, for small ϵ (i.e. $\epsilon \le r_{\min}$), R_k describes the curvature of the intersections of the sets $(S_i F)_\epsilon$:

$$R_k(\epsilon) = \sum_{\#I \ge 2} (-1)^{\#I - 1} C_k \left(\bigcap_{i \in I} (S_i F)_\epsilon \right), \tag{1.2.2}$$

where the sum is taken over all subsets I of $\{1, \dots, N\}$ with at least two elements. Hence the function R_k describes the k-th curvature of what we call the *overlap* of F_ϵ, namely the set $\bigcup_{i \ne j} (S_i F)_\epsilon \cap (S_j F)_\epsilon$.

Existence of fractal curvatures. If the scaling function R_k is well behaved, which is the case for sets satisfying OSC, then usually (average) k-th fractal total curvatures exist. The precise statement is derived from the following result, which characterizes the limiting behaviour of $\epsilon^{s-k}C_k(F_\epsilon)$.

Theorem 1.2.6. *Let F be a self-similar set satisfying OSC and $F_\epsilon \in \mathcal{R}^d$. Then for $k \in \{0, \ldots, d\}$ the following holds:*

(i) The limit $\lim\limits_{\delta\to 0}\frac{1}{|\log\delta|}\int_\delta^1 \epsilon^{s-k}C_k(F_\epsilon)\frac{d\epsilon}{\epsilon}$ exists and equals to the finite number

$$X_k = \frac{1}{\eta}\int_0^1 \epsilon^{s-k-1}R_k(\epsilon)\ d\epsilon, \tag{1.2.3}$$

where $\eta = -\sum_{i=1}^N r_i^s \log r_i$.

(ii) If F is non-arithmetic, then the limit $\lim\limits_{\epsilon\to 0}\epsilon^{s-k}C_k(F_\epsilon)$ exists and equals X_k.

The proof of this statement will be given in Section 4.2. The number X_k defined in (1.2.3) as an integral over the function R_k determines the average limit – and in the non-arithmetic case as well the limit – of the expression $\epsilon^{s-k}C_k(F_\epsilon)$. If for F we had that $s_k = s-k$, then, by definition, the average fractal curvature $\overline{C}_k^f(F)$ would coincide with X_k, and in case of a non-arithmetic set F also the fractal curvature $C_k^f(F)$. But this is not always true as we have seen in Example 1.2.5, where for some k, s_k was strictly smaller than $s-k$. Additional assumptions are required. A sufficient condition for $s_k = s-k$ is the requirement that X_k is different from 0.

Corollary 1.2.7. *If $X_k \neq 0$, then $s_k = s-k$. Consequently, $\overline{C}_k^f(F) = X_k$ and, in case F is non-arithmetic, also $C_k^f(F) = X_k$.*

Proof. The assumption $X_k \neq 0$ and (i) of Theorem 1.2.6 imply that

$$\begin{aligned} 0 < |X_k| &= \left|\lim_{\delta\to 0}\frac{1}{|\log\delta|}\int_\delta^1 \epsilon^{s-k}C_k(F_\epsilon)\frac{d\epsilon}{\epsilon}\right| \le \lim_{\delta\to 0}\frac{1}{|\log\delta|}\int_\delta^1 \epsilon^{s-k}\left|C_k(F_\epsilon)\right|\frac{d\epsilon}{\epsilon} \\ &\le \limsup_{\epsilon\to 0}\epsilon^{s-k}\left|C_k(F_\epsilon)\right| \le \limsup_{\epsilon\to 0}\epsilon^{s-k}C_k^{\mathrm{var}}(F_\epsilon) \end{aligned}$$

and thus for all $t > 0$,

$$\limsup_{\epsilon\to 0}\epsilon^{s-k-t}C_k^{\mathrm{var}}(F_\epsilon) \ge |X_k|\lim_{\epsilon\to 0}\epsilon^{-t} = \infty.$$

Hence there is no $t > 0$ such that $\epsilon^{s-k-t}C_k^{\mathrm{var}}(F_\epsilon) \to 0$ implying $s_k = s-k$. □

Formula (1.2.3) provides an explicit way to calculate X_k and therefore (average) fractal curvatures, at least if X_k turns out to be different from 0. For $k = d$ it can be shown that always $X_d > 0$ and thus $s_d = s-d$. This case is related to the Minkowski content and will be discussed separately below. So assume for the moment that $k \in \{0, \ldots, d-1\}$.

For those k it remains to clarify the situation when $X_k = 0$ for a self-similar set. First note that the condition $X_k \neq 0$ is not necessary for s_k to be $s-k$. For $X_k = 0$ both situations are possible - either $s_k < s-k$ or $s_k = s-k$. In Example 1.3.5 we will discuss a set of the latter type, while the cubes presented in Example 1.2.5 above are of the former type. Note that in the latter case, Theorem 1.2.6 provides the right values for the (average) k-th fractal curvature, namely $\overline{C}_k^f(F) = X_k = 0$ and in the non-arithmetic case as well $C_k^f(F) = 0$.

The next Theorem provides a tool to detect sets of the latter type, i.e. with $s_k = s-k$ (with or without $X_k = 0$). Here it is necessary to work locally rather then just with the total curvatures. For $\epsilon > 0$, define the *inner ϵ-parallel set* of a set A by

$$A_{-\epsilon} := \{x \in A : d(x, A^c) > \epsilon\} \tag{1.2.4}$$

or equivalently as the complement of the (outer) ϵ-parallel set of the complement of A, i.e. $A_{-\epsilon} = ((A^c)_\epsilon)^c$. Inner parallel sets only make sense for sets with nonempty interior, otherwise they are empty.

Theorem 1.2.8. *Let F be a self-similar set satisfying OSC and $F_\epsilon \in \mathcal{R}^d$, O some feasible open set of F, and $k \in \{0, \ldots, d\}$. Suppose there exist some constants $\epsilon_0, \beta > 0$ and some Borel set $B \subset O_{-\epsilon_0}$ such that*

$$C_k^{\mathrm{var}}(F_\epsilon, B) \geq \beta$$

for each $\epsilon \in (r_{\min}\epsilon_0, \epsilon_0]$. Then for all $\epsilon < \epsilon_0$

$$\epsilon^{s-k} C_k^{\mathrm{var}}(F_\epsilon) \geq c,$$

where $c := \beta \epsilon_0^{s-k} r_{\min}^s > 0$.

The rough idea is that curvature in some advantageous location B in a large parallel set F_ϵ, is exponentiated and spreaded by the self-similarity as ϵ tends to zero. A thorough proof of this statement is provided in Section 4.5. An immediate consequence is that, under the hypotheses of Theorem 1.2.8, $s-k$ is a lower bound for s_k and thus

Corollary 1.2.9. $s_k = s-k$

Proof. Theorem 1.2.8 implies that $\liminf_{\epsilon \to 0} \epsilon^{s-k} C_k^{\mathrm{var}}(F_\epsilon) \geq c$ and thus for each $t > 0$, $\liminf_{\epsilon \to 0} \epsilon^{s-k-t} C_k^{\mathrm{var}}(F_\epsilon) \geq c \lim_{\epsilon \to 0} \epsilon^{-t} = \infty$. Hence $s_k \geq s-k$. The validity of the reversed inequality has already been obtained in Corollary 1.2.3. □

Theorem 1.2.8 can be seen as a counterpart to Theorem 1.2.2. While the latter provides an upper bound for the scaling exponent s_k which is in a sense universal, the former gives the corresponding lower bound, though only under additional assumptions. It is a useful supplement to Corollary 1.2.7 for the treatment of self-similar sets with $X_k = 0$. Its power will be revealed in Example 1.3.5.

Minkowski content. For the case $k = d$ the above results hold in a more general setting, namely without the assumption of polyconvex parallel sets. For this case the results are already known. They have been obtained by Dimitris Gatzouras in [11]. We want to recall Gatzouras's results and discuss more carefully how they fit into our setting.

First recall that $C_d(F_\epsilon, \cdot\,) = \lambda_d(F_\epsilon \cap \cdot\,)$, whenever $C_d(F_\epsilon, \cdot\,)$ is defined. Therefore it is straightforward to generalize the definitions of the d-th scaling exponent and the d-th (average) fractal curvature to arbitrary compact sets $F \in \mathbb{R}^d$. The resulting quantities are

$$s_d = s_d(F) := \inf \left\{ t : \epsilon^t \lambda_d(F_\epsilon) \to 0 \text{ as } \epsilon \to 0 \right\},$$

$$M(F) := \lim_{\epsilon \to 0} \epsilon^{s_d} \lambda_d(F_\epsilon)$$

and

$$\overline{M}(F) := \lim_{\delta \to 0} \frac{1}{|\log \delta|} \int_\delta^1 \epsilon^{s_d} \lambda_d(F_\epsilon) \frac{d\epsilon}{\epsilon}.$$

Obviously, $d - s_d$ coincides with the *(upper) Minkowski dimension*, while $M(F)$ and $\overline{M}(F)$ are well known as *Minkowski content* and *average Minkowski content* of F, respectively, provided they are defined.

For self-similar sets F satisfying OSC, it is well known, that the Minkowski dimension coincides with the similarity dimension s of F. Hence $s_d = s - d$, which answers completely the question for the right scaling exponent for the volume of the parallel sets $\lambda_d(F_\epsilon)$. However, for a long time it had been an open problem, whether self-similar sets are *Minkowski measurable*, i.e. whether their Minkowski content exists, although this question aroused considerable interest. After partial answers for sets in $\mathbb{R}$ by Michel Lapidus and Carl Pomerance [19] and Kenneth Falconer [6], Gatzouras gave the following classification of Minkowski measurability of self-similar sets in $\mathbb{R}^d$, cf. [11, Theorems 2.3 and 2.4].

Theorem 1.2.10. (Gatzouras's theorem)
Let F be a self-similar set satisfying OSC. The average Minkowski content of F always exists and coincides with the strictly positive value

$$X_d = \frac{1}{\eta} \int_0^1 \epsilon^{s-d-1} R_d(\epsilon)\; d\epsilon.$$

If F is non-arithmetic, then also the Minkowski content $M(F)$ of F exists and equals X_d.

Here the function R_d is the d-th scaling function generalized in the obvious way:

$$R_d(\epsilon) = \lambda_d(F_\epsilon) - \sum_{i=1}^N \mathbf{1}_{(0,r_i]}(\epsilon) \lambda_d((S_i F)_\epsilon)\,. \tag{1.2.5}$$

This theorem contains all the results discussed before for the case $k = d$ and generalizes them to arbitrary self-similar sets satisfying OSC. Note in particular that the case $X_k = 0$, which causes a lot of trouble in the general discussion, does not occur for $k = d$, since it is possible to show explicitely that always $X_d > 0$.

With only little extra work we derive in Section 4.6 a proof of Gatzouras's theorem from the proof of Theorem 1.2.6, which differs in some parts from the one provided by Gatzouras in [11] and shows more clearly the close connection to curvature measures. Moreover, this proof prepares a strengthening of Gatzouras's theorem which is presented in Theorem 1.4.4 below. It characterizes the limiting behaviour of the parallel volume not only globally but also locally.

Fractal Euler numbers. In [22], the *fractal Euler number* of a set F was introduced as

$$\chi_f(F) := \lim_{\epsilon\to 0} \left(\frac{\epsilon}{b}\right)^{\sigma} \chi(F_\epsilon)\,,$$

where b is the diameter of F and σ the Euler exponent (cf. Remark 1.1.3). Similarly, the *average fractal Euler number* of F was defined as

$$\overline{\chi}_f(F) := \lim_{\delta\to 0} \frac{1}{|\log\delta|} \int_\delta^1 \left(\frac{\epsilon}{b}\right)^{\sigma} \chi(F_\epsilon)\,\frac{d\epsilon}{\epsilon}\,.$$

The normalizing factor $b^{-\sigma}$ was inserted to ensure the scaling invariance of these numbers. It does not affect the limiting behaviour.

We want to work out the relation between $C_0^f(F)$ and $\chi_f(F)$ more clearly and compare the results obtained here to those in [22]. Assume that the parallel sets of F are polyconvex. Then always $\sigma(F) \le s_0(F)$. If equality holds, then also the numbers $C_0^f(F)$ and $\chi_f(F)$ coincide up to the factor $b^{-\sigma}$, provided both are defined. The same is true for their averaged counterparts: $\overline{\chi}_f(F) = b^{-\sigma}\overline{C}_0^f(F)$. Therefore, the results obtained for the 0-th fractal curvatures of self-similar sets can be carried over to the fractal Euler numbers of these sets. From Theorem 1.2.6 and Corollary 1.2.7 we immediately deduce the following.

Corollary 1.2.11. *Let F be a self-similar set satisfying OSC and $F_\epsilon \in \mathcal{R}^d$. If $X_0 \neq 0$ then $\sigma = s$. Moreover, $\overline{\chi}_f(F)$ exists and equals the number $b^{-s}X_0$. If F is non-arithmetic, then also $\chi_f(F) = b^{-s}X_0$.*

For the considered class of sets this statement is a significant improvement of the results obtained in [22]. In Corollary 1.2.11, we have no additional assumptions on F apart from the OSC. In fact, it can be shown that the additional conditions in Theorem 2.1 in [22] are always satisfied in the situation of Corollary 1.2.11. It should be noted that on the contrary the results in [22] apply to a larger class of self-similar sets. We do not require their parallel sets to be polyconvex, since the Euler characteristic is defined more generally.

Remark 1.2.12. For sets F in $\mathbb{R}$ exactly two fractal curvatures are available, $C_1^f(F)$ and $C_0^f(F)$. Since in $\mathbb{R}$ for each F and $\epsilon > 0$, F_ϵ is a finite union of closed intervals, $C_1^f(F)$ is defined if and only if the Minkowski content $M(F)$ exists and both numbers coincide. Similarly, since $C_0(F_\epsilon, \cdot)$ is in this case a positive measure, we always have $s_0 = \sigma$ and $C_0^f(F) = b^{-\sigma}\chi_f(F)$ whenever one of these numbers exists. Corresponding relations hold for the average counterparts.

Fractal Euler numbers in $\mathbb{R}$ have been discussed in detail in [22]. Also the close relation to the gap counting function was outlined there. The limiting behaviour of the gap counting function and the Minkowski content have been studied extensively for sets in $\mathbb{R}$ - not only for self-similar sets. We refer in particular to the book by Michel L. Lapidus and Machiel van Frankenhuysen [20]. In this book some kind of Steiner formula was obtained for general sets F in $\mathbb{R}$, where Minkowski content and gap counting limit, i.e. in fact $C_0^f(F)$ and $C_1^f(F)$, appear as coefficients among others. This suggests that there are interesting relations between fractal curvatures and the theory of complex dimensions. These connections are still waiting for being studied in detail.

Open questions and conjectures. In all the considered examples, in particular in all the examples presented here and in the succeeding section, we can observe that always either $s_k = s - k$ or $s_k = 0$. It is a very interesting question, whether other values are possible for s_k. We conjecture that this is not the case. For $k = d$ the situation is clear. We always have $s_d = s - d$. Hence $s_d = 0$ only occurs when $s = d$, i.e. when the considered set is full-dimensional like the cubes in Example 1.2.5. For $k < d$, we conjecture that the case $s_k = 0$ occurs if and only if the set is a "classical" set, i.e. among the class of self-similar sets F satisfying OSC and $F_\epsilon \in \mathcal{R}^d$, exactly those sets have a scaling exponent $s_k = 0$ which are themselves polyconvex. All the other sets in this class have scaling exponents $s_k = s - k$, and should be regarded as the "true" fractals. Note that this classification would be independent of $k \in \{0, \ldots, d-1\}$.

Up to this point we have investigated the limiting behaviour of the total curvatures $C_k(F_\epsilon)$ for self-similar sets F. Before we continue with the discussion of rescaled limits of the curvature measures $C_k(F_\epsilon, \cdot)$, we illustrate the obtained results with some examples.

1.3. Examples

We want to illustrate the results of the previous section with some examples and determine the (average) fractal curvatures for some self-similar sets F. To compute the k-th fractal curvature of F, Theorem 1.2.6 and Corollary 1.2.7 suggest to determine first the scaling function R_k and then to compute the number X_k according to formula (1.2.3). If X_k is different from zero, then we have computed $C_k^f(F)$ (or $\overline{C}_k^f(F)$, respectively), if not, we can try to use Theorem 1.2.8. Recall the definition of R_k from (1.2.1) and also formula (1.2.2) which will prove very useful for small ϵ. We restrict ourselves to examples in $\mathbb{R}^2$, where we introduced three functionals $\overline{C}_2^f(F)$, $\overline{C}_1^f(F)$ and $\overline{C}_0^f(F)$. Roughly speaking they can be regarded as *fractal volume*, *fractal boundary length* and *fractal curvature*, respectively. Recall that $\overline{C}_2^f(F)$ coincides with the (average) Minkowski content $\overline{M}(F)$ and that $\overline{C}_0^f(F)$ is related to the (average) fractal Euler number.

Example 1.3.1. (Sierpinski gasket)
Let F be the Sierpinski gasket generated as usual by three similarities S_1, S_2 and S_3 with contraction ratios $\frac{1}{2}$ such that the diameter of F is 1 and the convex hull M of F is an equilateral triangle (cf. Figure 1.2). F satisfies the OSC and has polyconvex parallel sets F_ϵ. Since F is $\log 2$-arithmetic, the above results ensure only the existence of average fractal curvatures. (In fact, it can be shown that fractal curvatures do not exist for the Sierpinski gasket.) We determine the scaling functions R_k. It turns out that they have at most two discontinuities, namely at $\frac{1}{2}$, where the indicator functions in R_k switch from 0 to 1, and at $u = \frac{\sqrt{3}}{12}$, the radius of the incircle of the middle triangle (cf. Figure 1.2), where the intersection structure of the sets $(S_iF)_\epsilon$ changes. For the case $k = 0$, recall that $C_0(K)$ is the Euler characteristic of the set K, i.e. in $\mathbb{R}^2$ the number of connected components minus the number of "holes" of K. From Figure 1.3 it is easily

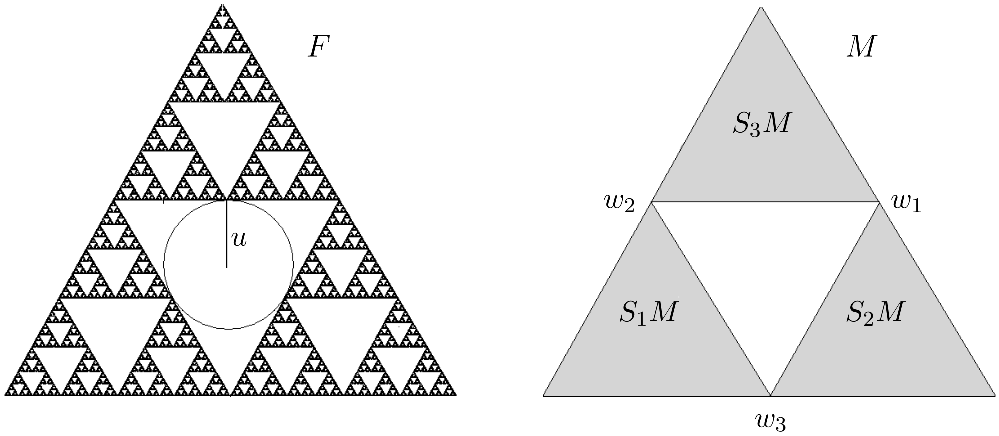

Figure 1.2.: The Sierpinski gasket F and a picture showing how the three similarities S_1, S_2 and S_3 generating F act on its convex hull M.

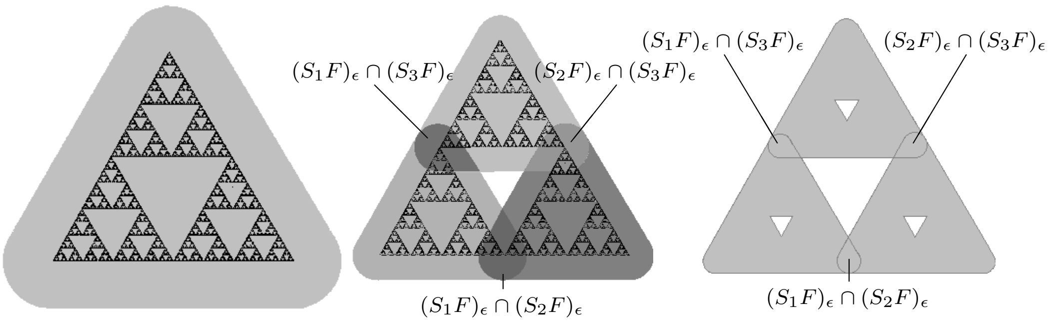

Figure 1.3.: Some ϵ-parallel sets of the Sierpinski gasket F for $\epsilon \geq u$ (left) and $\epsilon < u$ (middle and right). Note that for $\epsilon < u$ the sets $(S_iF)_\epsilon \cap (S_jF)_\epsilon$, $i \neq j$, remain convex and mutually disjoint as $\epsilon \to 0$.

seen that

$$R_0(\epsilon) = \begin{cases} C_0(F_\epsilon) & = \quad 1 \quad \text{for } \frac{1}{2} \le \epsilon \\ C_0(F_\epsilon) - \sum_i C_0((S_iF)_\epsilon) & = \ -2 \quad \text{for } u \le \epsilon < \frac{1}{2} \\ -\sum_{i\neq j} C_0((S_iF)_\epsilon \cap (S_jF)_\epsilon) & = \ -3 \quad \text{for } \epsilon < u \end{cases}. \tag{1.3.1}$$

Since here $s = \frac{\log 3}{\log 2}$ and $\eta = \log 2$, integration according to formula (1.2.3) yields

$$X_0 = -\frac{u^s}{\eta s} = -\frac{u^s}{\log 3} \approx -0.098\,.$$

Being different from zero, X_0 is, by Corollary 1.2.7, the value of the 0-th average fractal curvature $\overline{C}_0^f(F)$ of F.

For the case $k = 1$, we use the interpretation that $C_1(K)$ is half the boundary length of K. Looking at Figure 1.3, it is easily seen that

$$R_1(\epsilon) = \begin{cases} C_1(F_\epsilon) & = \frac{3}{2} + \pi\epsilon & \text{for } \frac{1}{2} \le \epsilon \\ C_1(F_\epsilon) - \sum_i C_1((S_iF)_\epsilon) & = -\frac{3}{4} - 2\pi\epsilon & \text{for } u \le \epsilon < \frac{1}{2} \\ -\sum_{i\neq j} C_1((S_iF)_\epsilon \cap (S_jF)_\epsilon) & = -(2\pi + 3\sqrt{3})\epsilon & \text{for } \epsilon < u \end{cases}. \tag{1.3.2}$$

Therefore, by formula (1.2.3),

$$X_1 = \frac{1}{\eta}\left(\frac{3}{4}\frac{u^{s-1}}{s-1} + 3\sqrt{3}\frac{u^s}{s}\right) = \frac{3}{4\log\frac{3}{2}}u^{s-1} + \frac{3\sqrt{3}}{\log 3}u^s \approx 0.82\,,$$

which is obviously non-zero and thus $\overline{C}_1^f(F) = X_1$. Similarly for $k = 2$,

$$R_2(\epsilon) = \begin{cases} C_2(F_\epsilon) & = \frac{\sqrt{3}}{4} + 3\epsilon + \pi\epsilon^2 & \text{for } \frac{1}{2} \le \epsilon \\ C_2(F_\epsilon) - \sum_i C_2((S_iF)_\epsilon) & = \frac{\sqrt{3}}{16} - \frac{3}{2}\epsilon - 2\pi\epsilon^2 & \text{for } u \le \epsilon < \frac{1}{2} \\ -\sum_{i\neq j} C_2((S_iF)_\epsilon \cap (S_jF)_\epsilon) & = -(2\pi + 3\sqrt{3})\epsilon^2 & \text{for } \epsilon < u \end{cases} \tag{1.3.3}$$

and so the average Minkowski content of F (which always exists) is

$$\overline{C}_2^f(F) = X_2 = -\frac{\sqrt{3}}{16\log\frac{3}{4}}u^{s-2} + \frac{3}{2\log\frac{3}{2}}u^{s-1} - \frac{3\sqrt{3}}{\log 3}u^s \approx 1.81.$$

A more explicit formula for X_k. Before we continue with further examples, we provide a more convenient formula for X_k, which can reduce the amount of calculation to be carried out. As in the previous example of the Sierpinski gasket the scaling functions R_k are often, though not always, piecewise polynomials of degree at most k. Here we simply assume such a behaviour for R_k.

Lemma 1.3.2. *Let F be a self-similar set with similarity dimension s and polyconvex parallel sets. Let $k \in \{0, \dots, d\}$ and, in case s is an integer, assume $k < s$. Suppose there are numbers $J \in \mathbb{N}$ and $0 = u_0 < u_1 < u_2 < \dots < u_J < u_{J+1} = 1$ such that the function R_k has a polynomial expansion of degree at most k in the interval (u_j, u_{j+1}) for each $j = 0, \dots, J$, i.e. there are coefficients $a_{j,l} \in \mathbb{R}$ such that*

$$R_k(\epsilon) = \sum_{l=0}^{k} a_{j,l}\epsilon^l \text{ for } \epsilon \in (u_j, u_{j+1}).$$

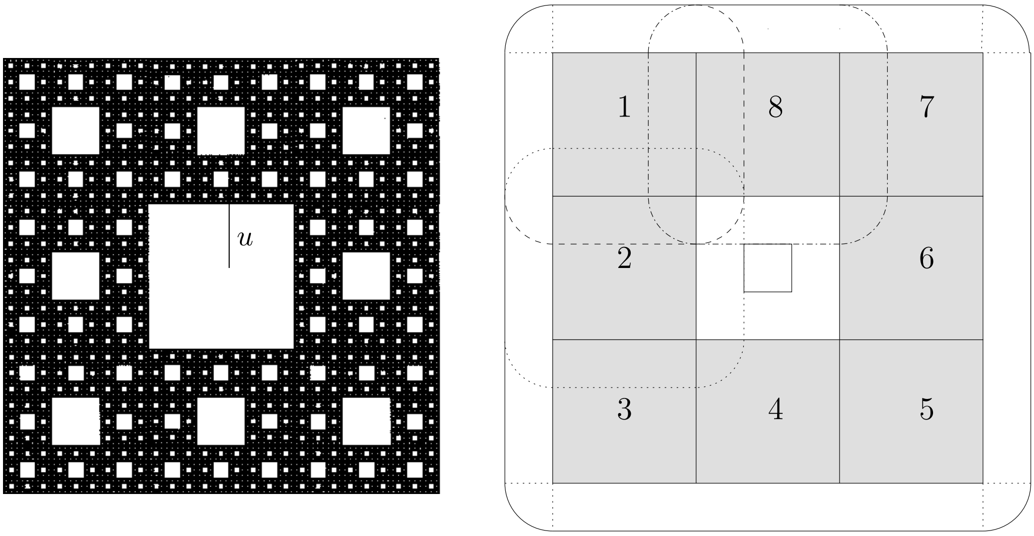

Figure 1.4.: Sierpinski carpet Q and a parallel set of Q for $\epsilon = \frac{1}{9}$.

Then, setting $a_{J+1,l} := 0$ *for each* $l = 0, \ldots, k$*, it holds*

$$X_k = \frac{1}{\eta} \sum_{l=0}^{k} \frac{1}{s-k+l} \sum_{j=0}^{J} (a_{j,l} - a_{j+1,l}) u_{j+1}^{s-k+l}. \tag{1.3.4}$$

Proof. For a proof of formula (1.3.4), write the integral in formula (1.2.3) as a sum of integrals over the intervals (u_{j+1}, u_j) and then plug in the polynomial expansions of R_k.

$$\eta X_k = \sum_{j=0}^{J} \int_{u_j}^{u_{j+1}} \epsilon^{s-k-1} \sum_{l=0}^{k} a_{j,l} \epsilon^l d\epsilon = \sum_{j=0}^{J} \sum_{l=0}^{k} a_{k,j,l} \int_{u_j}^{u_{j+1}} \epsilon^{s-k-1+l} d\epsilon$$

Integration yields $\frac{1}{s-k+l}(u_{j+1}^{s-k+l} - u_j^{s-k+l})$ for the term with indices j and l (provided that $s - k + l \neq 0$, which is the case since we assumed s being non-integer or $k < s$) and so, by exchanging the order of summation,

$$\eta X_k = \sum_{l=0}^{k} \frac{1}{s-k+l} \left(\sum_{j=0}^{J} a_{j,l} u_{j+1}^{s-k+l} - \sum_{j=0}^{J} a_{j,l} u_j^{s-k+l} \right).$$

By rearranging the index j in the second sum and summarizing the terms with equal j, formula (1.3.4) easily follows. □

Example 1.3.3. (Sierpinski carpet Q)

The Sierpinski carpet Q is the well known self-similar set in Figure 1.4 generated by 8 similarities S_i each mapping the unit square $[0,1]^2$ with contraction ratio $\frac{1}{3}$ to one of the smaller outer squares. Q has similarity dimension $s = \frac{\log 8}{\log 3}$ and is $\log 3$-arithmetic. Thus we can only expect the average fractal curvatures to exist. Indeed, all three of them exist and are different from zero as the computations below show.

In $(0,1)$ the scaling functions have discontinuities at $\frac{1}{3}$, the switching point of the indicator functions, and $\frac{1}{6}$, the inradius of the middle cut out square. Between these points, the intersection structure of the sets $Q_\epsilon^i := (S_i Q)_\epsilon$ and the symmetries suggest to compute R_k as follows:

$$R_k(\epsilon) = \begin{cases} C_k(Q_\epsilon) & \text{for } \frac{1}{3} \le \epsilon \\ C_k(Q_\epsilon) - \sum_i C_k(Q_\epsilon^i) & \text{for } \frac{1}{6} \le \epsilon < \frac{1}{3} \\ -\sum_{i\neq j} C_k(Q_\epsilon^i \cap Q_\epsilon^j) + \sum_{i\neq j\neq l} C_k(Q_\epsilon^i \cap Q_\epsilon^j \cap Q_\epsilon^l) & \text{for } \epsilon < \frac{1}{6} \end{cases}$$

By symmetry arguments, R_k simplifies for $\epsilon < \frac{1}{6}$ to

$$R_k(\epsilon) = -8C_k(Q_\epsilon^1 \cap Q_\epsilon^2) - 4C_k(Q_\epsilon^8 \cap Q_\epsilon^2) + 4C_k(Q_\epsilon^8 \cap Q_\epsilon^1 \cap Q_\epsilon^2)\,.$$

Now for each scaling function the polynomials for each interval are easily determined (cf. Figure 1.4) and we obtain

$$R_0(\epsilon) = \begin{cases} 1 \\ -7 \\ -8 \end{cases}, \quad R_1(\epsilon) = \begin{cases} 2 + \pi\epsilon \\ -\frac{10}{3} - 7\pi\epsilon \\ -\frac{8}{3} - (7\pi + 4)\epsilon \end{cases}$$

$$\text{and } R_2(\epsilon) = \begin{cases} 1 + 4\epsilon + \pi\epsilon^2 \\ \frac{1}{9} - \frac{20}{3}\epsilon - 7\pi\epsilon^2 \\ -\frac{16}{3}\epsilon - (7\pi+4)\epsilon^2 \end{cases} \quad \text{for} \begin{cases} \frac{1}{3} \le \epsilon \\ \frac{1}{6} \le \epsilon < \frac{1}{3} \\ \epsilon < \frac{1}{6} \end{cases}, \text{ respectively.}$$

Using formula (1.3.4) we can now compute X_0, X_1 and X_2. Note that $\eta = \log 3$.

$$\begin{aligned} X_0 &= -\tfrac{1}{\log 3}\tfrac{1}{s}\left(\tfrac{1}{6}\right)^s \approx -0.0162 \\ X_1 &= \tfrac{4}{\log 3}\left(\tfrac{1}{s-1} - \tfrac{1}{s}\right)\left(\tfrac{1}{6}\right)^s \approx 0.0725 \\ X_2 &= \tfrac{4}{\log 3}\left(\tfrac{1}{s-2} + \tfrac{2}{s-1} - \tfrac{1}{s}\right)\left(\tfrac{1}{6}\right)^s \approx 1.352 \end{aligned}$$

In the following example we modify the Sierpinski carpet to obtain a self-similar set with equal dimension but a different geometric and topological structure.

Example 1.3.4. (Modified carpet)
The self-similar set M in Figure 1.5 is generated by 8 similarities S_i each mapping the unit square with contraction ratio $\frac{1}{3}$ to one of the 8 small squares leaving out the upper middle one. This time some of the similarities include some rotation by $\pm\frac{\pi}{2}$ or π as indicated. Like the Sierpinski carpet, M has similarity dimension $s = \frac{\log 8}{\log 3}$ and is $\log 3$-arithmetic. We compute the average fractal curvatures of M and compare them to those of the Sierpinski carpet.

In $(0,1)$ the scaling function R_0 has a discontinuity at $\frac{1}{3}$and one at $\frac{1}{18}$, since for $\epsilon < \frac{1}{18}$ the first holes appear in M. With similar arguments as for the Sierpinski carpet we obtain that

$$R_0(\epsilon) = \begin{cases} 1 & \text{for } \frac{1}{3} \le \epsilon \\ -7 & \text{for } \frac{1}{18} \le \epsilon < \frac{1}{3} \\ -14 & \text{for } \epsilon < \frac{1}{18} \end{cases},$$

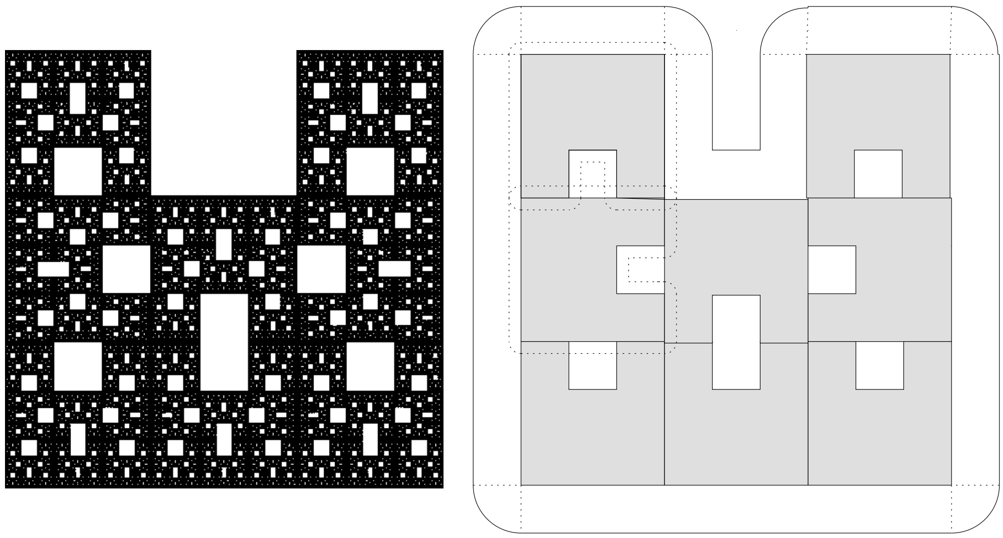

Figure 1.5.: Modified Sierpinski carpet M and some parallel sets

and so integration according to formula (1.2.3) yields

$$\overline{C}_0^f(M) = X_0 = -\frac{1}{s\log 3}\frac{7}{8}\left(\frac{1}{6}\right)^s \approx -0.014\,.$$

For the cases $k=1$ and $k=2$, the situation is more complicated. First observe that

$$C_1(F_\epsilon) = \begin{cases} \frac{11}{6} + (\pi + \arcsin\frac{1}{6\epsilon})\epsilon & \text{for } \frac{1}{6} \le \epsilon \\ \frac{7}{3} + (\frac{3}{2}\pi - 2)\epsilon & \text{for } \frac{1}{18} \le \epsilon < \frac{1}{6} \end{cases}$$

and similarly for $i = 1, \dots, 8$,

$$C_1((S_iM)_\epsilon) = \frac{11}{18} + \left(\pi + \arcsin\frac{1}{18\epsilon}\right)\epsilon \quad \text{for } \frac{1}{18} \le \epsilon\,.$$

From these two equations $R_1(\epsilon)$ can be determined in the interval $[\frac{1}{18}, 1)$ by means of the relation

$$R_1(\epsilon) = C_1(F_\epsilon) - 8\ C_1((S_iM)_\epsilon)\ \mathbf{1}_{(0,\frac{1}{3}]}(\epsilon)\,.$$

Obviously, this time R_1 is not piecewise a polynomial as in the previous examples. For $\epsilon < \frac{1}{18}$, we derive R_1 from the intersections of the $(S_iM)_\epsilon$ and obtain

$$R_1(\epsilon) = -\frac{20}{9} - 7\left(\frac{3}{2}\pi + 2\right)\epsilon\,.$$

Integrating R_1 according to (1.2.3) yields

$$\overline{C}_1^f(M) = X_1 \approx 0.0720\,.$$

Similarly, for $k=2$ we determine the area of F_ϵ:

$$C_2(F_\epsilon) = \begin{cases} 1 + \frac{11}{3}\epsilon + (\pi + \arcsin\frac{1}{6\epsilon})\epsilon^2 + \frac{1}{6}\sqrt{\epsilon^2 - \left(\frac{1}{6}\right)^2} & \text{for } \frac{1}{6} \le \epsilon \\ \frac{8}{9} + \frac{14}{3}\epsilon + (\frac{3}{2}\pi - 2)\epsilon^2 & \text{for } \frac{1}{18} \le \epsilon < \frac{1}{6} \end{cases}$$

and for $i = 1, \ldots, 8$,

$$C_2((S_iM)_\epsilon) = \frac{1}{9} + \frac{11}{9}\epsilon + \left(\pi + \arcsin\frac{1}{18\epsilon}\right)\epsilon^2 + \frac{1}{18}\sqrt{\epsilon^2 - \left(\frac{1}{18}\right)^2} \text{ for } \frac{1}{18} \le \epsilon\,.$$

Now $R_2(\epsilon)$ can be derived for $\epsilon \in [\frac{1}{18}, 1)$ using that

$$R_2(\epsilon) = C_2(F_\epsilon) - 8\ C_2((S_iM)_\epsilon)\ \mathbf{1}_{(0,\frac{1}{3}]}(\epsilon).$$

For $\epsilon < \frac{1}{18}$, we look again at the intersections of the $(S_iM)_\epsilon$ and obtain

$$R_2(\epsilon) = -\frac{40}{9}\epsilon - 7\left(\frac{3}{2}\pi + 2\right)\epsilon^2.$$

Integrating R_2 according to (1.2.3) yields

$$\overline{C}_2^f(M) = X_2 \approx 1.3439\,.$$

Comparison of the carpets. We summarize the approximate values determined above for the fractal curvatures of the two carpets:

	C_0^f	C_1^f	C_2^f
Sierpinski carpet	−0.016	0.0725	1.352
Modified carpet	−0.014	0.0720	1.344

The corresponding fractal curvatures are different. Hence they can be used to distinguish both sets. On the other hand the values are rather close to each other, which corresponds to the impression that geometrically the sets are not very different. More investigations are necessary to understand, whether fractal curvature are useful characteristics for the distinction and classification of fractal sets, and whether they have some more explicit geometric interpretation. In particular it would be interesting to see, whether sets with very similar geometric structure also have fractal curvatures which are very close to each other, i.e. whether there is some kind of continuity.

In contrast to the cube Q in Example 1.2.5 it can happen, that for a self-similar set X_k equals zero but nevertheless $s_k = s - k$. The next example is a set, for which $X_0 = 0$ but $s_0 = s$. It also clarifies the difference between s_0 and the Euler exponent σ, defined in [22].

Example 1.3.5. (Sierpinski tree)
Let the set F be generated by three similarities S_1, S_2 and S_3. S_1 has contraction ratio $\frac{4}{5}$ and shrinks an equilateral triangle of diameter 1 towards one of its corners, while the other two similarities S_2 and S_3 map the triangle with ratio $\frac{1}{5}$ to the remaining two corners, including a rotation by $\frac{2\pi}{3}$ and $-\frac{2\pi}{3}$, respectively (cf. Figure 1.6). F has similarity dimension s given by $5^s - 4^s = 2$ and is non-arithmetic, since $\frac{\log 5 - \log 4}{\log 5}$ is not rational. Therefore, this time we can try to determine the fractal curvatures rather than

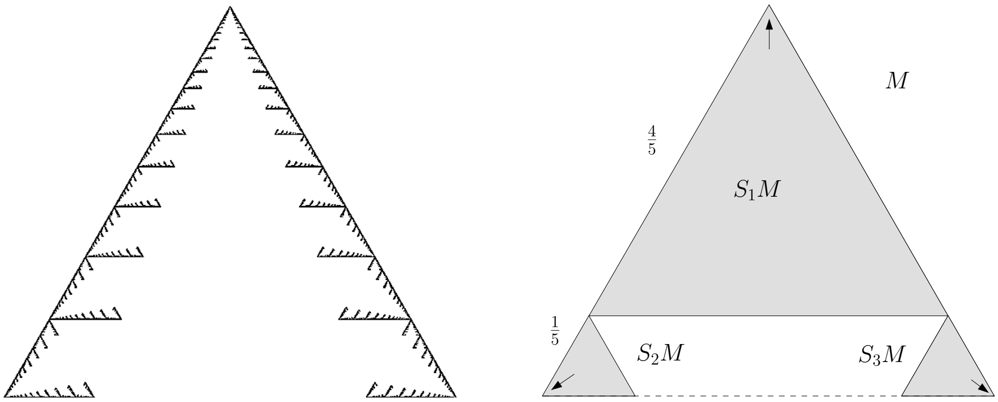

Figure 1.6.: The Sierpinski tree F and how it is generated. The right picture indicates how the similarities generating F act on its convex hull M. The arrows indicate to which points the upper corner of M is mapped.

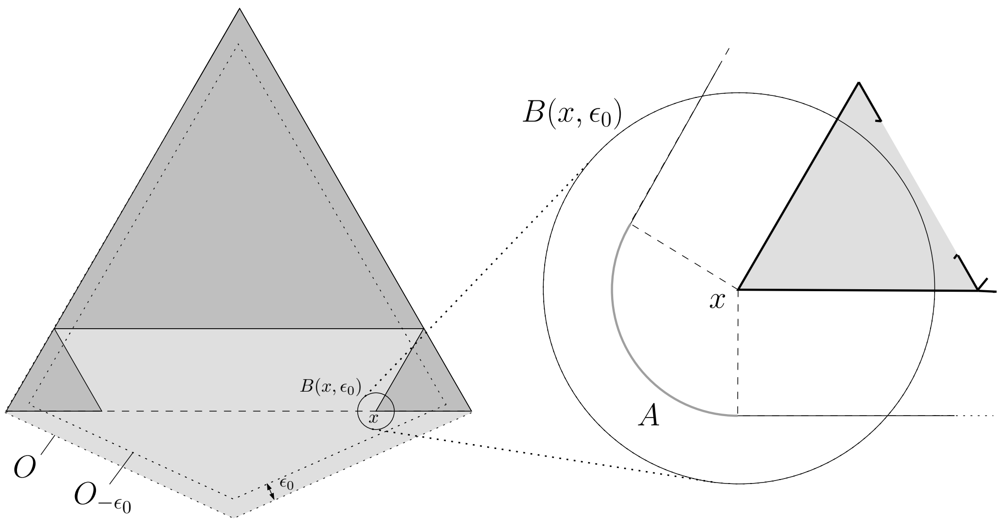

Figure 1.7.: Feasible open set O of the Sierpinski tree F and some inner parallel set $O_{-\epsilon_0}$. The enlarged part on the right shows some detail of the ϵ-parallel set of F near x for some $\epsilon < \epsilon_0$. The arc $A \subset \partial F_\epsilon$ contributes $\frac{1}{3}$ to the 0-th curvature of F_ϵ.

just the averaged counterparts. We compute $C_0^f(F)$, for which we first determine the scaling function R_0. The only discontinuities of $R_0(\epsilon)$ in $(0,1)$ are $\frac{4}{5}$ and $\frac{1}{5}$ and so

$$R_0(\epsilon) = \begin{cases} C_0(F_\epsilon) & = & 1 & \text{for } \frac{4}{5} \le \epsilon \\ C_0(F_\epsilon) - \sum_i C_0((S_iF)_\epsilon) & = & 0 & \text{for } \frac{1}{5} \le \epsilon < \frac{4}{5} \\ -C_0((S_1F)_\epsilon \cap (S_2F)_\epsilon) - C_0((S_1F)_\epsilon \cap (S_3F)_\epsilon) & = & -2 & \text{for } \epsilon < \frac{1}{5} \end{cases} .$$

By formula (1.3.4),

$$X_0 = \frac{1}{\eta s}\left(1 - \left(\frac{4}{5}\right)^s - 2\left(\frac{1}{5}\right)^s\right) = 0 \,.$$

Unfortunately, Corollary 1.2.7 does not allow to conclude now directly that $s_0 = s$ and thus $C_0^f(F) = 0$. But this is in fact true and will be derived from Theorem 1.2.8. Let x and O be as indicated in Figure 1.6. Note that O is a feasible open set of F. Choose some $\epsilon_0 < \frac{1}{20}$ and let $B = B(x, \epsilon_0)$ the ball with center x and radius ϵ_0. It is not difficult to see that $B \subset O_{-\epsilon_0}$, since $d(x, \partial O) = \frac{1}{10}$. Moreover, for $\epsilon \le \epsilon_0$, $C_0^{\text{var}}(F_\epsilon, B) \ge \frac{1}{3}$, since the set $\partial F_\epsilon \cap B$ does always contain the arc A whose length is $\frac{1}{3}$ of the perimeter of the circle with center x and radius ϵ. This contributes the amount of $\frac{1}{3}$ to the mass of $C_0^+(F_\epsilon, B)$ and thus to $C_0^{\text{var}}(F_\epsilon, B)$. Now Theorem 1.2.8 implies that $\epsilon^s C_0^{\text{var}}(F_\epsilon) \ge c$ for $\epsilon < \epsilon_0$ (where $c = \frac{1}{3}\left(\frac{\epsilon_0}{5}\right)^s$) and so, by Corollary 1.2.9, $s_0 = s$. Hence $C_0^f(F) = 0$ as claimed.

The above example contrasts Example 1.2.5, where we also had $X_0 = 0$ but $s_0 < s$. While for the parallel sets of the cube Q not only the 0-th total curvature $C_0(Q_\epsilon)$ remains bounded as $\epsilon \to 0$ but also the local 0-th curvature, here locally the curvature grows as $\epsilon \to 0$ and only the total curvature $C_0(F_\epsilon)$ remains bounded (in fact constant). For all ϵ, the positive curvature of F_ϵ equals out its negative curvature. In [22] we considered the Euler characteristic, which equals to the 0-th total curvature. It does not "see" the local behaviour of the curvature and therefore we obtain $\sigma = 0$ and $\chi^f(F) = 1$, which reflects somehow its topolocical structure (connected and simply connected) but not its "fractality" which is better revealed from $s_0 = s$ and $C_0^f(F) = 0$ (0-th curvature scales locally with ϵ^s but vanishes globally).

Remark 1.3.6. In Remark 1.1.4 we introduced scaling exponents s_k^+ and s_k^-, by taking in Definition 1.1.2 only the positive or the negative curvature, respectively, into account. In our examples in $\mathbb{R}^2$ this distinction does only make sense for $k = 0$. It is not difficult to see, that for the Sierpinski gasket or the carpets only the negative curvature $C_0^-(F_\epsilon)$ increases as $\epsilon \to 0$, while the positive curvature $C_0^+(F_\epsilon)$ remains bounded. Therefore, $s_0^- = s_0 = s$ and $s_0^+ = 0 < s_0$ in those examples. The situation is different for the Sierpinski tree. Here the negative and positive curvature grow with the same speed. Hence $s_0^+ = s_0^- = s_0 = s$.

1.4. Fractal curvature measures

With the convergence of rescaled total curvatures as discussed in Section 1.2 naturally the question arises how the corresponding (rescaled) curvature measures behave. It turns out that, for self-similar sets F, typically weak limits of rescaled curvature measures exist too. These limit measures should be regarded as the *fractal curvature measures* of F. We show that they are, in fact, all multiples of the same measure, namely the s-dimensional Hausdorff measure on F.

Rescaled curvature measures. As before, let $F \subset \mathbb{R}^d$ be a compact set such that for all $\epsilon > 0$, $F_\epsilon \in \mathcal{R}^d$. This ensures that the curvature measures $C_k(F_\epsilon, \cdot)$ are defined for each ϵ. We want to study the limiting behaviour of these measures as $\epsilon \to 0$. This is motivated by the question, whether it is possible to define some kind of curvature measures for fractals in this way. The weak convergence of signed measures seems to be the appropriate notion of convergence here. It is the straightforward generalization of the usual weak convergence of positive measures. For each $\epsilon > 0$, let μ_ϵ be a totally finite signed measure on $\mathbb{R}^d$. The measures μ_ϵ are said to *converge weakly* to a totally finite signed measure μ as $\epsilon \to 0$, $\mu_\epsilon \xrightarrow{w} \mu$, if and only if $\int_{\mathbb{R}^d} f d\mu_\epsilon \to \int_{\mathbb{R}^d} f d\mu$ for all bounded continuous functions f on $\mathbb{R}^d$. We refer to the Appendix for more details.

Since weak convergence is always accompanied by the convergence of the total masses of the measures, it is clear that the measures $C_k(F_\epsilon, \cdot)$ have to be rescaled with the factor ϵ^{s_k}, where s_k is the k-th scaling exponent as defined above. Therefore we make the following definitions: For each $\epsilon > 0$ and $k = 0, \ldots, d$ we define the *k-th rescaled curvature measure* $\nu_{k,\epsilon}$ of F_ϵ by

$$\nu_{k,\epsilon}(\,\cdot\,) := \epsilon^{s_k} C_k(F_\epsilon, \,\cdot\,). \tag{1.4.1}$$

In general, these measures need not converge weakly as $\epsilon \to 0$. We have seen above that often the *total mass* $\nu_{k,\epsilon}(\mathbb{R}^d) = \epsilon^{s_k} C_k(F_\epsilon)$ already fails to converge, which makes weak convergence impossible. Therefore, in analogy with Definition 1.1.6, we also define averaged versions $\overline{\nu}_{k,\epsilon}$ of the rescaled curvature measures $\nu_{k,\epsilon}$ by

$$\overline{\nu}_{k,\epsilon}(\,\cdot\,) := \frac{1}{|\log \epsilon|} \int_\epsilon^1 \widetilde{\epsilon}^{s_k} C_k(F_\epsilon, \,\cdot\,) \frac{d\widetilde{\epsilon}}{\widetilde{\epsilon}}. \tag{1.4.2}$$

For $k = d$ and $k = d-1$, the measure $C_k(F_\epsilon, \cdot)$ is known to be positive, hence so are $\nu_{k,\epsilon}$ and $\overline{\nu}_{k,\epsilon}$. In general, the measures $\nu_{k,\epsilon}$ and $\overline{\nu}_{k,\epsilon}$ are totally finite signed measures.

Weak convergence for self-similar sets. Let F be a self-similar set satisfying OSC. Denote by μ_F the normalized s-dimensional Hausdorff measure on F, i.e.

$$\mu_F = \frac{\mathcal{H}^s|_F(\,\cdot\,)}{\mathcal{H}^s(F)}. \tag{1.4.3}$$

It is well known that the OSC implies $0 < \mathcal{H}^s(F) < \infty$. Hence μ_F is well defined. As before, we additionally assume that the parallel sets of F are polyconvex.

We ask for the weak convergence of the rescaled curvature measures and their averaged versions and restrict this question to sets for which $s_k = s - k$, since for sets with $s_k < s - k$ we do not even have information on the convergence behaviour of the total masses of these measures. It is again necessary to distinguish between arithmetic and non-arithmetic self-similar sets F. The weak convergence of the measures $\nu_{k,\epsilon}$ as $\epsilon \to 0$

is only ensured in the non-arithmetic case, while the measures $\overline{\nu}_{k,\epsilon}$ converge in general. This is not very surprising, since the convergence of the total masses $\nu_{k,\epsilon}(\mathbb{R}^d)$ was ensured by Theorem 1.2.6 only for non-arithmetic sets, while the total masses $\overline{\nu}_{k,\epsilon}(\mathbb{R}^d)$ converge in general. So the value of the total mass of the limit measure (if it exists) must be $C_k^f(F)$ or $\overline{C}_k^f(F)$, respectively. The following statement gives an answer to when weak limits exist and what the limit measures are.

Theorem 1.4.1. *Let F be a self-similar set satisfying OSC and $F_\epsilon \in \mathcal{R}^d$, $k \in \{0, \dots, d\}$. Assume $s_k = s - k$. Then always*

$$\overline{\nu}_{k,\epsilon} \xrightarrow{\text{w}} \overline{C}_k^f(F)\ \mu_F \quad \text{as } \epsilon \to 0\,.$$

If $\{-\log r_1, \dots, -\log r_N\}$ is non-arithmetic, then

$$\nu_{k,\epsilon} \xrightarrow{\text{w}} C_k^f(F)\ \mu_F \quad \text{as } \epsilon \to 0\,.$$

For each k, the limit measure is some multiple of the measure μ_F. Note that the case $\overline{C}_k^f(F) = 0$ is included in this formulation. For this case the limit is the zero measure. Otherwise it is either a positve or a purely negative measure, depending on the signum of the factor $C_k^f(F)$ or $\overline{C}_k^f(F)$, respectively. The limit measure $C_k^f(F)\ \mu_F$ (or $\overline{C}_k^f(F)\ \mu_F$, respectively) should be regarded as the *k-th fractal curvature measure* of F. The theorem states that the $d+1$ fractal curvature measures of F all coincide up to some constant factors. Taking into account the self-similarity of the considered sets, it is not very surprising that all fractal curvature measures essentially coincide with μ_F. Any measure on a self-similar set F describing its geometry should respect the self-similar structure of F. But the self-similar measures on F are well known and so it is not surprising to retrieve them here. The proof of this theorem will be provided in Section 5.2.

Weak limits of the parallel volume. For $k = d$, Theorem 1.4.1 can be generalized to arbitrary self-similar sets satisfying OSC. By replacing $C_d(F_\epsilon, \cdot\,)$ with the Lebesgue measure, we can again drop the assumption of polyconvexity for the parallel sets F_ϵ. The definition of the rescaled measure $\nu_{d,\epsilon}$ in (1.4.1) generalizes to arbitrary compact sets $F \subset \mathbb{R}^d$ by setting

$$\nu_{d,\epsilon}(\,\cdot\,) := \epsilon^{s_d} \lambda_d(F_\epsilon \cap \,\cdot\,). \tag{1.4.4}$$

Similarly, (1.4.2) generalizes to

$$\overline{\nu}_{d,\epsilon}(\,\cdot\,) := \frac{1}{|\log \epsilon|} \int_\epsilon^1 \widetilde{\epsilon}^{s_d} \lambda_d(F_\epsilon \cap \,\cdot\,) \frac{d\widetilde{\epsilon}}{\widetilde{\epsilon}}. \tag{1.4.5}$$

We call $\nu_{d,\epsilon}$ the *rescaled ϵ-parallel volume* and $\overline{\nu}_{d,\epsilon}$ the *average rescaled ϵ-parallel volume* of F, respectively. If some weak limit of these measures exist, as $\epsilon \to 0$, then the total mass of the limit measure must coincide with the Minkowski content $M(F)$ or its average counterpart $\overline{M}(F)$, respectively. Indeed, such limit measures exist as the following statement shows.

Theorem 1.4.2. *Let F be a self-similar set satisfying OSC. Then always*

$$\overline{\nu}_{d,\epsilon} \xrightarrow{\text{w}} \overline{M}(F)\ \mu_F \quad \text{as } \epsilon \to 0\,.$$

If F is non-arithmetic, then also

$$\nu_{d,\epsilon} \xrightarrow{\text{w}} M(F)\ \mu_F \qquad \text{as } \epsilon \to 0\,.$$

This result extends Gatzouras's theorem. Not only the total (average) ϵ-parallel volume of self-similar sets converges, as $\epsilon \to 0$. The convergence happens even locally in every "nice" subset of $\mathbb{R}^d$. This is the meaning of weak convergence. More precisely, if $B \subset \mathbb{R}^d$ is a *μ_F-continuity set*, i.e. if $\mu_F(\partial B) = 0$, then $\nu_{d,\epsilon}(B) \to M(F)\ \mu_F(B)$ as $\epsilon \to 0$ for F non-arithmetic and $\overline{\nu}_{d,\epsilon}(B) \to \overline{M}(F)\mu_F(B)$ correspondingly for general F.

Normalized curvature measures We also want to explore another type of limit for the curvature measures which avoids the averaging and yields convergence nevertheless. Although in our situation averaging is a natural procedure to improve the convergence behaviour, it is not the only possible one. Another way to overcome the problem of oscillations, which prevent the convergence, is to normalize the measures. Since normalization is only possible for positive and finite measures and since the rescaled curvature measures $\nu_{k,\epsilon}$ are in general signed measures for $k \in \{0, \ldots, d-2\}$, this does only make sense for $k = d-1$ and $k = d$. We discuss both cases separately, since for $k = d$ we can again obtain more general results.

The case $k = d - 1$. Define the $d-1$-*th normalized curvature measure* of F_ϵ by

$$\nu^1_{d-1,\epsilon}(\,\cdot\,) := \frac{\nu_{d-1,\epsilon}(\,\cdot\,)}{\nu_{d-1,\epsilon}(\mathbb{R}^d)} = \frac{C_{d-1}(F_\epsilon, \,\cdot\,)}{C_{d-1}(F_\epsilon)}.$$

Theorem 1.4.3. *Let F be a self-similar set satisfying OSC and $F_\epsilon \in \mathcal{R}^d$. Assume that $\underline{X}_{d-1} := \liminf_{\epsilon\to 0} \epsilon^{s-d+1} C_{d-1}(F_\epsilon) > 0$. Then*

$$\nu^1_{d-1,\epsilon} \xrightarrow{\text{w}} \mu_F \qquad \text{as } \epsilon \to 0\,.$$

Observe that the measures $\nu^1_{d-1,\epsilon}$ converge weakly even in the arithmetic case. No distinction is necessary between arithmetic and non-arithmetic self-similar sets. The normalization has a similar effect as the averaging. The additional assumption $\underline{X}_{d-1} > 0$ implies that in particular $X_{d-1} > 0$, since $X_{d-1} \geq \underline{X}_{d-1}$. Hence $s_{d-1} = s - d + 1$. In the non-arithmetic case this assumption is equivalent to $X_{d-1} > 0$. For arithmetic sets it is slightly stronger. The proof of Theorem 1.4.3 is given in Section 5.3.

In general, this result does not carry over to the non-normalized counterparts $\nu_{d-1,\epsilon}$. Under the conditons of Theorem 1.4.3, we obviously have the relation

$$\nu_{d-1,\epsilon} = \epsilon^{s_{d-1}} C_{d-1}(F_\epsilon)\ \nu^1_{d-1,\epsilon}.$$

In the arithmetic case, the convergence of the prefactor $\epsilon^{s_{d-1}} C_{d-1}(F_\epsilon)$ is not assured and so in general the existence of a weak limit of these measures can not be derived from this result.

The case $k = d$. Define the *normalized parallel volume* $\nu_{d,\epsilon}$ of F_ϵ by

$$\nu^1_{d,\epsilon}(\,\cdot\,) = \frac{\nu_{d,\epsilon}(\,\cdot\,)}{\nu_{d,\epsilon}(\mathbb{R}^d)} = \frac{\lambda_d(F_\epsilon \cap \,\cdot\,)}{\lambda_d(F_\epsilon)}.$$

The measures $\nu_{d,\epsilon}$ are well defined for each compact set $F \subseteq \mathbb{R}^d$ and $\epsilon > 0$. For self-similar sets F satisfying OSC the normalized parallel volume converges weakly to μ_F as the following Theorem states.

Theorem 1.4.4. *Let F be a self-similar set satisfying OSC and $F_\epsilon \in \mathcal{R}^d$. Then*

$$\nu^1_{d,\epsilon} \xrightarrow{\text{w}} \mu_F \qquad \text{as } \epsilon \to 0\,.$$

Again it is not necessary to distinguish between arithmetic and non-arithmetic self-similar sets. Here no additional assumptions are required, since, by Gatzouras's theorem, always $\underline{X}_d > 0$. The proof of Theorem 1.4.4 can be found in Section 5.3.

2. Curvature measures in the convex ring

Curvature measures were introduced by Herbert Federer in 1959 in his famous paper *Curvature measures* [8] for sets with positive reach by means of a local Steiner formula. A set $K \subseteq \mathbb{R}^d$ is said to have *positive reach* if there exist some $\epsilon > 0$ such that each point $x \in K_\epsilon$ has a unique nearest point in K. The *reach* of K is the supremum over all such ϵ. Convex sets are special sets with positive reach, namely those with infinite reach and are thus included in his theory. For convex sets many additional results have been established, for instance an axiomatic characterization of curvature measures by Rolf Schneider [28].

The *total curvatures*, also known as *intrinsic volumes* or *Minkowski functionals*, had been studied long before by Herbert Minkowski and others. Much of the theory was collected and developed further by Hugo Hadwiger in his famous book [13]. A milestone was to interpret intrinsic volumes as valuations and use their additivity to extend them to the convex ring. The corresponding additive extension of curvature measures to polyconvex sets is due to Helmut Groemer [12], while a more explicit characterization of curvature measures in the convex ring was given by Rolf Schneider [29], who generalized the local Steiner formula by weighting the parallel volume with an index function.

There has also been considerable progress in the development of curvature measures for non-convex (and non-polyconvex) sets since the time of Federer, which has not yet come to an end. On this subject, which is not in the scope of the theory required here, we refer to the survey paper of Andreas Bernig [2] and the references given therein.

Here we want to collect some facts about curvature measures in the convex ring. In particular, we outline the most important properties of curvature measures and their variation measures. For the latter we also derive some useful estimates, which we require later on. Our main references for the mostly well known statements in this chapter are the books of Rolf Scheider [30] and Rolf Scheider and Wolfgang Weil [31]. Also compare the monograph by Daniel A. Klain and Gian-Carlo Rota [17].

2.1. Properties of curvature measures

Local Steiner formula. For convex bodies K, curvature measures are usually introduced as the coefficients in the polynomial expansion of the local parallel volume of K. The *metric projection* π_K onto a convex set $K \in \mathcal{K}^d$ maps each point $x \in \mathbb{R}^d$ to its (uniquely determined) nearest neighbour in K. For fixed $K \in \mathcal{K}^d$ and $\epsilon > 0$ the set $K_\epsilon \cap \pi_K^{-1}(B)$ is regarded as the *local parallel set* of K with respect to the Borel set B. It consists of the two sets $K \cap B$ and $\{x \in K_\epsilon \setminus K : \pi_K(x) \in B\}$. The Steiner formula describes the volume of the local parallel set of K as a polynomial in ϵ. Let κ_k denote the k-dimensional volume of the unit ball in $\mathbb{R}^k$ and $\mathcal{H}^d$ the d-dimensional Hausdorff measure.

Theorem 2.1.1. *For each $K \in \mathcal{K}^d$ there exist uniquely determined finite Borel measures $C_0(K,\cdot)$, ..., $C_d(K,\cdot)$ on $\mathbb{R}^d$, called the* curvature measures *of K, such that*

$$\mathcal{H}^d(K_\epsilon \cap \pi_K^{-1}(B)) = \sum_{k=0}^{d} \epsilon^k \kappa_k C_{d-k}(K,B)$$

for each Borel set B and $\epsilon > 0$.

The proof is done in two steps. By decomposing polytopes into their faces, explicit expressions for their curvature measures and parallel volume can be derived. The general case follows by approximation of convex sets with polytopes.

Additivity. Curvature measures have the property of being *additive*. If $K, L \in \mathcal{K}^d$ such that also $K \cup L \in \mathcal{K}^d$, then

$$C_k(K \cup L, B) = C_k(K,B) + C_k(L,B) - C_k(K \cap L, B) \tag{2.1.1}$$

for each Borel set $B \subseteq \mathbb{R}^d$. Repeated application of this relation leads to the so called *inclusion-exclusion principle*: If $K^1, \dots, K^m$ and $K = \bigcup_{i=1}^m K^i$ are in $\mathcal{K}^d$, then for all Borel sets $B \subseteq \mathbb{R}^d$

$$C_k(K,B) = \sum_{I \in N_m} (-1)^{\#I-1} C_k\Big(\bigcap_{i \in I} K^i, B\Big). \tag{2.1.2}$$

Here N_m denotes the family of all nonempty subsets I of $\{1,\dots,m\}$, so that the sum runs through all intersections of the K^i.

Further properties. Curvature measures of convex bodies have several further properties which we summarize now. Let K, L denote convex bodies in $\mathcal{K}^d$ and $B \subseteq \mathbb{R}^d$ an arbitrary Borel set. For each $k \in \{0,\dots,d\}$ it holds:

Motion covariance: If g is an Euclidean motion, then $C_k(gK, gB) = C_k(K,B)$.

Homogeneity of degree k: For $\lambda > 0$, $C_k(\lambda K, \lambda B) = \lambda^k C_k(K,B)$.

Locality: If $K \cap A = L \cap A$ for some open set $A \subseteq \mathbb{R}^d$, then $C_k(K,B) = C_k(L,B)$ for all Borel sets $B \subseteq A$.

Continuity in the first argument: If $K^1, K^2, \dots \in \mathcal{K}^d$ such that $K^i \to K$ (in the Hausdorff metric) as $i \to \infty$ then $C_k(K^i,\cdot) \xrightarrow{\text{w}} C_k(K,\cdot)$. In particular, $C_k(K^i) \to C_k(K)$.

Monotonicity of the total curvature: If $K, L \in \mathcal{K}^d$ and $K \subseteq L$, then $C_k(K) \le C_k(L)$.

The above properties allow to characterize curvature measures axiomatically. According to Schneider [28], any continuous, additive and motion covariant functional Φ assigning to each convex body $K \in \mathcal{K}^d$ a finite measure $\Phi(K,\cdot)$ is a linear combination of the curvature measures C_k, i.e. there exist constants $c_0, \dots, c_d$ such that $\Phi = \sum_k c_k C_k$. Demanding additionally homogeneity of degree k from Φ is sufficient to characterize it as a constant multiple of the k-th curvature measure.

Additive extension to the convex ring. In case the set K on the left hand side of (2.1.2) is not convex, the measure $C_k(K,\cdot)$ is not defined by the local Steiner formula. But then the right hand side could be regarded as its definition. This leads to the additive extension of curvature measures to the convex ring. Helmut Groemer showed in [12] that this extension is indeed possible, i.e. the so defined measures $C_k(K,\cdot)$ do not depend on the chosen representation of K by convex sets K^i.

Theorem 2.1.2. *For each $k \in \{0,\dots,d\}$, the k-th curvature measure C_k has a unique additive extension to $\mathcal{R}^d$.*

Curvature measures of polyconvex sets are in general signed measures, in contrast to the convex case. However, for $k = d$ and $d-1$, $C_k(K,\cdot)$ is a positive measure for each $K \in \mathcal{R}^d$. Most of the properties of curvature measures of convex sets generalize to polyconvex sets. By definition, they are additive and so the inclusion-exclusion principle (2.1.2) is valid for all $K, K^i \in \mathcal{R}^d$. Moreover, for each $k = 0,\dots,d$ the k-th curvature measure is motion covariant, homogeneous of degree k and has the locality property. Continuity and Monotonicity property do not generalize to polyconvex sets, which is easily seen from simple counter-examples.

Since each similarity S is the composition of an Euclidean motion and a homothety with some ratio $r > 0$, the k-th curvature measure has the following *scaling property* with respect to S:

$$C_k(SK, SB) = r^k C_k(K,B) \tag{2.1.3}$$

for $K \in \mathcal{R}^d$ and any Borel set $B \in \mathbb{R}^d$.

Curvature of parallel sets. Since any parallel set of a convex set is again convex, the parallel sets of polyconvex sets are polyconvex as well (compare Fact 1.1.1) and their curvature measures are defined. For sets $K \in \mathcal{R}^d$ we are particularly interested in the continuity properties of the total curvatures $C_k(K_\epsilon)$ as a function of ϵ. The statement below is a consequence of the weak continuity of curvature measures for convex sets.

Lemma 2.1.3. *For $K \in \mathcal{R}^d$ and $k \in \{0,\dots,d\}$, $C_k(K_\epsilon)$, as a function of ϵ, has a finite set of discontinuities in $(0,\infty)$ and $\lim_{\epsilon\to 0} C_k(K_\epsilon) = C_k(K)$. (For $k = d$, $C_k(K_\epsilon)$ is even continuous in $(0,\infty)$.)*

Proof. Let $K^1,\dots,K^m \in \mathcal{K}^d$ be sets such that $K = \bigcup_{i=1}^m K^i$. Then by the inclusion-exclusion principle,

$$C_k(K_\epsilon) = \sum_{I \in N_m} (-1)^{\#I-1} C_k\left(\bigcap_{i\in I} K^i_\epsilon\right), \tag{2.1.4}$$

where the sets $\bigcap_{i\in I} K^i_\epsilon$ are convex (possibly empty) for all $\epsilon \geq 0$. (Here $K_0 = K$.) More precisely, for each I there exists $\epsilon_I \geq 0$ such that $\bigcap_{i\in I} K^i_\epsilon = \varnothing$ for all $0 \leq \epsilon < \epsilon_I$ and $\bigcap_{i\in I} K^i_\epsilon \neq \varnothing$ for all $\epsilon \geq \epsilon_I$. Now the continuity property implies that $C_k(\bigcap_{i\in I} K^i_\epsilon)$ is continuous in (ϵ_I,∞) and continuous from the right in ϵ_I. Moreover, $C_k(\bigcap_{i\in I} K^i_\epsilon) \equiv 0$ in $[0,\epsilon_I)$ and thus the only possible discontinuity point in $(0,\infty)$ is ϵ_I. Since this holds for every $I \in N_m$, by (2.1.4), $C_k(K_\epsilon)$ has finitely many discontinuities in $(0,\infty)$ (at most $\#N_m$). In particular, since always $C_k(\bigcap_{i\in I} K^i_\epsilon) \to C_k(\bigcap_{i\in I} K^i)$ as $\epsilon \to 0$, we conclude that $C_k(K_\epsilon) \to C_k(K)$. □

2.2. Variation measures

Some of the above properties of curvature measures carry over to the corresponding variation measures. Recalling that the positive, negative and total variation of the measure $C_k(K, \cdot\,)$ are given by

$$C_k^+(K,B) = \sup_{B'\subseteq B} C_k(K,B'), \quad C_k^-(K,B) = -\inf_{B'\subseteq B} C_k(K,B')$$

and $C_k^{\mathrm{var}}(K,B) = C_k^+(K,B) + C_k^-(K,B)$ respectively, for each Borel set $B \subseteq \mathbb{R}^d$, the following is easily seen.

Proposition 2.2.1. *For $k \in \{0,\ldots,d\}$ and $K \in \mathcal{R}^d$ the measures $C_k^+(K,\cdot\,)$, $C_k^-(K,\cdot\,)$ and $C_k^{\mathrm{var}}(K,\cdot\,)$ are motion covariant, homogeneous of degree k and have the locality property.*

Proof. Let g a Euclidean motion. The motion covariance of $C_k^+(K,\cdot\,)$ follows from the same property of $C_k(K,\cdot\,)$ observing that for all Borel sets B

$$C_k^+(gK,gB) = \sup_{gB'\subseteq gB} C_k(gK,gB') = \sup_{B'\subseteq B} C_k(K,B') = C_k^+(K,B).$$

The remaining properties are verified in a similar way, as well for $C_k^-(K,\cdot\,)$. Then, for $C_k^{\mathrm{var}}(K,\cdot\,)$, they follow immediately from the relation $C_k^{\mathrm{var}}(K,B) = C_k^+(K,B) + C_k^-(K,B)$. □

Note that also the scaling property remains valid for the variation measures: If $k \in \{0,\ldots,d\}$, $\bullet \in \{+,-,\mathrm{var}\}$ and S is a similarity, then

$$C_k^\bullet(SK,SB) = r^k C_k^\bullet(K,B) \tag{2.2.1}$$

for each $K \in \mathcal{R}^d$ and any Borel set $B \subseteq \mathbb{R}^d$.

It would be very useful also to have a generalization of Lemma 2.1.3 to the variation measures of a curvature measure. Unfortunately, we have no idea how to prove this. Nevertheless, we believe that the following statement is true.

Conjecture 2.2.2. For $K \in \mathcal{R}^d$, $k \in \{0,\ldots,d-2\}$ and $\bullet \in \{+,-,\mathrm{var}\}$, $C_k^\bullet(K_\epsilon)$, as a function of ϵ, has a finite set of discontinuities in $(0,\infty)$ and $\lim_{\epsilon\to 0} C_k(K_\epsilon) = C_k(K)$.

Since for $k = d-1$ and $k = d$, $C_k(K,\cdot\,)$ is a positive measure, the situation is clear for those cases. A verification of this statement would lead to some extensions of the main results to the variation measures and simplify the proofs in Chapter 5. For more details confer Remark 4.5.2.

Estimates for the total variation measure. Above we derived some properties of the variation measures directly from the corresponding properties of the underlying curvature measure. Unfortunately, additivity is not among them. Indeed, the variation measures of a curvature measure fail to be additive in general. Therefore, we now derive some inequalities for the total variation measures, which, in a way, take over the role the inclusion-exclusion principle plays for curvature measures.

Let $K^j \in \mathcal{R}^d$ for $j = 1, \dots, m$ and $K = \bigcup_{j=1}^m K^j$. Recall that N_m was the family of all nonempty subsets of $\{1, \dots, m\}$. For each $I \in N_m$ write $K(I) := \bigcap_{j \in I} K^j$. The additivity of curvature measures allows to derive some estimates for their variation measures.

Lemma 2.2.3. *Let $K^j \in \mathcal{R}^d$ for $j = 1, \dots, m$ and $K = \bigcup_{j=1}^m K^j$. Then*

$$C_k^{\mathrm{var}}(K, B) \leq \sum_{I \in N_m} C_k^{\mathrm{var}}(K(I), B) \tag{2.2.2}$$

for each Borel set B. If the sets K^j are convex then

$$C_k^{\mathrm{var}}(K, B) \leq (2^m - 1) \max_j C_k(K^j). \tag{2.2.3}$$

Note that, since $C_k^{\pm}(K, \cdot) \leq C_k^{\mathrm{var}}(K, \cdot)$, estimate (2.2.3) remains valid if $C_k^{\mathrm{var}}(K, B)$ on the left hand side is replaced with $C_k^+(K, B)$ or $C_k^-(K, B)$. In general, inequality (2.2.2) does not remain valid when the total variation is replaced with the positive or negative variation measure.

Proof. Let $\mathbb{R}^d = K^+ \cup K^-$ be a Hahn decomposition of the signed measure $C_k(K, \cdot)$, i.e. K^+ and K^- are disjoint sets satisfying $C_k^+(K, K^-) = C_k^-(K, K^+) = 0$ (cf. Appendix, Theorem A.1.1). By the inclusion-exclusion formula, it holds

$$C_k^{\pm}(K, B) = (\pm 1) C_k(K, B \cap K^{\pm}) = (\pm 1) \sum_{I \in N_m} (-1)^{\# I - 1} C_k(K(I), B \cap K^{\pm}),$$

and, since $|C_k(K, \cdot)| \leq C_k^{\mathrm{var}}(K, \cdot)$,

$$C_k^{\pm}(K, B) \leq \sum_{I \in N_m} C_k^{\mathrm{var}}(K(I), B \cap K^{\pm}).$$

Hence we obtain

$$\begin{aligned} C_k^{\mathrm{var}}(K, B) &= C_k^+(K, B) + C_k^-(K, B) \\ &\leq \sum_{I \in N_m} C_k^{\mathrm{var}}(K(I), B \cap K^+) + \sum_{I \in N_m} C_k^{\mathrm{var}}(K(I), B \cap K^-) \\ &= \sum_{I \in N_m} C_k^{\mathrm{var}}(K(I), B) \end{aligned}$$

for each Borel set $B \subseteq \mathbb{R}^d$, as stated in (2.2.2). The second assertion follows from the first one by noting that for each $I \in N_m$, the set $K(I)$ is convex and thus $C_k^{\mathrm{var}}(K(I), \cdot) = C_k(K(I), \cdot)$. Since $K(I) \subseteq K^j$ for some $j \in \{1, \dots, m\}$, the monotonicity of the total curvatures in $\mathcal{K}^d$ yields $C_k(K(I), B) \leq C_k(K(I)) \leq \max_j C_k(K^j)$. Observing now that the number of summands, i.e. the number of sets in N_m, is $2^m - 1$, the second assertion follows. □

The above estimates are not satisfactory in case we have a large number m of sets K^j with comparably few mutual intersections. But in this situation it can be improved easily. Define the *intersection number* $\Gamma = \Gamma(\mathcal{X})$ of a finite family $\mathcal{X} = \{K^1, \ldots, K^m\}$ of sets as the maximum over all $l \in \{1, \ldots, m\}$ of the number of nonempty intersections $K^l \cap K^j$ with $K^j \in \mathcal{X}$, i.e.

$$\Gamma(\mathcal{X}) = \max_l \# \left\{ j : K^j \cap K^l \neq \varnothing \right\}. \tag{2.2.4}$$

If Γ is small compared to m, then the following estimate is useful.

Corollary 2.2.4. *Let $K^j \in \mathcal{R}^d$ for $j = 1, \ldots, m$ and $K = \bigcup_{j=1}^m K^j$, Γ the intersection number of the family $\{K^1, \ldots, K^m\}$ and $b > 0$ such that for all $I \in N_m$*

$$C_k^{\mathrm{var}}(K(I), B) \leq b.$$

Then

$$C_k^{\mathrm{var}}(K, B) \leq m 2^\Gamma b.$$

Proof. Let I_l be the set of all indices j such that $K^l \cap K^j \neq \varnothing$. For each K^l it suffices to consider its intersections with sets K^j with $j \in I_l$, all other intersections being empty. Therefore the sum on the right hand side of (2.2.2) is contained in

$$\sum_{l=1}^m \sum_{I \subseteq I_l} C_k^{\mathrm{var}}(K^l \cap K(I), B),$$

where we set $K(I) := \mathbb{R}^d$ in case $I = \varnothing$. By assumption, each term is bounded from above by b. Moreover, $\# I_l \leq \Gamma$ and so the number of subsets of I_l is not greater than 2^Γ. Hence the asserted estimate follows. □

Now assume that $L = \bigcap_{j=1}^m K^j$ is the intersection of a finite number of sets $K^j \in \mathcal{R}^d$. Again we ask for an upper bound of $C_k^{\mathrm{var}}(L, \cdot\,)$ in terms of the curvatures of the sets K^j. For convex sets K^j there is an obvious bound for the total curvature. In this case the set L is convex as well and the monotonicity implies

$$C_k(L) \leq \min_{j=1,\ldots,m} C_k(K^j).$$

If the K^j are polyconvex, we have at least some bound in terms of representations of K^j with convex sets.

Lemma 2.2.5. *Let $L = \bigcap_{j=1}^m K^j$. Assume that each K^j has a representation $K^j = \bigcup_{i=1}^P K^{j,i}$ as a union of (at most) P convex sets $K^{j,i}$. Then*

$$C_k^{\mathrm{var}}(L) \leq (2^{(P^m)} - 1) \max_{j,i} C_k(K^{j,i}).$$

Note that, since $C_k^{\pm}(L, \cdot\,) \leq C_k^{\mathrm{var}}(L, \cdot\,)$, the assertion also holds in case $C_k^{\mathrm{var}}(L)$ is replaced with $C_k^+(L)$ or $C_k^-(L)$.

Proof. We have

$$L = \bigcap_{j=1}^{m} K^j = \bigcap_{j=1}^{m} \bigcup_{i_j=1}^{P} K^{j,i_j} = \bigcup_{i_1,\dots,i_m=1}^{P} \left(\bigcap_{j=1}^{m} K^{j,i_j} \right),$$

i.e. L is the union of the P^m convex sets $\bigcap_j K^{j,i_j}$. Therefore, the assertion follows immediately from the second statement in Lemma 2.2.3. □

3. Adapted Renewal theorem

For the proofs of Theorem 1.2.6 and Theorem 1.2.10 we require the Renewal theorem which we recall and discuss now. Afterwards we will reformulate it in a way which is most convenient for our purposes. Later on we will only use this variant of the Renewal theorem. It is stated in Theorem 3.1.4.

The Renewal theorem. Let P be a Borel probability measure with support contained in $[0,\infty)$ and $\eta := \int_0^\infty tP(dt) < \infty$. Let $z : \mathbb{R} \to \mathbb{R}$ be a function with a discrete set of discontinuities satisfying

$$|z(t)| \leq c_1 e^{-c_2|t|} \qquad \text{for all } t \in \mathbb{R} \tag{3.1.1}$$

for some constants $0 < c_1, c_2 < \infty$. It is well known in probability theory that under these conditions on z the equation

$$Z(t) = z(t) + \int_0^\infty Z(t-\tau)P(d\tau) \tag{3.1.2}$$

has a unique solution $Z(t)$ in the class of functions satisfying $\lim_{t\to-\infty} Z(t) = 0$. Equation (3.1.2) is called a *renewal equation* and the asymptotic behaviour of its solution as $t \to \infty$ is given by the so-called Renewal theorem. The Renewal theorem is a standard tool in probability theory (cf. e.g. Feller [9]). In the last years, it has been discovered as a tool in fractal geometry too. Therefore, versions are available which are adapted to the fractal setting. In fractal applications, P usually is a probability measure supported by a finite set of points $y_1, \dots, y_N \in [0,\infty)$ such that $P(\{y_i\}) = p_i$ for $i = 1, \dots, N$ and therefore

$$\eta = \sum_{i=1}^{N} y_i p_i. \tag{3.1.3}$$

Discrete versions of the Renewal theorem are for instance provided in Kenneth Falconer's book [7, Corollary 7.3, p. 122]) or in the paper [21] of Michael Levitin and Dmitri Vassiliev.

A function $g : \mathbb{R} \to \mathbb{R}$ is said to be *asymptotic* to a function $f : \mathbb{R} \to \mathbb{R}$, $g \sim f$, if for all $\epsilon > 0$ there exists a number $D = D(\epsilon)$ such that

$$(1-\epsilon)f(t) \leq g(t) \leq (1+\epsilon)f(t) \text{ for all } t > D. \tag{3.1.4}$$

Recall from Section 1.2 that the set $\{y_1, ..., y_N\}$ is called *h-arithmetic* if h is the largest number such that $y_i \in h\mathbb{Z}$ for $i = 1, \dots, N$ and *non-arithmetic* if no such number h exists.

Theorem 3.1.1. (Renewal theorem) *Let* $0 < y_1 \leq y_2 \leq \ldots \leq y_N$ *and* $p_1, \ldots, p_N$ *be positive real numbers such that* $\sum_{i=1}^N p_i = 1$. *For a function* z *as defined in (3.1.1), let* $Z : \mathbb{R} \to \mathbb{R}$ *be the unique solution of the renewal equation*

$$Z(t) = z(t) + \sum_{i=1}^{N} p_i Z(t - y_i) \tag{3.1.5}$$

satisfying $\lim_{t\to-\infty} Z(t) = 0$. *Then the following holds:*

(i) If the set $\{y_1, ..., y_N\}$ *is non-arithmetic, then*

$$\lim_{t\to\infty} Z(t) = \frac{1}{\eta} \int_{-\infty}^{\infty} z(\tau) d\tau.$$

(ii) If $\{y_1, ..., y_N\}$ *is* h*-arithmetic for some* $h > 0$*, then*

$$Z(t) \sim \frac{h}{\eta} \sum_{k=-\infty}^{\infty} z(t - kh).$$

Moreover, Z *is uniformly bounded in* $\mathbb{R}$.

Theorem 3.1.1 implies that in the non-arithmetic case the limit $\lim_{t\to\infty} Z(t)$ exists, while in the h-arithmetic case Z is asymptotic to some periodic function of period $h > 0$ (i.e. to some function f with $f(t+h) = f(t)$ for all $t \in \mathbb{R}$). The latter is sufficient for the limit $\lim_{T\to\infty} \frac{1}{T} \int_0^T Z(t)dt$ to exist which is easily derived from the following observation.

Lemma 3.1.2. *Let* f *be a locally integrable periodic function with period* $h > 0$ *and let* $L := \int_0^h f(t)dt$.

(i) Then the limit $\lim\limits_{T\to\infty} \frac{1}{T} \int_0^T f(t)dt$ *exists and equals* $h^{-1}L$.

(ii) If $g : \mathbb{R} \to \mathbb{R}$ *is a function such that* $g \sim f$*, then also the limit* $\lim\limits_{T\to\infty} \frac{1}{T} \int_0^T g(t)dt$ *exists and equals* $h^{-1}L$.

As a direct consequence of the Renewal theorem and Lemma 3.1.2 we obtain

Corollary 3.1.3. *Under the assumptions of Theorem 3.1.1 the following limit always exists and is equal to the expression on the right hand side:*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T Z(t)dt = \frac{1}{\eta} \int_{-\infty}^{\infty} z(\tau) d\tau.$$

Proof. If $\{y_1, ..., y_N\}$ is h-arithmetic, just note that the function $f(t) = \frac{h}{\eta} \sum_{k=-\infty}^{\infty} z(t - kh)$ in Theorem 3.1.1(ii) is uniformly bounded and periodic, and apply Lemma 3.1.2(ii) to $g(t) = Z(t)$. In the non-arithmetic case the limit $\lim_{t\to\infty} Z(t)$ exists and the assertion follows by applying Lemma 3.1.2 to $g(t) = Z(t)$ which is asymptotic to the constant function $f \equiv \lim_{t\to\infty} Z(t)$. □

Reformulation of the Renewal theorem. Now we are ready to restate the Renewal theorem in a more convenient way. Here we will always consider some self-similar set F with contraction ratios r_i and similarity dimension s. Therefore we fix $p_i = r_i^s$ and $y_i = -\log r_i$. We substitute $t = -\log \epsilon$, since we are interested in the limiting behaviour of functions $f : (0, \infty) \to \mathbb{R}$ as the argument ϵ tends to zero. Moreover, by taking into account Corollary 3.1.3, we conclude the existence of average limits from the asymptotic periodicity.

Theorem 3.1.4. ***(Adapted Renewal theorem)***
Let F be a self-similar set with ratios $r_1, \ldots, r_N$ and similarity dimension s. For a function $f : (0, \infty) \to \mathbb{R}$, suppose that for some $k \in \mathbb{R}$ the function φ_k defined by

$$\varphi_k(\epsilon) = f(\epsilon) - \sum_{i=1}^{N} r_i^k \mathbf{1}_{(0,r_i]}(\epsilon) f(\epsilon/r_i) \tag{3.1.6}$$

has a discrete set of discontinuities and satisfies

$$|\varphi_k(\epsilon)| \leq c\epsilon^{k-s+\gamma} \tag{3.1.7}$$

for some constants $c, \gamma > 0$ and all $\epsilon > 0$. Then $\epsilon^{s-k} f(\epsilon)$ is uniformly bounded in $(0, \infty)$ and the following holds:

(i) The limit $\lim\limits_{\delta \to 0} \frac{1}{|\log \delta|} \int_\delta^1 \epsilon^{s-k} f(\epsilon) \frac{d\epsilon}{\epsilon}$ exists and equals

$$\frac{1}{\eta} \int_0^1 \epsilon^{s-k-1} \varphi_k(\epsilon)\, d\epsilon\,, \tag{3.1.8}$$

where $\eta = -\sum_{i=1}^{N} r_i^s \log r_i$.

(ii) If $\{-\log r_1, \ldots, -\log r_N\}$ is non-arithmetic, then the limit of $\epsilon^{s-k} f(\epsilon)$ as $\epsilon \to 0$ exists and equals the average limit.

Proof. The definition (3.1.6) of φ_k implies that

$$f(\epsilon) = \sum_{i=1}^{N} r_i^k \mathbf{1}_{(0,r_i]}(\epsilon) f(\epsilon/r_i) + \varphi_k(\epsilon)\,. \tag{3.1.9}$$

Define

$$Z(t) = \begin{cases} e^{(k-s)t} f(e^{-t}) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}. \tag{3.1.10}$$

Taking into account (3.1.9), for all $t \geq 0$ we have

$$\begin{aligned} Z(t) &= e^{(k-s)t} \left(\sum_{i=1}^{N} r_i^k \mathbf{1}_{(0,r_i]}(e^{-t}) f(e^{-(t+\log r_i)}) + \varphi_k(e^{-t}) \right) \\ &= \sum_{i=1}^{N} r_i^s e^{(k-s)(t+\log r_i)} \mathbf{1}_{[-\log r_i, \infty)}(t) f(e^{-(t+\log r_i)}) + e^{(k-s)t} \varphi_k(e^{-t}) \end{aligned}$$

The i-th term of the sum can be replaced by $r_i^s Z(t+\log r_i)$. (For $t < -\log r_i$ this expression equals zero as well as the corresponding i-th term in the sum.) Thus we have the renewal equation

$$Z(t) = \sum_{i=1}^{N} r_i^s Z(t+\log r_i) + z(t)\,, \tag{3.1.11}$$

where the function $z : \mathbb{R} \to \mathbb{R}$ is defined by

$$z(t) = \begin{cases} e^{(k-s)t}\varphi_k(e^{-t}) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}. \tag{3.1.12}$$

Observing that the assumptions on φ_k ensure z to have a discrete set of discontinuities and to satisfy

$$|z(t)| = e^{(k-s)t}\left|\varphi_k(e^{-t})\right| \leq ce^{-t\gamma}$$

for some constants $c, \gamma > 0$ and all $t \geq 0$, we can apply the Theorem 3.1.1 with $p_i = r_i^s$ and $y_i = -\log r_i$. There are two cases to discuss.

The non-arithmetic case. If $\{-\log r_1, \ldots, -\log r_N\}$ is non-arithmetic, then the limit

$$\lim_{t\to\infty} Z(t) = \lim_{t\to\infty} e^{-t(s-k)} f(e^{-t}) = \lim_{\epsilon\to 0} \epsilon^{s-k} f(\epsilon)$$

exists and is equal to the integral

$$\frac{1}{\eta}\int_0^\infty z(\tau)d\tau\,. \tag{3.1.13}$$

By (3.1.12) and with the substitution $r = e^{-\tau}$ we obtain

$$\lim_{\epsilon\to 0} \epsilon^{s-k} f(\epsilon) \quad = \quad \frac{1}{\eta}\int_0^1 r^{s-k-1}\varphi_k(r)dr\,.$$

This completes the proof of (ii) of Theorem 3.1.4. For the non-arithmetic case, (i) immediately follows from (ii), since the average limit exists whenever the limit exists and both coincide.

The arithmetic case. If $\{-\log r_1, \ldots, -\log r_N\}$ is h-arithmetic for some $h > 0$, Corollary 3.1.3 states that the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T Z(t)\; dt = \lim_{\delta\to 0} \frac{1}{|\log\delta|}\int_\delta^1 \epsilon^{s-k} f(\epsilon)\frac{d\epsilon}{\epsilon}$$

exists and equals the integral in (3.1.13). Therefore we obtain formula (3.1.8) also for the h-arithmetic case. This completes the proof of Theorem 3.1.4. □

Remark 3.1.5. In the h-arithmetic case, Theorem 3.1.4 concentrates on the existence of average limits. However, some additional information on the limiting behaviour can be derived from the original Renewal theorem, which stated the existence of some (additively) periodic function of period h, to which $Z(t)$ is asymptotic as $t \to \infty$. In the situation of Theorem 3.1.4 this is translated into the existence of a (multiplicatively) periodic function $G(\epsilon)$ of period $\zeta = e^{-h}$, i.e. $G(\zeta\epsilon) = G(\epsilon)$ for all $\epsilon > 0$, to which the function $g(\epsilon) = \epsilon^{s-k} f(\epsilon)$ is asymptotic as $\epsilon \to 0$.

4. Fractal curvatures – proofs

The purpose of this chapter is to provide proofs of the results presented in Section 1.2. After some preparations and the proof of Proposition 1.2.1 in the first section, we state some key estimate (Lemma 4.2.1) which we will then use to prove Theorem 1.2.6. The problem left is the verification of the key estimate, which will be done in Sections 4.3 and 4.4. In 4.5 we then turn to the proofs of Theorem 1.2.2 and Theorem 1.2.8. Finally, in Section 4.6 we reproof Gatzouras's theorem on the existence of the (average) Minkowski content.

Throughout the chapter we assume F to be a self-similar set in $\mathbb{R}^d$ satisfying OSC. Moreover, O will always denote some feasible open set of F such that the SOSC is satisfied, i.e. in particular $F \cap O \neq \varnothing$.

4.1. Preparations

We introduce some more notation required in the proofs below and collect some simple facts. On the way we prove Proposition 1.2.1.

Code space and level sets. Set $\Sigma := \{1, \dots, N\}$ and for $n = 0, 1, 2, \dots$ let Σ^n denote the set of all sequences $w_1 w_2 \dots w_n$ such that $w_i \in \Sigma$. n is called the *length* of the sequence $w = w_1 w_2 \dots w_n$. We set

$$\Sigma^* := \bigcup_{n=0}^{\infty} \Sigma^n.$$

Observe that Σ^* contains all finite sequences including the *empty word* $w \in \Sigma^0$. Finite sequences $w \in \Sigma^*$ are also called *words* over the alphabet Σ. If $v = v_1 \dots v_m$ and $w = w_1 \dots w_n$ are words in Σ^* , then vw simply denotes the word $v_1 \dots v_m w_1 \dots w_n$. Moreover, let Σ^∞ denote the family of all infinite sequences $w_1 w_2 w_3 \dots$ such that $w_i \in \Sigma$. For $w = w_1 \dots w_n \in \Sigma^*$, we introduce the abbreviations

$$r_w := r_{w_1} r_{w_2} \dots r_{w_n}$$

and

$$S_w := S_{w_1} \circ S_{w_2} \circ \dots \circ S_{w_n} .$$

The sets $S_w F$ are called the *level sets* of F . Since the S_i are contractions and since $S_{wi} F \subset S_w F$ for each $w \in \Sigma^*$ and $i \in \Sigma$, the mapping $\pi : \Sigma^\infty \to F$ given by

$$w_1 w_2 w_3 \dots \mapsto x := \bigcap_{n=1}^{\infty} S_{w_1 w_2 \dots w_n} F$$

is well defined. Observe that π is surjective but not necessarily injective, i.e. for each $x \in F$ there exists some sequence $w \in \Sigma^\infty$ such that $x = \pi(w)$ but it might not be unique.

The families $\Sigma(r)$**.** For $0 < r \leq 1$, let $\Sigma(r)$ be the family of all finite words $w = w_1 \dots w_n \in \Sigma^*$ such that

$$r_w < r \leq r_w r_{w_n}^{-1}. \tag{4.1.1}$$

For convenience, we define $\Sigma(r)$, for $r > 1$, to be the set containing only the empty word. It is clear that for fixed $r \leq 1$ for each sequence $w_1 w_2 w_3 \ldots \in \Sigma^\infty$ there is exactly one n such that $w = w_1 \dots w_n$ satisfies (4.1.1) (and, for $r > 1$, $n = 0$, correspondingly). Therefore, on the one hand

$$F = \bigcup_{w \in \Sigma(r)} S_w F \tag{4.1.2}$$

for each $r > 0$. On the other hand the words in $\Sigma(r)$ are mutually *incompatible*, i.e. there is no pair of words $v, w \in \Sigma(r)$ such that $v = ww'$ for some nonempty $w' \in \Sigma^*$. Moreover, for each $r > 0$,

$$\sum_{w \in \Sigma(r)} r_w^s = 1, \tag{4.1.3}$$

which is easily seen from the definition of the similarity dimension s. Due to (4.1.1), $\Sigma(r)$ consists of words w, for which the corresponding level sets $S_w F$ are approximately of the same size $r \cdot \operatorname{diam} F$. Note that (4.1.1) implies in particular

$$r_w < r \leq r_w r_{\min}^{-1} \tag{4.1.4}$$

for each $w \in \Sigma(r)$, where $r_{\min} = \min_i r_i$.

The cardinalities $\#\Sigma(r)$ of these finite families of words are bounded as follows. By (4.1.4), $r r_{\min} \leq r_w < r$ for each $w \in \Sigma(r)$. Hence, by (4.1.3), on the one hand

$$1 = \sum_{w \in \Sigma(r)} r_w^s < \sum_{w \in \Sigma(r)} r^s = r^s \#\Sigma(r)$$

and on the other hand

$$1 = \sum_{w \in \Sigma(r)} r_w^s \geq \sum_{w \in \Sigma(r)} (r r_{\min})^s = r_{\min}^s r^s \#\Sigma(r)\,.$$

Therefore,

$$r^{-s} < \#\Sigma(r) \leq r_{\min}^{-s} r^{-s}. \tag{4.1.5}$$

The families $\Sigma(r)$ will play an important role in the proofs later on. As a first application of these families, we present a proof of Proposition 1.2.1.

Proof of Proposition 1.2.1. Assume $F_\epsilon \in \mathcal{R}^d$. Then, by Fact 1.1.1, $F_\delta \in \mathcal{R}^d$ for all $\delta \geq \epsilon$. Let now $\delta < \epsilon$ and set $r = \epsilon^{-1}\delta$. By (4.1.2), we have

$$F_\delta = \bigcup_{w \in \Sigma(r)} (S_w F)_\delta = \bigcup_{w \in \Sigma(r)} S_w F_{\delta / r_w}\,.$$

Since $\delta / r_w > \delta / r = \epsilon$, F_{δ/r_w} is a parallel set of F_ϵ and thus, again by Fact 1.1.1, polyconvex. Hence each set $S_w F_{\delta/r_w}$ in the finite union above is polyconvex implying the same for F_δ. $\square$

Definition of u, ρ and γ. Above we fixed some feasible open set O of F such that the SOSC is satisfied. The condition $F \cap O \neq \varnothing$ implies that there exists a sequence $u = u_1 \dots u_p \in \Sigma^*$ such that

$$S_u F \subset O \tag{4.1.6}$$

and, since $S_u F$ is compact, some constant $\alpha > 0$ such that

$$d(x, \partial O) > \alpha \text{ for all } x \in S_u F.$$

Applying the similarity S_w, $w \in \Sigma^*$, the above inequality yields

$$d(x, \partial S_w O) > \alpha r_w \text{ for all } x \in S_{wu} F. \tag{4.1.7}$$

Define

$$\rho := r_{\min} \frac{\alpha}{2}. \tag{4.1.8}$$

Moreover, for each $\epsilon > 0$ we set $\epsilon^* = \rho^{-1}\epsilon$. As will become clear later, it is very convenient to look at the level sets with $w \in \Sigma(\epsilon^*)$ when investigating ϵ-parallel sets.

Finally we introduce the following numbers. Choose some r such that $u \in \Sigma(r)$ and let $\overline{s}$ be the unique solution of

$$\sum_{v \in \Sigma(r),\, v \neq u} r_v^{\overline{s}} = 1 \tag{4.1.9}$$

and $\gamma := s - \overline{s}$. By (4.1.3), $\overline{s} < s$ and so $\gamma > 0$. Obviously, γ and $\overline{s}$ depend on the word u we fixed above. Throughout Chapters 4 and 5 we consider the word u and the constant γ together with the set O as being once and for all fixed for the self-similar set F.

4.2. Proof of Theorem 1.2.6

Throughout this section we assume that the self-similar set F has polyconvex parallel sets, as demanded in the hypothesis of Theorem 1.2.6. Moreover, let $k \in \{0, 1, \dots, d\}$ be fixed.

The key estimate. For each $r > 0$, we define the set

$$O(r) := \bigcup_{v \in \Sigma(r)} S_v O, \tag{4.2.1}$$

where O is the feasible open set for F we fixed above. Observe that $O(r)$ is again a feasible open set of F for each $r > 0$. In particular, $O = O(r)$ for any $r > 1$ and $O(1) = \mathbf{S}O = \bigcup_i S_i O$. For the complement $O(r)^c$ of these sets the following estimate holds.

Lemma 4.2.1. *For each $r > 0$, there exists a constant $c > 0$ such that for all $\epsilon \leq \delta \leq \rho r$*

$$C_k^{\text{var}}(F_\epsilon, (O(r)^c)_\delta) \leq c \epsilon^{k-s} \delta^\gamma.$$

This estimate roughly means that, as δ and ϵ approach 0, the k-th curvature concentrates more and more in the set $O(r)_{-\delta}$, the inner parallel set of $O(r)$, while the curvature in the complement $(O(r)_{-\delta})^c = (O(r)^c)_\delta$ vanishes. The constants ρ and γ are those we fixed in (4.1.8) and (4.1.9). c depends on r (and the k fixed above) but is independent of ϵ and δ. Since $C_k^{\pm}(F_\epsilon, (O(r)^c)_\delta) \leq C_k^{\text{var}}(F_\epsilon, (O(r)^c)_\epsilon)$ and $|C_k(F_\epsilon, (O(r)^c)_\epsilon)| \leq C_k^{\text{var}}(F_\epsilon, (O(r)^c)_\epsilon)$, the above Lemma provides also upper bounds for these expressions.

We require the key estimate in this full generality in the proofs on the weak convergence of curvature measures in Chapter 5. For the moment the following special version is sufficient, where we set $\epsilon = \delta$ and also fix $r = 1$.

Corollary 4.2.2. *There exist some constant $c > 0$ such that for all $0 < \epsilon \leq 1$*

$$C_k^{\text{var}}(F_\epsilon, ((\mathbf{S}O)^c)_\epsilon) \leq c\epsilon^{k-s+\gamma}.$$

Proof. Setting in Lemma 4.2.1 $r = 1$, i.e. $O(r) = \mathbf{S}O$, and $\epsilon = \delta$, the validity of the stated inequality follows immediately for all $\epsilon \leq \rho$. If necessary, the constant c can be enlarged such that it also holds for $\rho < \epsilon \leq 1$. □

Note that, for ϵ fixed, the estimate remains valid with $(\mathbf{S}O)^c)_\epsilon$ replaced by any of its subsets. In particular, since $\mathbf{S}O \subseteq O$ and thus $(O^c)_\epsilon \subseteq ((\mathbf{S}O)^c)_\epsilon$, the estimate holds as well for the sets $(O^c)_\epsilon$.

We postpone the proof of Lemma 4.2.1 for the moment and first discuss how it can be used to prove Theorem 1.2.6. The first step is the investigation of the scaling functions.

Scaling functions. Recall from (1.2.1) that the k-th scaling function R_k is defined by

$$R_k(\epsilon) = C_k(F_\epsilon) - \sum_{i=1}^{N} \mathbf{1}_{(0,r_i]}(\epsilon) C_k((S_i F)_\epsilon),$$

for $\epsilon > 0$. We investigate the properties of R_k to see that the Renewal theorem can be applied. On the one hand we require an upper bound for the growth of $|R_k|$ as $\epsilon \to 0$, which will be derived from Corollary 4.2.2, and on the other hand a statement on the continuity of R_k.

Lemma 4.2.3. *There is a constant $c > 0$ such that for all $0 < \epsilon \leq 1$*

$$|R_k(\epsilon)| \leq c\epsilon^{k-s+\gamma}. \tag{4.2.2}$$

Proof. For $\epsilon > 0$, let $U(\epsilon) = \bigcup_{i \neq j} (S_i F)_\epsilon \cap (S_j F)_\epsilon$ and $B^j(\epsilon) = (S_j F)_\epsilon \setminus U(\epsilon)$. Then $F_\epsilon = \bigcup_j B^j(\epsilon) \cup U(\epsilon)$ is a disjoint union and so

$$C_k(F_\epsilon) = \sum_{j=1}^{N} C_k(F_\epsilon, B^j(\epsilon)) + C_k(F_\epsilon, U(\epsilon)).$$

Similarly,

$$C_k((S_j F)_\epsilon) = C_k((S_j F)_\epsilon, B^j(\epsilon)) + C_k((S_j F)_\epsilon, U(\epsilon)),$$

since $B^j(\epsilon) \cap (S_iF)_\epsilon = \varnothing$ for $j \neq i$. Thus the function R_k can be written as

$$R_k(\epsilon) = \sum_{j=1}^{N} \left(C_k(F_\epsilon, B^j(\epsilon)) - C_k((S_jF)_\epsilon, B^j(\epsilon))\right) + C_k(F_\epsilon, U(\epsilon)) - \sum_{j=1}^{N} C_k((S_jF)_\epsilon, U(\epsilon)).$$

Observe that the set $A^j(\epsilon) = (\bigcup_{i\neq j}(S_iF)_\epsilon)^c$ is open and that $F_\epsilon \cap A^j(\epsilon) = (S_jF)_\epsilon \cap A^j(\epsilon)$. Since $B^j(\epsilon) \subseteq A^j(\epsilon)$, the locality property of C_k implies,

$$C_k(F_\epsilon, B^j(\epsilon)) = C_k((S_jF)_\epsilon, B^j(\epsilon)).$$

Hence all terms of the first sum on the right hand side equal zero and can be omitted. Taking absolute values, we infer that

$$|R_k(\epsilon)| \leq |C_k(F_\epsilon, U(\epsilon))| + \sum_{j=1}^{N} |C_k((S_jF)_\epsilon, U(\epsilon))| . \tag{4.2.3}$$

For the first term on the right hand side we claim that $U(\epsilon) \subseteq ((\mathbf{S}O)^c)_\epsilon$ and conclude from Corollary 4.2.2, the existence of $c > 0$ such that for all $\epsilon > 0$

$$|C_k(F_\epsilon, U(\epsilon))| \leq C_k^{\mathrm{var}}(F_\epsilon, U(\epsilon)) \leq c\epsilon^{k-s+\gamma}. \tag{4.2.4}$$

For a proof of the set inclusion $U(\epsilon) \subseteq ((\mathbf{S}O)^c)_\epsilon$, let $x \in U(\epsilon)$. We show that $d(x, (\mathbf{S}O)^c) \leq \epsilon$ and thus $x \in ((\mathbf{S}O)^c)_\epsilon$. Assume $d(x, (\mathbf{S}O)^c) > \epsilon$. Since the union $\mathbf{S}O = \bigcup_i S_iO$ is disjoint, there is a unique j such that $x \in S_jO$. Moreover, $d(x, \partial S_jO) > \epsilon$. Since $x \in U(\epsilon)$, there is at least one index $i \neq j$ such that $x \in (S_iF)_\epsilon$ and consequently a point $y \in S_iF$ with $d(x, y) \leq \epsilon$. But then $y \in S_iF \cap S_jO$, a contradiction to OSC. Hence, $d(x, (\mathbf{S}O)^c) \leq \epsilon$.

For the remaining terms in (4.2.3) observe that for each j

$$|C_k((S_jF)_\epsilon, U(\epsilon))| = r_j^k \left| C_k(F_{\epsilon/r_j}, S_j^{-1}U(\epsilon))\right| \leq r_j^k C_k^{\mathrm{var}}(F_{\epsilon/r_j}, S_j^{-1}U(\epsilon)).$$

We show that $S_j^{-1}U(\epsilon) \cap F_{\epsilon/r_j} \subseteq (O^c)_{\epsilon/r_j}$. Let $x \in S_j^{-1}U(\epsilon) \cap F_{\epsilon/r_j}$. Then $S_jx \in U(\epsilon)$ and so there exists at least one index $i \neq j$ with $S_jx \in (S_iF)_\epsilon$. Hence $d(S_jx, \partial S_jO) \leq \epsilon$ since otherwise there would exist a point $y \in S_iF \cap S_jO$, a contradiction to OSC. Therefore, $d(x, \partial O) \leq \epsilon/r_j$, i.e. $x \in (O^c)_{\epsilon/r_j}$.

By the set inclusion just proved and Corollary 4.2.2, there exists a constant $c > 0$ such that for all ϵ, $C_k^{\mathrm{var}}(F_{\epsilon/r_j}, S_j^{-1}U(\epsilon))$ is bounded from above by $c(\epsilon/r_j)^{k-s+\gamma}$ and thus

$$|C_k((S_jF)_\epsilon, U(\epsilon))| \leq c_j\epsilon^{k-s+\gamma}$$

where $c_j := cr_j^{s-\gamma}$.

Since each of the terms in (4.2.3) is bounded from above by $c\epsilon^{k-s+\gamma}$ for some constant $c > 0$, we can also find such a constant for $|R_k(\epsilon)|$ and so the assertion follows. □

The last missing ingredient for the application of Theorem 3.1.4 is a statement on the continuity properties of R_k. We require that R_k is continuous except for a discrete set, i.e. the discontinuities are well separated from each other and do not accumulate inside the domain $(0, \infty)$ of R_k.

Lemma 4.2.4. *The function R_k has a discrete set of discontinuities in* $(0,\infty)$.

Proof. From Lemma 2.1.3 it is easily seen that $C_k(F_\epsilon)$ and $C_k((S_iF)_\epsilon)$ have this property, since they have at most finitely many discontinuities in each interval $[\epsilon_0,\infty)$, $\epsilon_0>0$. Thus R_k has at most finitely many discontinuities in each interval $[\epsilon_0,\infty)$ and the assertion follows. □

Note that the discontinuities of R_k possibly accumulate at 0.

Proof of Theorem 1.2.6. Since

$$C_k((S_iF)_\epsilon)=r_i^kC_k(F_{\epsilon/r_i}),$$

the functions $f(\epsilon):=C_k(F_\epsilon)$ and $\varphi_k(\epsilon):=R_k(\epsilon)$ satisfy a renewal equation

$$\varphi_k(\epsilon)=f(\epsilon)-\sum_{i=1}^N r_i^k\mathbf{1}_{(0,r_i]}(\epsilon)f(\epsilon/r_i)$$

as in (3.1.6). Now Lemma 4.2.3 and Lemma 4.2.4 ensure that the hypotheses of Theorem 3.1.4 are satisfied. Therefore the average limit of the expression $\epsilon^{s-k}C_k(F_\epsilon)$ exists and in case of a non-arithmetic set F also the limit. □

To complete the proof of Theorem 1.2.6, it remains to verify the key estimate Lemma 4.2.1. This is the agenda for the succeeding two sections.

4.3. Convex representations of F_ϵ

In this section we translate the problem of proving the key estimate into the problem of estimating cardinalities of certain families of level sets. For getting control over the behaviour of the measure $C_k^{\mathrm{var}}(F_\epsilon,\cdot)$ as $\epsilon\to 0$, we decompose F_ϵ into convex sets for each $\epsilon>0$. The idea is to use small copies from a fixed collection of convex sets K^i for the decomposition, such that only the number of convex sets used in the decomposition increases as $\epsilon\to 0$ while their curvatures are "fixed" (up to scaling). This allows to reduce the problem to estimating the number of sets involved in such representations.

First we fix the collection of convex sets that will be used. It is convenient to use a representation of F_ρ by convex sets, where ρ is the constant we defined in (4.1.8).

Decomposition of parallel sets. Let $K^i, i=1,\dots,P$ be convex sets such that

$$F_\rho=\bigcup_{i=1}^P K^i.$$

Note that for each $\epsilon>\rho$ this provides a decomposition of F_ϵ into convex sets, namely into parallel sets of the K^i. If $\epsilon=\rho+\delta$ then

$$F_\epsilon=\bigcup_{i=1}^P K^i_\delta. \tag{4.3.1}$$

For $\epsilon < \rho$ the decomposition is done in two steps. First we decompose F_ϵ into small copies of F_ρ:

$$F_\epsilon = \bigcup_{w\in\Sigma(\epsilon^*)} (S_w F)_\epsilon \,.$$

The choice of w from the family $\Sigma(\epsilon^*)$ ensures that $\rho < \frac{\epsilon}{r_w} \le \rho r_{\min}^{-1}$ (cf. (4.1.4)) and so $(S_w F)_\epsilon = S_w F_{\epsilon/r_w} = S_w(F_\rho)_\delta$ for some $0 \le \delta < \delta_{\max}$ where $\delta_{\max} := \rho(r_{\min}^{-1} - 1)$. Hence $(S_w F)_\epsilon$ has a representation by small copies of δ-parallel sets of K^i. For each $w \in \Sigma(\epsilon^*)$,

$$(S_w F)_\epsilon = \bigcup_{i=1}^{P} S_w K_\delta^i \quad \text{for some} \quad 0 \le \delta < \delta_{\max}. \tag{4.3.2}$$

This also provides a representation of F_ϵ as a union of convex sets.

Intersection numbers. Now the first task is to investigate finite intersections of decomposition sets $(S_w F)_\epsilon$. While for the existence of a representation (4.3.2) it was only important that ϵ is not too small compared to r_w ($r_w < \epsilon^*$), the reversed relation that ϵ is also not too large compared to r_w ($\epsilon^* \le r_w r_{\min}^{-1}$) will now be essential for the control of intersection numbers. Recall the definition of the intersection number of a finite family of sets from (2.2.4).

The first statement says, that the intersection number of the family of sets $(S_w F)_\epsilon$ with $w \in \Sigma(\epsilon^*)$ is uniformly bounded (independent of ϵ).

Lemma 4.3.1. *There exists a constant $\Gamma_{\max}$ such that for each $\epsilon > 0$ and $r \ge \epsilon^*$*

$$\Gamma(\{(S_w F)_\epsilon : w \in \Sigma(r)\}) \le \Gamma_{\max}.$$

Proof. Note that it suffices to prove the assertion for $r = \epsilon^*$, since choosing for fixed $r > 0$ some $\epsilon < \rho r$ (i.e. $r > \epsilon^*$) does not increase the intersection number compared to the choice $\epsilon = \rho r$. Fix $\epsilon > 0$ and recall the definition of the word u from (4.1.6). First we show that

(i) *The sets $(S_{wu} F)_\epsilon, w \in \Sigma(\epsilon^*)$, are pairwise disjoint.*

By (4.1.7), we have

$$d(x, \partial S_w O) > \alpha r_w \ge \alpha r_{\min} \epsilon^* \ge \epsilon$$

for each $w \in \Sigma(\epsilon^*)$ and $x \in S_{wu} F$. This implies

$$(S_{wu} F)_\epsilon \subseteq S_w O.$$

Since, by OSC, the sets $S_w O$, $w \in \Sigma(\epsilon^*)$, are pairwise disjoint, assertion (i) follows.

Fix some $v \in \Sigma(\epsilon^*)$ and a point $x \in S_{vu} F$. Let $\Gamma(v)$ denote the number of sequences $w \in \Sigma(\epsilon^*)$ with $(S_w F)_\epsilon \cap (S_v F)_\epsilon \ne \varnothing$. Then it holds

(ii) *For each sequence w counted in $\Gamma(v)$, the set $(S_{wu} F)_\epsilon$ is contained in the ball $B(x, c\epsilon)$ where* $c := 2\rho^{-1} \mathrm{diam}\, F + 3$. Note that c is independent of v or ϵ.

Let $y \in (S_{wu} F)_\epsilon$. Since $(S_w F)_\epsilon \cap (S_v F)_\epsilon \ne \varnothing$, there is a point x' in this intersection. Therefore

$$d(x', y) \le \mathrm{diam}\,(S_w F)_\epsilon = r_w \mathrm{diam}\, F + 2\epsilon \le \epsilon^* \mathrm{diam}\, F + 2\epsilon = (\rho^{-1} \mathrm{diam}\, F + 2)\epsilon,$$

since $w \in \Sigma(\epsilon^*)$, and similarly

$$d(x,x') \leq \operatorname{diam}(S_v F) + \epsilon \leq (\rho^{-1}\operatorname{diam} F + 1)\epsilon.$$

Thus $d(x,y) \leq (2\rho^{-1}\operatorname{diam} F + 3)\epsilon = c\epsilon$ for all $y \in (S_{wu}F)_\epsilon$, and so $(S_{wu}F)_\epsilon \subseteq B(x, c\epsilon)$ as stated in (ii).

Observing that each ϵ-parallel set contains an ϵ-ball and has thus Lebesgue measure at least $\kappa_d \epsilon^d$, where κ_j denotes the volume of the j-dimensional unit ball in $\mathbb{R}^j$, and taking into account (i) and (ii), we obtain

$$\Gamma(v)\kappa_d \epsilon^d \leq \lambda_d(B(x,c\epsilon)) = \kappa_d (c\epsilon)^d.$$

Hence $\Gamma(v) \leq c^d =: \Gamma_{\max}$, where $\Gamma_{\max}$ is independent of ϵ, as desired. □

Curvature of intersections of level sets. Lemma 4.3.1 allows to bound the variation measures of arbitrary intersections of sets $(S_w F)_\epsilon$ with $w \in \Sigma(\epsilon^*)$.

Lemma 4.3.2. *There is a constant $c > 0$ such that for all $\epsilon > 0$ and all Borel sets $B \subseteq \mathbb{R}^d$*

$$C_k^{\text{var}}((S_{w(1)}F)_\epsilon \cap \ldots \cap (S_{w(m)}F)_\epsilon, B) \leq c\epsilon^k$$

whenever $m \in \mathbb{N}$ and $w(1), \ldots, w(m) \in \Sigma(\epsilon^*)$.

Proof. It suffices to show that the total masses $C_k^{\text{var}}((S_{w(1)}F)_\epsilon \cap \ldots \cap (S_{w(m)}F)_\epsilon)$ satisfy the inequality for some $c > 0$. Moreover, we can assume $m \leq \Gamma_{\max}$, since, by Lemma 4.3.1, $(S_{w(1)}F)_\epsilon \cap \ldots \cap (S_{w(m)}F)_\epsilon = \varnothing$ for $m > \Gamma_{\max}$.

Applying Lemma 2.2.5 to the sets $X^j := (S_{w(j)}F)_\epsilon$, which have representations (4.3.2) by P convex sets $K^{j,i} := S_{w(j)} K^i_{\delta(j)}$ for some $\delta(j)$ with $0 \leq \delta(j) < \delta_{\max}$, we obtain for the set $X := \bigcup_{j=1}^m X^j$

$$C_k^{\text{var}}(X) \leq (2^{(P^m)} - 1) \max_{j,i} C_k(K^{j,i}). \tag{4.3.3}$$

Observe now that $K^i_{\delta(j)} \subset K^i_{\delta_{\max}}$ and so the monotonicity of C_k for convex sets and the scaling property imply

$$C_k(K^{j,i}) = r^k_{w(j)} C_k(K^i_{\delta(j)}) \leq r^k_{w(j)} C_k(K^i_{\delta_{\max}}).$$

Since $w(j) \in \Sigma(\epsilon^*)$, $r_{w(j)} \leq \epsilon^*$ and so

$$C_k(K^{j,i}) \leq \rho^{-k}\epsilon^k C_k(K^i_{\delta_{\max}}).$$

Since the right hand side does not depend on j, the maximum in (4.3.3) is bounded from above by $\rho^{-k}\epsilon^k \max_i C_k(K^i_{\delta_{\max}})$. Noting that $m \leq \Gamma_{\max}$ it follows that the asserted inequality is satisfied for the constant $c := (2^{(P^{\Gamma_{\max}})} - 1)\rho^{-k} \max_i C_k(K^i_{\delta_{\max}})$, which does neither depend on ϵ nor on m or the choice of the sequences $w(j)$. This completes the proof. □

Curvature estimates via cardinalities. Using the above estimate for finite intersections of level sets and the intersection number $\Gamma_{\max}$ we can now reduce the task of estimating $C_k^{\text{var}}(F_\epsilon, \cdot)$ to the problem of determining the cardinalities of certain families of level sets. For a closed set $B \subseteq \mathbb{R}^d$ and $\epsilon > 0$, let

$$\Sigma(B, \epsilon) = \{w \in \Sigma(\epsilon^*) : (S_w F)_\epsilon \cap B \neq \varnothing\}. \tag{4.3.4}$$

Lemma 4.3.3. *There is a constant $c' > 0$ such that for all closed sets $B \subseteq \mathbb{R}^d$ and all $\epsilon > 0$*

$$C_k^{\text{var}}(F_\epsilon, B) \leq c' \, \#\Sigma(B, \epsilon)\epsilon^k.$$

Proof. Fix $\epsilon > 0$. Observe that each $(S_w F)_\epsilon$ with $w \in \Sigma(\epsilon^*)$ which does not intersect B has some positive distance to B. Hence there is an open set A containing B such that $F_\epsilon \cap A = \bigcup_w (S_w F)_\epsilon \cap A$ where the union is taken over all $w \in \Sigma(B, \epsilon)$. Hence the locality of the curvature measure implies

$$C_k^{\text{var}}(F_\epsilon, B) = C_k^{\text{var}}(\bigcup_{w \in \Sigma(B,\epsilon)} (S_w F)_\epsilon, B).$$

The union on the right hand side consists of $m = \#\Sigma(B, \epsilon)$ polyconvex sets. It satisfies the conditions of Corollary 2.2.4, since, by the Lemma 4.3.2, there exist upper bounds $b := c\epsilon^k$ (ϵ is fixed) for the variation measures with respect to finite intersections. Moreover, by Lemma 4.3.1, $\Gamma_{\max}$ is an upper bound for the intersection number of the family $\{(S_w F)_\epsilon : w \in \Sigma(B, \epsilon)\}$. Thus, by Corollary 2.2.4, the assertion holds for the constant $c' = 2^{\Gamma_{\max}} c$, where c is the constant of Lemma 4.3.2. □

4.4. Cardinalities of level set families

In view of Lemma 4.3.3 it is evident that in order to prove Lemma 4.2.1 we require upper bounds for the cardinalities of the families $\Sigma(B, \epsilon)$ for the sets $B = (O(r)^c)_\delta$. Such bounds will be discussed now. Note that in this section no curvature is involved. Therefore, here the assumption of polyconvex parallel sets for F is not required. The main result of this section is the following:

Lemma 4.4.1. *For each $r > 0$, there is a constant $c > 0$ such that for all $0 < \epsilon \leq \delta \leq \rho r$*

$$\#\Sigma\left((O(r)^c)_\delta, \epsilon\right) \leq c\epsilon^{-s}\delta^\gamma.$$

Again the constant γ is as defined in (4.1.9). Note that the above estimate remains valid with the set $(O(r)^c)_\delta$ replaced by any of its subsets. Lemma 4.2.1 follows immediately.

Proof of Lemma 4.2.1. Combine Lemma 4.3.3 and Lemma 4.4.1 and note that the constant c' in Lemma 4.3.3 is independent of the choice of the set B. □

Splitting the proof of Lemma 4.4.1. It remains to provide a proof of Lemma 4.4.1, which we will divide into several steps. For this purpose we introduce some more notation. We say that a word $v \in \Sigma^*$ *occurs* in a word $w \in \Sigma^*$, in symbols $v \subset w$, if there are words $w', w'' \in \Sigma^*$ such that $w = w'vw''$. We write $v \not\subset w$, if v does *not occur* in w.

Recall the definition of the word $u = u_1 \dots u_p$ we defined in (4.1.6). The main idea of the proof is that some level set $(S_w F)_\epsilon$ for which u occurs in w lies sufficiently far away from the boundary of $O(r)$ and is thus not counted in the family $\#\Sigma\left((O(r)^c)_\delta, \epsilon\right)$ if ϵ, δ, r are arranged appropriately. The problem then reduces to counting the number of words w in certain families such that u does not occur in w.

For $\epsilon > 0$ let $\Xi(\epsilon)$ be the family of all words $w \in \Sigma(\epsilon)$ such that u does not occur in w, i.e.

$$\Xi(\epsilon) = \{w \in \Sigma(\epsilon) : u \not\subset w\}.$$

For $0 < \epsilon \leq \delta$ and $w \in \Sigma(\epsilon)$ there exists a subword $w' \in \Sigma(\delta)$ such that $w = w'w''$ for some $w'' \in \Sigma^*$. So, for $0 < \epsilon \leq \delta$, let

$$\Omega(\epsilon, \delta) = \left\{w \in \Sigma(\epsilon) : w = w'w'', w' \in \Sigma(\delta), u \not\subset w'\right\}. \tag{4.4.1}$$

Similarly, for $0 < \epsilon \leq \delta \leq r$, let

$$\Lambda(\epsilon, \delta, r) = \left\{w \in \Sigma(\epsilon) : w = w^0 w' w'', w^0 \in \Sigma(r), w^0 w' \in \Sigma(\delta), u \not\subset w'\right\}. \tag{4.4.2}$$

Then the following relations hold for the cardinalities of these families. Recall that $\epsilon^* = \rho^{-1}\epsilon$.

I.	For all $\epsilon^* \leq \delta^* \leq r$,	$\#\Sigma\left((O(r)^c)_\delta, \epsilon\right)$	$\leq$	$\#\Lambda(\epsilon^*, \delta^*, r)$.
II.	For all $\epsilon \leq \delta \leq r$,	$\#\Lambda(\epsilon, \delta, r)$	$\leq$	$\#\Sigma(r)\#\Omega(\frac{\epsilon}{r}, \frac{\delta}{r r_{\min}})$.
III.	For all $\epsilon \leq \delta$,	$\#\Omega(\epsilon, \delta)$	$\leq$	$r_{\min}^{-s}\left(\frac{\epsilon}{\delta}\right)^{-s}\#\Xi(\delta)$.
IV.	There exist $c_1, \gamma > 0$ such that,	$\#\Xi(\epsilon)$	$\leq$	$c_1 \epsilon^{\gamma - s}$.

Combining these estimates, Lemma 4.4.1 is easily derived. Fix $r > 0$. Combining the inequalities II. – IV., we derive for $\epsilon \leq \delta \leq r$

$$\begin{aligned}
\#\Lambda(\epsilon, \delta, r) &\leq \#\Sigma(r)\#\Omega(\frac{\epsilon}{r}, \frac{\delta}{r r_{\min}}) \\
&\leq \#\Sigma(r) r_{\min}^{-s}\left(\frac{\epsilon r_{\min}}{\delta}\right)^{-s}\#\Xi(\frac{\delta}{r r_{\min}}) \\
&\leq \#\Sigma(r) r_{\min}^{-2s}\epsilon^{-s}\delta^s c_1 (r r_{\min})^{s-\gamma}\delta^{\gamma-s} \\
&\leq c_2 \epsilon^{-s}\delta^\gamma,
\end{aligned}$$

where the constant c_2 only depends on r ($c_2 = c_1 r_{\min}^{-s-\gamma}\#\Sigma(r) r^{s-\gamma}$). Applying this to the right hand side of I, we obtain for $\epsilon^* \leq \delta^* \leq r$

$$\#\Sigma\left((O(r)^c)_\delta, \epsilon\right) \leq \#\Lambda(\epsilon^*, \delta^*, r) \leq c_2(\rho^{-1}\epsilon)^{-s}(\rho^{-1}\delta)^\gamma = c\epsilon^{-s}\delta^\gamma$$

where $c = c_2 \rho^{s-\gamma}$ is independent of δ and ϵ. Hence we have derived a constant c satisfying the assertion of Lemma 4.4.1.

It remains to provide proofs of the four inequalities I. – IV.

Proof of I. Let $0 < \epsilon^* \le \delta^* \le r$ and $w \in \Sigma(\epsilon^*) \setminus \Lambda(\epsilon^*, \delta^*, r)$, i.e $w \in \Sigma(\epsilon^*)$ and $w = w^0 w' w''$ such that $w^0 \in \Sigma(r)$ and $w^0 w' \in \Sigma(\delta^*)$ but $u \subset w'$. We show that this implies $(S_w F)_\epsilon \cap (O(r)^c)_\delta = \varnothing$ and thus $w \notin \Sigma((O(r)^c)_\delta, \epsilon)$, proving the assertion.

Let $x \in (S_w F)_\epsilon$. There exists $y \in S_w F$ with $d(x, y) \le \epsilon$. The assumption $u \subset w'$ implies that there exist $u', u'' \in \Sigma^*$ such that $w = u' u u''$. By definition of u, the set inclusions

$$S_w F \subseteq S_{w^0 u' u} F \subset S_{w^0 u'} O \subseteq S_{w^0} O \subseteq O(r)$$

hold and so $y \in S_w F$ is an interior point of the open set $O(r)$. We estimate its distance to the boundary and thus to the complement of $O(r)$. Taking into account (4.1.7) and $w^0 w' \in \Sigma(\delta^*)$, we infer

$$\begin{aligned} d(y, \partial O(r)) &\ge d(y, \partial S_{w^0 u'} O) > r_{w^0 u'} \alpha \\ &\ge 2\rho r_{\min}^{-1} r_{w^0 w'} \ge 2\rho\delta^* = 2\delta. \end{aligned}$$

Hence $d(x, O(r)^c) \ge d(y, O(r)^c) - d(x, y) > 2\delta - \epsilon \ge \delta$, implying $x \notin (O(r)^c)_\delta$. □

Proof of II. From the definitions it is easily seen that

$$\#\Lambda(\epsilon, \delta, r) = \sum_{w^0 \in \Sigma(r)} \#\Omega(\frac{\epsilon}{r_{w^0}}, \frac{\delta}{r_{w^0}}).$$

Now observe that for fixed δ, $\#\Omega(\epsilon, \delta)$ is a decreasing function of ϵ (provided $0 < \epsilon \le \delta$), and for fixed ϵ, $\#\Omega(\epsilon, \delta)$ is increasing in δ (as long as $\epsilon \le \delta$). Therefore, in $\#\Omega(\frac{\epsilon}{r_{w^0}}, \frac{\delta}{r_{w^0}})$, $\frac{\epsilon}{r_{w^0}}$ can be replaced by the smaller value $\frac{\epsilon}{r}$, and independently $\frac{\delta}{r_{w^0}}$ by the larger value $\frac{\delta}{r_{\min} r}$ to provide the upper bound $\#\Omega(\frac{\epsilon}{r}, \frac{\delta}{r r_{\min}})$, for each $w^0 \in \Sigma(r)$. ($w^0 \in \Sigma(r)$ implies $r_{w^0} < r \le r_{w^0} r_{\min}^{-1}$ and so $\frac{1}{r} < \frac{1}{r_{w^0}} \le \frac{1}{r r_{\min}}$.) Since the obtained upper bound is independent of $w^0 \in \Sigma(r)$, we conclude that $\#\Lambda(\epsilon, \delta, r) \le \#\Sigma(r) \#\Omega(\frac{\epsilon}{r}, \frac{\delta}{r r_{\min}})$ as asserted above. □

Proof of III. Let $\epsilon \le \delta$. If $w = w' w'' \in \Omega(\epsilon, \delta)$, then necessarily $w' \in \Xi(\delta)$ and $w'' \in \Sigma(\frac{\epsilon}{r_{w'}})$. Therefore,

$$\#\Omega(\epsilon, \delta) = \sum_{w' \in \Xi(\delta)} \#\Sigma(\frac{\epsilon}{r_{w'}}).$$

Observing now that $\#\Sigma(\epsilon$ is a nonincreasing function of ϵ, we can replace $\frac{\epsilon}{r_{w'}}$ by the smaller value $\frac{\epsilon}{\delta}$ ($w' \in \Xi(\delta)$ implies $r_{w'} < \delta$ and so $\frac{\epsilon}{\delta} < \frac{\epsilon}{r_{w'}}$) to see that each term in the above sum is bounded from above by $\#\Sigma(\frac{\epsilon}{\delta})$, which is independent of w' and can thus be taken out of the sum. Hence

$$\#\Omega(\epsilon, \delta) \le \#\Xi(\delta) \#\Sigma(\frac{\epsilon}{\delta}).$$

Now we infer from (4.1.5), that $\#\Sigma(\frac{\epsilon}{\delta}) \le r_{\min}^{-s} \left(\frac{\epsilon}{\delta}\right)^{-s}$, proving assertion III. □

Proof of IV. We have to show that the expression

$$\xi(\epsilon) := \epsilon^{\overline{s}} \# \Xi(\epsilon)$$

is bounded as $\epsilon \to 0$, where $\overline{s} = s - \gamma$ as in (4.1.9). Observe that $\#\Xi(\epsilon)$ is a nonincreasing, nonnegative function of ϵ. Fix some $r > 0$, such that $u \in \Sigma(r)$. For $\epsilon \leq r$, each word $w \in \Xi(\epsilon)$ begins with some word $v \in \Sigma(r)$ different from u and so

$$\#\Xi(\epsilon) \leq \sum_{v \in \Sigma(r), v \neq u} \#\Xi(\frac{\epsilon}{r_v}) \tag{4.4.3}$$

for all $\epsilon \leq r$.

By (4.4.3), $\xi(\epsilon)$ satisfies for all $\epsilon \leq r$

$$\xi(\epsilon) \leq \sum_{v \in \Sigma(r), v \neq u} r_v^{\overline{s}} \xi(\frac{\epsilon}{r_v}),$$

and so, for all $\epsilon' \leq r$,

$$\begin{aligned} \sup_{\epsilon \geq \epsilon'} \xi(\epsilon) &\leq \sup_{\epsilon \geq \epsilon'} \sum_{v \in \Sigma(r), v \neq u} r_v^{\overline{s}} \xi(\frac{\epsilon}{r_v}) \\ &\leq \sum_{v \in \Sigma(r), v \neq u} r_v^{\overline{s}} \sup_{\epsilon \geq \epsilon'} \xi(\frac{\epsilon}{r_v}) \\ &\leq \sup_{\epsilon \geq r^{-1}\epsilon'} \xi(\epsilon). \end{aligned}$$

Since $\#\Xi(\epsilon)$ and thus $\xi(\epsilon)$ are bounded on each interval $[\epsilon', \infty)$, by the above inequality, $\xi(\epsilon)$ is bounded as $\epsilon \to 0$. This completes the proof of IV. □

Remark 4.4.2. Families of finite sequences similar to $\Xi(\epsilon)$ have been studied by Steven P. Lalley in [18]. In the above proofs, especially in the proof of IV, we adopted some of his ideas.

4.5. Bounds for the total variations

In this section we prove the Theorems 1.2.2 and 1.2.8, which provide upper and lower estimates, respectively, for the total mass $C_k^{\mathrm{var}}(F_\epsilon)$ of the total variation measure of F_ϵ. While the proof of Theorem 1.2.2 is based on the key estimate Lemma 4.2.1, for the one of Theorem 1.2.8 we only require appropriate decompositions of the parallel sets F_ϵ. Throughout the section we assume that $F_\epsilon \in \mathcal{R}^d$.

More scaling functions. In analogy with the k-th scaling function R_k we define functions R_k^+, R_k^- and R_k^{var} by

$$R_k^\bullet(\epsilon) = C_k^\bullet(F_\epsilon) - \sum_{i=1}^{N} \mathbf{1}_{(0,r_i]}(\epsilon) C_k^\bullet((S_i F)_\epsilon), \tag{4.5.1}$$

for $\bullet \in \{+, -, \mathrm{var}\}$ and each $\epsilon > 0$. With similar arguments as in the proof of Lemma 4.2.3, which provided an upper bound for $|R_k(\epsilon)|$, we can show the following estimate to hold for $|R_k^{\mathrm{var}}(\epsilon)|$.

Lemma 4.5.1. *There are constants $c, \gamma > 0$ such that for all $\epsilon \in (0,1]$*

$$|R_k^{\text{var}}(\epsilon)| \leq c\epsilon^{k-s+\gamma}.$$

Corresponding estimates hold for R_k^+ and R_k^-. Based on this lemma, we will now prove the boundedness of the expression $\epsilon^{s-k} C_k^{\text{var}}(F_\epsilon)$.

Proof of Theorem 1.2.2. Since, by the scaling property, $C_k^{\text{var}}((S_i F)_\epsilon) = r_i^k C_k^{\text{var}}(F_{\epsilon/r_i})$, by multiplying ϵ^{s-k}, we derive from (4.5.1) that

$$\epsilon^{s-k} C_k^{\text{var}}(F_\epsilon) = \sum_{i=1}^N r_i^s \mathbf{1}_{(0,r_i]}(\epsilon) \left(\frac{\epsilon}{r_i}\right)^{s-k} C_k^{\text{var}}(F_{\epsilon/r_i}) + \epsilon^{s-k} R_k^{\text{var}}(\epsilon).$$

Define the function $g_k : (0,\infty) \to \mathbb{R}$ by setting $g_k(\epsilon) = \epsilon^{s-k} C_k^{\text{var}}(F_\epsilon)$ for $\epsilon \in (0,1]$ and $g_k(\epsilon) = 0$ for $\epsilon > 1$. Then, for each $\epsilon \in (0,1]$, the above equation can be rewritten as

$$g_k(\epsilon) = \sum_{i=1}^N r_i^s g_k(\frac{\epsilon}{r_i}) + \epsilon^{s-k} R_k^{\text{var}}(\epsilon),$$

and so, by Lemma 4.5.1, the following inequality holds for some $c > 0$

$$g_k(\epsilon) \leq \sum_{i=1}^N r_i^s g_k(\frac{\epsilon}{r_i}) + c\epsilon^\gamma.$$

Note that this inequality is also trivially satisfied for all $\epsilon > 1$. Hence for each $\epsilon_0 > 0$,

$$\begin{aligned} \sup_{\epsilon \in (\epsilon_0, \epsilon_0 r_{\max}^{-1}]} g_k(\epsilon) &\leq \sum_{i=1}^N r_i^s \sup_{\epsilon \in (\epsilon_0, \epsilon_0 r_{\max}^{-1}]} g_k(\frac{\epsilon}{r_i}) + \sup_{\epsilon \in (\epsilon_0, \epsilon_0 r_{\max}^{-1}]} c\epsilon^\gamma \\ &\leq \sup_{\epsilon \in (\epsilon_0 r_{\max}^{-1}, \infty]} g_k(\epsilon) + c(\epsilon_0 r_{\max}^{-1})^\gamma \end{aligned}$$

and so

$$\sup_{\epsilon \in (\epsilon_0, \infty)} g_k(\epsilon) \leq \sup_{\epsilon \in (\epsilon_0 r_{\max}^{-1}, \infty)} g_k(\epsilon) + c r_{\max}^{-\gamma} \epsilon_0^\gamma.$$

Iterating this inequality, i.e. applying it repeatedly to the first term on the right hand side, after n steps we arrive at

$$\sup_{\epsilon \in (\epsilon_0, \infty)} g_k(\epsilon) \leq \sup_{\epsilon \in (\epsilon_0 r_{\max}^{-n}, \infty)} g_k(\epsilon) + c \sum_{j=1}^n (r_{\max}^\gamma)^{-j} \epsilon_0^\gamma.$$

Now set $\epsilon_0 = r_{\max}^n$. Then the first term on the right hand side $\sup_{\epsilon \in (1,\infty)} g_k(\epsilon)$ equals zero and so for each $n \in \mathbb{N}$:

$$\sup_{\epsilon \in (r_{\max}^n, \infty)} g_k(\epsilon) \leq c \sum_{j=1}^n (r_{\max}^\gamma)^{n-j} = c \sum_{j=1}^n (r_{\max}^\gamma)^j.$$

Letting $n \to \infty$, the right hand side converges to $M := \frac{c}{1 - r_{\max}^\gamma}$ and so $\sup_{\epsilon \in (0,\infty)} g_k(\epsilon) \leq M$. Hence the expression $\epsilon^{s-k} C_k^{\text{var}}(F_\epsilon)$ is uniformly bounded by M in the interval $(0,1]$, as we asserted in Theorem 1.2.2. □

Remark 4.5.2. As recorded in Conjecture 2.2.2, we are convinced that, for $K \in \mathcal{R}^d$ and $\bullet \in \{+.-, \mathrm{var}\}$, $C_k^\bullet(K_\epsilon)$ has at most finitely many discontinuities in $(0,\infty)$. If this conjecture was true then one could easily prove an analogue of Lemma 4.2.4 for the functions $R_k^\bullet$, i.e. one could show that these functions have a discrete set of discontinuities. This statement combined with Lemma 4.5.1 would allow to apply the Renewal theorem directly to the functions $f^\bullet(\epsilon) := C_k^\bullet(F_\epsilon)$ and $\varphi_k^\bullet(\epsilon) := R_k^\bullet(\epsilon)$. We could derive a statement similar to Theorem 1.2.6 on the existence of rescaled (average) limits of the total masses $C_k^\bullet(F_\epsilon)$ of the variation measures. This would allow to strengthen as well the results on fractal curvature measures. The rescaled variation measures would converge weakly in a similar way as stated in Theorem 1.4.1 for the rescaled curvature measures themselves.

Proof of Theorem 1.2.8. Fix some $k \in \{0,\dots,d\}$. Let B be a set as in the hypotheses of Theorem 1.2.8, i.e. assume that there are constants $\epsilon_0, \beta > 0$ such that $B \subseteq O_{-\epsilon_0}$ and $C_k^{\mathrm{var}}(F_\epsilon, B) \geq \beta$ for $\epsilon \in (r_{\min}\epsilon_0, \epsilon_0]$.

Since for each $r > 0$ the sets S_wO, $w \in \Sigma(r)$, are pairwise disjoint, the same holds for their subsets S_wB and so for arbitrary $\epsilon > 0$

$$C_k^{\mathrm{var}}(F_\epsilon) \geq \sum_{w\in\Sigma(r)} C_k^{\mathrm{var}}(F_\epsilon, S_wB).$$

Fix some $\epsilon < \epsilon_0$ and choose $r = r_{\min}^{-1}\epsilon_0^{-1}\epsilon$. Then, for each $w \in \Sigma(r)$, $r_w < r_{\min}^{-1}\epsilon_0^{-1}\epsilon \leq r_w r_{\min}^{-1}$, i.e. in particular $\epsilon \leq r_w\epsilon_0$, and so, since $B \subseteq O_{-\epsilon_0}$,

$$S_wB \subseteq S_wO_{-\epsilon_0} = (S_wO)_{-r_w\epsilon_0} \subseteq (S_wO)_{-\epsilon}.$$

Hence, by the locality property of C_k^{var} in the open set $(S_wO)_{-\epsilon}$ (where $F_\epsilon \cap (S_wO)_{-\epsilon} = (S_wF)_\epsilon \cap (S_wO)_{-\epsilon}$) and the scaling property,

$$C_k^{\mathrm{var}}(F_\epsilon, S_wB) = C_k^{\mathrm{var}}(S_w(F_{\epsilon r_w^{-1}}), S_wB) = r_w^k C_k^{\mathrm{var}}(F_{\epsilon r_w^{-1}}, B).$$

Since $\epsilon r_w^{-1} \in (r_{\min}\epsilon_0, \epsilon_0]$, the hypothesis implies that $C_k^{\mathrm{var}}(F_{\epsilon r_w^{-1}}, B) \geq \beta$ and therefore,

$$C_k^{\mathrm{var}}(F_\epsilon) \geq \sum_{w\in\Sigma(r)} r_w^k C_k^{\mathrm{var}}(F_{\epsilon r_w^{-1}}, B) \geq \sum_{w\in\Sigma(r)} (\epsilon_0^{-1}\epsilon)^k\beta = \beta\epsilon_0^{-k}\epsilon^k \#\Sigma(r).$$

Recalling from (4.1.5) that $\#\Sigma(r) \geq r^{-s} = r_{\min}^s\epsilon_0^s\epsilon^{-s}$, we obtain

$$C_k^{\mathrm{var}}(F_\epsilon) \geq \beta r_{\min}^s \epsilon_0^{s-k}\epsilon^{-s+k} = c\epsilon^{-s+k}.$$

Since $\epsilon < \epsilon_0$ was arbitrary, the assertion of Theorem 1.2.8 immediately follows.

4.6. Proof of Gatzouras's theorem

Towards the end of Section 1.2 we discussed Gatzouras's results on Minkowski measurability, which we want to prove now. In the proof we will use the results we have already obtained, in particular Theorems 1.2.6 and 1.2.8. Going again through the proofs of these theorems for $k = d$, we will check that, when replacing $C_d(F_\epsilon, \cdot)$ with $\lambda_d(F_\epsilon \cap \cdot\,)$, most arguments remain valid even if the assumption $F_\epsilon \in \mathcal{R}^d$ is dropped. Only few arguments have to be modified.

Let F be a self-similar set satisfying OSC. We emphasize that now we do not assume the parallel sets F_ϵ to be polyconvex. The main step towards a proof of Gatzouras's theorem is a generalization of Lemma 4.3.3. Recall the definition of the family $\Sigma(B, \epsilon)$ from (4.3.4).

Lemma 4.6.1. *There is a constant $c > 0$ such that for each closed sets $B \subseteq \mathbb{R}^d$ and all $\epsilon > 0$,*

$$\lambda_d(F_\epsilon \cap B) \leq c\,\#\Sigma(B, \epsilon)\,\epsilon^d.$$

Proof. For $\epsilon > 0$, the set inclusion $F_\epsilon \cap B \subseteq \bigcup_{w\in\Sigma(B,\epsilon)}(S_wF)_\epsilon$ implies that

$$\begin{aligned}
\lambda_d(F_\epsilon \cap B) &\leq \lambda_d(\bigcup_{w\in\Sigma(B,\epsilon)} (S_wF)_\epsilon) \\
&\leq \sum_{w\in\Sigma(B,\epsilon)} \lambda_d((S_wF)_\epsilon) \\
&\leq \sum_{w\in\Sigma(B,\epsilon)} r_w^d \lambda_d(F_{\epsilon/r_w}) \\
&\leq \sum_{w\in\Sigma(B,\epsilon)} (\rho^{-1}\epsilon)^d \lambda_d(F_{\rho r_{\min}^{-1}}) \\
&= \#\Sigma(B,\epsilon)\rho^{-d}\lambda_d(F_{\rho r_{\min}^{-1}})\epsilon^d,
\end{aligned}$$

where the third inequality is due to the scaling property and the last one to the fact that $r_w < \rho^{-1}\epsilon \leq r_w r_{\min}^{-1}$ for each $w \in \Sigma(B, \epsilon)$. Therefore the constant $c := \rho^{-d}\lambda_d(F_{\rho r_{\min}^{-1}})$ satisfies the assertion. □

Since in Lemma 4.4.1 (as in the whole Section 4.4) we did not assume F to have polyconvex parallel sets, we immediately obtain a generalization of the key estimate Lemma 4.2.1, by combining Lemma 4.4.1 and the just derived Lemma 4.6.1.

Lemma 4.6.2. *For each $r > 0$, there exists $c > 0$ such that for all $\epsilon \leq \delta \leq \rho r$*

$$\lambda_d(F_\epsilon \cap (O(r)^c)_\delta) \leq c\epsilon^{d-s}\delta^\gamma.$$

Note that this estimate will also be the key to the proof of Theorem 1.4.4, the localized version of Gatzouras's theorem. Here, by setting $r = 1$ and $\delta = \epsilon$, we immediately derive an analogue of Corollary 4.2.2.

Corollary 4.6.3. *There exist some constant $c > 0$ such that for all $0 < \epsilon \leq 1$*

$$\lambda_d(F_\epsilon, ((\mathbf{S}O)^c)_\epsilon) \leq c\epsilon^{d-s+\gamma}.$$

It follows at once that Lemma 4.2.3 generalizes in a similar way. Just note that for the Lebesgue measure the locality property trivially holds for arbitrary Borel sets $K, L, A \subseteq \mathbb{R}^d$, i.e. if $A \cap K = A \cap L$, then $\lambda_d(K \cap B) = \lambda_d(L \cap B)$ for all Borel sets

$B \subseteq A$. Recall from (1.2.5) that in the general case the scaling function R_d was given by

$$R_d(\epsilon) = \lambda_d(F_\epsilon) - \sum_{i=1}^{N} \mathbf{1}_{(0,r_i]}(\epsilon)\lambda_d((S_iF)_\epsilon)\,.$$

Lemma 4.6.4. *There exists $c > 0$ such that for all $0 < \epsilon \leq 1$*

$$|R_d(\epsilon)| \leq c\epsilon^{d-s+\gamma}.$$

Since $\lambda_d(F_\epsilon)$ is continuous in ϵ for each closed set F, the function $R_d(\epsilon)$ has at most finitely many discontinuities (at the points r_i). Therefore the Renewal theorem 3.1.4 applies. Taking into account that $s_d = s - d$, it follows that $\overline{M}(F) = X_d$ and for non-arithmetic sets F also $M(F) = X_d$.

It remains to show $X_d > 0$. For this observe that for $k = d$ Theorem 1.2.8 generalizes as follows when the assumption $F_\epsilon \in \mathcal{R}^d$ is dropped.

Proposition 4.6.5. *Let F be a self-similar set satisfying OSC and O some feasible open set of F. Suppose there exist some constants $\epsilon_0, \beta > 0$ and some Borel set $B \subset O_{-\epsilon_0}$ such that*

$$\lambda_d(F_\epsilon \cap B) \geq \beta$$

for each $\epsilon \in (r_{\min}\epsilon_0, \epsilon_0]$. Then for all $\epsilon < \epsilon_0$

$$\epsilon^{s-d}\lambda_d(F_\epsilon) \geq c,$$

where $c := \beta\epsilon_0^{s-d}r_{\min}^s > 0$.

Proof. Observe that the arguments in the proof of Theorem 1.2.8 remain valid in the general case with $C_d^{\mathrm{var}}(F_\epsilon, \cdot) = C_d(F_\epsilon, \cdot)$ replaced by $\lambda_d(F_\epsilon \cap \cdot)$. □

It should be noted that the bound $\epsilon^{s-d}\lambda_d(F_\epsilon) \geq c > 0$ in Proposition 4.6.5 immediately implies $X_d > 0$. Therefore it suffices to show that there is always some set B satisfying the hypothesis of this proposition. Let O be some feasible open set of F such that the SOSC holds. Then there exists a point x in $F \cap O \neq \varnothing$ and, since O is open, some constant $\alpha' > 0$ such that $d(x, \partial O) > \alpha'$. Let $\epsilon_0 = \frac{\alpha'}{2}$ and $B := B(x, \epsilon_0)$. Then $B \subset O_{-\epsilon_0}$, since for each $y \in B(x, \epsilon_0)$,

$$d(y, \partial O) \geq d(x, \partial O) - d(x, y) > \alpha' - \frac{\alpha'}{2} = \epsilon_0\,.$$

Moreover, for each $\epsilon \in (r_{\min}\epsilon_0, \epsilon_0]$,

$$\lambda_d(F_\epsilon \cap B) \geq \lambda_d(B(x, r_{\min}\epsilon_0)) = \kappa_d(r_{\min}\epsilon_0)^d =: \beta\,.$$

Hence the hypothesis of Proposition 4.6.5 is satisfied and $X_d \geq c > 0$ follows. This completes the proof of Gatzouras's theorem.

5. Fractal curvature measures – proofs

In this chapter we will provide proofs of the results of Section 1.4, where we discussed weak limits of rescaled curvature measures. In the first section we will construct a family of sets which on the one hand is adapted to the structure of F and on the other hand is a separating class. This set family is the key to the proofs of all weak convergence results discussed here, since it allows to determine the limit measures uniquely, by computing their values for the sets of the family. The most part of Section 5.2 is reserved for the proof of Theorem 1.4.1. In the end we will briefly outline how the arguments in the proof of this theorem have to be adapted to obtain a proof of Theorem 1.4.2. In the last section we turn our attention to normalized curvature measures and prove Theorems 1.4.3 and 1.4.4. As before, F is some self-similar set satisfying OSC. The set O, the word u and the constants ρ and γ are as defined in Section 4.1.

5.1. A separating class for F

Let $\mathfrak{B}^d$ denote the Borel σ-algebra of $\mathbb{R}^d$. A family $\mathcal{A}$ of Borel sets is called a *separating class* if two measures that agree on $\mathcal{A}$ necessarily agree on $\mathfrak{B}^d$. We now introduce some separating class $\mathcal{A}_F$, which is adapted to the structure of F. Define the set family

$$\mathcal{C}_F := \left\{C \in \mathfrak{B}^d : \exists r > 0 \text{ such that } C \subseteq O(r)^c\right\} ,$$

where $O(r)$ is as defined in (4.2.1), and let

$$\mathcal{A}_F := \{S_w O : w \in \Sigma^*\} \cup \mathcal{C}_F \,.$$

For the family $\mathcal{A}_F$ the following holds.

Lemma 5.1.1. *$\mathcal{A}_F$ is an intersection stable generator of $\mathfrak{B}^d$.*

Proof. The stability of the family $\mathcal{A}_F$ with respect to intersections is easily seen. Either the intersection of two sets S_vO and S_wO, $v, w \in \Sigma^*$, is empty or one of the sets is contained in the other. Moreover, any intersection $A \cap C$ of a set $C \in \mathcal{C}_F$ and a set $A \in \mathcal{A}_F$ is again an element of the family $\mathcal{C}_F$.

Since $\mathcal{A}_F$ consists of Borel sets, the σ-algebra $\sigma(\mathcal{A}_F)$ generated by $\mathcal{A}_F$ is contained in $\mathfrak{B}^d$. It remains to prove the reversed inclusion: $\mathfrak{B}^d \subseteq \sigma(\mathcal{A}_F)$. This is done by showing that each open set is a countable union of sets of $\mathcal{A}_F$.

Let B be an open set and $x \in B$. There exists some $r > 0$ such that $B(x, r) \subset B$. Set $l := (\operatorname{diam} O)^{-1}$ and let

$$\Sigma_x = \left\{w \in \Sigma(lr) : x \in \overline{S_w O}\right\} .$$

By definition of $\Sigma(lr)$, for all $w \in \Sigma_x$, $\operatorname{diam} S_w O = r_w \operatorname{diam} O \le r$ and thus $S_w O \subset B(x, r)$.

For each $w \in \Sigma(lr) \setminus \Sigma_x$, $S_w O$ has some positive distance to x. Therefore we can find some positive constant c such that $d(x, \overline{S_w O}) > c$ for all $w \in \Sigma(lr) \setminus \Sigma_x$.

Let $C_x = B(x,c) \setminus \bigcup_{w \in \Sigma_x} S_w O$, which is obviously a subset of $O(lr)^c$ and thus an element of $\mathcal{C}_F$. Moreover, let $A_x = C_x \cup \bigcup_{w \in \Sigma_x} S_w O$. By construction, A_x is a finite union of sets from $\mathcal{A}_F$ and $A_x \subseteq B(x,r) \subset B$. On the other hand the family $\{A_x : x \in B\}$ covers the set B and thus $B = \bigcup_{x \in B} A_x$. Since $x \in \operatorname{int} A_x$, the family $\{\operatorname{int} A_x : x \in B\}$ forms an open cover of B, which, by the Lindelöf Theorem has an countable open subcover, i.e. there are $x_1, x_2, \dots$ in B such that $B = \bigcup_i \operatorname{int} A_{x_i}$. But then also $B = \bigcup_i A_{x_i}$ and so, since each set A_{x_i} is a finite union of sets from $\mathcal{A}_F$, we obtain a representation of B as a countable union of sets from $\mathcal{A}_F$, as desired. This completes the proof. □

The properties derived in Lemma 5.1.1 are sufficient for $\mathcal{A}_F$ to be a separating class for the family of totally finite (signed) measures. We recall the uniqueness theorem which is well known for positive measures and easily generalized to signed measures.

Theorem 5.1.2. *Let μ and ν totally finite signed measures on $\mathfrak{B}^d$, and $\mathcal{A}$ an intersection stable generator of $\mathfrak{B}^d$ such that $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$. Then $\mu = \nu$.*

Although we can not give a direct reference for this theorem for signed measures, we believe that it is well known. For a proof of this statement for positive measures see for instance Elstrodt [5, p.60] or Jacobs [16, p.51]. The arguments of the proof in [5] extend to signed measures, once one has verified the fact that for any increasing sequence of sets A_n converging to A and any signed measure μ, $\mu(A_i) \to \mu(A)$ as $n \to \infty$.

By Lemma 5.1.1, the family $\mathcal{A}_F$ satisfies the hypotheses of Theorem 5.1.2 and is thus a separating class. $\mathcal{A}_F$ is constructed in such a way that the measures considered can easily be computed for the sets of this family.

In the proof of Theorem 1.4.1 we will have to show that certain measures coincide with some multiple of the measure μ_F defined in (1.4.3). Since by the above uniqueness theorem it is sufficient to compare the measures for sets $A \in \mathcal{A}_F$, we collect the values $\mu_F(A)$ for those sets. Recall that μ_F is the self-similar measure with weights $\{r_1^s, \dots, r_N^s\}$, i.e. it is the unique probability measure satisfying the invariance relation

$$\mu_F = \sum_{i=1}^{N} r_i^s \mu_F \circ S_i^{-1}.$$

It is well known that, provided the OSC is satisfied, $\mu_F(O) = \mu_F(F)$ and similarly $\mu_F(S_w O) = \mu_F(S_w F)$ for each $w \in \Sigma^*$. Since $\mu_F(F) = 1$, the invariance relation implies $\mu_F(S_w F) = r_w^s$. Therefore, we record:

$$\mu_F(S_w O) = r_w^s \quad \text{for all} \quad w \in \Sigma^* \tag{5.1.1}$$

and

$$\mu_F(C) = 0 \quad \text{for all} \quad C \in \mathcal{C}_F. \tag{5.1.2}$$

5.2. Proof of Theorem 1.4.1 and 1.4.2

First we prove Theorem 1.4.1, for which we assume that F has polyconvex parallel sets. Fix some $k \in \{0, \dots, d\}$ and assume that the k-th scaling exponent of F satisfies $s_k = s - k$. The first step is to deduce some technical estimates, which provide bounds for the curvature measure of F_ϵ in the sets $(S_w O)_\delta$ and $S_w O$ for $w \in \Sigma^*$. They are based on the key lemma (Lemma 4.2.1). The exponent γ which will occur in all the estimates below, is the one we defined in (4.1.9).

Lemma 5.2.1. *Let $w \in \Sigma^*$. There exists $c > 0$ such that for all $0 < \epsilon \le \delta \le \rho r_w$ and $\bullet \in \{+, -, \mathrm{var}\}$*

$$C_k^\bullet(F_\epsilon, (S_w O)_\delta) \le r_w^k C_k^\bullet(F_{\epsilon r_w^{-1}}) + c\epsilon^{k-s}\delta^\gamma \tag{5.2.1}$$

and

$$C_k^\bullet(F_\epsilon, S_w O) \ge r_w^k C_k^\bullet(F_{\epsilon r_w^{-1}}) - c\epsilon^{k-s}\delta^\gamma. \tag{5.2.2}$$

Proof. Fix $w \in \Sigma^*$. Observe that $(S_w O)_\delta = (S_w O)_{-\delta} \cup (\partial S_w O)_\delta$. Provided that $\epsilon \le \delta$, for the first set in this union, the locality of $C_k^\bullet$ (here in the open set $(S_w O)_{-\delta}$ we have $(S_w F)_\epsilon \cap (S_w O)_{-\delta} = F_\epsilon \cap (S_w O)_{-\delta}$) and the scaling property imply

$$\begin{aligned} C_k^\bullet(F_\epsilon, (S_w O)_{-\delta}) &= C_k^\bullet((S_w F)_\epsilon, (S_w O)_{-\delta}) \\ &= r_w^k C_k^\bullet(F_{\epsilon r_w^{-1}}, O_{-\delta r_w^{-1}}) \\ &\le r_w^k C_k^\bullet(F_{\epsilon r_w^{-1}}). \end{aligned} \tag{5.2.3}$$

Choosing r such that $w \in \Sigma(r)$, i.e. $r_w < r \le r_w r_{\min}^{-1}$, the second set is a subset of $(O(r)^c)_\delta$, since $\partial S_w O \subseteq O(r)^c$. Therefore, by Lemma 4.2.1 , there are constants $c, \gamma > 0$ such that

$$C_k^\bullet(F_\epsilon, (\partial S_w O)_\delta) \le C_k^\bullet(F_\epsilon, (O(r)^c)_\delta) \le c\epsilon^{k-s}\delta^\gamma$$

for all $\epsilon \le \delta \le \rho r$ (and so in particular for $\delta \le \rho r_w$). Hence the first estimate (5.2.1) follows immediately from the relation

$$C_k^\bullet(F_\epsilon, (S_w O)_\delta) \le C_k^\bullet(F_\epsilon, (S_w O)_{-\delta}) + C_k^\bullet(F_\epsilon, (\partial S_w O)_\delta).$$

For the second estimate choose r such that $O = O(r)$, i.e. $1 < r \le r_{\min}^{-1}$. Then, again by Lemma 4.2.1, there are $c', \gamma > 0$ such that for all $\epsilon \le \delta \le \rho r$

$$\begin{aligned} C_k^\bullet(F_\epsilon) &\le C_k^\bullet(F_\epsilon, O_{-\delta}) + C_k^\bullet(F_\epsilon, (O^c)_\delta) \\ &\le C_k^\bullet(F_\epsilon, O_{-\delta}) + c'\epsilon^{k-s}\delta^\gamma. \end{aligned}$$

Bringing $c'\epsilon^{k-s}\delta^\gamma$ onto the other side of this inequality and taking into account (5.2.3) we infer that for all $\epsilon \le \delta \le \rho r$

$$\begin{aligned} C_k^\bullet(F_\epsilon, S_w O) &\ge C_k^\bullet(F_\epsilon, (S_w O)_{-\delta}) \\ &= r_w^k C_k^\bullet(F_{\epsilon r_w^{-1}}, O_{-\delta r_w^{-1}}) \\ &\ge r_w^k \left(C_k^\bullet(F_{\epsilon r_w^{-1}}) - c'(\epsilon r_w^{-1})^{k-s}(\delta r_w^{-1})^\gamma \right). \end{aligned}$$

Hence the estimate (5.2.2) holds for the constant $c = c' r_w^{s-\gamma}$ for all $\epsilon \le \delta \le \rho r$ and thus in particular for $\epsilon \le \delta \le \rho r_w$. For the maximum of the two constants c derived for the two estimates, both inequalities are satisfied, completing the proof of Lemma 5.2.1. □

For convenience we introduce the abbreviation

$$\nu(f) := \int_{\mathbb{R}^d} f d\nu \tag{5.2.4}$$

for the integral of a function f with respect to a (signed) measure ν. For $w \in \Sigma^*$ and $\delta > 0$, let $f_\delta^w : \mathbb{R}^d \to [0,1]$ be a continuous function such that

$$f_\delta^w(x) = 1 \text{ for } x \in S_w O \quad \text{ and } \quad f_\delta^w(x) = 0 \text{ for } x \text{ outside } (S_w O)_\delta.$$

For simplicity, assume that $f_\delta^w \leq f_{\delta'}^w$ for all $\delta < \delta'$. Obviously, f_δ^w has compact support and satisfies $\mathbf{1}_{S_w O} \leq f_\delta^w \leq \mathbf{1}_{(S_w O)_\delta}$. Moreover, as $\delta \to 0$, the functions f_δ^w converge (pointwise) to $\mathbf{1}_{S_w O}$, implying in particular the convergence of the integrals $\nu(f_\delta^w) \to \nu(\mathbf{1}_{S_w O}) = \nu(S_w O)$ with respect to any (signed) Radon measure ν.

Now recall the definition of the rescaled curvature measures $\nu_{k,\epsilon}$ from (1.4.1). Using the above estimates, we derive some bounds for the integrals $\nu_{k,\epsilon}(f_\delta^w)$.

Lemma 5.2.2. *Let $w \in \Sigma^*$. Then for all $0 < \epsilon \leq \delta \leq \rho r_w$*

$$\left| \nu_{k,\epsilon}(f_\delta^w) - r_w^s \nu_{k,\epsilon r_w^{-1}}(\mathbb{R}^d) \right| \leq 2c\delta^\gamma,$$

where $c = c(w)$ is the constant in Lemma 5.2.1.

Proof. Fix $w \in \Sigma^*$. Since $\mathbf{1}_{S_w O} \leq f_w^\delta \leq \mathbf{1}_{(S_w O)_\delta}$, Lemma 5.2.1 implies that there exist $c, \gamma > 0$ such that for all $\epsilon \leq \delta \leq \rho r$

$$\nu_{k,\epsilon}^{\pm}(f_\delta^w) \leq \nu_{k,\epsilon}^{\pm}((S_w O)_\delta) \leq r_w^s \nu_{k,\epsilon r_w^{-1}}^{\pm}(\mathbb{R}^d) + c\delta^\gamma \tag{5.2.5}$$

and similarly

$$\nu_{k,\epsilon}^{\pm}(f_\delta^w) \geq \nu_{k,\epsilon}^{\pm}(S_w O) \geq r_w^s \nu_{k,\epsilon r_w^{-1}}^{\pm}(\mathbb{R}^d) - c\delta^\gamma. \tag{5.2.6}$$

Applying these inequalities to $\nu_{k,\epsilon}(f_\delta^w) = \nu_{k,\epsilon}^+(f_\delta^w) - \nu_{k,\epsilon}^-(f_\delta^w)$, we obtain that on the one hand

$$\begin{aligned} \nu_{k,\epsilon}(f_\delta^w) &\leq r_w^s \nu_{k,\epsilon r_w^{-1}}^+(\mathbb{R}^d) + c\delta^\gamma - \left(r_w^s \nu_{k,\epsilon r_w^{-1}}^-(\mathbb{R}^d) - c\delta^\gamma \right) \\ &= r_w^s \nu_{k,\epsilon r_w^{-1}}(\mathbb{R}^d) + 2c\delta^\gamma \end{aligned}$$

and on the other hand

$$\begin{aligned} \nu_{k,\epsilon}(f_\delta^w) &\geq r_w^s \nu_{k,\epsilon r_w^{-1}}^+(\mathbb{R}^d) - c\delta^\gamma - \left(r_w^s \nu_{k,\epsilon r_w^{-1}}^-(\mathbb{R}^d) + c\delta^\gamma \right) \\ &= r_w^s \nu_{k,\epsilon r_w^{-1}}(\mathbb{R}^d) - 2c\delta^\gamma. \end{aligned}$$

Combining both estimates, the assertion of Lemma 5.2.2 follows immediately. □

Proof of the convergence $\nu_{k,\epsilon} \xrightarrow{\text{w}} C_k^f(F)\ \mu_F$. Assume that F is non-arithmetic. The total masses of the variation measures $\nu_{k,\epsilon}^+$ and $\nu_{k,\epsilon}^-$ of $\nu_{k,\epsilon}$ are uniformly bounded. Moreover, since for each $\epsilon \leq 1$ the support of $\nu_{k,\epsilon}^\pm$ is contained in the 1-parallel set of F, the families $\{\nu_{k,\epsilon}^+\}_{\epsilon\in(0,1]}$ and $\{\nu_{k,\epsilon}^-\}_{\epsilon\in(0,1]}$ are tight. Therefore, by Prohorov's Theorem, they are relatively compact, i.e. every sequence has a weakly convergent subsequence. In particular, every null sequence has a subsequence $\{\epsilon_n\}$ such that the measures ν_{k,ϵ_n}^+ converge weakly, and this subsequence has a further subsequence, again denoted $\{\epsilon_n\}$, such that also the measures ν_{k,ϵ_n}^- converge weakly.

Now let $\{\epsilon_n\}$ such a sequence, i.e. assume that

$$\nu_{k,\epsilon_n}^+ \xrightarrow{\text{w}} \nu_k^+ \quad \text{and} \quad \nu_{k,\epsilon_n}^- \xrightarrow{\text{w}} \nu_k^- \quad \text{as} \quad n \to \infty,$$

for some limit measures ν_k^+ and ν_k^-. The weak convergence of the variation measures $\nu_{k,\epsilon_n}^\pm$ implies the weak convergence of the signed measures $\nu_{k,\epsilon_n} = \nu_{k,\epsilon_n}^+ - \nu_{k,\epsilon_n}^-$ and the limit measure is given by $\nu_k = \nu_k^+ - \nu_k^-$. Note that the measures ν_k^+ and ν_k^- are not necessarily the positive and negative variation of ν_k. They are just some representation of ν_k as a difference of two positive measures and do not necessarily live on two disjoint sets. In general, the limit measures ν_k might depend on the chosen sequence $\{\epsilon_n\}$. However, it is our aim to show that here this is not the case, i.e. we want to prove that for any such sequence $\{\epsilon_n\}$, the limit measure ν_k is the same, namely that ν_k coincides with $\mu_k := C_k^f(F)\ \mu_F$, which implies at once that the weak limit $\nu_{k,\epsilon}$ as $\epsilon \to 0$ exists and coincides with μ_k, as stated in Theorem 1.4.1 for the non-arithmetic case.

In Section 5.1 we constructed the family $\mathcal{A}_F$ and showed that it is an intersection stable generator of $\mathcal{B}^d$. By Theorem 5.1.2, the measures ν_k and μ_k coincide if they coincide for all sets $A \in \mathcal{A}_F$. The measure μ_k is known. For sets $C \in \mathcal{C}_F$, by (5.1.2), $\mu_F(C) = 0$ and so $\mu_k(C) = C_k^f(F)\mu_F(C) = 0$. For sets S_wO, $w \in \Sigma^*$, by (5.1.1), $\mu_F(S_wO) = r_w^s$ and thus $\mu_k(S_wO) = C_k^f(F)r_w^s$.

Therefore, we have to show that for all $w \in \Sigma^*$

$$\nu_k(S_wO) = C_k^f(F)r_w^s, \tag{5.2.7}$$

and for all $C \in \mathcal{C}_F$

$$\nu_k(C) = 0. \tag{5.2.8}$$

Proof of (5.2.7). We approximate the measure of the sets S_wO with the integrals of the functions f_δ^w defined in (5.2.4 and use Lemma 5.2.2. Fix $w \in \Sigma^*$ and let $r = r_w$. By Lemma 5.2.2, we have for all n and δ such that $\epsilon_n \leq \delta \leq \rho r$

$$\left|\nu_{k,\epsilon_n}(f_\delta^w) - r_w^s \nu_{k,\epsilon_n r_w^{-1}}(\mathbb{R}^d)\right| \leq 2c\delta^\gamma. \tag{5.2.9}$$

Keeping δ fixed and letting $n \to \infty$, the weak convergence implies $\nu_{k,\epsilon_n}(f_\delta^w) \to \nu_k(f_\delta^w)$, since f_δ^w is continuous. Moreover, $\nu_{k,\epsilon_n r_w^{-1}}(\mathbb{R}^d) = (\epsilon r_w^{-1})^{s-k} C_k(F_{\epsilon r_w^{-1}}) \to C_k^f(F)$, by Theorem 1.2.6. Hence the above inequality yields

$$\left|\nu_k(f_\delta^w) - r_w^s C_k^f(F)\right| \leq 2c\delta^\gamma \tag{5.2.10}$$

for each $\delta \leq \rho r$. Letting now $\delta \to 0$, the integrals $\nu_k(f_\delta^w)$ converge to $\nu_k(\mathbf{1}_{S_wO}) = \nu_k(S_wO)$, while the right hand side of the inequality vanishes. Therefore, $|\nu_k(S_wO) - r_w^s C_k^f(F)| \leq 0$ which implies $\nu_k(S_wO) = r_w^s C_k^f(F)$, as claimed in (5.2.7). □

Proof of (5.2.8). Fix $r > 0$. It suffices to show $\nu_k^{\pm}(O(r)^c) = 0$, since this immediately implies that $\nu_k(C) = \nu_k^+(C) - \nu_k^-(C) = 0$ for all $C \subseteq O(r)^c$. Similarly as before we approximate the indicator function of $O(r)^c$ by continuous functions. For $\delta > 0$, let $g_\delta : \mathbb{R}^d \to [0,1]$ a continuous function such that

$$g_\delta(x) = 1 \quad \text{for} \quad x \in O(r)^c \quad \text{and} \quad g_\delta(x) = 0 \quad \text{for} \quad x \in (O(r))_{-\delta}. \tag{5.2.11}$$

Since $g_\delta \leq \mathbf{1}_{(O(r)^c)_\delta}$, by Lemma 4.2.1, for all $\epsilon_n \leq \delta \leq \rho r$,

$$\nu_{k,\epsilon_n}^{\pm}(g_\delta) \leq c\delta^\gamma.$$

Keeping δ fixed and letting $n \to \infty$, the weak convergence implies that $\nu_{k,\epsilon_n}^{\pm}(g_\delta) \to \nu_k^{\pm}(g_\delta)$ while the right hand side remains unchanged. Letting now $\delta \to 0$, the functions g_δ converge pointwise to $\mathbf{1}_{O(r)^c}$ and thus $\nu_k^{\pm}(g_\delta) \to \nu_k^{\pm}(O(r)^c)$, while $c\delta^\gamma$ vanishes. Hence $\nu_k^{\pm}(O(r)^c) = 0$, completing the proof of (5.2.8). □

This completes the proof of the weak convergence of $\nu_{k,\epsilon}$ to μ_k as $\epsilon \to 0$ for the non-arithmetic case.

Now we want to discuss the weak convergence of the averaged measures $\overline{\nu}_{k,\epsilon}$ as defined in (1.4.2). For this purpose we first derive some estimate for the integrals $\overline{\nu}_{k,\epsilon}(f_\delta^w)$, which is the analogue to Lemma 5.2.2 for the averaged measures $\overline{\nu}_{k,\epsilon}$.

Lemma 5.2.3. *Let $w \in \Sigma^*$. For all ϵ and δ such that $0 < \epsilon \leq \delta \leq \rho r_w$ it holds*

$$\left|\overline{\nu}_{k,\epsilon}(f_\delta^w) - r_w^s \overline{\nu}_{k,\epsilon r_w^{-1}}(\mathbb{R}^d)\right| \leq 2c\delta^\gamma + \frac{\log\delta}{\log\epsilon} 2(c\delta^\gamma + M),$$

where $c = c(w)$ is the constant of Lemma 5.2.1 and M is the one of Lemma 1.2.2.

Proof. Fix $w \in \Sigma^*$. Since $f_w^\delta \leq \mathbf{1}_{(S_w O)_\delta}$,

$$\overline{\nu}_{k,\epsilon}^{\pm}(f_\delta^w) \leq \overline{\nu}_{k,\epsilon}^{\pm}((S_w O)_\delta) = \frac{1}{|\log\epsilon|}\int_\epsilon^1 \tilde{\epsilon}^{s-k} C_k^{\pm}(F_{\tilde{\epsilon}}, (S_w O)_\delta)\frac{d\tilde{\epsilon}}{\tilde{\epsilon}}$$

For $\epsilon \leq \delta$, we split the integral into two parts, one over the interval $[\epsilon, \delta]$ and one over $(\delta, 1)$, and apply the first inequality of Lemma 5.2.1 to the first part and Lemma 1.2.2 to the second part. Then for all $\epsilon \leq \delta \leq \rho r_w$ we obtain

$$\begin{aligned}
\overline{\nu}_{k,\epsilon}^{\pm}(f_\delta^w) &\leq \frac{1}{|\log\epsilon|}\int_\epsilon^\delta (r_w^s \nu_{k,\tilde{\epsilon} r_w^{-1}}^{\pm}(\mathbb{R}^d) + c\delta^\gamma)\frac{d\tilde{\epsilon}}{\tilde{\epsilon}} + \frac{1}{|\log\epsilon|}\int_\delta^1 M\frac{d\tilde{\epsilon}}{\tilde{\epsilon}} \\
&\leq r_w^s \overline{\nu}_{k,\epsilon r_w^{-1}}^{\pm}(\mathbb{R}^d) + \frac{c\delta^\gamma}{|\log\epsilon|}\int_\epsilon^\delta \frac{d\tilde{\epsilon}}{\tilde{\epsilon}} + \frac{M}{|\log\epsilon|}\int_\delta^1 \frac{d\tilde{\epsilon}}{\tilde{\epsilon}} \\
&\leq r_w^s \overline{\nu}_{k,\epsilon r_w^{-1}}^{\pm}(\mathbb{R}^d) + c\delta^\gamma + \frac{\log\delta}{\log\epsilon}(c\delta^\gamma + M).
\end{aligned} \tag{5.2.12}$$

In a similar way we derive the corresponding lower bounds. Since $\mathbf{1}_{S_w O} \leq f_w^\delta$, it holds

$$\overline{\nu}_{k,\epsilon}^{\pm}(f_\delta^w) \geq \overline{\nu}_{k,\epsilon}^{\pm}(S_w O) \geq \frac{1}{|\log\epsilon|}\int_\epsilon^\delta \tilde{\epsilon}^{s-k} C_k^{\pm}(F_{\tilde{\epsilon}}, S_w O)\frac{d\tilde{\epsilon}}{\tilde{\epsilon}}.$$

Applying the second estimate of Lemma 5.2.1, we infer that for all $\epsilon \le \delta \le \rho r_w$

$$\begin{aligned}\overline{\nu}^{\pm}_{k,\epsilon}(f^w_\delta) &\ge \frac{1}{|\log\epsilon|}\int_\epsilon^\delta (r_w^s \nu^{\pm}_{k,\tilde\epsilon r_w^{-1}}(\mathbb{R}^d) - c\delta^\gamma)\frac{d\tilde\epsilon}{\tilde\epsilon} \\ &\ge r_w^s \overline{\nu}^{\pm}_{k,\tilde\epsilon r_w^{-1}}(\mathbb{R}^d) - \frac{1}{|\log\epsilon|}\int_\delta^1 r_w^s \nu^{\pm}_{k,\tilde\epsilon r_w^{-1}}(\mathbb{R}^d)\frac{d\tilde\epsilon}{\tilde\epsilon} - \frac{c\delta^\gamma}{|\log\epsilon|}\int_\epsilon^\delta \frac{d\tilde\epsilon}{\tilde\epsilon}.\end{aligned}$$

Since, by Lemma 1.2.2, $\nu^{\pm}_{k,\tilde\epsilon r_w^{-1}}(\mathbb{R}^d) \le M$ and $r_w^s \le 1$, the second term is bounded from below by $-\frac{\log\delta}{\log\epsilon}M$. Therefore we obtain

$$\overline{\nu}^{\pm}_{k,\epsilon}(f^w_\delta) \ge r_w^s \overline{\nu}^{\pm}_{k,\tilde\epsilon r_w^{-1}}(\mathbb{R}^d) - c\delta^\gamma - \frac{\log\delta}{\log\epsilon}(c\delta^\gamma + M) \tag{5.2.13}$$

Applying inequalities (5.2.12) and (5.2.13) to $\overline{\nu}_{k,\epsilon}(f^w_\delta) = \overline{\nu}^+_{k,\epsilon}(f^w_\delta) - \overline{\nu}^-_{k,\epsilon}(f^w_\delta)$, the asserted inequality follows in a similar way as the one for $\nu_{k,\epsilon}(f^w_\delta)$ in the proof of Lemma 5.2.2. □

Proof of the convergence $\overline{\nu}_{k,\epsilon} \xrightarrow{\text{w}} \overline{C}^f_k(F)\ \mu_F$. The proof for the averaged measures $\overline{\nu}_{k,\epsilon}$ is almost the same as the one for $\nu_{k,\epsilon}$ in the non-arithmetic case. It is easily seen that also the families $\{\overline{\nu}^+_{k,\epsilon}\}_{\epsilon\in(0,1]}$ and $\{\overline{\nu}^-_{k,\epsilon}\}_{\epsilon\in(0,1]}$ are tight and thus, by Prohorov's Theorem, relatively compact. Let $\{\epsilon_n\}$ be a null sequence such that

$$\overline{\nu}^+_{k,\epsilon_n} \xrightarrow{\text{w}} \overline{\nu}^+_k \quad \text{and} \quad \overline{\nu}^-_{k,\epsilon_n} \xrightarrow{\text{w}} \overline{\nu}^-_k \quad \text{as} \quad \epsilon \to 0,$$

for some limit measures $\overline{\nu}^+_k$ and $\overline{\nu}^-_k$. Then $\overline{\nu}_{k,\epsilon_n} \xrightarrow{\text{w}} \overline{\nu}_k := \overline{\nu}^+_k - \overline{\nu}^-_k$. Now we have to show that the limit measure $\overline{\nu}_k$ coincides with $\overline{\mu}_k := \overline{C}^f_k(F)\ \mu_F$, from which we can conclude the asserted convergence as $\epsilon \to 0$.

Again we work with the family $\mathcal{A}_F$. By Theorem 5.1.2, the measures $\overline{\nu}_k$ and $\overline{\mu}_k$ coincide if they coincide for all sets $A \in \mathcal{A}_F$. Since, by (5.1.2), $\overline{\mu}_k(C) = \overline{C}^f_k(F)\mu_F(C) = 0$ for sets $C \in \mathcal{C}_F$ and, by (5.1.1), $\mu_k(S_wO) = \overline{C}^f_k(F) r_w^s$ for $w \in \Sigma^*$, we have to show that for all $w \in \Sigma^*$

$$\overline{\nu}_k(S_wO) = \overline{C}^f_k(F) r_w^s, \tag{5.2.14}$$

and for all $C \in \mathcal{C}_F$

$$\overline{\nu}_k(C) = 0 \tag{5.2.15}$$

in order to complete the proof that $\overline{\nu}_{k,\epsilon} \xrightarrow{\text{w}} \overline{C}^f_k(F)\ \mu_F$.

Proof of (5.2.14). The arguments are very similar to those in the proof of (5.2.7). Fix $w \in \Sigma^*$ and let $r = r_w$. Lemma 5.2.3 ensures that for all n and δ such that $\epsilon_n \le \delta \le \rho r$

$$\left|\overline{\nu}_{k,\epsilon_n}(f^w_\delta) - r_w^s \overline{\nu}_{k,\epsilon_n r_w^{-1}}(\mathbb{R}^d)\right| \le 2c\delta^\gamma + \frac{\log\delta}{\log\epsilon_n} 2(c\delta^\gamma + M).$$

Keeping δ fixed and letting $n \to \infty$, the weak convergence implies $\overline{\nu}_{k,\epsilon_n}(f^w_\delta) \to \overline{\nu}_k(f^w_\delta)$, while $\overline{\nu}_{k,\epsilon_n r_w^{-1}}(\mathbb{R}^d) \to \overline{C}^f_k(F)$, by Theorem 1.2.6. On the right hand side the second term vanishes. Hence the above inequality yields

$$\left|\overline{\nu}_k(f^w_\delta) - r_w^s \overline{C}^f_k(F)\right| \le 2c\delta^\gamma$$

for each $\delta \le \rho r$. Letting now $\delta \to 0$, the integrals $\overline{\nu}_k(f^w_\delta)$ converge to $\overline{\nu}_k(S_wO)$, while the right hand side of the inequality vanishes, proving assertion (5.2.14). □

Proof of (5.2.15). Fix $r > 0$. For the functions g_δ defined in (5.2.11) there exists $c > 0$ such that for all $\epsilon \leq \delta \leq \rho r$,

$$\overline{\nu}_{k,\epsilon}^{\pm}(g_\delta) \leq c\delta^\gamma + \frac{\log \delta}{\log \epsilon}(c\delta^\gamma + M),$$

which is derived in a similar way as (5.2.12).

Having established this estimate, the arguments are the same as those in the proof of (5.2.8). Set $\epsilon = \epsilon_n$ in the above inequality and let first $n \to \infty$ and afterwards $\delta \to 0$. Then the right hand side converges to 0, while on the left hand side we end up with $\overline{\nu}_k^{\pm}(O(r))$, which therefore equals zero. Now the assertion (5.2.15) is an immediate consequence. □

This completes the proof of Theorem 1.4.1.

Proof of Theorem 1.4.2. Suppose now that F is an arbitrary self-similar set satisfying OSC, i.e. the parallel sets F_ϵ are not necessarily polyconvex. We will briefly outline, how the arguments in the above proof have to be modified to obtain a proof of Theorem 1.4.2.

By Gatzouras's theorem, the average Minkowski content $\overline{M}(F)$ does always exist and is strictly positive implying that always $s_d = s - d$. Going again through the proof of Theorem 1.4.1, it is easily seen that for $k = d$ most of the arguments remain valid in the general case. Some parts even simplify due to the positivity of the measure. In Lemma 5.2.1 we simply replace $C_d^{\mathrm{var}}(F_\epsilon, \cdot)$ by $\lambda_d(F_\epsilon \cap \cdot)$ and use Lemma 4.6.2 instead of Lemma 4.2.1 to obtain the corresponding general estimate. Lemma 5.2.2 and 5.2.3 remain valid as stated, when the generalized definitions (1.4.4) and (1.4.5) of $\nu_{d,\epsilon}$ and $\overline{\nu}_{d,\epsilon}$ are used. The proofs of both lemmas even simplify, since these measures are not signed.

For proving the convergence $\nu_{d,\epsilon} \xrightarrow{\mathrm{w}} M(F)\ \mu_F$ for non-arithmetic sets F, Prohorov's Theorem can now be applied directly to $\nu_{d,\epsilon}$, without switching to the signed measures first. Choose any sequence $\{\epsilon_n\}$ such that $\nu_{d,\epsilon_n} \xrightarrow{\mathrm{w}} \nu_d$ and show that the limit measure ν_d of this sequence equals $\mu_d := M(F)\ \mu_F$ and is thus independent of the chosen sequence. The equivalence of the measures is obtained with the same arguments as in in the proof of Theorem 1.4.1 taking into account Lemma 4.6.2 and the generalized Lemma 5.2.2. The proof of the convergence $\overline{\nu}_{d,\epsilon} \xrightarrow{\mathrm{w}} \overline{M}(F)\ \mu_F$ is adapted in a similar way.

5.3. Proof of Theorem 1.4.3 and 1.4.4

For $k = d-1$ and $k = d$, we show the weak convergence of the normalized curvature measures $\nu_{k,\epsilon}^1 \xrightarrow{\mathrm{w}} \mu_F$. First we discuss the case $k = d-1$. The arguments for $k = d$ are very similar as we will briefly outline afterwards.

Proof of Theorem 1.4.3. Let F be a self-similar set as assumed in Theorem 1.4.3. In particular it was assumed that there exists a constant $b > 0$ such that

$$\liminf_{\epsilon \to 0} \epsilon^{s-d+1} C_{d-1}(F_\epsilon) = b, \tag{5.3.1}$$

which immediately implies $s_{d-1} = s - d + 1$. By definition, for all $\epsilon > 0$,

$$\nu^1_{d-1,\epsilon} = (\epsilon^{s-d+1} C_{d-1}(F_\epsilon))^{-1} \nu_{d-1,\epsilon}.$$

If F is non-arithmetic, then the weak convergence $\nu^1_{d-1,\epsilon} \xrightarrow{\text{w}} \mu_F$ follows immediately, since, by the Theorems 1.2.6 and 1.4.1, $\epsilon^{s-d+1} C_{d-1}(F_\epsilon)$ converges to $C^f_{d-1}(F)$, while $\nu_{d-1,\epsilon} \xrightarrow{\text{w}} C^f_{d-1}(F)\ \mu_F$. By the assumptions, $C^f_{d-1}(F) > 0$.

So assume now that F is arithmetic. In this case we have to work a bit more. Similarly as in the above proofs, let $\{\epsilon_n\}$ a null sequence such that $\nu^1_{d-1,\epsilon_n} \xrightarrow{\text{w}} \nu^1_{d-1}$ as $n \to \infty$ for some limit measure ν^1_{d-1}. We prove that ν^1_{d-1} is independent of the choice of the sequence $\{\epsilon_n\}$ by showing that $\nu^1_{d-1} = \mu_F$. By Theorem 5.1.2 and the considerations in Section 5.1, it suffices to show that for all $w \in \Sigma^*$

$$\nu^1_{d-1}(S_w O) = r^s_w, \tag{5.3.2}$$

and for all $C \in \mathcal{C}_F$,

$$\nu^1_{d-1}(C) = 0. \tag{5.3.3}$$

Proof of (5.3.2). Fix $w \in \Sigma^*$. Dividing the inequality in Lemma 5.2.2 by $\nu_{d-1,\epsilon}(\mathbb{R}^d)$, we infer that for all $\epsilon_n \le \delta \le \rho r_w$

$$\left|\nu^1_{d-1,\epsilon_n}(f^w_\delta) - r^s_w q_w(\epsilon_n)\right| \le 2c\delta^\gamma (\nu_{d-1,\epsilon_n}(\mathbb{R}^d))^{-1},$$

where

$$q_w(\epsilon) := \frac{\nu_{d-1,\epsilon r_w^{-1}}(\mathbb{R}^d)}{\nu_{d-1,\epsilon}(\mathbb{R}^d)} = \frac{(\epsilon r_w^{-1})^{s-d+1} C_{d-1}(F_{\epsilon r_w^{-1}})}{\epsilon^{s-d+1} C_{d-1}(F_\epsilon)}.$$

We show that this quotient converges to 1 as $\epsilon \to 0$. Since F was assumed to be arithmetic, there is some $h > 0$ such that, for each $i = 1, \dots, N$, there exists $n_i \in \mathbb{N}$ such that $-\log r_i = n_i h$. We infer from Remark 3.1.5 that the expression $g(\epsilon) = \epsilon^{s-d+1} C_k(F_\epsilon)$ is asymptotic to some periodic function $G(\epsilon)$ of (multiplicative) period $\zeta = e^{-h}$. Noting that $r_w = \zeta^n$ is some integer power n of the period, we conclude that numerator and denominatior of $q_w(\epsilon)$ are asymptotic to the same function $G(\epsilon r_w^{-1}) = G(\epsilon)$. This implies convergence of the quotient $q_w(\epsilon)$ to 1.

Letting now $n \to \infty$ in the above inequality, the quotient $q_w(\epsilon_n)$ converges to 1 while the right hand side is bounded from above by b^{-1}, by the assumption (5.3.1). Hence

$$\left|\nu^1_{d-1}(f^w_\delta) - r^s_w\right| \le 2cb^{-1}\delta^\gamma,$$

and, by taking the limit $\delta \to 0$, we derive that $\nu^1_{d-1}(S_w O) = r^s_w$, completing the proof of (5.3.2). □

Proof of (5.3.3). It suffices to show that $\nu^1_{d-1}(O(r)^c) = 0$ for each $r > 0$. So fix some r. Analogously to the proof of (5.2.8), we conclude from Lemma 4.2.1 that the functions g_δ (defined in (5.2.11)) satisfy the inequality

$$\nu^1_{d-1,\epsilon_n}(g_\delta) = \frac{\nu_{d-1,\epsilon_n}(g_\delta)}{\nu_{d-1,\epsilon_n}(\mathbb{R}^d)} \le c\delta^\gamma (\nu_{d-1,\epsilon_n}(\mathbb{R}^d))^{-1}$$

for all $\epsilon_n \le \delta \le \rho r$. Hereby note that $C^{\text{var}}_{d-1}(F_\epsilon, \cdot\,) = C_{d-1}(F_\epsilon, \cdot\,)$. Now the assertion easily follows by letting first $n \to \infty$ and then $\delta \to 0$ and taking into account (5.3.1). □

We have shown that for each null sequence $\{\epsilon_n\}$, for which as $n \to \infty$ the measures ν^1_{d-1,ϵ_n} converge at all, they converge to μ_F. Hence $\nu^1_{d-1,\epsilon} \xrightarrow{\mathrm{w}} \mu_F$ as $\epsilon \to 0$, as we stated in Theorem 1.4.3.

Proof of Theorem 1.4.4. Note that, by Gatzouras's theorem, a condition similar to (5.3.1) is always satisfied: There exists some $b > 0$ such that $\liminf_{\epsilon\to 0} \epsilon^{s-d}\lambda_d(F_\epsilon) = b$. Hence there is no extra assumption required in this case. Now the arguments of the above proof carry over to the case $k = d$ and to arbitrary self-similar sets satisfying OSC, when Theorem 1.2.10 and Lemma 4.6.2 are taken into account.

A. Appendix: Signed measures and weak convergence

We summarize a few facts and definitions concerning signed measures, which we always regard as totally finite signed measures here. In particular, we clarify the notion of weak convergence of a sequence of signed measures. For more details on signed measures we refer to the text books on measure theory, e.g. the ones by Jürgen Elstrodt [5], Joseph L. Doob [4] or Konrad Jacobs [16], and for the weak convergence of measures to Patrick Billingsley [3].

Signed measures. Let X a metric space and $\mathcal{X}$ the σ-algebra of Borel sets of X. A function $\mu : \mathcal{X} \to \mathbb{R}$ is called a *signed measure* if it is σ-additive, i.e. for any sequence of pairwise disjoint sets $A_1, A_2, \ldots \in \mathcal{X}$ it holds

$$\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i).$$

In particular, this definition implies $\mu(\varnothing) = 0$ and $|\mu(X)| < \infty$. μ is called a *measure* or *positive measure* if $\mu(A) \geq 0$ for all $A \in \mathcal{X}$.

We define the set functions μ^+, μ^- and μ^{var} by setting for each $A \in \mathcal{X}$

$$\mu^+(A) := \sup_{B \subseteq A} \mu(B), \quad \mu^-(A) := -\inf_{B \subseteq A} \mu(B) \quad \text{and} \quad \mu^{\text{var}}(A) := \mu^+(A) + \mu^-(A)\,.$$

It can be shown that μ^+, μ^- and μ^{var} are finite positive measures on $\mathcal{X}$. They are called respectively the *positive, negative* and *total variation measures* of μ.

Theorem A.1.1. *Let μ be a signed measure on a σ-algebra $\mathcal{X}$. Then*

(i) (Jordan decomposition) $\mu = \mu^+ - \mu^-$.

(ii) (Hahn decomposition) X is the disjoint union of two sets $X^+, X^- \in \mathcal{X}$ such that $\mu^-(X^+) = \mu^+(X^-) = 0$. The sets X^+ and X^- are unique up to μ^{var}-null sets.

Integration with respect to a signed measure. Recall that for a measurable function $f : X \to \mathbb{R}$ the *integral* with respect to a positive measure μ is defined as follows. For nonnegative simple functions $g = \sum_{i=1}^n c_i \mathbf{1}_{A_i}$, where $A_i \subseteq X$ and $c_i > 0$, set

$$\int_X g d\mu = \sum_{i=1}^n c_i \mu(A_i)\,.$$

Any nonnegative measurable function f is approximated from below by a sequence $g_1, g_2, \ldots$ of simple functions and the integral is then defined as the limit

$$\int_X f d\mu = \lim_{j\to\infty} \int_X g_j d\mu \,.$$

It can be shown that the limit does not depend on the choice of the sequence $g_1, g_2, \ldots$. f is *μ-integrable* if the limit is finite. Finally, arbitrary measurable functions f are *μ-integrable* if the integrals $\int_X f_+ d\mu$ and $\int_X f_- d\mu$ are both finite, and for such functions f the integral is defined as

$$\int_X f d\mu = \int_X f_+ d\mu - \int_X f_- d\mu \,.$$

Here $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = f_+(x) - f(x)$ denote the positive and negative part of f, respectively.

The integral with respect to a signed measure μ is now defined with respect to its Jordan decomposition. A measurable function $f : X \to \mathbb{R}$ is *μ-integrable* if and only if it is μ^{var}-integrable. Then the *integral* with respect to μ is defined as

$$\int_X f d\mu = \int_X f d\mu^+ - \int_X f d\mu^- \,.$$

Weak convergence of signed measures. Having defined the integral with respect to a signed measure, the generalization of the concept of weak convergence to signed measures is straightforward. Let $\mu, \mu_1, \mu_2, \ldots$ be signed measures on $\mathcal{X}$. The sequence $\{\mu_n\}$ is said to *converge weakly* to the limit measure μ, $\mu_n \xrightarrow{\mathrm{w}} \mu$ as $n \to \infty$, if

$$\lim_{n\to\infty} \int_X f d\mu_n = \int_X f d\mu$$

for all bounded continuous functions f.

It is obvious from the definition, that weak convergence of the variation measures μ_n^+ and μ_n^- of μ_n to the variation measures μ^+ and μ^- of μ is sufficient for weak convergence of a sequence of signed measures:

$$\mu_n^+ \xrightarrow{\mathrm{w}} \mu^+ \quad \text{and} \quad \mu_n^- \xrightarrow{\mathrm{w}} \mu^- \Longrightarrow \mu_n \xrightarrow{\mathrm{w}} \mu$$

This implication suggests to investigate the variation measures, to which the theory of weak convergence of (positive) measures applies, instead of studying the signed measures themselves. Note that the converse implication is not true. This is illustrated by a simple example.

Example A.1.2. Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be two disjoint sequences in X, i.e. in particular $x_n \neq y_n$ for all $n \in \mathbb{N}$, converging both to the same point $x \in X$. For each n let $\mu_n^+ := \delta_{x_n}$ and $\mu_n^- := \delta_{y_n}$ the Dirac measures of x_n and y_n, respectively. Set $\mu_n := \mu_n^+ - \mu_n^-$. Obviously, $\mu_n^\pm \xrightarrow{\mathrm{w}} \delta_x$ and therefore $\mu_n \xrightarrow{\mathrm{w}} \delta_x - \delta_x = 0$. Thus the limit measure μ is the zero measure and so its positive and negative variations μ^+ and μ^- are zero as well. Hence we have $\mu_n \xrightarrow{\mathrm{w}} \mu$ but not $\mu_n^+ \xrightarrow{\mathrm{w}} \mu^+$ and not $\mu_n^- \xrightarrow{\mathrm{w}} \mu^-$.

Bibliography

[1] Christoph Bandt, Nguyen Viet Hung, and Hui Rao, *On the open set condition for self-similar fractals*, preprint (2004).

[2] Andreas Bernig, *Aspects of Curvature*, preprint (2003).

[3] Patrick Billingsley, *Convergence of probability measures. 2nd ed.*, Chichester: Wiley, 1999.

[4] Joseph L. Doob, *Measure theory*, New York: Springer, 1994.

[5] Jürgen Elstrodt, *Maß- und Integrationstheorie. 3., erweiterte Auflage*, Berlin: Springer, 2002.

[6] Kenneth J. Falconer, *On the Minkowski measurability of fractals*, Proc. Am. Math. Soc. **123** (1995), no. 4, 1115–1124.

[7] ______, *Techniques in fractal geometry*, Chichester: Wiley, 1997.

[8] Herbert Federer, *Curvature measures*, Trans. Am. Math. Soc. **93** (1959), 418–491.

[9] William Feller, *An introduction to probability theory and its applications. Vol II. 2nd ed.*, New York: Wiley, 1971.

[10] Joseph H.G. Fu, *Monge-Ampère functions. I and II*, Indiana Univ. Math. J. **38** (1989), no. 3, 745–789.

[11] Dimitris Gatzouras, *Lacunarity of self-similar and stochastically self-similar sets*, Trans. Amer. Math. Soc. **352** (2000), no. 5, 1953–1983.

[12] Helmut Groemer, *On the extension of additive functionals on classes of convex sets*, Pac. J. Math. **75** (1978), 397–410.

[13] Hugo Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Berlin, Göttingen, Heidelberg: Springer, 1957.

[14] Daniel Hug, Günter Last, and Wolfgang Weil, *A local Steiner-type formula for general closed sets and applications*, Math. Z. **246** (2004), no. 1-2, 237–272.

[15] John E. Hutchinson, *Fractals and self similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.

[16] Konrad Jacobs, *Measure and integral*, New York: Academic Press, 1978.

[17] Daniel A. Klain and Gian-Carlo Rota, *Introduction to geometric probability*, Lezioni Lincee. Cambridge: Cambridge University Press. Rome: Accademia Nazionale dei Lincei, 1997.

[18] Steven P. Lalley, *The packing and covering functions of some self-similar fractals*, Indiana Univ. Math. J. **37** (1988), no. 3, 699–710.

[19] Michel L. Lapidus and Carl Pomerance, *The Riemann zeta-function and the one-dimensional Weyl-Berry conjecture for fractal drums*, Proc. London Math. Soc. (3) **66** (1993), no. 1, 41–69.

[20] Michel L. Lapidus and Machiel van Frankenhuysen, *Fractal geometry and number theory. Complex dimensions of fractal strings and zeros of zeta functions*, Boston: Birkhäuser, 2000.

[21] Michael Levitin and Dmitri Vassiliev, *Spectral asymptotics, renewal theorem, and the Berry conjecture for a class of fractals*, Proc. London Math. Soc. (3) **72** (1996), no. 1, 188–214.

[22] Marta Llorente and Steffen Winter, *A notion of Euler characteristic for fractals*, to appear in: Math. Nachr. (2005).

[23] Benoit B. Mandelbrot, *Measures of fractal lacunarity: Minkowski content and alternatives*, in: Fractal geometry and stochastics I, Basel: Birkhäuser, 1995, pp. 15–42.

[24] Jan Rataj and Martina Zähle, *Curvatures and currents for unions of sets with positive reach II*, Ann. Global Anal. Geom. **20** (2001), no. 1, 1–21.

[25] ———, *Normal cycles of Lipschitz manifolds by approximation with parallel sets*, Differ. Geom. Appl. **19** (2003), no. 1, 113–126.

[26] ———, *General normal cycles and Lipschitz manifolds of bounded curvature*, Ann. Global Anal. Geom. **27** (2005), no. 2, 135–156.

[27] Andreas Schief, *Separation properties for self-similar sets*, Proc. Amer. Math. Soc. **122** (1994), no. 1, 111–115.

[28] Rolf Schneider, *Curvature measures of convex bodies*, Ann. Mat. Pura Appl., IV. Ser. **116** (1978), 101–134.

[29] ———, *Parallelmengen mit Vielfachheit und Steiner-Formeln*, Geom. Dedicata **9** (1980), 111–127.

[30] ———, *Convex bodies: the Brunn-Minkowski theory*, Cambridge: Cambridge University Press, 1993.

[31] Rolf Schneider and Wolfgang Weil, *Integralgeometrie*, Stuttgart: Teubner, 1992.

[32] Martina Zähle, *Integral and current representation of Federer's curvature measures*, Arch. Math. **46** (1986), 557–567.

[33] ———, *Curvatures and currents for unions of sets with positive reach*, Geom. Dedicata **23** (1987), 155–171.

[34] ———, *Nonosculating sets of positive reach*, Geom. Dedicata **76** (1999), no. 2, 183–187.

Erklärung

Ich erkläre, daß ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt habe.

Jena, 29. Mai 2006