Geometric measures for fractals

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1 Introduction

Curvature measures are an important tool in geometry. Introduced for sets with positive reach by Federer [2], they have been extended to various classes of sets in $\mathbb{R}^d$ using methods like additive extension, approximation by parallel sets or axiomatic approaches. A central role play Steiner type formulas, which describe the volume of the parallel sets and in which curvature measures appear as "coefficients".

The attempt to introduce some notion of curvature for fractal sets by means of approximation with parallel sets in [17] led to the definition of *fractal curvatures* and *fractal curvature measures*. In this survey article, we recall this concept and discuss the main results obtained so far for the class of self-similar sets. There are close relations to the well known Minkowski content, which we want to emphasize here, since the analogies may be helpful for understanding the new concept. But the relation goes beyond analogy. Viewing Minkowski content as a curvature measure, leads to a localization of this notion.

Our aim is to illustrate the main ideas in [17] in a concise and introductory way, to make the results more accessible. Most statements are presented without proofs, occasionally we try to sketch the main ideas. For details the reader is refered to [17]. Only the proof of the upper bound for the scaling exponents is discussed in more detail. It illustrates very well the kind of arguments required in this theory and yet is not too long to be included here.

We start by recalling Minkowski content and explaining its 'localization'. Curvature measures are in general signed measures. In Section 3, we discuss the difficulties arising when passing on from measures to signed measures. In Section 4, we recall curvature measures for polyconvex sets and define fractal curvatures. Then we are ready to discuss the results for self-similar sets in Sections 5 and 6. Finally, in Section 7 we give some references to very recent advances and discuss possible generalizations and extensions.

2 Minkowski content

For $A \subset \mathbb{R}^d$ and $\varepsilon > 0$, the ε-*parallel* set A_ε of A is given by

$$A_\varepsilon := \{x \in \mathbb{R}^d : \inf_{y \in A} \|x - y\| \leq \varepsilon\}.$$

We write $V(A_\varepsilon)$ for the volume or Lebesgue measure of A_ε. The s-*dimensional upper and lower Minkowski content* of A are defined by

$$\overline{\mathcal{M}}^s(A) := \limsup_{\varepsilon \to 0} \varepsilon^{s-d} V(A_\varepsilon). \qquad \text{and} \qquad \underline{\mathcal{M}}^s(A) := \liminf_{\varepsilon \to 0} \varepsilon^{s-d} V(A_\varepsilon).$$

If $\underline{\mathcal{M}}^s(A) = \overline{\mathcal{M}}^s(A)$, then the common value $\mathcal{M}^s(A)$ is the s-dimensional Minkowski content of A. The Minkowski content gives rise to (upper and lower) *Minkowski* or *box dimension*, which characterize the "optimal" scaling exponents s for A in the above definition:

$$\overline{D}(A) = \inf\{t \geq 0 : \overline{\mathcal{M}}^t(A) = 0\} = \sup\{t \geq 0 : \overline{\mathcal{M}}^t(A) = \infty\}$$
$$\underline{D}(A) = \inf\{t \geq 0 : \underline{\mathcal{M}}^t(A) = 0\} = \sup\{t \geq 0 : \underline{\mathcal{M}}^t(A) = \infty\}$$

If $\underline{D}(A) = \overline{D}(A)$, the common value $D = D(A)$ is called Minkowski dimension of A. We will omit the dimension index D and write $\mathcal{M}(A)$ for the Minkowski content $\mathcal{M}^D(A)$ of dimension D. In many situations the Minkowski dimension is known, while the computation of the Minkowski content is a difficult problem (similarly as for the Hausdorff dimension and the exact value of the corresponding Hausdorff measure). Often it is not even known whether the limit $\mathcal{M}(A)$ exists. A set is called *Minkowski measurable* if and only if $\mathcal{M}(A)$ exists and is positive and finite.

Due to applications in spectral theory and as a "lacunarity" parameter, cf. [9, 5], the question of Minkowski measurability aroused considerable interest. However, even for self-similar sets $F \subset \mathbb{R}^d$ satisfying open set condition (OSC; see definitions in Section 5) the question remained open for some time. For subsets of $\mathbb{R}$, first results were obtained by Lapidus [5] and Falconer [1]. The following general characterization for self-similar sets in $\mathbb{R}^d$ was given by Gatzouras [3, Theorems 2.3 and 2.4]: Non-arithmetic sets are Minkowski measurable, while for arithmetic sets, only an averaged limit $\widetilde{\mathcal{M}}(F)$ can be shown to exist. $\widetilde{\mathcal{M}}(F)$ is called *average Minkowski content* and defined by

$$\widetilde{\mathcal{M}}(F) := \lim_{\delta \searrow 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{D-d} V(F_\varepsilon) \frac{d\varepsilon}{\varepsilon}.$$

Theorem 2.1 (Gatzouras's theorem)
Let F be a self-similar set satisfying OSC. Then the average Minkowski content $\widetilde{\mathcal{M}}(F)$ of F exists and is positive and finite. If F is non-arithmetic, then also the Minkowski content $\mathcal{M}(F)$ of F exists and $0 < \mathcal{M}(F) < \infty$, i.e. F is Minkowski measurable.

The result is based on the Renewal Theorem. From Gatzouras's result explicit formulas for the computation of $\mathcal{M}(F)$ and $\widetilde{\mathcal{M}}(F)$ can be derived (see Theorem 5.4 below).

It is well known that the Minkowski content $\mathcal{M}(\,\cdot\,)$ is finitely additive but not σ-additive, i.e. it is a content but not a measure. However, changing the point of view slightly, the Minkowski content of F can be interpreted as a uniform mass distribution on F, hence a measure. The key to this is to "localize" the approximation by parallel sets. Let $C_d(F_\varepsilon,\,\cdot\,) := V(F_\varepsilon \cap \,\cdot\,)$. $C_d(F_\varepsilon,\,\cdot\,)$ is a measure on F_ε and it is natural to ask for the limiting behaviour of these measures as $\varepsilon \to 0$. If they are appropriately rescaled, they do indeed converge in the weak sense to a limit measure on F, provided their total masses converge, i.e. provided the Minkowski content of F exists. Otherwise, only some average versions of these measures can be shown to converge. Let μ_F be the normalized s-dimensional Hausdorff measure on F (where s is the dimension of F) i.e. $\mu_F = (\mathcal{H}^s(F))^{-1}\mathcal{H}^s|_F(\,\cdot\,)$.

Theorem 2.2 *Let F be a self-similar set satisfying OSC. If F is non-arithmetic, then the measures $\varepsilon^{D-d}C_d(F_\varepsilon,\,\cdot\,)$ converge weakly to $\mathcal{M}(F)\ \mu_F =: \mathcal{C}_d(F,\,\cdot\,)$ as $\varepsilon \searrow 0$. The measures*

$$\widetilde{C}_d(F_\delta,\,\cdot\,) := \frac{1}{|\ln \delta|}\int_\delta^1 \varepsilon^{D-d}C_d(F_\varepsilon,\,\cdot\,)\frac{d\varepsilon}{\varepsilon} \tag{1}$$

always converge weakly to $\widetilde{\mathcal{C}}_d(F,\,\cdot\,) = \widetilde{\mathcal{M}}(F)\ \mu_F$ as $\delta \searrow 0$.

Theorem 2.2 localizes Gatzouras's theorem. Not only the total (average) ε-parallel volume of self-similar sets converges, as $\varepsilon \searrow 0$. The convergence even takes place locally for 'nice' subsets of $\mathbb{R}^d$. More precisely, if $B \subset \mathbb{R}^d$ is a *μ_F-continuity set*, i.e. if $\mu_F(\partial B) = 0$, then $\varepsilon^{D-d}C_d(F_\varepsilon, B) \to \mathcal{M}(F)\ \mu_F(B)$ as $\varepsilon \searrow 0$ for F non-arithmetic (and $\widetilde{C}_d(F_\varepsilon,\,\cdot\,) \to \widetilde{\mathcal{M}}(F)\mu_F(B)$ for general F).

The approximation with parallel sets induces a measure on F. The uniformity of the measures $C_d(F_\varepsilon,\,\cdot\,)$ carries over to the limit measure (which is by no means obvious). However, any uniform measure on F is necessary a multiple of the Hausdorff measure on F. The Minkowski content comes in naturally as the total mass of the limit measure. The theorem parallels known results on the continuity of curvature measures and, in particular, the volume (see e.g. Prop. 4.2(6)) and extends them to the fractal setting. (The classical results only give $C_d(F_\varepsilon,\,\cdot\,) \xrightarrow{\text{w}} C_d(F,\,\cdot\,) = 0$ as $\varepsilon \searrow 0$ for fractal sets F of dimension less than d.)

In the light of the above results for Minkowski content and its localization, it is natural to ask for the limiting behaviour of other geometric measures associated to the parallel sets, for instance the surface area $\mathcal{H}^{d-1}(\partial A_\varepsilon)$ of the boundary of A_ε, its Euler characteristic $\chi(A_\varepsilon)$, or, more generally, curvature measures, as $\varepsilon \searrow 0$. In fact, it was the study of curvature measures that suggested the local interpretation of Minkowski content.

3 Sequences of signed measures

Curvature measures are signed measures in general and this is one of the main reasons why things become more difficult compared to the situation for the Minkowski content. In order to see the difficulties arising from the non-positivity of the measures, we briefly discuss some of the phenomena arising in the study of sequences of signed measures.

Let $(\mu_\varepsilon)_{\varepsilon>0}$ be a sequence of finite signed measures (i.e. with finite total variation) on some metric space X. Denote by $\mu_\varepsilon^+, \mu_\varepsilon^-$ and $\mu_\varepsilon^{\text{var}}$ the positive, negative and total variation measure of μ_ε, respectively, and let $M(\varepsilon)$ and $M^{\text{var}}(\varepsilon)$ be the total masses of μ_ε and $\mu_\varepsilon^{\text{var}}$.

Recall that the sequence (μ_ε) is said to converge weakly to a limit measure μ as $\varepsilon \searrow 0$, in symbols $\mu_\varepsilon \xrightarrow{\text{w}} \mu$, iff for each bounded continuous function f the integrals $\mu_\varepsilon(f) := \int_X f d\mu_\varepsilon$ converge to the value of the integral $\mu(f) := \int_X f d\mu$. (The integral with respect to a signed measure μ is defined by $\int_X f d\nu = \int_X f d\nu^+ - \int_X f d\nu^-$.)

In general, the weak convergence $\mu_\varepsilon \xrightarrow{\text{w}} \mu$ does not imply the convergence of the variation measures.

Example 3.1 *For $\varepsilon > 0$, let the measures μ_ε on $\mathbb{R}$ be defined by $\mu_\varepsilon := \delta_0 - \delta_\varepsilon$ for $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$, and $\mu_\varepsilon := 0$ otherwise. (Here δ_x denotes the Dirac measure at x.) Then $\mu_{1/n}^{\text{var}} = \delta_0 + \delta_{1/n}$ and so $M^{\text{var}}(1/n) = 2$, while $M^{\text{var}}(\varepsilon) = 0$ otherwise. Hence $\mu_\varepsilon \xrightarrow{\text{w}} 0$, while $\mu_\varepsilon^{\text{var}}$ does not converge as $\varepsilon \searrow 0$.*

If additionally the convergence of $M^{\text{var}}(\varepsilon)$ is assumed, then the variation measures converge weakly to some limit measures $\mu^{(+)}, \mu^{(-)}$ and $\mu^{(\text{var})}$ and the relations $\mu = \mu^{(+)} - \mu^{(-)}$ and $\mu^{(\text{var})} = \mu^{(+)} + \mu^{(-)}$ carry over to the limits. However, in general, $\mu^{(+)}, \mu^{(-)}$ and $\mu^{(\text{var})}$ are not the variation measures of μ. The limit $\mu^{(\text{var})}$ of the total variation measures is only an upper bound for μ^{var}, the total variation of the limit measure μ (and similarly for μ^+, μ^-); hence the notation $\mu^{(\text{var})}$ with parentheses. The reason is that the measures μ_ε^+ and μ_ε^- have essentially disjoint supports, a property which is not necessarily preserved in the limit. $\mu^{(+)}$ and $\mu^{(-)}$ may even have equal support.

Example 3.2 *For $\varepsilon > 0$, let the measures μ_ε on $\mathbb{R}$ be defined by $\mu_\varepsilon = \delta_\varepsilon - \delta_1$. Then $\mu_\varepsilon^{\text{var}} = \mu_\varepsilon^+ + \mu_\varepsilon^- = \delta_\varepsilon + \delta_1$ and so $M^{\text{var}}(\varepsilon) = 2$, while $M(\varepsilon) = 0$ for each $\varepsilon > 0$. The total mass zero gives no hint on the real "size" of the underlying measures μ_ε. For $\varepsilon \searrow 0$, one has $\mu_\varepsilon \xrightarrow{\text{w}} \mu = \delta_0 - \delta_1$ and $\mu_\varepsilon^{\text{var}} \xrightarrow{\text{w}} \mu^{(\text{var})} = \delta_0 + \delta_1$. In this case, $\mu^{(\text{var})}$ is the variation of the limit measure, i.e. $\mu^{(\text{var})} = \mu^{\text{var}}$.*

Now let $\tilde{\mu}_\varepsilon = \delta_\varepsilon - \delta_0$. Then $\tilde{\mu}_\varepsilon^{\text{var}} = \delta_\varepsilon + \delta_0$, $\widetilde{M}(\varepsilon) = 0$ and $\widetilde{M}^{\text{var}}(\varepsilon) = 2$ as before. But now, as $\varepsilon \searrow 0$, the situation is different: $\tilde{\mu}_\varepsilon \xrightarrow{\text{w}} \tilde{\mu} = \delta_0 - \delta_0 = 0$ and $\tilde{\mu}_\varepsilon^{\text{var}} \xrightarrow{\text{w}} \tilde{\mu}^{(\text{var})} = 2\delta_0$. The limit $\tilde{\mu}$ is the zero measure (with $\tilde{\mu}^{\text{var}} = 0$) and so $\tilde{\mu}^{(\text{var})}$ is not the total variation measure of $\tilde{\mu}$.

The above examples show that for the limiting behaviour of the measures μ_ε, the behaviour of the mass $M^{\text{var}}(\varepsilon)$ is essential. In case, $M^{\text{var}}(\varepsilon) \to \infty$ or

$M^{\text{var}}(\varepsilon) \to 0$, rescaling with a factor ε^t for some $t \in \mathbb{R}$ may help to obtain non-trivial limits. We define the *(upper) mass scaling exponent* m of the sequence $(\mu_\varepsilon)_{\varepsilon>0}$ by

$$m := \inf\{t \in \mathbb{R} : \lim_{\varepsilon \searrow 0} \varepsilon^t M^{\text{var}}(\varepsilon) = 0\}. \tag{2}$$

Remark 3.3 *There are several other scaling exponents associated to the sequence (μ_ε) which are sometimes useful. The numbers m', m^+ and m^- are defined, by replacing $M^{\text{var}}(\varepsilon)$ in the above definition with $|M(\varepsilon)|$, $M^+(\varepsilon)$ and $M^-(\varepsilon)$, respectively (where $M^+(\varepsilon)$ ($M^-(\varepsilon)$) is the total mass of μ_ε^+ (μ_ε^-)). In general, $m' \leq m = \max\{m^+, m^-\}$. Similarly, lower scaling exponents $\underline{m}, \underline{m}', \underline{m}^+$ and $\underline{m}^-$ can be introduced by replacing the limits with $\liminf$'s (which in general may differ from the corresponding upper exponents). Often the interrelations of these eight exponents allow conclusions on the existence of limits and their properties (see below).*

Let

$$M := \lim_{\varepsilon \searrow 0} \varepsilon^m M(\varepsilon) \quad \text{and} \quad M^{\text{var}} := \lim_{\varepsilon \searrow 0} \varepsilon^m M^{\text{var}}(\varepsilon)$$

be the rescaled limits of the total masses, provided they exist. Of course, in general, these limits do not exist and then $\liminf$'s and $\limsup$'s can be considered, but in this case the corresponding (rescaled) measures can not converge, either. Provided they exist, the weak limit of the rescaled sequence $(\varepsilon^m \mu_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \searrow 0$ will be denoted by μ and the limit of $(\varepsilon^m \mu_\varepsilon^{\text{var}})_{\varepsilon>0}$ by $\mu^{(\text{var})}$,

$$\varepsilon^m \mu_\varepsilon \xrightarrow{\text{w}} \mu \qquad \text{and} \qquad \varepsilon^m \mu_\varepsilon^{\text{var}} \xrightarrow{\text{w}} \mu^{(\text{var})}.$$

If positive and negative parts scale with different exponents, i.e. if $m^+ \neq m^-$, then the limit measure is purely positive or purely negative, depending on which of these exponents is larger. For instance, if $m^+ > m^-$, then $\mu = \mu^+ = \mu^{\text{var}}$ and $\mu^- = 0$. The most interesting case is when positive and negative parts scale with the same exponent. Then the limit measure can have positive and negative part. However, it can also happen in this case that positive and negative variation measures cancel out each other in the limit.

Example 3.4 *For $\varepsilon > 0$ and $s, t \in \mathbb{R}$, let $\mu_\varepsilon^{s,t} := \varepsilon^{-s}\delta_\varepsilon - \varepsilon^{-t}\delta_1$. If $s > t$, then $m = s$ is the correct scaling exponent and $\varepsilon^m \mu_\varepsilon^{s,t} = \delta_\varepsilon - \varepsilon^{s-t}\delta_1 \to \delta_0$ as $\varepsilon \searrow 0$. Hence the limit measure is purely positive. If $s < t$, then $m = t$ and $\varepsilon^m \mu_\varepsilon^{s,t} = \varepsilon^{t-s}\delta_\varepsilon - \delta_1 \to -\delta_1 =: \mu$ as $\varepsilon \searrow 0$, i.e. the limit measure is purely negative. The most interesting case is $s = t$. Then $m = s = t$ and $\varepsilon^m \mu_\varepsilon^{s,t} = \delta_\varepsilon - \delta_1 \to \delta_0 - \delta_1$, i.e. the limit measure has positive and negative part. If the measures $\hat{\mu}_\varepsilon^{s,t} := \varepsilon^{-s}\delta_\varepsilon - \varepsilon^{-t}\delta_0$ are considered instead, then, for $s = t$, still $m = s$ but now $\varepsilon^m \hat{\mu}_\varepsilon^{s,t} \to \delta_0 - \delta_0 = 0$ as $\varepsilon \searrow 0$, i.e. the limit measure is the zero measure, although s is the optimal scaling exponent.*

If the total masses of the measures μ_ε do not converge (even if appropriately rescaled), then (Cesàro) averaging may improve the convergence behaviour, in particular, if self-similar sets are considered (compare e.g. the results for the averaged Minkowski content). Let

$$\widetilde{M} := \lim_{\delta \searrow 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^m M(\varepsilon) \frac{d\varepsilon}{\varepsilon}$$

and if $\widetilde{M}$ exists, one can ask for the weak convergence of the sequence $(\tilde{\mu}_\varepsilon)_{\varepsilon>0}$ defined by

$$\tilde{\mu}_\delta(\,\cdot\,) := \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^m \mu_\varepsilon(\,\cdot\,) \frac{d\varepsilon}{\varepsilon}$$

for $\delta > 0$. The limit measure will be denoted by $\tilde{\mu}$. If M exists, then $\widetilde{M} = M$ and if the weak limit μ exists, then $\tilde{\mu} = \mu$. Thus the average limits are a reasonable extension of the corresponding limits.

Remark 3.5 *If the measure μ_ε is positive (as for instance the volume of the ε-parallel set), then $\mu_\varepsilon^{\mathrm{var}} = \mu_\varepsilon$ and $M^{\mathrm{var}}(\varepsilon) = M(\varepsilon)$, thus $m' = m$, $M = M^{\mathrm{var}}$ and the limit measure of the sequence $(\varepsilon^m \mu_\varepsilon)$ is non-negative (if it exists). Hence the situation simplifies and the number of exponents and limits to look at reduces. In particular, if μ_ε is the volume of the ε-parallel set F_ε (of a compact set $F \subset \mathbb{R}^d$), then $m = \overline{D} - d$ (where $\overline{D}$ was the (upper) Minkowski dimension) and M (if it exists) is the Minkowski content of F.*

4 Curvature measures and fractal curvatures

First we recall the notion of curvature measure and discuss some properties. For simplicity, we restrict considerations to polyconvex sets. For more details see for instance Scheider [14].

Curvature measures. Recall that a set $K \subseteq \mathbb{R}^d$ is *convex* iff for any two points $x, y \in K$ the line segment connecting them is contained in K. We write $\mathcal{K}^d$ for the family of all *convex bodies*, i.e. of all nonempty compact convex sets in $\mathbb{R}^d$. A set K is called *polyconvex* if it has a representation as a finite union of convex bodies. The convex ring $\mathcal{R}^d$ is the family of all polyconvex sets in $\mathbb{R}^d$. It is called a *ring* because of its stability with respect to finite unions and intersections.

For each set $K \in \mathcal{R}^d$ *curvature measures* $C_0(K,\,\cdot\,), C_1(K,\,\cdot\,), \ldots, C_d(K,\,\cdot\,)$ can be defined. For convex bodies K, they are characterized by the *Local Steiner formula*. Let π_K denote the *metric projection* onto the convex set $K \in \mathcal{K}^d$, mapping a point $x \in \mathbb{R}^d$ to its (unique) nearest point in K. For $\varepsilon > 0$, the set $K_\varepsilon \cap \pi_K^{-1}(B)$ is the *local ε-parallel set* of K with respect to the Borel set B.

Theorem 4.1 *For each $K \in \mathcal{K}^d$, there are uniquely determined finite Borel measures $C_0(K,\,\cdot\,)$, …, $C_d(K,\,\cdot\,)$ on $\mathbb{R}^d$, such that*

$$V(K_\varepsilon \cap \pi_K^{-1}(B)) = \sum_{k=0}^{d} \varepsilon^{d-k} \kappa_{d-k} C_k(K, B)$$

for each Borel set $B \subseteq \mathbb{R}^d$ and $\varepsilon > 0$.

Here κ_i is the i-dimensional volume of the unit ball in $\mathbb{R}^i$.

Curvature measures of convex bodies are measures in the second argument and they are *additive* in the first. If $K, L, K \cup L \in \mathcal{K}^d$, then

$$C_k(K \cup L, B) = C_k(K, B) + C_k(L, B) - C_k(K \cap L, B). \tag{3}$$

The additivity allows to extend curvature measures to sets $K \in \mathcal{R}^d$, by using representations with convex sets. Curvature measures of polyconvex sets are in general signed measures. However, for $k = d$ and $d-1$, $C_k(K, \cdot)$ is always non-negative. $C_d(K, \cdot) = V(K \cap \cdot)$ is the volume restricted to K and $C_{d-1}(K, \cdot)$ is half the surface area of K, provided K is the closure of its interior. Except for $k = d$, the measures $C_k(K, \cdot)$ are concentrated on the boundary ∂K of K. If the boundary of K is sufficiently smooth, curvature measures have a representation as integals of the symmetric functions of principal curvatures. The total mass $C_k(K) := C_k(K, \mathbb{R}^d)$ of the measure $C_k(K, \cdot)$ is called the *k-th total curvature* of K. Total curvatures are also known as *intrinsic volumes* or *Minkowski functionals*. $C_0(K)$ coincides with the *Euler characteristic* of K, by the Gauss-Bonnet Theorem. For $K \in \mathcal{K}^d$, $C_0(K) = 1$. We collect some important properties of curvature measures and total curvatures.

Proposition 4.2 *Let $K, L \in \mathcal{R}^d$ and $B \subseteq \mathbb{R}^d$ be an arbitrary Borel set. For each $k \in \{0, \ldots, d\}$ we have:*

1. Additivity: $C_k(K \cup L, B) = C_k(K, B) + C_k(L, B) - C_k(K \cap L, B)$.
2. Motion invariance: *If g is a rigid motion, then $C_k(gK, gB) = C_k(K, B)$.*
3. Homogeneity: *For $\lambda > 0$, $C_k(\lambda K, \lambda B) = \lambda^k C_k(K, B)$.*
4. Locality: *If $K \cap A = L \cap A$ for some open set $A \subseteq \mathbb{R}^d$, then $C_k(K, B) = C_k(L, B)$ for all Borel sets $B \subseteq A$.*
5. Continuity: *If $K, K^1, K^2, \ldots \in \mathcal{K}^d$ with $K^i \to K$ as $i \to \infty$ (w.r.t. the Hausdorff metric) then the measures $C_k(K^i, \cdot)$ converge weakly to $C_k(K, \cdot)$, $C_k(K^i, \cdot) \xrightarrow{\text{w}} C_k(K, \cdot)$. In particular, $C_k(K^i) \to C_k(K)$.*
6. Continuity II: *If $K \in \mathcal{R}^d$, then $C_k(K_\varepsilon, \cdot) \xrightarrow{\text{w}} C_k(K, \cdot)$ as $\varepsilon \searrow 0$. In particular, $\lim_{\varepsilon \searrow 0} C_k(K_\varepsilon) = C_k(K)$.*
7. Monotonicity of the total curvatures: *If $K, L \in \mathcal{K}^d$ and $K \subseteq L$, then $C_k(K) \leq C_k(L)$.*

The additivity leads to the following useful formula called the *inclusion-exclusion principle*. If $K^1, \ldots, K^n \in \mathcal{R}^d$ and $K := \bigcup_{i=1}^n K^i$, then for all Borel sets $B \subseteq \mathbb{R}^d$

$$C_k(K, B) = \sum_{I \in N_n} (-1)^{\#I-1} C_k\big(\bigcap_{i \in I} K^i, B\big). \tag{4}$$

Here N_n is the family of all nonempty subsets I of $\{1, \ldots, n\}$. Hence the sum is over all intersections of the K^i. $\#I$ denotes the cardinality of the set I.

Remark 4.3 *The properties of motion invariance, homogeneity and locality in the above Proposition carry over to the variation measures. The continuity II only if K itself is already a parallel set (of some other set). Unfortunately, additivity does not hold for the variation measures.*

Fractal curvature measures. Let $A \subset \mathbb{R}^d$ be a compact set. To define fractal curvatures for A we need the curvature measures $C_k(A_\varepsilon, \cdot\,)$ of the parallel sets A_ε to be well defined for all $\varepsilon > 0$ (or at least for small ε). This can, for instance, be ensured by requiring that all the parallel sets are polyconvex and, for simplicity, we will assume this in the sequel. Keep in mind that more general notions of curvature exist to which the following concepts similarly apply.

If the $C_k(A_\varepsilon, \cdot\,)$ are well defined, then they form a sequence of signed measures (μ_ε) as in the previous section. Hence the notions of scaling exponents, total mass limits and weak limits specialize to the situation here. We first discuss the appropriate scaling exponents.

Definition 4.4 *Let $A \subseteq \mathbb{R}^d$ be compact with $A_\varepsilon \in \mathcal{R}^d$ for $\varepsilon > 0$ and let $k \in \{0, 1, \ldots, d\}$. The* (upper) k-th curvature scaling exponent *of A is the number*

$$s_k = s_k(A) := \inf \left\{ t : \varepsilon^t C_k^{\mathrm{var}}(A_\varepsilon) \to 0 \text{ as } \varepsilon \searrow 0 \right\}.$$

Hence s_k is the exponent m of Section 3 for the k-th curvature measures. The total masses $M(\varepsilon)$ specialize to the total curvatures $C_k(A_\varepsilon)$ (and $M^{\mathrm{var}}(\varepsilon)$ to $C_k^{\mathrm{var}}(A_\varepsilon)$). Their limits are the *fractal curvatures* of the set A.

Definition 4.5 *For $k \in \{0, \ldots, d\}$, let the* k-th fractal curvature *of A be defined by*

$$\mathcal{C}_k(A) := \lim_{\varepsilon \searrow 0} \varepsilon^{s_k} C_k(A_\varepsilon)$$

and let the corresponding average k-th fractal curvature *be the number*

$$\widetilde{\mathcal{C}}_k(A) := \lim_{\delta \searrow 0} \frac{1}{|\ln \delta|} \int_\delta^1 , \varepsilon^{s_k} C_k(A_\varepsilon) \frac{d\varepsilon}{\varepsilon}$$

provided these limits exists.

Note that if $\mathcal{C}_k(A)$ exists, then $\widetilde{\mathcal{C}}_k(A) = \mathcal{C}_k(A)$. Finally, we introduce the *fractal curvature measures* A as weak limits of the curvature measures of A_ε.

Definition 4.6 *For $k \in \{0, \ldots, d\}$, we call a (signed) measure $\mathcal{C}_k(A, \cdot\,)$ the* k-th fractal curvature measure *of A, iff*

$$\varepsilon^{s_k} C_k(A_\varepsilon, \cdot\,) \xrightarrow{\mathrm{w}} \mathcal{C}_k(A, \cdot\,),$$

and we call $\widetilde{\mathcal{C}}_k(A, \cdot\,)$ the average k-th fractal curvature measure *of A, iff it is the weak limit of the measures*

$$\widetilde{C}_k(A_\delta, \cdot\,) := \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{s_k} C_k(A_\varepsilon, \cdot\,) \frac{d\varepsilon}{\varepsilon}$$

as $\delta \searrow 0$.

We state some general properties of fractal curvatures and their scaling exponents. $\mathcal{C}_k(A)$ is the total mass of the fractal curvature measure, i.e. $\mathcal{C}_k(A, \mathbb{R}^d) = \mathcal{C}_k(A)$. The support of $\mathcal{C}_k(A, \cdot)$ is contained in ∂A, if $k < d$. For $k = d$, $s_d = \overline{D} - d$ and $\mathcal{C}_d(F) = \mathcal{M}(F)$, i.e. it is just the Minkowski content. The motion invariance and homogeneity of $C_k^{\text{var}}(A_\varepsilon)$ (cf. Proposition 4.2) imply that $s_k(A)$ is motion and scaling invariant, i.e. we have $s_k(gA) = s_k(A)$ for rigid motions g and $s_k(rA) = s_k(A)$ for $r > 0$. Similarly, the fractal curvatures are motion invariant and homogeneous: $\mathcal{C}_k(gA) = \mathcal{C}_k(A)$ and $\mathcal{C}_k(rA) = r^{k+s_k}\mathcal{C}_k(rA)$. Note that the order of homogeneity is $k + s_k$. If $s_k = D - k$, where D is the Minkowski dimension of A (this is what one would expect for s_k; also cf. the results for self-similar sets below), then $k + s_k = D$, i.e. the fractal curvatures are homgeneous of order D. Finally, by the continuity of curvature measures, fractal curvatures and fractal curvature measures are consistent with their classical counterparts. For polyconvex sets K with $C_k^{\text{var}}(K) \neq 0$, one has $s_k(K) = 0$ and thus $\mathcal{C}_k(K, \cdot) = C_k(K, \cdot)$ (cf. Proposition 2.2.10 in [17]).

5 Curvature measures for self-similar sets

Self-similar sets. Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, \dots, N$, be contracting similarities. Denote the contraction ratio of S_i by $r_i \in (0, 1)$. It is well known that there is a unique non-empty and compact *invariant set* $F \subset \mathbb{R}^d$ for the system $\{S_1, \dots, S_N\}$, i.e. a set F satisfying the equation

$$F = \bigcup_{i=1}^{N} S_i F.$$

F is called the *self-similar set* generated by $\{S_1, \dots, S_N\}$. Throughout we will assume that F (or, more precisely, the system $\{S_1, \dots, S_N\}$) satisfies the *open set condition* (OSC), i.e. there exists an open, non-empty, bounded subset $O \subset \mathbb{R}^d$ such that $\bigcup_i S_i O \subseteq O$ and $S_i O \cap S_j O = \varnothing$ for all $i \neq j$. If OSC holds, then the set O is not unique and, by [13], it is always possible to choose O such that $O \cap F \neq \varnothing$. (Then F is said to satisfy the *strong open set condition* (SOSC) for O.) The unique solution s of $\sum_{i=1}^N r_i^s = 1$ is called the *similarity dimension* of F. It is well known that under OSC s coincides with Hausdorff and Minkowski dimension of F, i.e. $s = D$.

Let $h > 0$. A finite set of positive real numbers $\{y_1, ..., y_N\}$ is called *h-arithmetic* if h is the largest number such that $y_i \in h\mathbb{Z}$ for $i = 1, \dots, N$. If no such number h exists for $\{y_1, ..., y_N\}$, the set is called *non-arithmetic*. We call a self-similar set F (non-)arithmetic if the set $\{-\ln r_1, \dots, -\ln r_N\}$ is.

For a self-similar set $F \in \mathbb{R}^d$, it is sufficient to assume that there exists some parallel set $F_\varepsilon, \varepsilon > 0$ which is polyconvex, since this implies already that all the parallel sets of F have this property (cf. [17, Proposition 2.3.1]). So for self-similar sets we have the dichotomy that either all its parallel sets are polyconvex or none. A set with all parallel sets polyconvex is for instance the Sierpinski gasket and an example with no polyconvex parallel sets is the Koch curve. For subsets of $\mathbb{R}$, all parallel sets are polyconvex.

For self-similar sets F, an upper bound for the (upper) scaling exponents $s_k = s_k(F)$ is given by the following result.

Theorem 5.1 [17, Theorem 2.3.2] *Let F be a self-similar set satisfying OSC and $F_\varepsilon \in \mathcal{R}^d$, and let $k \in \{0,\dots,d\}$. The expression $\varepsilon^{s-k} C_k^{\mathrm{var}}(F_\varepsilon)$ is uniformly bounded in $(0,1]$, i.e. there is a constant M such that for all $\varepsilon \in (0,1]$, $\varepsilon^{s-k} C_k^{\mathrm{var}}(F_\varepsilon) \le M$.*

Corollary 5.2 $s_k \le s - k$

Another immediate consequence is that the expression $\varepsilon^{s-k} |C_k(F_\varepsilon)|$ is also bounded by M for $\varepsilon \in (0,1]$. A proof of Theorem 5.1 is given in Section 6.

Lower bounds for s_k are considerably harder to establish. For most self-similar sets, the equality $s_k = s-k$ holds. However, equality is not true in general, which is easily seen from the following example.

Example 5.3 *The unit cube $Q = [0,1]^d \subset \mathbb{R}^d$ (considered as a self-similar set generated by 2^d similarities, each with contraction ratio $\frac{1}{2}$) has similarity dimension $s = d$. For the curvature measures of its parallel sets no rescaling is necessary. Q is convex and so are its parallel sets Q_ε. The continuity implies that, for $k = 0,\dots,d$, $C_k(Q_\varepsilon, \cdot\,) \to C_k(Q, \cdot\,)$ as $\varepsilon \searrow 0$. Therefore, $s_k(Q) = 0$, which, for $k < d$, is certainly different to $d-k$.*

Theorem 5.4 [17, Theorem 2.3.6] *Let F be a self-similar set satisfying OSC and $F_\varepsilon \in \mathcal{R}^d$. Let $k \in \{0,\dots,d\}$ and assume that $s_k = s-k$. Then the following holds:*

(i) The average k-th fractal curvature $\widetilde{\mathcal{C}}_k(F)$ exists and equals

$$X_k := \frac{1}{\eta} \int_0^1 \epsilon^{s-k-1} R_k(\varepsilon)\; d\varepsilon, \tag{5}$$

where $\eta = -\sum_{i=1}^N r_i^s \ln r_i$ and the function $R_k : (0,1] \to \mathbb{R}$ is given by

$$R_k(\varepsilon) = C_k(F_\varepsilon) - \sum_{i=1}^N \mathbf{1}_{(0,r_i]}(\varepsilon) C_k((S_iF)_\varepsilon). \tag{6}$$

(ii) If F is non-arithmetic, then the k-th fractal curvature $\mathcal{C}_k(A)$ exists and equals X_k.

The formula (5) allows explicit calculations of fractal curvatures. Many examples are considered in [17, Section 2.4].

Remark 5.5 *The result extends to the case $s_k < s-k$, in that $\lim_{\varepsilon \searrow 0} \varepsilon^{s-k} C_k(F_\varepsilon)$ (or its averaged counterpart, respectively) equal X_k. But in this case, we have*

$X_k = 0$ and obtain no information on the existence of fractal curvatures, since we are looking at the wrong scaling exponent ($s-k$ instead of s_k). However, it is very useful to know that the formula holds in general. Typically, the scaling exponent is not a priori known. Computing X_k with the given formula allows to verify that $s_k = s-k$. Namely, if $X_k \neq 0$, then necessarily $s_k = s-k$. If $X_k = 0$, then both cases are possible for s_k, either $s_k = s-k$ or $s_k < s-k$ (cf. Example 2.4.6 in [17]). In this situation, the computation of $X_k^{\rm var}$ or the study of the local behaviour of the curvature measures helps (cf. Theorem 2.3.8 in [17]).

For the proof we used a Renewal Theorem; a version suitable for taking limits as $\varepsilon \searrow 0$ (rather than to ∞) is stated in [17, Thm. 4.1.4]. The main observation is that the function $f(\varepsilon) := C_k(F_\varepsilon)$ satisfies a renewal equation with error term $R_k(\varepsilon)$:

$$f(\varepsilon) = \sum_{i=1}^{N} r_i^k \mathbf{1}_{(0,r_i]}(\varepsilon) f(\varepsilon/r_i) + R_k(\varepsilon),$$

which is due to the equality $C_k((S_iF)_\varepsilon) = r_i^k C_k(F_{\varepsilon/r_i})$. The difficulty is to verify that the hypotheses of the Renewal Theorem are satisfied. We require some bound on the growth of $R_k(\varepsilon)$ as $\varepsilon \searrow 0$ and the continuity of $C_k(F_\varepsilon)$ and thus of $R_k(\varepsilon)$ in ε up to a discrete set of exceptions. The latter is easily derived from the properties of the curvature measures, while for the bounds on R_k some considerable efforts are required. The following lemma is the key to most of the results on fractal curvatures and fractal curvature measures obtained so far.

Let $\Sigma^* = \bigcup_{n\in\mathbb{N}\cup\{0\}} \{1,\ldots,N\}^n$ and, for $0 < r \leq 1$, let $\Sigma(r)$ be the family of all finite words $w = w_1 \ldots w_n \in \Sigma^*$ such that

$$r_w < r \leq r_w r_{w_n}^{-1}. \tag{7}$$

Choose a set O such that F satisfies SOSC for O, i.e. $F \cap O \neq \varnothing$. For $0 < r \leq 1$, we define the set $O(r)$ by

$$O(r) := \bigcup_{v\in\Sigma(r)} S_v O \tag{8}$$

and, for $r > 1$, by $O(r) := O$. In particular, $O(1) = \mathbf{S}O = \bigcup_i S_i O$. Note that F satisfies OSC with the open set $O(r)$, $r > 0$. For the complement $O(r)^c$ of these sets the following estimate holds.

Lemma 5.6 [17, Lemma 5.2.1] *For each $r > 0$, there exist constants $c, \gamma, \rho > 0$ such that for all $\varepsilon \leq \delta \leq \rho r$*

$$C_k^{\rm var}(F_\varepsilon, (O(r)^c)_\delta) \leq c\varepsilon^{k-s}\delta^\gamma.$$

In fact, only the constant c depends on r, while ρ and γ merely depend on the choice of O. The estimate roughly means that, as ε approaches 0, the mass of $C_k^{\mathrm{var}}(F_\varepsilon,\cdot)$ close to the boundary of $O(r)$ is small compared to its total mass. The bound is obtained by careful decomposition of the parallel sets into convex pieces of approximately equal size using the family $\Sigma(r)$, and by estimating the number of mutual intersections of these pieces and the total number of pieces involved as $\varepsilon \searrow 0$.

If for some self-similar set F, $s_k = s - k$, then by the above result the existence of the fractal curvatures of F is only ensured in the non-arithmetic case. So the best one can hope for is the existence of fractal curvature measures in this case. Indeed, if $\mathcal{C}_k(F)$ exists, then the corresponding fractal curvature measure $\mathcal{C}_k(F,\cdot)$ exists.

Theorem 5.7 *Let F be a self-similar set satisfying OSC and $F_\varepsilon \in \mathcal{R}^d$. Let $k \in \{0,\dots,d\}$ and assume $s_k = s-k$. Then the average fractal curvature measures of F exist and $\widetilde{\mathcal{C}}_k(F,\cdot) = \widetilde{\mathcal{C}}_k(F)\ \mu_F$. If additionally F is non-arithmetic, then the fractal curvature measures exist and $\mathcal{C}_k(F,\cdot) = \mathcal{C}_k(F)\ \mu_F$.*

The idea of proof is as follows. Since the families $(C_k(F_\varepsilon,\cdot))_{\varepsilon\in(0,1]}$ (and $(\widetilde{C}_k(F_\varepsilon,\cdot))_{\varepsilon\in(0,1]}$) are tight, by Prohorov's Theorem, there exist converging subsequences and the task is to show that the limit measures of all these subsequences are the same. This is done proving that the limit measure μ_k of each fixed converging subsequence coincides with the measure $\mathcal{C}_k(F)\ \mu_F$. For the equivalence of two measures it is sufficient that they coincide on an intersection stable generating class of the Borel σ-algebra. Since the computation of the limit $\lim_{\varepsilon\searrow 0}\varepsilon^{s-k}C_k(F_\varepsilon,B)$ is difficult for arbitrary sets B, the generator has to be adapted to the structure of F to include only sets B for which the limit can be determined. The generator $\mathcal{A}$ used in the proofs consists of copies of the open set O, i.e. the family $\{S_w O : w \in \Sigma^*\}$, and of all subsets C of the complement of some $O(r)$ (cf. (8)). For these sets μ_F is known ($\mu_F(S_w O) = r_w^s$ and $\mu_F(C) = 0$) and the values for μ_k can be computed by approximation of the sets with continuous functions and using the weak convergence of the chosen (sub-)sequence. Here the estimate of Lemma 5.6 is used.

Remark 5.8 *Results analogous to those in Theorems 5.4 and 5.7 hold for the limiting behaviour of the corresponding variation measures $C_k^{\mathrm{var}}(F_\varepsilon,\cdot)$ and their total masses $C_k^{\mathrm{var}}(F_\varepsilon)$.*

6 Proof of Theorem 5.1

We give a proof of the boundedness of the expression $\varepsilon^{s-k}C_k^{\mathrm{var}}(F_\varepsilon)$, which is self-contained except for an application of Lemma 5.6. The proof shows the kind of arguments required to obtain results on fractal curvatures. The first step is the following general observation regarding parallel sets. If for a compact set $A \subset \mathbb{R}^d$, $A_\varepsilon \in \mathcal{R}^d$ for some $\varepsilon > 0$, then $A_{\varepsilon+r} \in \mathcal{R}^d$ for all $r > 0$.

Lemma 6.1 *Let $A \subset \mathbb{R}^d$ compact and $0 < a < b$. Assume that $A_a \in \mathcal{R}^d$. Then there is a constant $c > 0$ such that $C_k^{\mathrm{var}}(A_\varepsilon) \le c$ for $\varepsilon \in [a,b]$.*

Proof. Let $A_a = \bigcup_{j=1}^n K^j$ be a representation of A_a by convex sets K^j. Then $A_\varepsilon = \bigcup_{j=1}^n (K^j)_{\varepsilon-a}$ is a representation of A_ε by convex sets for $\varepsilon > a$. Let $\mathbb{R}^d = H^+ \cup H^-$ be a Hahn decomposition of $\mathbb{R}^d$ for the measure $C_k(A_\varepsilon, \cdot)$. Then, by the inclusion-exclusion formula (4), we have for $a \le \varepsilon \le b$

$$\begin{aligned}
C_k^{\mathrm{var}}(A_\varepsilon) &= C_k(A_\varepsilon, H^+) + C_k(A_\varepsilon, H^-)\\
&= \sum_{I \in N_n} (-1)^{\#I-1} \left(C_k\big(\bigcap_{i\in I} K^i_{\varepsilon-a}, H^+\big) + C_k\big(\bigcap_{i\in I} K^i_{\varepsilon-a}, H^-\big) \right)\\
&\le \sum_{I \in N_n} C_k\big(\bigcap_{i\in I} K^i_{\varepsilon-a}\big)\\
&\le \#N_n \cdot C_k(\mathrm{conv}(F_b)) =: c
\end{aligned}$$

Here the last inequality is due to the fact that $K^i_{\varepsilon-a} \subset \mathrm{conv}(F_b)$ and the monotonicity of the total curvatures for convex sets.

Setting $r = 1$ and $\varepsilon = \delta$, we have $O(1) = \bigcup_i S_i O$ and Lemma 5.6 specializes as follows.

Corollary 6.2 *There are constants $c, \gamma > 0$ such that for all $\varepsilon \le 1$*

$$C_k^{\mathrm{var}}(F_\varepsilon, (O(1)^c)_\varepsilon) \le c\varepsilon^{k-s+\gamma}.$$

Proof. Setting $r = 1$ and $\varepsilon = \delta$ in Lemma 5.6, we get the assertion for $0 < \varepsilon \le \rho$. By Lemma 6.1, $C_k^{\mathrm{var}}(F_\varepsilon) \ge C_k^{\mathrm{var}}(F_\varepsilon, (O(1)^c)_\varepsilon)$ is bounded by some constant for $\varepsilon \in [\rho, 1]$. Hence, if the constant c is suitably enlarged, the estimate holds for $\varepsilon \in (0, 1]$.

The next step towards the proof of Theorem 5.1 is the following inequality.

Lemma 6.3 *For $\varepsilon > 0$ and $0 < r \le 1$, we have*

$$C_k^{\mathrm{var}}(F_\varepsilon) \le \sum_{w \in \Sigma(r)} C_k^{\mathrm{var}}((S_w F)_\varepsilon) + C_k^{\mathrm{var}}(F_\varepsilon, (O(r)^c)_\varepsilon)\,.$$

Proof. Fix $\varepsilon > 0$. Let $U = \bigcup_{v,w \in \Sigma(r)} (S_v F)_\varepsilon \cap (S_w F)_\varepsilon$ and $B_w = (S_w F)_\varepsilon \setminus U$ for $w \in \Sigma(r)$. Then $F_\varepsilon = U \cup \bigcup_{w \in \Sigma(r)} B^w$ and this union is disjoint. Thus

$$C_k^{\mathrm{var}}(F_\varepsilon) = \sum_{w \in \Sigma(r)} C_k^{\mathrm{var}}(F_\varepsilon, B^w) + C_k^{\mathrm{var}}(F_\varepsilon, U)\,.$$

The set $A^w := \left(\bigcup_{v \in \Sigma(r)\setminus\{w\}} (S_v F)_\varepsilon\right)^c$ is open (the complement is a finite union of closed sets). Moreover, $B^w \subseteq A^w$ and $F_\varepsilon \cap A^w = (S_w F)_\varepsilon \cap A^w$. Hence, by locality (cf. Prop. 4.2), we have $C_k^{\mathrm{var}}(F_\varepsilon, B^w) = C_k^{\mathrm{var}}((S_w F)_\varepsilon, B^w) \le$

$C_k^{\mathrm{var}}((S_w F)_\varepsilon)$. It remains to show that $U \subset (O(r)^c)_\varepsilon$. Let $x \in U$. We show that $d(x, O(r)^c) \le \varepsilon$ and thus $x \in (O(r)^c)_\varepsilon$. Assume $d(x, O(r)^c) > \varepsilon$. Since the union $O(r) = \bigcup_{w \in \Sigma(r)} S_w O$ is disjoint, there is a unique $v \in \Sigma(r)$ such that $x \in S_v O$. Moreover, $d(x, \partial S_v O) > \varepsilon$. Since $x \in U$, there is at least one word $w \in \Sigma(r), w \ne v$ such that $x \in (S_w F)_\varepsilon$ and thus a point $y \in S_w F$ with $d(x,y) \le \varepsilon$. But then $y \in S_w F \cap S_v O$, a contradiction to OSC. Hence, $d(x, O(r)^c) \le \varepsilon$.

Combining the above statements, the upper bound for s_k is now easily derived.

Proof of Theorem 5.1: Set $g(\varepsilon) := \varepsilon^{s-k} C_k^{\mathrm{var}}(F_\varepsilon)$. We have to show that $\sup\{g(\varepsilon) : \varepsilon \in (0,1]\}$ is bounded by some constant $M > 0$. Observe that $C_k^{\mathrm{var}}((S_i F)_\varepsilon) = r_i^k C_k^{\mathrm{var}}(F_{\varepsilon/r_i}) = \varepsilon^{k-s} r_i^s g(\varepsilon/r_i)$. Combining Lemma 6.3 and Corollary 6.2, there exist $c, \gamma > 0$ such that, for $0 < \varepsilon \le 1$,

$$g(\varepsilon) \le \sum_{i=1}^N r_i^s g(\varepsilon/r_i) + c\varepsilon^\gamma. \tag{9}$$

Let $r_{\max} := \max\{r_i | i = 1, \ldots, N\}$. For $n \in \mathbb{N}$ set $I_n := (r_{\max}^n, 1]$ and $M_1 := \max\{\sup_{\varepsilon \in I_1} g(\varepsilon), c\}$. Note that $M_1 < \infty$ is ensured by Lemma 6.1. We claim that for $n \in \mathbb{N}$,

$$\sup_{\varepsilon \in I_n} g(\varepsilon) \le M_n := M_1 \sum_{j=0}^{n-1} (r_{\max}^\gamma)^j, \tag{10}$$

which we show by induction. For $n = 1$, the statement is obvious. So assume that (10) holds for $n = k$. Then for $\varepsilon \in I_k$, we have $g(\varepsilon) \le M_k \le M_{k+1}$ and for $\varepsilon \in I_{k+1} \setminus I_k$ we have $\varepsilon/r_i \ge \varepsilon/r_{\max} \ge r_{\max}^k$, i.e. $\varepsilon/r_i \in I_k$ for all i. Hence, by (9),

$$g(\varepsilon) \le \sum_{i=1}^N r_i^s g(\varepsilon/r_i) + c\varepsilon^\gamma \le \sum_{i=1}^N r_i^s M_k + M_1 r_{\max}^{\gamma k} = M_{k+1},$$

proving (10) for $n = k+1$ and hence for all $n \in \mathbb{N}$. Now observe that the sequence $(M_n)_{n \in \mathbb{N}}$ is bounded ($M_n = M_1 \sum_{j=0}^n (r_{\max}^\gamma)^j \to (1 - r_{\max}^\gamma)^{-1}$ as $n \to \infty$). Hence $g(\varepsilon)$ is bounded in $(0,1]$, completing the proof of Theorem 5.1.

7 Generalizations

The results obtained in [17] and discussed here should be seen as the outset of a larger project to find geometric measures and characteristics suitable to describe the geometry of fractal sets. The current research aims at several directions:

Generalization of the results to other classes of self-similar sets. In particular, the assumption of polyconvexity for the parallel sets is a serious restriction of the applicability of the results in [17]. Some progress in getting rid of this assumption has been made by Zähle [18]. Here self-similar sets are considered, for which the closed complements of almost all parallel sets are sets with positive reach (for the cost of replacing limits by essential limits). Up to now it is not clear whether all self-similar sets have this property. Moreover, in this paper the concepts are extended to self-similar random sets and results on *random fractal curvatures* are obtained.

In [11], the limiting behaviour of the surface area of parallel sets is shown to be closely tied to the Minkowski content. The relations are derived from the fact that for most parallel sets the surface area is the derivative of the volume. Some results apply to arbitrary compact sets and thus go beyond the self-similar setting.

Curvature-direction measures. For self-similar sets, the fractal curvature measures turn out to be multiples of the Hausdorff measure and so only the total curvatures may give new insights into the geometry of the sets. One possible approach to obtain finer information, is to work with generalized curvature measures (or curvature-direction measures) of the parallel sets. They live on the normal bundle and take, additionally to the boundary points, also the normal directions into account. First results have been obtained by Rothe in his diploma thesis [12]. He showed the existence of *fractal curvature direction measures* for self-similar sets with polyconvex parallel sets and that they have a product structure. Examples suggest that the directional components of the limit measures carry non-trivial geometric information.

Other classes of fractals. In [4], Kombrink studied the fractal curvatures of self-conformal sets. Some bounds for the scaling exponents have been obtained. The approach is slightly different to ours. Not the parallel sets of the fractals are used for the approximation but the parallel sets of certain covers by convex sets.

Steiner type formulas. For certain self-similar sets (and other sets that can be described by fractal sprays) tube formulas have been obtained, which describe the volume of the parallel sets and relate it to the complex dimensions of the set, cf. [6, 7, 8, 10]. The classical Steiner formulas suggest, that the coefficients should be interpreted as curvatures in some way. Up to now the relations to fractal curvatures are not clear.

Estimation of fractal dimension and fractal curvatures from digital images. Fractal curvatures and the associated scaling exponents provide a whole set of characteristics which may be used to distinguish and classify fractal sets. They can easily be estimated from digital images, some methods have been implemented by Straka [16] and tested for self-similar sets. Using several geometric characteristics instead of just one (usually the volume) may also improve estimates of fractal dimension, see [15].

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