Volumes of representations of 3-manifold groups

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Contents

CHAPTER 1

Introduction

1.1. Preface

This thesis consists of two parts. In the first part, which is contained in Chapter 2, I discuss the article "'Chern-Simons Theory and the Volume of 3-manifolds"' by Derbez and Wang ([**11**]) in detail and check its overall correctness. To increase readability and make it possible to compare this work to the original work by Derbez and Wang Chapter 2 is structured in the same order and basically follows the same notation as [**11**]. Major changes and extensions are mentioned in the introductory chapter, whereas for better readability some minor changes will only become visible when explicitly comparing the two works.

The second part, which is contained in Chapter 3, presents my own results. These include a generalization of the additivity formula presented in Section 2.5.6 to representations into $Iso_e\widetilde{SL_2}(\mathbb{R})$ and a discussion of volumes of representations for 1-edged graph manifolds. In particular we prove the existence of a representation of non-zero volume for a large class of 1-edged graph manifolds.

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1.2. Overview

I want to give an overview of the main results of both, the article of Derbez and Wang [**11**], and my own results.

Since a more detailed overview of the results of [**11**] will be given in Section 2.1, I only want to mention the most important results without giving many definitions and details. Readers that are not familiar with volumes of representations and related topics are adviced to skip this part and proceed straight to Chapter 2.

Let M^n be a compact closed orientable manifold of real dimension n and G a Lie group with maximal compact subgroup K such that the contractible space $X^n = G/K$ has dimension n. Let $\rho : \pi_1 M^n \to G$ be a representation. We can assign a volume to it which is denoted by $\mathrm{vol}_G(M, \rho)$. A definition will be given in Section 2.3.

The theory of volumes of representation has connections to many branches of mathematics. An example for a recent application is the proof that for M a graph manifold and N a closed prime non-trivial graph manifold the set

$$\mathcal{D}(M,N) = \{d \in \mathbb{Z} \,|\, \exists f : M \to N \text{ continuous}, \ \deg(f) = d\}$$

of degrees of continuous maps $M \to N$ is finite ([**9**]).

Although the notion of a volume of a representation has already been known for a long time, many of its properties are remarkably unknown and there are still some interesting and attractive questions to study.

Here we consider the case where M is a closed oriented 3-manifold and G is either the identity component $Iso_e\widetilde{SL_2(\mathbb{R})}$ of the isometry group of $\widetilde{SL_2(\mathbb{R})}$ or $PSL(2;\mathbb{C})$, the orientation preserving isometry group of the hyperbolic 3-space $\mathbb{H}^3$.

For a fixed closed oriented 3-manifold M denote by $\mathrm{vol}(M,G)$ the subset of $\mathbb{R}$ consisting of all volumes of representations $\rho : \pi_1 M \to G$, that is

$$\mathrm{vol}(M,G) = \{\mathrm{vol}_G(M,\rho) \,|\, \rho : \pi_1 M \to G \text{ a representation}\},$$

and by $SV(M)$, resp. $HV(M)$, the maximal value in $\mathrm{vol}(M,G)$ for $G = Iso_e\widetilde{SL_2(\mathbb{R})}$, resp. $G = PSL(2;\mathbb{C})$.

The *covering property* is satisfied by a non-negative 3-manifold invariant η, if for any finite covering $p : \widetilde{M} \to M$, we have $\eta(\widetilde{M}) = |\deg(p)|\eta(M)$.

The following questions are still to a great extent open and are a main motivation for [**11**].

QUESTION 1.

(1) *How can we tell whether there are non-zero elements in* $\mathrm{vol}(M,G)$*?*
or weakly,
When is there a finite cover $\widetilde{M}$ *of* M *so that are non-zero elements in* $\mathrm{vol}(\widetilde{M},G)$*?*
(2) *Does* HV *or* SV *satisfy the covering property in the sense of Thurston?*
(3) *What kind of topological information is captured by the non-zero elements in* $\mathrm{vol}(N,G)$*?*

REMARK 1.1. The question of finding 3-manifold invariants satisfying the covering property was asked by Thurston in [**27**, Problem 3.16] and Milnor and Thurston gave the first example of such an invariant [**35**]. Other examples are the simplicial volume (Gromov-Thurston-Soma, [**20**]) and an invariant for graph manifolds which can be found in [**47**].

The first important result in [**11**] is an explicit description of the set $\mathrm{vol}\left(M, Iso_e\widetilde{SL_2(\mathbb{R})}\right)$ for each 3-manifold supporting the $\widetilde{SL_2(\mathbb{R})}$-geometry.

PROPOSITION. *Let* M *be a 3-manifold supporting the* $\widetilde{SL_2(\mathbb{R})}$*-geometry with base 2-orbifold* $O(M)$ *of positive genus* g*. Then*

$$\mathrm{vol}\left(M, Iso_e\widetilde{SL_2(\mathbb{R})}\right) = \left\{ \frac{4\pi^2}{|e(M)|} \left(\sum_{i=1}^{r} \left(\frac{n_i}{a_i}\right) - n \right)^2 \right\}$$

where $n_1, \cdots, n_r, n$ *are integers such that*

$$\sum_{i=1}^{r} \lfloor \frac{n_i}{a_i} \rfloor - n \leq 2g-2 \ \text{and} \ \sum_{i=1}^{r} \lceil \frac{n_i}{a_i} \rceil - n \geq 2-2g$$

and $a_1, \cdots, a_r$ *are the indices of the singular points of the orbifold* M.

A proof can be found in Section 2.4.

A partial answer to Question 1(1) for non-geometric manifolds was recently given in [**10**]:

For each non-trivial graph manifold M there exists a finite cover $\widetilde{M}$ so that $\mathrm{vol}(\widetilde{M}, Iso_e\widetilde{SL_2(\mathbb{R})}) \neq \{0\}$.

That reduced question 1(1)(ii) to non-geometric manifolds containing a hyperbolic piece. In view of Theorem 2.1(2),(3), as well as the result of [**10**], and in an attempt to find a relation between the Gromov simplicial volume and the hyperbolic volume, Prof. M. Boileau and some others [**11**] posed the following more specific version of Question 1(1):

QUESTION 2. *Suppose* M *has positive simplicial volume, i.e. contains some hyperbolic piece.*

(1) Is there a representation $\rho : \pi_1 M \to PSL(2;\mathbb{C})$ *with positive volume?, or weakly*
(2) Is there a representation $\rho : \pi_1\widetilde{M} \to PSL(2;\mathbb{C})$ *with positive volume for some finite covering* $\widetilde{M}$ *of* M*?*

Questions 1(2) and 2 are partially answered by the following two results which will be proved in Sections 2.6 and 2.7:

PROPOSITION. *Let* M *be a closed irreducible non-geometric 3-manifold. Assume that* M *contains a hyperbolic piece* Q *such that each boundary component of* Q *that is non-separating in* M *is shared by a Seifert piece of* M*. Then there is a finite covering* $\widetilde{M} \to M$ *which admits a representation* $\rho : \pi_1\widetilde{M} \to PSL(2;\mathbb{C})$ *of positive volume.*

COROLLARY. *The hyperbolic volume* HV *does not satisfy the covering property. Namely there are finite coverings* $p : \widetilde{M} \to M$ *with* $HV(\widetilde{M}) > |\deg p| HV(M) = 0$.

Thurston pointed out a relation between Chern-Simons invariants and the hyperbolic volume of hyperbolic 3-manifolds for discrete and faithful representations [**44**]. In [**28**] it was extended to hyperbolic 3-manifolds for discrete and faithful representations into $PSL(2;\mathbb{C})$ and by [**26**] to closed manifolds for representations into the subgroup $\widetilde{SL_2(\mathbb{R})}$ of $Iso_e\widetilde{SL_2(\mathbb{R})}$.

Derbez and Wang [**11**] give a new, easier proof of the statement for representations into $\widetilde{SL_2(\mathbb{R})}$ and extend the statement for $PSL(2;\mathbb{C})$ to all representations that admit a lift into $SL(2;\mathbb{C})$.

PROPOSITION. *Let* $\rho : \pi_1 M \to G = Iso_e\widetilde{SL_2(\mathbb{R})}$ *be a representation and let* A *be the corresponding flat* G*-connection in the principal bundle* $P = M \times_\rho G$*. If* P *admits a section over* M *then*

$$\mathfrak{cs}_M(A, \delta) = 2\mathrm{vol}_G(M, \rho)$$

PROPOSITION. *Let* $\rho : \pi_1 M \to G = PSL(2;\mathbb{C})$ *be a representation and* A *the corresponding flat* G*-connection over* M*. Assume that* $M \times_\rho G$ *admits* *a section δ over M. Then*

$$\mathcal{I}(\mathfrak{cs}_M(A,\delta)) = -\frac{1}{\pi^2} vol_G(M,\rho)$$

The proofs for both statements can be found in Section 2.5.

Every prime non-geometric manifold is completely determined by its pieces and the isotopy classes of the gluings between them. It turns out that for non-geometric manifolds the volumes of representations into $PSL(2;\mathbb{C})$ and $Iso_e\widetilde{SL_2(\mathbb{R})}$ give some information about the gluing maps.

Derbez and Wang use the relation between volumes of representations and Chern-Simons theory to study this behaviour. They compute volumes of concrete representatiosn for one-edged graph manifolds and show that they indeed depend on the gluing parameters. The computations can be found in Section 2.8 and the results are summarized in Propositons 2.13 and 2.14.

In Chapter 3 my own results are presented. The main motivation was to proceed on an answer to Question 1 for $G = Iso_e\widetilde{SL_2(\mathbb{R})}$ and to make an attempt to find analogous results to Proposition 2.3.

In Chapter 2.5 an additivity result for volumes of representations of non-geometric 3-manifolds is derived which allows to compute the volume of a representation for a manifold M by restricting the representation to the geometric pieces of its JSJ-decomposition. The result is based on the relation between volumes of representations and Chern-Simons theory and thus holds only for $G = PSL(2;\mathbb{C})$ or $G = \widetilde{SL_2(\mathbb{R})}$. In Section 3.2 we generalize this result to representations into $Iso_e\widetilde{SL_2(\mathbb{R})}$. Namely we prove the following proposition:

Proposition 1.2. *Let $M = M_1 \cup_\tau M_2$ be a 1-edged manifold and $\rho : M \to G = Iso_e\widetilde{SL_2(\mathbb{R})}$ a representation. Let (s_1, h_1) be a basis of $H_1(\partial M_1;\mathbb{Z}) = H_1(T^2;\mathbb{Z})$ and $(a,b) \in \mathbb{Z}^2$ with $gcd(a,b) = 1$ and $\rho(as_1 + bh_1) = \overline{(0,1)}$.*

Then for $\widehat{M_i} = M_i(a,b)$, $i = 1,2$, ρ extends to $\widehat{\rho}_1 : \pi_1\widehat{M_1} \to G$ and $\widehat{\rho}_2 : \pi_1\widehat{M_2} \to G$ and

$$vol(M,\rho) = vol(\widehat{M_1}, \widehat{\rho}_1) + vol(\widehat{M_2}, \widehat{\rho}_2).$$

Remark 1.3. Additivity can be generalized in an obvious way to non-geometric manifolds that contain more than one essential torus $\{T_1, \cdots T_k\}$, since all arguments in the proof only depend on neighbourhoods $T_i \times (-1,1)$ of the tori.

The additivity result is used in Section 3.3 to study volumes of representations for graph manifolds in general and in particular for one-edged graph manifolds, leading to a result similar to Proposition 2.3 for one-edged graph manifolds. As a consequence we are able to proof the following non-vanishing result for a large class of one-edged graph manifolds.

Proposition 1.4. *Let $M = M_1 \cup_\tau M_2$ be a one-edged graph manifold, where M_j, $j = 1,2$, have Euler numbers $e_j = \frac{p_{e_j}}{q_{e_j}}$, $gcd(p_{e_j}, q_{e_j}) = 1$, and* *genus* $g^{(j)}$, $j = 1, 2$. *Let* $(s^{(j)}, h^{(j)})$, $j = 1, 2$, *be a section/fiber basis of the toral boundary* ∂M_j, $j = 1, 2$, *and denote the gluing with respect to this basis by the matrix* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *that is,* $(s^{(1)}, h^{(1)}) = (s^{(2)}, h^{(2)})A$.

Assume that

$$gcd(aq_{e_2}p_{e_1} + cq_{e_2}q_{e_1} - dp_{e_2}q_{e_1}, b) = 1, \tag{1.2.1}$$

and that $g^{(1)}, g^{(2)} \geq 2$.

Then there is a representation $\rho : \pi_1 M \to Iso_e\widetilde{SL_2(\mathbb{R})}$ *with non-zero volume.*

The Proposition follows immediately from Proposition 3.15 and gives a partial answer to Question 1(1)(i).

CHAPTER 2

Chern-Simons Theory and the Volume of 3-manifolds

2.1. Introduction

In this section we want to state some important notions and facts about volumes of represenations and give an overview of the results that were proved in the work of Derbez and Wang [**11**]. For readers that are not familiar with the topic we will give rigorous definitions and a better explanation of the background material in later sections.

Let M^n be a compact closed orientable manifold of real dimension n and G a Lie group with maximal compact subgroup K such that the contractible space $X^n = G/K$ has dimension n. Let $\rho : \pi_1 M^n \to G$ be a representation. We can assign a volume to it which is denoted by $\mathrm{vol}_G(M, \rho)$. A definition will be given in Section 2.3.

Although the notion of a volume of a representation has already been known for a long time, many of its properties are remarkably unknown and there are still some interesting and attractive questions to study.

Here we consider the case where M is a closed oriented 3-manifold and G is either the identity component $Iso_e\widetilde{SL_2(\mathbb{R})}$ of the isometry group of $\widetilde{SL_2(\mathbb{R})}$ or $PSL(2;\mathbb{C})$, the orientation preserving isometry group of the hyperbolic 3-space $\mathbb{H}^3$.

For a fixed closed oriented 3-manifold M denote by $\mathrm{vol}(M, G)$ the subset of $\mathbb{R}$ consisting of all volumes of representations $\rho : \pi_1 M \to G$, that is

$$\mathrm{vol}(M, G) = \{\mathrm{vol}_G(M, \rho) \,|\, \rho : \pi_1 M \to G \text{ a representation}\} .$$

Suppose that M supports the hyperbolic, resp. $\widetilde{SL_2(\mathbb{R})}$-geometry. Then the geometry of M endows M with a metric which naturally defines the *hyperbolic volume* $\mathrm{vol}_{\mathbb{H}^3}(M)$, resp. *Seifert volume* $\mathrm{vol}_{\widetilde{SL_2(\mathbb{R})}}(M)$, of M.

Denote by $||M||$ the (Gromov) simplicial volume of M, introduced by Gromov in [**20**]. The simplicial volume and the hyperbolic volume of a hyperbolic manifold coincide up to a constant which only depends on the dimension [**43**, Prop. 6.1.4]. Furthermore the simplicial volume is additive under connected sums of closed oriented connected manifolds of dimension at least 3 [**20**, p.10] and for 3-manifolds it is additionally additive under decomposition along essential tori [**31**, Chapter 3 and Satz 2]. Consequently, using the geometrization conjecture, we obtain that the simplicial volume of M is proportional to the hyperbolic volume of the hyperbolic pieces of M if M is a 3-manifold.

We summarize some basic results of the theory of volumes of representations in the following theorem. As references and for its development, see [**5**], [**16**], [**38**], and [**39**], as well as their references.

THEOREM 2.1. *Let M be a closed oriented 3-manifold.*

(1) *Both* $\mathrm{vol}(M, PSL(2;\mathbb{C}))$ *and* $\mathrm{vol}(M, Iso_e\widetilde{SL_2}(\mathbb{R}))$ *contain at most finitely many values and we denote by $HV(M)$ and $SV(M)$ the maximum value for $PSL(2;\mathbb{C})$ and $Iso_e\widetilde{SL_2}(\mathbb{R})$ respectively.*
(2) *Suppose M supports the hyperbolic geometry. Then* $\mathrm{vol}_{\mathbb{H}^3}(M)$ *is reached by* $\mathrm{vol}_{PSL(2;\mathbb{C})}(M,\rho)$ *for some discrete faithful representation. The analogous statement ist true when M supports the $\widetilde{SL_2}(\mathbb{R})$-geometry.*
(3) $\mathrm{vol}_{PSL(2;\mathbb{C})}(M,\rho) \leq \mu_3||M||$, *where μ_3 denotes the volume of any ideal regular tetrahedron in $\mathbb{H}^3$.*
(4) *Let $f : M \to N$ be a map of degree d and let $\rho : \pi_1 N \to G$ denote a representation. Then we get a representation $\rho \circ f_* : \pi_1 M \to G$ such that* $\mathrm{vol}_G(\rho \circ f_*, M) = d \cdot \mathrm{vol}_G(\rho, N)$.

This yields the inequalities

$$HV(M) \geq |\deg f| HV(N) \text{ and } SV(M) \geq |\deg f| SV(N).$$

PROOF. The proofs can be found in the following references:

(1) is proved explicitly in [**6**, Theorem 1] for $Iso_e\widetilde{SL_2}(\mathbb{R})$ and a more general proof that applies to both Lie groups is given in [**38**, pp. 550].

(2) For $Iso_e\widetilde{SL_2}(\mathbb{R})$ see [**6**, Theorem 3] and for $PSL(2;\mathbb{C})$ see [**17**, Theorem 7.1].

(3) is proved in [**39**, 5.1].

(4) is an easy consequence of the definition of Volumes of representations in Section 2.3.1 by developing maps, since a continuous map $f : M \to N$ is covered by a continuous map $\widetilde{f} : \widetilde{M} \to \widetilde{N}$ between the universal coverings and for a representation $\rho : \pi_1 N \to G$ together with the induced representation $\widehat{\rho} = \rho \circ f_* : \pi_1 M \to G$ we observe that the developing maps satisfy $D_{\widehat{\rho}} = D_\rho \circ \widetilde{f}$. This implies that $\mathrm{vol}_G(\rho \circ f_*, M) = d \cdot \mathrm{vol}_G(\rho, N)$ and the other part of the statement follows immediately. □

REMARK 2.2. A prime 3-manifold M admits no self-map of degree greater 1 if and only if either M has a non-trivial geometric decomposition, or supports an $\widetilde{SL_2}(\mathbb{R})$ or a hyperbolic geometry [**42**, Theorem 1.0]. Combined with Theorem 2.1 (2)-(4) we obtain that if $\mathrm{vol}(M, Iso_e\widetilde{SL_2}(\mathbb{R})) \neq \{0\}$ then necessarily either M has non-trivial geometric decomposition, or supports the $\widetilde{SL_2}(\mathbb{R})$, or the hyperbolic geometry and if $\mathrm{vol}(M, PSL(2;\mathbb{C})) \neq \{0\}$ then M contains some hyperbolic pieces.

The *covering property* is satisfied by a non-negative 3-manifold invariant η, if for any finite covering $p : \widetilde{M} \to M$, we have $\eta(\widetilde{M}) = |\deg(p)|\eta(M)$.

We do not know much about the extent to which HV and SV satisfy the covering property. It seems that we only know that HV, resp. SV, satisfy the covering property for hyperbolic, resp. Seifert manifolds. For

HV the property is due to the relation between the simplicial volume and HV and for SV it is due to SV being a function of the Euler class of the Seifert manifold and the Euler characteristic of its base 2-orbifold which both behave naturally under covering maps. That is, $SV(M) = 4\pi^2\chi^2_{O(M)}/|e(M)|$ for M a Seifert manifold and $O(M)$ its base 2-orbifold.

In the following we give an overview of the main results that are proved in this work. From now on all manifolds are assumed to be closed oriented and irreducible.

2.1.1. Volumes of Seifert manifolds. Using the works of Brooks-Goldman [**6**], Milnor [**34**], Wood, [**48**] and Eisenbud-Hirsch-Neumann [**16**], Derbez and Wang [**11**] give an explicit description of the set $\operatorname{vol}\left(M, Iso_e\widetilde{SL_2(\mathbb{R})}\right)$ for each 3-manifold M supporting the $\widetilde{SL_2(\mathbb{R})}$-geometry.

It is known that M supports the $\widetilde{SL_2(\mathbb{R})}$-geometry if and only if M is a Seifert manifold with non-zero Euler number $e(M)$ and its base 2-orbifold $O(M)$ has negative Euler characteristic $\chi_{O(M)}$. For $a \in \mathbb{R}$ denote by $\lfloor a \rfloor$ and $\lceil a \rceil$ the greatest integer $\leq a$, resp. least integer $\geq a$.

Proposition 2.3. *Let M be a 3-manifold supporting the $\widetilde{SL_2(\mathbb{R})}$-geometry with base 2-orbifold $O(M)$ of positive genus g. Then*

$$\operatorname{vol}\left(M, Iso_e\widetilde{SL_2(\mathbb{R})}\right) = \left\{ \frac{4\pi^2}{|e(M)|}\left(\sum_{i=1}^{r}\left(\frac{n_i}{a_i}\right) - n\right)^2 \right\}$$

where $n_1, \cdots, n_r, n$ are integers such that

$$\sum_{i=1}^{r}\lfloor\frac{n_i}{a_i}\rfloor - n \leq 2g-2 \text{ and } \sum_{i=1}^{r}\lceil\frac{n_i}{a_i}\rceil - n \geq 2-2g$$

and $a_1, \cdots, a_r$ are the indices of the singular points of the orbifold M.

Remark 2.4. Proposition 2.3 will be checked by describing all representations with non-zero volume in Proposition 2.25. The proposition explicitly proves the rationality of elements in $\operatorname{vol}\left(M, Iso_e\widetilde{SL_2(\mathbb{R})}\right)$, which was first proved in [**38**]. As a consequence of the Proposition and its proof we compute volumes for 3-manifolds with non-trivial geometric decomposition.

We corrected a small mistake in the statement of Proposition 2.25 with respect to [**11**]. Some changes and extensions have been done in its proof. We directly refer to the work of Milnor-Wood [**34**], [**48**] when computing the volume of a certain representation by using transverse foliations instead of refering to [**29**]. Furthermore we carry out some details for the proof that a representation with non-zero volume always maps the fiber h to an element of type $\overline{(\zeta, 1)}$ and that we may assume that $\rho(\alpha_i), \rho(\beta_i) \in \widetilde{SL_2(\mathbb{R})}$ for $\alpha_i, \beta_i \in \pi_1 M$ the generators corresponding to the holes in the base 2-orbifold.

2.1.2. Volumes of nongeometric manifolds. A partial answer to Question 1(1) for non-geometric manifolds was recently given in [**10**]:

For each non-trivial graph manifold M there exists a finite cover $\widetilde{M}$ so that $\mathrm{vol}(\widetilde{M}, Iso_e\widetilde{SL_2}(\mathbb{R})) \neq \{0\}$.

That reduced question 1(1)(ii) to non-geometric manifolds containing a hyperbolic piece and thus lead to Question 2.

Let M be a closed irreducible non-geometric 3-manifold. Let $\mathcal{T}_M \subset M$ be a minimal union of disjoint essential tori and Klein bottles in M, which is unique up to isotopy, such that each piece of $M \setminus \mathcal{T}_M$ is either Seifert or hyperbolic. We call M *one-edged manifold* if $\mathcal{T}_M$ consists of precisely one torus T.

The work [**11**] of Derbez and Wang gives a negative answer to Question 2(1) and a partially positive answer to Question 2(2). They are direct consequences of the following propositions.

PROPOSITION 2.5. *Let M be a closed irreducible non-geometric 3-manifold. Assume that M contains a hyperbolic piece Q such that each boundary component of Q that is non-separating in M is shared by a Seifert piece of M. Then there is a finite covering $\widetilde{M} \to M$ which admits a representation $\rho : \pi_1\widetilde{M} \to PSL(2;\mathbb{C})$ of positive volume.*

If each component of $\mathcal{T}_M$ is separating in M, then the condition in Proposition 2.5 is automatically satisfied, and we obtain the following corollary

COROLLARY 2.6. *A prime rational homology sphere $\mathbf{S^3}_{\mathbb{Q}}$ has a positive simplicial volume if and only if it admits a finite covering with positive hyperbolic volume.*

PROOF. If $\mathbf{S^3}_{\mathbb{Q}}$ is geometric then this is a direct consequence of Theorem 2.1 (2) and Remark 2.2.

If $\mathbf{S^3}_{\mathbb{Q}}$ is non-geometric then every component of $\tau_{\mathbf{S^3}_{\mathbb{Q}}}$ must be separating in $\mathbf{S^3}_{\mathbb{Q}}$ and thus the statement follows immediately from Proposition 2.5. □

PROPOSITION 2.7. *There are infinitely many 1-edged 3-manifolds M with non-vanishing simplicial volume but* $\mathrm{vol}(M, PSL(2;\mathbb{C})) = \{0\}$.

Another consequence of the two propositions is a negative answer to Question 1(2) for the hyperbolic volume, that is:

COROLLARY 2.8. *The hyperbolic volume HV does not satisfy the covering property. Namely there are finite coverings $p : \widetilde{M} \to M$ with $HV(\widetilde{M}) > |\deg p| HV(M) = 0$.*

There were no major changes in the proof of Proposition 2.7. The proof of Proposition 2.5 was extended by an explanation that the Chern-Simons invariant of a certain path of representations does not change under homotopies. Besides we give a more detailed explanation for the statements preliminary to the proof of Proposition 2.5.

2.1.3. Volumes as Chern-Simons invariants. It seems that the reason for Question 1 being still unanswered is due to the problem that it is not at all clear from the definition of the volume of a representation how to find significant representation, that is representations with non-zero volume, or representations whose volume contains information about the geometric structure of the manifold, and that given a concrete representation it is still hard to compute its volume.

When the manifold is geometric there is a natural choice for a significant representation. Namely a discrete faithful representation of the fundamental group into the Lie group of its geometry. In contrast in the case of a non-geometric manifold it is not clear how to obtain a good choice for a representation. One can try to construct it componentwise using the geometry of the geometric pieces. Doing that one faces several problems:

- We must choose the representations in such a way that they are compatible on the toral boundaries of the geometric pieces under the gluing maps
- It is in general not easy to compute the volume of a representation for a non-closed manifold, since some manipulations that work for closed manifolds do not carry over
- The volume is not additive with respect to the geometric decomposition (unlike the Gromov simplicial volume) and depends on the gluing.

However, the last point could also be advantageous, since the volumes of the representations seem to contain some topological information about the gluings.

One way to solve these problems for a relevant class of representations is to apply results from Chern-Simons theory. That works, since for sufficiently nice representations into the semi-simple Lie groups $G = Iso_e\widetilde{SL_2(\mathbb{R})}$ or $G = PSL(2;\mathbb{C})$ we obtain an easy way to compute the volume of the representation from the Chern-Simons invariant of the corresponding flat connection. It will be used later, when we compute volumes for concrete one-edged manifolds.

In the following we denote by X the homogeneous space associated to G, which is $\widetilde{SL_2(\mathbb{R})}$ or $\mathbb{H}^3$ in the case of $Iso_e\widetilde{SL_2(\mathbb{R})}$ or $PSL(2;\mathbb{C})$. We fix a closed G-invariant volume form ω_X on X and denote by $\mathfrak{g}$ the Lie algebra of G.

Recall that the Chern-Simons classes with structure group $PSL(2;\mathbb{C})$ are based on the first Pontrjagin class and we base the Chern-Simons classes with structure group $Iso_e\widetilde{SL_2(\mathbb{R})}$ on the invariant polynomial defined by $\mathcal{R}(A\otimes A) = Tr(X^2)+t^2$ where $A = X+t$ is an element of the Lie algebra of $Iso_e\widetilde{SL_2(\mathbb{R})}$ decomposed into an element of the Lie algebra of $\widetilde{SL_2(\mathbb{R})}$ and $t \in \mathbb{R}$.

PROPOSITION 2.9. *Let $\rho : \pi_1 M \to G = Iso_e\widetilde{SL_2(\mathbb{R})}$ be a representation and let A be the corresponding flat G-connection in the principal bundle*

$P = M \times_\rho G$. If P admits a section over M then

$$\mathfrak{cs}_M(A,\delta) = \int_M \delta^*\mathcal{R}\left(dA \wedge A + \frac{1}{3}A \wedge [A,A]\right) = 2\mathrm{vol}_G(M,\rho)$$

In particular the Chern-Simons invariant of a flat $Iso_e\widetilde{SL_2(\mathbb{R})}$-connection is gauge invariant.

REMARK 2.10. $P = M \times_\rho G$ admits a section if and only if ρ admits a lift into $\widetilde{SL_2(\mathbb{R})}$ which equivalently means that the bundle admits a reduction to a $\widetilde{SL_2(\mathbb{R})}$-bundle. The correspondence in Proposition 2.9 is pointed out in [**38**] and verified in [**26**]. In [**11**] an alternative very simple proof is given which also leads to a better understanding of the naturality of the correspondence.

The correspondence for $PSL(2;\mathbb{C})$ is a consequence of [**29**]. For $z \in \mathbb{C}$ denote by $\mathcal{I}(z)$ its imaginary part.

PROPOSITION 2.11. *Let $\rho : \pi_1 M \to G = PSL(2;\mathbb{C})$ be a representation and A the corresponding flat G-connection over M. Assume that $M \times_\rho G$ admits a section δ over M. Then*

$$\mathcal{I}(\mathfrak{cs}_M(A,\delta)) = -\frac{1}{\pi^2} vol_G(M,\rho)$$

REMARK 2.12. Since the volume of the representation does not depend on the choice of a section it is gauge invariant and thus also the imaginary part of the Chern-Simons invariant of a flat $PSL(2;\mathbb{C})$-connection is gauge invariant. We do not want to give any geometric interpretation of the real part of the Chern-Simons invariant, since it will not be used in our proofs, but we just want to mention that it is not gauge invariant. Consider the special case where the developing map $D_\rho : \widetilde{M} \to \mathbb{H}^3$ is an isometry with respect to the pull-back metric of $\mathbb{H}^3$. Then $\mathfrak{R}(\mathfrak{cs}_M(A,\delta))$ is the $\mathbb{R}/\mathbb{Z}$-valued Chern-Simons invariant of the Levi-Civita connection corresponding to the Riemannian metric in $\widetilde{M}$ pull-backed from the hyperbolic metric by D_ρ^*. This applies for instance when M is a complete hyperbolic manifold and ρ is faithful and discrete (see [**29**]).

We have added some computational details to the proofs of Propsitions 2.9 and 2.11 and got a different constant for the relation between Chern-Simons invariant and the volume of the representation. The different constant is due to different conventions in the definitions of operations on $\mathfrak{g}$-valued forms such as the wedge product or the Lie bracket.

2.1.4. Complexity of the sewing involution. Given a non-geometric manifold M, its geometry depends only on the geometry of the pieces and on the isotopy class of the gluing maps among them. As was already mentioned earlier, in contrast to the simplicial volume the volumes of representations into $G = PSL(2;\mathbb{C})$ or $Iso_e\widetilde{SL_2(\mathbb{R})}$ somehow incorporate some information about the gluing maps.

To observe this behaviour it is sufficient to consider the easiest cases of non-geometric manifolds. Namely, we will look at 1-edged manifolds

$M = Q_- \cup_\tau Q_+$, where $\tau : \partial Q_- = T_- \to \partial Q_+ = T_+$ is the gluing map between the two geometric pieces Q_- and Q_+.

We will consider two cases: The one where M is a *graph manifold*, that is, both Q_- and Q_+ are Seifert, with $G = Iso_e\widetilde{SL_2}(\mathbb{R})$, and the one where Q_- is Seifert and Q_+ is hyperbolic with $G = PSL(2;\mathbb{C})$. We call a finite covering $p : \widetilde{M} \to M$ $q \times q$*-characteristic*, if for any component $\widetilde{T}$ over T the map p induces the covering $p|\widetilde{T} : \widetilde{T} \to T$ corresponding to the subgroup $q\mathbb{Z} \oplus q\mathbb{Z} \subset \pi_1 T$. If the specific value of q is of no further importance we sometimes omit it and simply call the covering *characterisic*.

After fixing a basis s_ϵ, h_ϵ for $H_1(\partial Q_\epsilon;\mathbb{Z})$, $\epsilon = \pm$, the isotopy class of the map τ is uniquely determined by the integral matrix $\tau_* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant -1 and $\tau(s_-) = as_+ + ch_+$, $\tau(h_-) = bs_+ + dh_+$.

We choose the basis as follows:

(1) when Q_ϵ is Seifert, then the basis (s_ϵ, h_ϵ) is chosen to be a section-fiber basis for $T_\epsilon \subset Q_\epsilon$. In particular when both Q_- and Q_+ are Seifert manifolds, then $b \neq 0$, else M would be a Seifert manifold.

(2) when Q_ϵ is hyperbolic, then the basis (s_ϵ, h_ϵ) is chosen to consist of the first and second shortest simple closed geodesic on the Euclidean boundary on the maximal cusp. Then the norm of a curve $as_\epsilon + bh_\epsilon$ defined by $\sqrt{a^2+b^2}$ is equivalent to the Euclidean norm in the cusp. Thus there exist real constants $0 < k \leq K$ such that

$$k\sqrt{a^2+b^2} \leq \text{length}(as_\epsilon + bh_\epsilon) \leq K\sqrt{a^2+b^2}.$$

PROPOSITION 2.13. *Let $N = Q_- \cup_\tau Q_+$ be an one-edged graph manifold and $G = Iso_e\widetilde{SL_2}(\mathbb{R})$. Then there is a n-fold $q \times q$-characteristic covering $\widetilde{N} \to N$, where n, q depend only on Q_- and Q_+, and a representation $\rho : \pi_1\widetilde{N} \to G$ such that*

$$\text{vol}_G(\widetilde{N},\rho) = 8\pi^2\frac{n}{q^2} \text{ if } a = d = 0,\ \text{vol}_G(\widetilde{N},\rho) = 4\pi^2\frac{n}{q^2|b|} \text{ if } c = 0,$$
$$\text{vol}_G(\widetilde{N},\rho) = 4\pi^2\frac{n}{q^2|ac|} \text{ if } ac \neq 0,\ \text{vol}_G(\widetilde{N},\rho) = 4\pi^2\frac{n}{q^2|cd|} \text{ if } cd \neq 0.$$

Let Q be a one-cusped hyperbolic 3-manifold. Then a deformation of the hyperbolic structure on Q can be extended to a complete hyperbolic metric on the surgered manifold $Q(a,b)$ if the length of the curve $as_\epsilon + bh_\epsilon$ is greater than 2π. Thus for $C = 2\pi/k$ we obtain that a deformation of the hyperbolic structure on Q extends to $Q(a,b)$ if $\sqrt{a^2+b^2} > C$. Here a, b are coprime and $Q(a,b)$ is obtained from Q by identifying ∂Q with the boundary of a solid torus such that the slope $as_\epsilon + bh_\epsilon$ is identified with the meridian of the solid torus.

PROPOSITION 2.14. *Let $N = Q_- \cup_\tau Q_+$ be an one-edged 3-manifold, where Q_- is Seifert and Q_+ is hyperbolic and let $G = PSL(2;\mathbb{C})$. Then there exists a n-fold $q \times q$-characteristic covering $\widetilde{N} \to N$, where n, q depend only on Q_- and Q_+, and a representation $\rho : \pi_1\widetilde{N} \to G$ such that for any*

$\sqrt{a^2+c^2} > C$

$$\mathrm{vol}_G(\widetilde{N}, \rho) = n \cdot \mathrm{vol}(Q_+(a,c)) + \frac{\pi n(q-1)}{2q}\mathrm{length}(\gamma) \tag{2.1.1}$$

where γ is the geodesic added to Q_+ to complete the cusp with respect to the (a,c)-Dehn filling and $\mathrm{length}(\gamma)$ *denotes its length in the complete hyperbolic structure of $Q_+(a,c)$. The same statement is true for (b,d).*

REMARK 2.15. By Theorem 1B in [**37**] we obtain

$$\mathrm{vol}Q_+(a,c) = \mathrm{vol}Q_+ - \frac{\pi}{2}\mathrm{length}(\gamma) + \mathcal{O}\left(\frac{1}{a^4+c^4}\right)$$

where

$$\mathrm{length}(\gamma) = 2\pi\frac{\mathcal{I}(z_0)}{|a+z_0c|^2} + \mathcal{O}\left(\frac{1}{a^4+c^4}\right)$$

with z_0 a complex number with $\mathcal{I}(z_0) > 0$ giving the modulus of the Euclidean structure on the torus T corresponding to the cusp of Q_+. Substituting these equations in Proposition 2.14 we obtain

$$\frac{\mathrm{vol}_{PSL(2;\mathbb{C})}}{n} = \mathrm{vol}Q_+ - \pi^2\frac{\mathcal{I}(z_0)}{q|a+z_0c|^2} + \mathcal{O}\left(\frac{1}{a^4+c^4}\right).$$

We added several details and explanations to the proof of Propostion 2.13 and 2.14. In particular we gave arguments for the vanishing of the volume of some representations and we augmented the proof of Proposition 2.14 by an explanation of how the length(γ) term comes into the formula. Furthermore we gave an explanation why the preliminary assumptions on Q_- and Q_+ in the beginning of Section 2.8 can be made.

2.1.5. Organization of the thesis and further changes. The thesis follows the structure of [**11**]. Sections 2.2, 2.3, and 2.5 provide the necessary background information. Section 2.4 contains the proof of Proposition 2.3. Section 2.5 contains the proofs of Proposition 2.9 and 2.11. Proposition 2.5 is verified in Section 2.6, Proposition 2.40 in Section 2.7 and Propositions 2.13 and 2.14 are verified in Section 2.8.

In Section 2.2 the most important results from 3-manifold theory are summarized. With respect to [**11**], we added an argument for the fact that all Seifert pieces of a non-geometric oriented prime 3-manifold have geometry $\mathbb{H}^2 \times \mathbb{R}$, since we couldn't find it in literature, and augmented the introduction to Seifert geometry, so that one can read it more independently of other sources.

In Section 2.3 we introduce three different approaches to volumes of representations. That is, we can define the volume of a representation using developing maps, or continuous cohomology, or transversely projective foliations. The latter only works for representations into $\widetilde{SL_2}(\mathbb{R})$ and is used in Section 2.4.

In Section 2.5 we recapitulate the necessary topics from gauge theory and Chern-Simons theory. There is a change in Section 2.5.2, where we prove the correspondence between the Chern-Simons classes for $SU(2;\mathbb{C})$ and $SO(3;\mathbb{R})$ by direct computation instead of using Chern-Weil theory.

2.2. 3-manifolds

2.2.1. Geometrization of 3-manifolds. Let M be an orientable compact, but not necessarily closed, 3-manifold. An embedded closed surface $\sigma \subset M$, where σ is not a 2-sphere, is called *incompressible* if every embedded disk $D \subset M$ with $D \cap \sigma = \partial D$ bounds a disk inside σ. If σ is incompressible and not parallel to any component of ∂M, then we call σ *essential*. An *incompressible 2-sphere* in M is a 2-sphere that does not bound a ball in M.

We call a 3-manifold *prime* if it can not be decomposed as a connected sum of two closed manifolds both different from $\mathbf{S}^3$ and we call a 3-manifold *irreducible* if it does not contain any incompressible 2-sphere. Irreducible and prime are almost the same. That is, every irreducible manifold is prime and the only orientable prime manifold that is not irreducible is $\mathbf{S}^2 \times \mathbf{S}^1$.

An orientable 3-manifold is called *Haken* if it is compact, irreducible and contains an incompressible surface.

The geometrization for 3-manifolds was conjectured and partially proven by W. Thurston and its proof was completed by G. Perelman. Using decomposition results of Thurston, Johansson and Jaco-Shalen it can be stated as follows:

Let M be a closed orientable 3-manifold. There is a, up to order, unique decomposition $M = M_1 \# M_2 \# \cdots \# M_k$ into a connected sum of closed orientable prime 3-manifolds $M_1, \cdots, M_k$ and each of the M_i, $i = 1, \cdots, k$ satisfies precisely one of the following:

(1) M_i supports one of the following 8 geometries: $\mathbb{H}^3$, Sol, $\mathbf{S}^2 \times \mathbb{R}$, $\mathbf{S}^3$, $\mathbb{R}^3$, Nil, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{SL_2(\mathbb{R})}$. In this case M_i is called *geometric*.

(2) There is a minimal non-empty union of essential tori $\mathcal{T}_{M_i} \subset M_i$ of M_i, unique up to isotopy, such that each connected component of $M_i \setminus \mathcal{T}_{M_i}$ is either a Seifert fibered manifold, supporting the $\mathbb{H}^2 \times \mathbb{R}$-geometry, or supports the $\mathbb{H}^3$-geometry. The components of $M_i \setminus \mathcal{T}_{M_i}$ are called geometric pieces and M_i is called *non-geometric*.

We want to give a short explanation for (2). Let M satisfy the assumptions of (2). Note that every manifold satisfying (2) is by definition compact, prime, not equal to $\mathbf{S}^2 \times \mathbf{S}^1$, and contains an incompressible torus. Thus it is Haken.

We can apply Theorem 2.6 in [**32**] and obtain that there is a finite covering $\widetilde{M} \to M$ such that $\widetilde{M}$ is either a torus-bundle over the circle, in which case it would be geometric with one of the structures $\mathbb{R}^3$, Sol, and Nil, or each connected component in the torus decomposition of $\widetilde{M}$ either has hyperbolic structure or is Seifert fibered over an orientable surface with negative Euler characterstic (see Section 2.2.4 for details on Seifert fibered spaces).

Let $\widetilde{M_i}$ be one of the connected Seifert fibered components with base surface $O(\widetilde{M_i})$ satisfying $\chi(O(\widetilde{M_i})) < 0$. It has non-empty boundary and thus by Lemma 4.2 in [**33**] there exists a characteristic covering $\widehat{M_i} \to \widetilde{M_i}$ such that $\widehat{M_i}$ is the product of a surface with a circle. It is known that a surface with negative Euler characteristic can always be endowed with a smooth hyperbolic metric which we obtain by applying pants decomposition and equipping each pant with a hyperbolic metric [**7**, p.111] Thus $\widehat{M_i}$ and consequently also M_i have geometry $\mathbb{H}^2 \times \mathbb{R}$.

A prime closed orientable 3-manifold M is called a (non-trivial) *graph manifold* if it is non-geometric and each of its geometric pieces is Seifert.

2.2.2. Thurston Hyperbolic Dehn filling Theorem. Denote by $\mathbb{H}^n$ the n-dimensional hyperbolic space. It can be seen as the upper half-space $\mathbb{R}^{n-1} \times (0, \infty)$ endowed with the metric $\frac{dx_1^2+dx_2^2+\cdots+dx_n^2}{x_n^2}$. Its group of orientation preserving isometries $Iso_+(\mathbb{H}^n)$ can be viewed as the unit tangent bundle of $\mathbb{H}^n$ and can be identified with $PSL(2;\mathbb{R})$ for $n = 2$ and with $PSL(2;\mathbb{C})$ for n=3.

Let M be a compact orientable 3-manifold with toral boundary $\partial M = T_1 \cup \cdots \cup T_l$ whose interior admits a complete (finite volume) hyperbolic structure. That is, int$M$ is isometric to $\mathbb{H}^3/\Gamma$ where $\Gamma \subset PSL(2;\mathbb{C})$ is a discrete, torsion-free subgroup. Fix a basis (μ_i, λ_i) for $H_1(T_i;\mathbb{Z})$, $i = 1, \cdots, l$ and let $(a_1, b_1), \cdots, (a_l, b_l) \in \mathbb{Z}^2 \cup \{\infty\}$. We define the surgered manifold $M((a_1, b_1), \cdots, (a_l, b_l))$ as follows:

(1) If $(a_i, b_i) = \infty$ then T_i is left unfilled.
(2) If $\gcd(a_i, b_i) = 1$, let $V = D^2 \times \mathbf{S}^1$ be the solid torus and (m, l) be a meridian/longitude basis of $H_1(\partial V = \partial D^2 \times \mathbf{S}^1;\mathbb{Z})$. We glue V into M by identifying ∂V with T_i such that the isotopy class of the simple closed curve $a_i\mu_i + b_i\lambda_i$ is identified with m. This uniquely determines the resulting manifold.
(3) If $\gcd(a_i, b_i) = q_i$ then let $a_i = q_i r_i$ and $b_i = q_i s_i$ with (r_i, s_i) coprime. Denote by $\mathcal{R}_{q_i}$ the rotation of the unit disk D^2 by $2\pi/q_i$ and let V_{q_i} be the orbifold $V_{q_i} = D^2/\mathcal{R}_{q_i} \times \mathbf{S}^1$. Glue V_{q_i} into M by identifying the isotopy class of the simple closed curve $a_i\mu_i + b_i\lambda_i$ with m. Note that in this case $M((a_1, b_1), \cdots, (a_l, b_l))$ is an orbifold.

Thurston's Hyperbolic Dehn surgery theorem (see [**43**, 5.8.2] and also [**4**] and [**13**]) gives a criterion for the resulting orbifold to be hyperbolic:

THEOREM 2.16. *Let M be a compact oriented hyperbolic 3-manifold with toral boundary $\partial M = T_1 \cup \cdots \cup T_l$, whose interior admits a complete hyperbolic structure. Then there exists $C > 0$ such that for $\sqrt{a_i^2 + b_i^2} > C$, $i = 1, \cdots, l$, $M((a_1, b_1), \cdots, (a_l, b_l))$ is a complete hyperbolic orbifold.*

More explicitly [**43**, Chapter 5] tells us that the complete hyperbolic metric on M corresponds to a faithful, discrete representation $\rho : \pi_1 M \to PSL(2;\mathbb{C})$, such that, up to conjugation, $\rho(\mu_i) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $\rho(\lambda_i) = \begin{pmatrix} 1 & \eta_i \\ 0 & 1 \end{pmatrix}$.

If (a_i, b_i) satisfies $\sqrt{a_i^2 + b_i^2} > C$, for $i = 1, \cdots, p$, then the representation can be modified to a representation ρ_d so that, up to conjugation,

$\rho_d(\mu_i) = \begin{pmatrix} e^{2\pi i \alpha_i} & 0 \\ 0 & e^{-2\pi i \alpha_i} \end{pmatrix}$ and $\rho_d(\lambda_i) = \begin{pmatrix} e^{2\pi i \beta_i} & 0 \\ 0 & e^{-2\pi i \beta_i} \end{pmatrix}$ and the induced structure on int$M$ is a non-complete hyperbolic metric that can be completed in the surgered hyperbolic orbifold $M((a_1, b_1), \cdots, (a_p, b_p))$.

Furthermore Thurston's Theorem shows that, if there exists i such that $(a_i, b_i) \neq \infty$, then

$$\mathrm{vol}_{\mathbb{H}^3} M((a_1, b_1), \cdots, (a_p, b_p)) < \mathrm{vol}_{\mathbb{H}^3} M$$

and

$$\lim_{(a_1,b_1)\to\infty, \cdots, (a_p,b_p)\to\infty} \mathrm{vol}_{\mathbb{H}^3} M((a_1, b_1), \cdots, (a_p, b_p)) = \mathrm{vol}_{\mathbb{H}^3} M.$$

REMARK 2.17. For $\epsilon > 0$ we can decompose the interior M° of M as $M^\circ = M_{(0,\epsilon]} \cup M_{[\epsilon,\infty)}$, where $M_{(0,\epsilon]}$, respectively $M_{[\epsilon,\infty)}$, are all points $v \in M$ for which the shortest non-contractible loop at v has length smaller than or equal to ϵ, respectively larger than or equal to ϵ. For small ϵ, the set $M_{(0,\epsilon]}$ consists precisely of the cusps of M (see [**43**, Chapters 5, 6] and also [**19**]).

A *horoball* in hyperbolic 3-space $\mathbb{H}^3$, viewed as the upper half-space in $\mathbb{R}^3$, is a subset of $\mathbb{H}^3$ that is either a plane parallel to the $x-y$-plane or a ball that lies in the upper half plane and whose boundary touches the $x-y$-plane. Identify the universal covering of the interior of M with hyperbolic 3-space $\mathbb{H}^3$. Then the preimage of every cusp corresponds to an infinite union of disjoint horoballs in $\mathbb{H}^3$ and thus in particular for small $\epsilon > 0$ the preimage of $M_{(0,\epsilon]}$ consists of an infinite union of disjoint horoballs. Now we increase ϵ and consider the preimage of the subset of $M_{(0,\epsilon]}$ consisting of all cusps of M in $\mathbb{H}^3$. There is a minimal $\epsilon > 0$ for which first two horoballs become tangent in $\mathbb{H}^3$. The cusps corresponding to the horoballs that first became tangent in M are called *maximal cusps* [**3**, Theorem 11].

Denote by M_{max} the interior of M with a system of maximal cusps removed, identify M with M_{max} and equip M with the induced metric. Then ∂M has a Euclidean metric induced from the hyperbolic metric and each closed Euclidean geodesic in ∂M has the induced length. By the 2π-Lemma (see [**3**]) $M((a_1, b_1), \cdots, (a_p, b_p))$ is hyperbolic if the geodesic corresponding to (a_i, b_i) has length $> 2\pi$ for $i = 1, \cdots, p$. The first and second shortest geodesic on each component T_i of ∂M must form a basis and under this basis the norm of a curve (a, b) defined by $\sqrt{a^2 + b^2}$ is equivalent to the Euclidean norm in T_i (see [**21**, p.309]). Thus under such a basis there is a universal constant $C > 0$ such that, for any one cusped hyperbolic manifold, if $\sqrt{a_i^2 + b_i^2} > C$ then $M(a_i, b_i)$ is hyperbolic.

2.2.3. The Geometry $\widetilde{SL_2}\mathbb{R}$. In this section we give an overview of the most important properties of the Lie group $\widetilde{SL_2}\mathbb{R}$. It is based on [**41**] and [**10**].

It is well-known that the group of orientation preserving isometries of $\mathbb{H}^2$, viewed as the upper-half plane, can be identified with $PSL_2(\mathbb{R})$. Using this identification we obtain left-invariant metric on $PSL_2(\mathbb{R})$:

Consider the tangent bundle $T\mathbb{H}^2$ of $\mathbb{H}^2$ with projection $\pi : T\mathbb{H}^2 \to \mathbb{H}^2$. The metric g on $\mathbb{H}^2$ endows it with a natural metric $\widehat{g}$. Given curves α, β in

$T\mathbb{H}^2$ with $\alpha(0) = \beta(0)$, it is defined by

$$\widehat{g}(\dot{\alpha}(0), \dot{\beta}(0)) := g(d\pi(\dot{\alpha}(0)), d\pi(\dot{\beta}(0))) + g(\frac{D\alpha}{dt}(0), \frac{D\beta}{dt}(0)),$$

where $\frac{D}{dt}$ is the covariant derivative. This construction works for arbitrary Riemannian manifolds. A more explicit construction for $\mathbb{H}^2$ can be found in [**41**]. From now on $T\mathbb{H}^2$ is always equipped with the metric $\widehat{g}$.

Let $f : \mathbb{H}^2 \to \mathbb{H}^2$ be an isometry. Then $df : T\mathbb{H}^2 \to T\mathbb{H}^2$ is also an isometry. Thus df restricts to a well-defined isometry $df : UT\mathbb{H}^2 \to UT\mathbb{H}^2$, where $UT\mathbb{H}^2$ is the unit tangent bundle of $T\mathbb{H}^2$. It is shown in [**41**] that the left action of $PSL_2(\mathbb{R})$ on $UT\mathbb{H}^2$ is simply transitive and since it is by isometries, it induces a left invariant metric on $PSL_2(\mathbb{R})$ and consequently also on $\widetilde{SL_2(\mathbb{R})}$. There are two immediate consequences:

(1) $\widetilde{SL_2(\mathbb{R})}$ is a subgroup of the group $Iso_e\widetilde{SL_2(\mathbb{R})}$ (the identity component of the group of isometries of $\widetilde{SL_2(\mathbb{R})}$)
(2) the circle bundle structure of $PSL_2(\mathbb{R})$ endows $\widetilde{SL_2(\mathbb{R})}$ with a line bundle structure

Furthermore $\widetilde{SL_2(\mathbb{R})}$ is not isometric to the trivial bundle $\mathbb{H}^2 \times \mathbb{R}$, as one can see by considering an appropiate distribution (cf. [**41**]).

A better imagination of $PSL_2(\mathbb{R})$ is obtained by identifying it with a solid torus $V = D^2 \times S^1$ through an isomorphism of Lie groups, that is:

$$SL_2(\mathbb{R}) = \{(\gamma, \omega) | |\gamma| < 1, -\pi \le \omega < \pi\}$$

The multiplication structure is rather complicated and we don't want to write it down explicitly here. A more detailed description of the identification can be found in [**26**]. From that it becomes clear that $\widetilde{SL_2(\mathbb{R})} = D^2 \times \mathbb{R}$ as topological space, where the group structure is again more complicated.

Fix the identity $\mathbb{1} \in PSL_2(\mathbb{R})$ as base-point for the construction of the universal covering and denote by $sh(\alpha)$ the element of $\widetilde{SL_2(\mathbb{R})}$ that corresponds to the homotopy class of the path $\gamma : [0,1] \to PSL_2(\mathbb{R})$, $t \mapsto \begin{pmatrix} cos(t\pi\alpha) & sin(t\pi\alpha) \\ -sin(t\pi\alpha) & cos(t\pi\alpha) \end{pmatrix}$ in $PSL(2,\mathbb{R})$.

It is well-known that the center of $PSL_2(\mathbb{R})$ is given by $\mathbb{1}$ and the center of $\widetilde{SL_2(\mathbb{R})}$ is thus given by the elements that project to $\mathbb{1}$, i.e. $\{sh(n) | n \in \mathbb{Z}\}$.

Next we want to give a brief discussion of the Lie group $Iso_e\widetilde{SL_2(\mathbb{R})}$, that is, of the identity component of the isometry group of $\widetilde{SL_2(\mathbb{R})}$. We obtain it by extending the $\mathbb{Z}$-action of $sh(n)$ on the lines of $\widetilde{SL_2(\mathbb{R})}$ to an $\mathbb{R}$-action. Such an extension exists and is described in details in [**41**]. It is constructed by considering the S^1-action on $UT\mathbb{H}^2$ which is defined by rotating each fiber by a fixed angle. It covers the identity on $\mathbb{H}^2$ and lifts to an $\mathbb{R}$-action by isometries on $\widetilde{SL_2(\mathbb{R})}$, given by translation of lines.

The $\mathbb{R}$-action commutes with the $\widetilde{SL_2(\mathbb{R})}$-action on $\widetilde{SL_2(\mathbb{R})}$ and their intersection as subgroups of $Iso_e\widetilde{SL_2(\mathbb{R})}$ is $\{sh(n) | n \in \mathbb{Z}\}$. It is proved in

[**41**] that they generate $Iso_e\widetilde{SL_2(\mathbb{R})}$, which gives us the following useful description:

REMARK 2.18. We obtain $Iso_e\widetilde{SL_2(\mathbb{R})}$ by taking the quotient of $\mathbb{R} \times \widetilde{SL_2(\mathbb{R})}$ under the equivalence relation $\sim$, defined by: $(x, A) \sim (x', A')$ iff there exists $n \in \mathbb{Z}$ with $x' - x = n$ and $A' = sh(-n) \circ A$. We denote this quotient by $\mathbb{R} \times_{\mathbb{Z}} \widetilde{SL_2(\mathbb{R})}$.

2.2.4. Seifert fibered spaces. The geometry $\widetilde{SL_2(\mathbb{R})}$ is a specific example of the more general concept of a Seifert fibered space. A *Seifert fibered space* is a space which is fibered by circles with only finitely many exceptional fibers. That is, all but finitely many fibers admit neighbourhoods such that there is a fiber-preserving homeomorphism to $D^2 \times S^1$. We don't want to go into the details of the exceptional fibers here, but rather give a construction from which we obtain all orientable Seifert fibered spaces.

Let $F_{g,n}$ be an oriented n-punctured surface of genus $g \geq 0$. Denote the boundary components by $s_1, \cdots, s_n$. An orientation on S^1 induces an orientation on $N' = F_{g,n} \times S^1$. Let h_i be the oriented S^1-fiber on the torus $T_i = s_i \times h_i$. Let $0 \leq s \leq n$. Attach s solid tori V_i along the boundary components $T_1, \cdots, T_s$ of N' such that the meridian of V_i is identified with the slope $a_i s_i + b_i h_i$, where $a_i > 0$ and $\gcd(a_i, b_i) = 1$ for $i = 1, \cdots, s$. Denote the resulting manifold by $\left(g, n-s; \frac{b_1}{a_1}, \cdots, \frac{b_s}{a_s}\right)$. Each orientable Seifert fibered space N with orientable base $F_{g,n-s}$ and $\leq s$ exceptional fibers is obtained in that way. The base orbifold of N is the orbifold obtained as the quotient space $O(N) = N/S^1$.

Six of the eight geometries of 3-manifolds are Seifert fibered spaces. Let N be a closed oriented Seifert fibered space, i.e. $s = n$. Then the geometry of N is uniquely determined by two real numbers.

The *Euler number* of the Seifert fibration

$$e(N) = \sum_{i=1}^{s} \frac{b_i}{a_i} \in \mathbb{Q}$$

and the *Euler characteristic* of the base orbifold O(N)

$$\chi_{O(N)} = 2 - 2g - \sum_{i=1}^{s} \left(1 - \frac{1}{a_i}\right) \in \mathbb{Q}.$$

In terms of these two numbers, the geometry of a closed oriented Seifert fibered space is given by the following table(cf. [**41**]):

$e \backslash \chi$	> 0	$= 0$	< 0
$= 0$	$S^2 \times \mathbb{R}$	$\mathbb{R}^3$	$H^2 \times \mathbb{R}$
$\neq 0$	S^3	Nil	$\widetilde{SL_2(\mathbb{R})}$

2.3. Volumes of representations

Let G be a semi-simple, connected Lie group, K a maximal compact subgroup and M^n a closed orientable manifold of dimension n equal to the dimension of G/K. Let $\rho : \pi_1 M^n \to G$ be a representation, that is, a group homomorphism. Then we can assign a volume to ρ which we denote by $vol_G(M, \rho)$ and which will be explained in the following. If it is clear from the context which Lie group G is meant we sometimes simply write $vol(M, \rho)$.

There are different ways in which the volume of a representation can be defined which eventually turn out to be the same.

2.3.1. Developing maps. The first way to define the volume of a representation is by using the notion of a developing map, but before saying what a developing map is we need to fix a few notations. Recall that the group G acts on X by left-multiplication. With respect to this action we choose a G-invariant metric g_X on X and denote by ω_X the corresponding G-invariant volume form on X.

Note that also $\pi_1 M$ acts on X through the representation ρ and ω_X is invariant under the $\pi_1 M$-action. Let $\widetilde{M}$ be the universal cover of M and think of a point $\widetilde{x}$ in $\widetilde{M}$ as a homotopy class of paths in M starting at a fixed base-point $x_0 \in M$. Then $\pi_1(M, x_0)$ also acts on $\widetilde{M}$ by concatenation of paths. That is, for $[\gamma] \in \pi_1 M$, and $\widetilde{x} = [c]$ an equivalence class of paths: $[\gamma] . \widetilde{x} = [\gamma . c]$.

Now a map

$$D_\rho : \widetilde{M} \to X$$

is called a *developing map* if it is equivariant under the $\pi_1 M$-action:

$$D_\rho([\gamma] . \widetilde{x}) = \rho(\gamma)^{-1} D_\rho(\widetilde{x}) \ \forall \widetilde{x} \in \widetilde{M}, [\gamma] \in \pi_1 M.$$

The existence of a developing map is proved by explicit construction in [**2**]. For a better understanding of D_ρ we want to sketch the construction here:

Fix a triangulation Δ_M of M and lift it to a triangulation $\Delta_{\widetilde{M}}$ of $\widetilde{M}$ which is clearly $\pi_1 M$-invariant. Choose a fundamental domain Ω in $\widetilde{M}$ whose boundary does not intersect the zero-skeleton $\Delta^0_{\widetilde{M}}$ of the triangulation $\Delta_{\widetilde{M}}$ and let $\{x_1, \cdots, x_l\} = \Delta^0_{\widetilde{M}} \cap \Omega$. Let furthermore $y_1, \cdots, y_l$ be any collection of l pairwise distinct points in X and set

$$D_\rho(x_i) = y_i, i = 1, \cdots, l.$$

There is a unique extension of D_ρ to $\Delta^0_{\widetilde{M}}$ by $\pi_1 M$-equivariance and finally we extend D_ρ to $\Delta_{\widetilde{M}}$ by extending first to edges, then to faces, and so on, by straightening the images to geodesics using the homogeneous metric on the contractible space X.

The map D_ρ is unique up to equivariant homotopy. Thus $D_\rho^*(\omega_X)$ is a well-defined $\pi_1 M$-invariant closed n-form on $\widetilde{M}$ and can therefore be considered as a closed n-form on M. In particular the cohomology class of $D_\rho^*(\omega_X)$

is independent of the choice of D_ρ and thus we define:

$$vol_G(M,\rho) = \left| \int_M D_\rho^*(\omega_X) \right| = \left| \sum_{i=1}^{s} \epsilon_i vol_X(D_\rho(\widetilde{\Delta_i})) \right|$$

where $\Delta_1, \cdots, \Delta_s$ are the n-simplices of the triangulation Δ_M, the $\widetilde{\Delta}_i$ are lifts of the Δ_i and $\epsilon_i = \pm 1$, depending on whether $D_\rho|_{\widetilde{\Delta_i}}$ is orientation preserving or reversing.

2.3.2. Continuous cohomology. The second way to approach the volume of a representation is via continuous cohomology. Let $o = \{K\}$ denote the base point of G/K, and denote for $\gamma_1, \cdots, \gamma_l \in G$ by $\Delta(\gamma_1, \cdots, \gamma_l)$ the geodesic l-simplex spanned by the vertices $o, \gamma_1 o, \cdots \gamma_l \cdots \gamma_2 \gamma_1 o$.

Since it is not necessary and would complicate things, we don't want to go into the details of continuous cohomology and just state an important result. For further background see [**45**] and [**15**].

THEOREM 2.19. *There is a natural isomorphism*

$$H^*(\mathfrak{g}, \mathfrak{k}; \mathbb{R}) = H^*(G\text{-invariant forms on } X) \to H^*_{cont}(G; \mathbb{R})$$

defined by

$$\eta \mapsto \left((\gamma_1, \cdots, \gamma_l) \mapsto \int_{\Delta(\gamma_1, \cdots, \gamma_l)} \eta \right).$$

The proof of the existence of a natural homomorphism can be found in [**15**] and it is an isomorphism by the Van-Est Theorem which can be found in [**45**].

We want to recall that to a representation $\rho : \pi_1 M \to G$ we can associate a flat X-bundle $\widetilde{M} \times_\rho X$ with structure group G over M as follows: The group $\pi_1 M$ acts on $\widetilde{M} \times X$ by $g.(\widetilde{x}, h) = (g.\widetilde{x}, \rho(g)^{-1} h)$. Define $\widetilde{M} \times_\rho X$ as the quotient $\widetilde{M} \times X / \pi_1 M$.

Let $q : \widetilde{M} \times X \to X$ be the natural projection and $s : M \to \widetilde{M} \times_\rho X$ be a section. Note that by contractability of X a section always exists and all sections are homotopic. Let ω be a closed G-invariant form on X. Then $q^*(\omega)$ is a closed $\pi_1 M$-invariant form on $\widetilde{M} \times X$ and thus induces a closed form ω' on $\widetilde{M} \times_\rho X$ and ω' induces a closed form $s^*\omega'$ on M. Alltogether that gives us a natural homomorphism

$$\rho^* : H^*_{cont}(G; \mathbb{R}) = H^*(G\text{-invariant differential forms on } X) \to H^*(M; \mathbb{R}),$$

induced by $\rho^*(\omega) = s^*(\omega')$.

The volume of the representation ρ is then defined by

$$vol_G(M,\rho) = \left| \int_M \rho^*(\omega_X) \right|.$$

The two definitions are equivalent. To see the equivalence, observe that the map $Id \times D_\rho : \widetilde{M} \to \widetilde{M} \times X$ is $\pi_1 M$-equivariant and thus descends to a section $M \to \widetilde{M} \times_\rho X$.

2.3.3. Volumes of representations by transverse foliations. Let M be a closed smooth manifold and $\mathcal{F}$ a codimension one foliation defined as the kernel of some 1-form ω. By the Frobenius theorem there is a 1-form δ with $d\omega = \omega \wedge \delta$. Godbillon and Vey proved in [**18**] that $\delta \wedge d\delta$ is closed and the cohomology class $GV(\mathcal{F}) = [\delta \wedge d\delta] \in H^3(M;\mathbb{R})$ is independent of the choice of ω. The class $GV(\mathcal{F})$ is called the *Godbillon-Vey class* of $\mathcal{F}$.

Let now M be a closed oriented 3-manifold. In [**5**] Brooks and Goldman related the Godbillon-Vey invariant to the volume of a representation $\widetilde{\phi} : \pi_1 M \to \widetilde{SL_2(\mathbb{R})}$.

The representation $\widetilde{\phi}$ induces a representation $\phi : \pi_1 M \to PSL_2(\mathbb{R})$ with Euler class zero and conversely. $PSL_2(\mathbb{R})$ acts on $\mathbf{S}^1$ and thus we can consider the associated bundle $M \times_\phi \mathbf{S}^1$ over M on which ϕ defines a transverse $(PSL_2(\mathbb{R}), \mathbf{S}^1)$-foliation $\mathcal{F}_\phi$. Since ϕ has Euler class zero the bundle admits a section δ. Brooks and Goldman observed in [**5**], Lemma 2, that $\delta^* GV(\mathcal{F}_\phi)$ depends only on ϕ and not on the choice of section δ and defined the *Godbillon-Vey invariant* of ϕ by

$$GV(\phi) = \int_M \delta^* GV(\mathcal{F}_\phi).$$

Assume that $\mathcal{F}$ defines a $(PSL_2(\mathbb{R}), \mathbf{S}^1)$-transverse foliation on M. Then there is a canonical way to define a circle bundle $E \to M$ with structure group $PSL_2(\mathbb{R})$, a transverse foliation $\mathcal{F}_\phi$, and a section $\delta : M \to E$ such that $\mathcal{F} = \delta^* \mathcal{F}_\phi$. Here ϕ denotes the associated representation. It was proved in [**5**], Lemma 1, that

$$GV(\phi) = \int_M GV(\mathcal{F}).$$

The following relation between the volume of a representation and the Godbillon-Vey invariant has been verified in [**5**].

PROPOSITION 2.20. *Let M be a closed oriented 3-manifold, $\phi : \pi_1 M \to PSL_2(\mathbb{R})$ a representation with zero Euler class and $\widetilde{\phi} : \pi_1 M \to \widetilde{SL_2(\mathbb{R})}$ a lift of ϕ. Then*

$$GV(\phi) = vol_{\widetilde{SL_2(\mathbb{R})}}(M, \widetilde{\phi})$$

where $\widetilde{SL_2(\mathbb{R})}$ is viewed as a semi-simple Lie group acting on itself by multiplication with the corresponding homogeneous space $\widetilde{SL_2(\mathbb{R})}$.

Another important result from [**5**] relates the Godbillon-Vey class of a transverse foliation of a circle bundle to its Euler class.

PROPOSITION 2.21. *Let $\mathbf{S}^1 \to E \to M$ be a circle bundle with structure group $PSL_2(\mathbb{R})$ and $\mathcal{F}$ a transverse foliation. Then*

$$\int_{\mathbf{S}^1} GV(\mathcal{F}) = 4\pi^2 e(E)$$

where $\int_{\mathbf{S}^1} : H^3(E) \to H^2(M)$ denotes integration along the fiber $\mathbf{S}^1$ and $e(E)$ denotes the Euler class of the bundle E.

2.4. Proof of Proposition 2.3

In the following let N be a closed oriented $\widetilde{SL_2(\mathbb{R})}$-manifold whose base 2-orbifold is an orientable hyperbolic 2-orbifold $\mathcal{O}$ with positive genus g and p singular points. Then we have a presentation

$$\pi_1 N = \left\langle \begin{array}{l} \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \\ s_1, \ldots, s_p, h \end{array} \,\middle|\, \begin{array}{l} [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = s_1 \cdots s_p, \\ s_1^{a_1} h^{b_1} = \cdots = s_p^{a_p} h^{b_p} = 1, [h, *] = 1 \end{array} \right\rangle$$

with the condition $e = \sum_i \frac{b_i}{a_i} \neq 0$, where h is the fiber.

The following result of Milnor [**34**] is needed to apply the results from Section 2.3.3.

THEOREM 2.22. *Let $\mathbf{S}^1 \to M \to S$ be an oriented circle bundle over a surface S with structure group $PSL_2(\mathbb{R})$. Then the following are equivalent:*

(1) The structure group of M can be reduced to a totally disconnected structure group
(2) $|e(M)| \leq |\chi(S)|$ for $\chi(M) \leq 0$ and $e(M) = 0$ for $\chi(M) \geq 0$.
(3) M is induced by a representation $\phi : \pi_1 S \to PSL_2(\mathbb{R})$ whose extension to $\pi_1 M$ admits a lift $\widetilde{\phi} : \pi_1 M \to \widetilde{SL_2(\mathbb{R})}$ with $\widetilde{\phi}(h) = sh(1)$.
(4) There is a transverse foliation of M.

REMARK 2.23. The Theorem was generalized by Wood [**48**] to topological groups G with $PSL_2(\mathbb{R}) \subset G \subset \mathrm{Top}_2\mathbf{S}^1/\{\pm\mathbb{1}\}$, where $Top_2\mathbf{S}^1$ is the group of homeomorphisms of $\mathbf{S}^1$ that commute with the antipodal map. For the generalized version replace $\widetilde{SL_2(\mathbb{R})}$ by the universal covering $\widetilde{G}$ of G. The foliation is smooth if G is a subgroup of the group of diffeomorphisms of $\mathbf{S}^1$ that commute with the antipodal map modulo $\pm\mathbb{1}$.

It follows immediately from the proof of Theorem 1.1 and 1.2 in [**48**] that the representations inducing M are precisely the ones satisfying the properties in (3).

LEMMA 2.24. *Let $g_1, \cdots, g_s$ denote elements in $\widetilde{SL_2(\mathbb{R})}$ such that each g_i is conjugated to an element of the form $sh(\alpha_i)$, for some $\alpha_i \in \mathbb{R}$. If the product $g_1 \cdots g_s$ can be represented as a product of $g > 0$ commutators $\prod_{i=1}^{g} [v_i, w_i]$, then*

$$\sum_{i=1}^{s} \lfloor \alpha_i \rfloor \leq 2g - 2 \text{ and } \sum_{i=1}^{s} \lceil \alpha_i \rceil \geq 2 - 2g. \tag{2.4.1}$$

Conversely, if inequality 2.4.1 holds, then there exist conjugates $\widetilde{g}_i$ of the g_i, $i = 1, \cdots s$ and $v_1, w_1, \cdots, v_g, w_g \in \widetilde{SL_2(\mathbb{R})}$ such that

$$\widetilde{g}_1 \cdots \widetilde{g}_s = \prod_{i=1}^{g} [v_i, w_i] \,.$$

PROOF. The Lemma is a consequence of Lemma 2.1, Theorem 2.3, Theorem 2.5, and Theorem 4.1 in [**16**]. The proof is included in the proof of Theorem 3.2 in [**16**]. □

PROPOSITION 2.25. *If a representation* $\rho : \pi_1(N) \to Iso_e\widetilde{SL_2(\mathbb{R})} = \mathbb{R} \times_{\mathbb{Z}} \widetilde{SL_2(\mathbb{R})}$ *has non-zero volume, then there are integers* $n, n_1, \ldots, n_p$ *subject to the conditions*

$$\sum \lfloor n_i/a_i \rfloor - n \leq 2g-2 \text{ and } \sum \lceil n_i/a_i \rceil - n \geq 2-2g \tag{2.4.2}$$

such that

$$\rho(s_i) = \overline{\left(\frac{n_i}{a_i} - \frac{b_i}{a_i}\frac{1}{e}\left(\sum_i \left(\frac{n_i}{a_i}\right) - n\right), g_i sh\left(\frac{-n_i}{a_i}\right) g_i^{-1}\right)} \tag{2.4.3}$$

where $g_i \in \widetilde{SL_2(\mathbb{R})}$ *and*

$$\rho(h) = \overline{\left(\frac{1}{e}\left(\sum_i \left(\frac{n_i}{a_i}\right) - n\right), 1\right)} \tag{2.4.4}$$

The volume of ρ *is given by*

$$\mathrm{vol}(N, \rho) = 4\pi^2 \frac{1}{|e|}\left(\sum_i \left(\frac{n_i}{a_i}\right) - n\right)^2. \tag{2.4.5}$$

Conversely, if there are integers satisfying the ineqalities 2.4.2 then there is a representation ρ *satisfying 2.4.3 and 2.4.4 with volume given by 2.4.5.*

Moreover the ρ*-image of* $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ *can be chosen to lie in* $\widetilde{SL_2(\mathbb{R})} = \left\{\overline{(0,x)} | x \in \widetilde{SL_2(\mathbb{R})}\right\}$.

For the proof of the proposition we use the following Lemma

LEMMA 2.26. *Let* N *be a compact manifold and* $\rho : \pi_1 N \times [0,1] \to Iso_e\widetilde{SL_2(\mathbb{R})}$ *be a smooth family of representations. Denote by* ρ_t *the restriction of* ρ *to* $\pi_1 N \times \{t\}$*. Then* $vol(N, \rho_t)$ *also depends smoothly on* t*.*

PROOF. Consider the universal covering $\pi : \widetilde{N} \to N$. $\widetilde{N}$ is a principal $\pi_1 N$-bundle. Define a $\pi_1 N$-left action on $X \times [0,1]$, with $X = Iso_e\widetilde{SL_2(\mathbb{R})}/K$, where $K \cong S^1$ is a maximal compact subgroup, by:

$$\begin{array}{rccc} \mu : & \pi_1 N \times (X \times [0,1]) & \to & X \times [0,1] \\ & (\bar{\gamma}, (x,t)) & \mapsto & (\rho_t(\bar{\gamma}) \cdot x, t) \end{array}$$

Taking the associated bundle with respect to this action yields a smooth fiber bundle Q with fiber $X \times [0,1]$ over N:

Let $Q = \widetilde{N} \times X \times [0,1] / \equiv$, where $(p,x,t) \equiv (\bar{\gamma} \cdot p, \rho_t(\bar{\gamma})^{-1} \cdot x, t)$ for $\bar{\gamma} \in \pi_1 N$. Then the $X \times [0,1]$-bundle over N is defined by $\pi_Q : Q \to N$ with $\pi_Q([(p,x,t)]) = \pi(p)$.

Since the μ-action leaves $X \times \{t\}$ invariant, we can in fact view Q as a smooth fiber bundle over $N \times [0,1]$ with projection map $\widehat{\pi}_Q : Q \to N \times [0,1]$, $[(p,x,t)] \mapsto (\pi(p), t)$.

Obviously the restriction Q_t of Q to $\widehat{\pi}_Q^{-1}(N \times \{t\})$ is precisely the fiber bundle associated to the representation ρ_t, with fiber X over N, where $t \in [0,1]$.

As in Section 2.2 we can now map a G-invariant closed form ω on X to a closed form $s^*(\omega')$ on $N \times [0,1]$ as described follows:

First take its pull-back form $q^*(\omega)$ under $q : \widetilde{N} \times X \to X$ and observe that it is $\pi_1 N$-invariant. Then take the closed form ω' induced by the projection onto Q. Finally take a section $s : N \times [0,1] \to Q$, which exists since X is contractible and consider the pull-back $s^*(\omega')$ which is again smooth and closed.

Clearly that restriction to $N \times \{t\}$ coincides with the form that we obtain from ω, when applying the same construction with respect to the restricted bundle Q_t, yielding the form $\omega'_t = \omega'|_{Q_t}$ and then using the restricted section $s_t := s|_{N\times\{t\}}$, yielding the form $s_t^*(\omega'_t)$. Thus $\alpha_t(\omega) := s_t^*(\omega_t)')$ varies smoothly in t.

On the other hand the cohomology class of $s_t^*(\omega'_t)$ by construction coincides with $\rho_t^*\omega$. Hence for a given G-invariant form ω on X the family $\{\rho_t^*\omega\}_{t\in[0,1]}$ of cohomology classes admits a family of representatives $\alpha_t(\omega)$ varying smoothly in t. Now let ω_X be the G-invariant volume form on X. Then, by compactness of M,

$$vol_G(N, \rho_t) = \left| \int_N \rho_t^*(\omega_X) \right| = \left| \int_M \alpha_t(\omega_X) \right|$$

also varies smoothly in t. That completes the proof of the Lemma. □

Proof of Proposition 2.25. First assume that we are given a representation ρ with non-zero volume. This implies that $\rho(h)$ is in the center of G, i.e. $\rho(h) = \overline{(\zeta, 1)} \in G = \mathbb{R} \times_{\mathbb{Z}} \widetilde{SL_2(\mathbb{R})}$. The argument for this is almost the same as the one given in [**5**] on page 663: Suppose $\rho(h) \neq \overline{(\zeta, 1)}$ for any $\zeta \in \mathbb{R}$. Since h is an element in the center of G, $\rho(\pi_1 N)$ must lie in the centralizer C of $\rho(h)$. Hence the map $\rho^* : H^3_{cont}(\widetilde{SL_2(\mathbb{R})}) \to H^3(N)$ factors through $H^3_{cont}(C) = 0$, where the latter vanishes since C is two-dimensional. Two-dimensionality of C is proved in Section 3.2.1. But, since $\mathrm{vol}(N, \rho) = \left|\int_M \rho^*\omega_X\right|$, for ω_X the volume form on $X = \widetilde{SL_2(\mathbb{R})}$, it is immediate that $\mathrm{vol}(N, \rho) = 0$, a contradiction.

Now let $\rho(s_i) = \overline{(z_i, x_i)}$. Then $s_i^{a_i} h^{b_i} = 1$ implies that:

$$\overline{(a_i z_i, x_i^{a_i})(b_i\zeta, 1)} = \overline{(a_i z_i + b_i\zeta, x_i^{a_i})} = 1$$

and therefore there is an $n_i \in \mathbb{Z}$ such that:

$$a_i z_i + b_i\zeta = n_i \text{ in } \mathbb{R} \text{ and } x_i \text{ is conjugate in } \widetilde{SL_2(\mathbb{R})} \text{ to } sh\left(-\frac{n_i}{a_i}\right) \quad (2.4.6)$$

Since the composition in the first factor of G is commutative we obtain that every product of commutators in G must lie in $\widetilde{SL_2(\mathbb{R})}$ and hence $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = s_1 \cdots s_p$ implies that:

$$\overline{(z_1 + \cdots + z_p, x_1 \dots x_p)} = \overline{\left(0, \prod_{j=1}^{g} [\rho(\alpha_j), \rho(\beta_j)]\right)}.$$

Consequently there is an $n \in \mathbb{Z}$ such that

$$z_1 + \cdots + z_p = n \text{ and } \prod_{j=1}^{g} [\rho(\alpha_j), \rho(\beta_j)] = sh(n) x_1 \dots x_p. \quad (2.4.7)$$

Observe that after possibly changing the representatives (z_i, x_i) for the $\rho(s_i)$ we may assume that $n = 0$. Applying 2.1, 2.3, and 2.4 of [**16**] yields condition 2.4.2.

Solving 2.4.6 and 2.4.7 for n_i and ζ we obtain

$$z_i = \frac{n_i}{a_i} - \frac{b_i}{a_i}\zeta, \zeta = \frac{1}{e}\left(\sum_{i=1}^{p}\frac{n_i}{a_i} - n\right), \tag{2.4.8}$$

which implies equations 2.4.3 and 2.4.4.

The statement, that for a given representation $\rho : \pi_1 N \to \mathbb{R} \times_{\mathbb{Z}} \widetilde{SL_2(\mathbb{R})}$ of non-zero volume we can always assume that the image of $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n$ lies in $\widetilde{SL_2\mathbb{R}}$, is a direct consequence of Lemma 2.26 and the structure of the fundamental group of a Seifert manifold.

Indeed, as we see from the structure of the fundamental group and the already proven fact that $\rho(h)$ is an element of the center, the only obstructions on the choice of $\rho(\alpha_1), \cdots, \rho(\alpha_n), \rho(\beta_1), \cdots, \rho(\beta_n)$ are given by a commutator relation, which does not impose any restrictions on the $\mathbb{R}$-component. Hence we can always choose a smooth path of representations ρ_t from ρ to a representation with image of $\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g$ in $\widetilde{SL_2(\mathbb{R})} = \left\{[(x, a)] \in \mathbb{R} \times_{\mathbb{Z}} \widetilde{SL_2(\mathbb{R})} | x = 0\right\}$. Then by the Lemma the volume $vol(N, \rho_t)$ also varies smoothly in t and is therefore constant, since it can attain only finitely many values.

Now we compute the volume of such a representation. Let $p_1 : \widetilde{N} \to N$ be a finite degree covering of Seifert fibered spaces with fiber degree 1, such that $\widetilde{N}$ is a circle bundle over a surface $\widetilde{F}$. According to Theorem 1.2 in [**36**] we obtain

$$\widetilde{e} = e(\widetilde{N}) = (deg p_1)e.$$

Let $\widetilde{t}$ be the fiber of $\widetilde{N}$ and $\widetilde{\rho} := p_1^{\#}\rho$. Then $\left(\widetilde{t}\right)^{\widetilde{e}} = \prod_{j=1}^{\widetilde{g}}\left[\widetilde{\alpha_j}, \widetilde{\beta_j}\right]$ in $\pi_1\widetilde{N}$ (cf. [**41**, p. 435]), and therefore $\widetilde{\rho}\left(\left(\widetilde{t}\right)^{\widetilde{e}}\right) = \overline{(\widetilde{e}\zeta, 1)} \in Z(G) \cap \widetilde{SL_2(\mathbb{R})}$, since the image of the fibre must lie in the center (see above) and the image of the product of commutators must lie in $\widetilde{SL_2(\mathbb{R})}$. Hence $\widetilde{e}\zeta = \widetilde{n} \in \mathbb{Z}$.

Let $p_2 : \widehat{N} \to \widetilde{N}$ be the covering along the fiber direction of degree $\widetilde{e}$, and then $\widehat{e} = e(\widehat{N}) = 1$. Then $\widehat{\rho} := \widetilde{\rho} \circ p_{2\#}$ sends $\pi_1\widehat{N}$ into $\widetilde{SL_2(\mathbb{R})}$ and the fiber $\widehat{t}$ of $\widehat{N}$ is sent to $sh(\widetilde{n})$.

And finally there is a covering $p_* : \widehat{N} \to N^*$ along the fiber direction of degree $\widetilde{n}$, and where N^* is a circle bundle over a hyperbolic surface F with $e^* = e(N^*) = \widetilde{n}$. Then $\widehat{\rho}$ induces a representation $\rho^* : \pi_1 N^* \to \widetilde{SL_2(\mathbb{R})}$ such that $\widehat{\rho} = p_*^{\#}\rho^*$. In particular $\rho^*(h^*) = sh(1)$, where h^* denotes the fiber of N^*. By Theorem 2.22 and Remark 2.23, ρ^* induces a $(PSL_2(\mathbb{R}), \mathbf{S^1})$-transverse foliation on N^*.

Hence according to Proposition 2.20 and 2.21, $\mathrm{vol}(N^*, \rho^*) = |4\pi^2 e^*| = 4\pi^2|\widetilde{n}|$. But then by Theorem 2.1 we obtain that:

$$\mathrm{vol}(\widehat{N}, \widehat{\rho}) = 4\pi^2\widetilde{n}^2 = 4\pi^2\widetilde{e}^2\zeta^2.$$

Hence since

$$deg p_1 deg p_2 = \frac{\widetilde{(e)}}{e}\widetilde{e},$$

we obtain:

$$\mathrm{vol}(N,\rho) = \frac{\mathrm{vol}(\widehat{N},\widehat{\rho})}{deg p_1 deg p_2} = \frac{4\pi^2\widetilde{e}^2\zeta^2|e|}{\widetilde{e}^2} = 4\pi^2|e|\zeta^2 = 4\frac{\pi^2}{|e|}\left(\sum_{i=1}^{p}\frac{n_i}{a_i} - n\right)^2.$$

□

2.5. Background on Chern-Simons Theory

In this section we give a short revision of Chern-Simons Theory, focusing on its connection to Volumes of Representations.

2.5.1. Principal G-bundles, Connections and Curvatures.

2.5.1.1. *Principal G-bundles.* Let G be a Lie group. First we want to quickly review the notion of a Principal G-bundle and state some properties of Connections on Principal G-bundles. The main purpose is to fix notations for the following sections. A *principal G-bundle* over a smooth manifold M is a smooth manifold P together with:

- a smooth map $\pi : P \to M$,
- a right G-action,

$$\begin{array}{ccc} P \times G & \to & P \\ (p, g) & \mapsto & p \cdot g =: r_g(p) \end{array}$$

satisfying that there is a covering $\{U_\alpha\}$ of M by open sets together with diffeomorphisms

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \to & U_\alpha \times G \\ p & \mapsto & (\pi(p), \phi_\alpha(p)) \end{array}$$

such that $\phi_\alpha(r_g(p)) = r_g(\phi_\alpha(p))$ for all $p \in U_\alpha$.

Under these conditions we call P the *total space*, M the *base space*, G the *structure group*, π the *projection map* and the diffeomorphisms are called *local trivializations*. A G-equivariant bundle automorphism $\phi : P \to P$ is called a *Gauge transformation* and the group of all Gauge transformations $\mathcal{G}(P)$ is called *Gauge group*.

For a given Lie group G we denote by $\mathfrak{g}$ its by Lie algebra and by $exp : \mathfrak{g} \to G$ its exponential map. Let $\pi : P \to M$ be a principal G-bundle. For $X \in \mathfrak{g}$, $(p, t) \mapsto p \cdot exp(tX)$ defines a flow on P. Its generating vector field is denoted by X^* and is called the *fundamental vector field* generated by X.

The map $\mathfrak{g} \to T_pP$, $X \mapsto X^*_p$ is injective with image $ker(D_p\pi)$. We call $ker(D_p\pi) =: V_p$ the *vertical tangent space* at p and define $V := \bigcup_{p \in P} V_p$.

2.5.1.2. *Differential forms with values in a Lie algebra* $\mathfrak{g}$ *and connections.* For a manifold M, the vector space of differential k-forms with values in $\mathfrak{g}$ is denoted by $\Omega^k(M; \mathfrak{g})$. Define the *exterior product*

$$\cdot \wedge \cdot : \Omega^k(M; \mathfrak{g}) \times \Omega^l(M; \mathfrak{g}) \to \Omega^{k+l}(M; \mathfrak{g} \otimes \mathfrak{g})$$

by

$$\omega \wedge \eta(X_1, \cdots, X_{k+l}) := \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} sgn(\sigma) \omega(X_{\sigma(1)}, \cdots, X_{\sigma(k)}) \otimes \eta(X_{\sigma(k+1)}, \cdots X_{\sigma(k+l)}),$$

the *Lie bracket*

$$[.,.] : \Omega^k(M; \mathfrak{g}) \times \Omega^l(M; \mathfrak{g}) \to \Omega^{k+l}(M; \mathfrak{g})$$

by

$$[\omega, \eta]\,(X_1, \cdots, X_{k+l}) := \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} sgn(\sigma) \left[\omega(X_{\sigma(1)}, \cdots, X_{\sigma(k)}), \eta(X_{\sigma(k+1)}, \cdots X_{\sigma(k+l)})\right],$$

and the *differential*

$$d : \Omega^k(M;\mathfrak{g}) \to \Omega^{k+1}(M;\mathfrak{g})$$

by

$$\begin{aligned} d\omega(X_1,\cdots,X_{k+l}) :=& \sum_{i=1}^{k+1} X_i(\omega(X_1,\cdots,\widehat{X_i},\cdots,X_{k+1})) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1,\cdots,\widehat{X_i},\cdots,\widehat{X_j},\cdots,X_{k+1}). \end{aligned}$$

A *connection* on P is a smooth distribution $H := \bigcup_{p\in P} H_p$ satisfying $TP = H \oplus V$ and $D_p r_g(H_p) = H_{pg}$, $\forall p \in P$, $g \in G$. Every connection can uniquely be identified with a *connection 1-form* ω on P, that is, $\omega \in \Omega^1(P;\mathfrak{g})$ with:

(1) $r_g^*\omega = Ad_{g^{-1}}(\omega)\ \forall g \in G$,
(2) $\omega(A^*) = A\ \forall A \in \mathfrak{g}$.

Given a connection H on P we obtain the corresponding connection 1-form ω on P by definining:

$$\begin{cases} \omega(X) = 0 & \text{if } X \in H \\ \omega(A^*) = A & \text{if } A \in \mathfrak{g} \end{cases}$$

Given a connection 1-form ω on P we obtain the corresponding connection on P as $H := ker(\omega)$. Due to this correspondence we usually do not explicitly distinguish between a connection and its connection 1-form, but mean both when talking of a connection.

Two connections H_1 and H_2 on P are called Gauge equivalent if there is a Gauge transformation ϕ with $D\phi(H_1) = H_2$.

2.5.1.3. *Curvature and holonomy.* Let now ω be a connection 1-form on a principal G-bundle $\pi : P \to M$. Then the *curvature* F^ω of ω is defined by

$$F^\omega := d\omega + \frac{1}{2}[\omega,\omega].$$

The connection $H = ker(\omega)$ is called *flat* if $F^\omega = 0$ and H is integrable if and only if it is flat. The set of connections on P is denoted by $\mathcal{A}(P)$ and the subset of flat connections by $\mathcal{FA}(P)$.

A smooth curve $c : [0,1] \to P$ is called *horizontal* (with respect to H) if $\dot{c}(t) \in H_{c(t)}\ \forall t \in [0,1]$. Let $\gamma : [0,1] \to M$ be a curve in M. Then for every $p \in \pi^{-1}(c(0))$ there is a unique horizontal lift $\bar{\gamma} : [0,1] \to P$ of γ with $\bar{\gamma}(0) = p$.

There is a correspondence between flat connections on P up to Gauge equivalence and representations $\pi_1 M \to G$ up to conjugation.

Given a flat connection H, a point $p \in P$, and a loop $\gamma : [0,1]$ in M starting at $\pi(p)$. Then the endpoint $p_\gamma := \bar{\gamma}(1) = p \cdot g(\gamma)$ of the horizontal lift of γ with $\bar{\gamma}(0) = p$ only depends on the homotopy class of γ in M and we obtain a representation $\rho : \pi_1 M \to G$, $[\gamma] \to g(\gamma)^{-1}$, called the *holonomy representation* of H.

Conversely, given a representation $\rho : \pi_1 M \to G$, $\pi_1 M$ acts on $\widetilde{M} \times G$ by $[\gamma]\cdot(\widetilde{m},g) := ([\gamma]\cdot\widetilde{m}, \rho([\gamma])\cdot g)$. The quotient of $\widetilde{M} \times G$ under this $\pi_1 M$-action, denoted by $\widetilde{M} \times_\rho G$, is isomorphic to P and we obtain a connection by taking the push-forward of the trivial connection on $\widetilde{M} \times G$ under the projection map.

The correspondences are inverse to each other if M is connected. We close this section with a short discussion of the so-called Maurer Cartan connection:

EXAMPLE 2.27. Consider a trivial principal G-bundle $P := M \times G$. The tangent spaces to the submanifolds $m \times G$ with $m \in M$ define a connection on G. It is the kernel of the connection 1-form $\omega_{M.C.} := d(l_{g^{-1}} \circ \pi_2)$, where l_g denotes left multiplication on G. This connection is called the Maurer Cartan connection. The connection is clearly integrable and thus flat, that is, it satisfies the equation

$$d\omega_{M.C.} = -\frac{1}{2}\left[\omega_{M.C.}, \omega_{M.C.}\right].$$

2.5.2. Chern-Weil theory and the Chern-Simons action. For a Lie group G a symmetric linear map $f : \otimes_{i=1}^{l}\mathfrak{g} \to \mathbf{K}$ is called a *polynomial of degree* l. The Polynomials of degree l that are invariant under the adjoint G-action $Ad : G \times \mathfrak{g} \to \mathfrak{g}$ are called *(Ad-)invariant Polynomials* and the set of all such Polynomials is denoted $I^l(G)$.

Given a principal G-bundle $\pi : P \to M$ and a connection ω on P, then every $f \in I^l(G)$ defines a $\mathbf{K}$-valued $2l$-form $f(\wedge^l F^\omega)$ defined by

$$f(\wedge^l F^\omega)(X_1, \cdots X_{2l}) := f(\wedge^l F^\omega(X_1, \cdots, X_{2l})).$$

By Chern-Weil Theory the form $f(\wedge^l F^\omega)$ is closed and is the pull-back of a unique form $\alpha^\omega(f)$ on M under π. The Chern-Weil Theorem shows that the class of $\alpha^\omega(f)$ in $H^{2l}(M; \mathbf{K})$ does not depend on ω, inducing the Chern-Weil homomorphism

$$W_P : I^l(G) \to H^{2l}(M; \mathbf{K}).$$

For a Lie group G a principal G-bundle $EG \to BG$ is called *universal for* G if EG is contractible. Universal bundles always exist when we allow EG and BG to be infinite dimensional manifolds and the base BG is unique up to homotopy equivalence. An important property of the universal bundle is that every principal G-bundle $\pi : P \to M$ is obtained as the induced bundle of a continuous map $\xi : M \to BG$ that is unique up to homotopy. The map is called *classifying map* and the corresponding bundle homomorphism is also denoted $\xi : P \to EG$. The universal Chern-Weil homomorphism $\widetilde{W} : I^l(G) \to H^{2l}(BG)$ exists and satisfies $\xi^*\widetilde{W}(f) = W_P(f)$. Thus W_P is completely determined by $\widetilde{W}$ and ξ. This property will be useful later.

In [**8**] Chern and Simons showed that for $f \in I^l(G)$ the r_g-invariant form $f(\wedge^l F^\omega)$ is exact with primitive given by

$$Tf(\omega) = l\int_0^1 f(\omega \wedge F_t^{l-1})dt,$$

where $\phi_t = tF^\omega + 1/2(t^2 - t)\left[\omega, \omega\right]$. The form can be written without integration as

$$Tf(\omega) = \sum_{i=0}^{l-1}(-1)^i \frac{l!(l-1)!}{2^i(l+i)!(l-1-i)!} f(\omega \wedge [\omega, \omega]^i \wedge F^{\omega l-i-1}).$$

In particular for $l = 2$ we get the 3-form

$$Tf(\omega) = f(d\omega \wedge \omega) + \frac{1}{3} f(\omega \wedge [\omega, \omega]).$$

Assume further that $P \to M$ admits a global section $s \in \Gamma(P)$, i.e. is trivial.

The *Chern-Simons invariant* of ω and s is defined by

$$\mathfrak{cs}(\omega, s) := \int_M s^* Tf(\omega).$$

The map

$$\begin{array}{ccc} \mathcal{A}_P \times \Gamma(P) & \to & \mathbf{K} \\ (\omega, s) & \mapsto & \mathfrak{cs}(\omega, s) \end{array}$$

is called *Chern-Simons action.*

Consider now the case where $dimM = 2l - 1$, e.g. $l = 2$, $dimM = 3$. Then $Tf(\omega)$ is closed, because $dTf(\omega)$ is the pull-back of a 2l-form on M, as was mentioned above. Thus under these conditions, the Chern-Simons invariant does not depend on the homotopy class of s.

Considering G as a principal G-bundle over a point and equipping it with the Maurer-Cartan equation we define the set $I_0^l(G)$ of *integral polynomials* of degree l by

$$I_0^l(G) = \left\{ f \in I^l(G) | Tf(\omega_{M.C.}) \in H^{2l-1}(G; \mathbb{Z}) \right\}.$$

For $f \in I_0(G)$ and M a closed manifold the Chern-Simons invariant of two different sections only differs by an integer and thus we obtain a well defined functional

$$\mathfrak{cs}_M^* : \mathcal{A}_{M \times G} \to \mathbf{K}/\mathbb{Z}.$$

This is wrong for $\partial M \neq \emptyset$.

EXAMPLE 2.28. For our purposes important examples are $G = SU(2; \mathbb{C})$ and $G = SO(3; \mathbb{R})$ and we restrict to the case $l = 2$, since we are interested in 3-manifolds.

For the group $SU(2; \mathbb{C})$ the only Ad-invariant polynomial of degree 2, up to multiplication by scalars, is the polynomial $Tr(A \cdot B)$, with $A, B \in \mathfrak{sl}_2(\mathbb{C})$. On the other hand the second Chern class C_2 is given by

$$C_2(A \otimes A) = \frac{1}{8\pi^2} Tr(A^2)$$

and is in $I_0^2(SU(2; \mathbb{C}))$. Thus it is natural to base the Chern-Simons classes for the group $SU(2; \mathbb{C})$ on the second Chern class.

For the same reasons the Chern-Simons classes of the special orthogonal group $G = SO(3; \mathbb{R})$ are based on the first Pontrjagin class $f = P_1 \in I_0^2(SO(3; \mathbb{R}))$, where

$$P_1(A, B) = -\frac{1}{8\pi^2} Tr(AB).$$

From now the Chern-Simons invariants of principal bundles with structure group $SU(2; \mathbb{C})$ or $SO(3; \mathbb{R})$ will always be defined with respect to these polynomials.

Given a closed orientable Riemannian manifold M of dimension $dimM = n$, then the positive unit frame bundle $SO(M)$ is canonically a $SO(n)$-bundle over M and the Levi-Civita connection on TM uniquely defines a connection on $SO(M)$ (cf. for instance [**1**]) which is also called Levi Civita connection. If $dimM = 3$, then SO(M) is trivial and thus $SO(M) \to M$ admits sections s. The Chern-Simons invariant of the Levi Civita connection evaluated on a section s is denoted by $\mathfrak{cs}_{L.C.}(M, s)$.

There is a 2-fold covering $\phi : SU(2;\mathbb{C}) \to SO(3;\mathbb{R})$ by a Lie group homomorphism and there is only one Ad-invariant polynomial of degree 2 up to scalar multiples for each of the two groups. Thus the question comes naturally, whether there is a relation between the Chern-Simons invariant of a connection ω on a trivial principal $SU(2;\mathbb{C})$-bundle $P = M \times SU(2;\mathbb{C})$ and the Chern-Simons invariant of the induced connection ω' on the corresponding $SO(3;\mathbb{R})$-bundle $P' = M \times SO(3;\mathbb{R})$, obtained via the surjective bundle homomorphism $f := (id_M, \phi) : P \to P'$. f is a smooth covering of degree 2 and thus in particular a local diffeomorphism.

Let $\phi_* : \mathfrak{su}(2) \to \mathfrak{so}(3)$ be the induced Lie algebra homomorphism. Then, for $X', Y' \in T_{p'}P'$, $p' \in P'$, the connection 1-form ω' is defined by

$$\omega'(X') = \phi_*(\omega(X))$$

where $X \in T_pP$ with $p' = f(p)$ and $X' = Df(X)$. It is not hard to check that ω' is indeed a well-defined connection 1-form. Choosing a neighbourhood $U \subset P$ of p such that $f|U$ is a diffeomorphism we observe that $\omega' = \phi_* \circ ((f|U)^{-1*}\omega)$.

LEMMA 2.29. *Let P, P', ω and ω' be defined as above. Then for a section $\delta : M \to P$ and the corresponding section $\delta' := f \circ \delta$ the Chern-Simons invariants satisfy:*

$$\mathfrak{cs}_M(\omega', \delta') = -4\mathfrak{cs}_M(\omega, \delta).$$

PROOF. Since $\phi_* : \mathfrak{su}(2) \to \mathfrak{so}(3)$ is a Lie algebra homomorphism it is a linear map between finite-dimensional vectorspaces and respects the Lie algebra structure. In particular it commutes with the exterior derivative, the Lie bracket, and the wedge product. Thus commutativity of pull-back and exterior derivative yield:

$$d\omega' = d(\phi_* \circ ((f|U)^{-1*}\omega)) = \phi_* \circ (d((f|U)^{-1*}\omega)) = \phi_* \circ (f|U)^{-1*}(d\omega)$$

which means that

$$d\omega'(X', Y') = \phi_*(d\omega(X, Y))$$

for $X', Y' \in T_{p'}P'$, $f(p') = p$, and $X, Y \in T_pP$ with $Df(X) = X'$, $Df(Y) = Y'$.

Also note that

$$[\omega', \omega'](X', Y') = [\phi_*(\omega(X, Y)), \phi_*(\omega(X, Y))] = \phi_*([\omega(X, Y), \omega(X, Y)]).$$

Recall the explicit form of ϕ_*. Remember that

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} ix & -y + iz \\ y + iz & -ix \end{pmatrix} | x, y, z \in \mathbb{R} \right\},$$

$$\mathfrak{so}(3) = \left\{ \begin{pmatrix} 0 & y & x \\ -y & 0 & -z \\ -x & z & 0 \end{pmatrix} | x, y, z \in \mathbb{R} \right\}.$$

Then it is known that

$$\phi_* \begin{pmatrix} ix & -y+iz \\ y+iz & -ix \end{pmatrix} = \begin{pmatrix} 0 & 2y & 2x \\ -2y & 0 & -2z \\ -2x & 2z & 0 \end{pmatrix}$$

Furthermore observe that

$$Tr\left(\begin{pmatrix} 0 & 2y & 2x \\ -2y & 0 & -2z \\ -2x & 2z & 0 \end{pmatrix}^2\right) = -8(x^2+y^2+z^2) = 4Tr\left(\begin{pmatrix} ix & -y+iz \\ y+iz & -ix \end{pmatrix}^2\right)$$

Putting these results together we obtain that

$$\begin{aligned} f^*TP_1(\omega')(X,Y,Z) &= P_1(\phi_*((d\omega' \wedge \omega')(Df(X), Df(Y), Df(Z)))) \\ &+ \frac{1}{3}P_1(\phi_*(\omega' \wedge [\omega', \omega'])(Df(X), Df(Y), Df(Z))) \\ &= P_1(\phi_*((d\omega \wedge \omega)(X,Y,Z))) \\ &+ \frac{1}{3}P_1(\phi_*(\omega \wedge [\omega, \omega])(X,Y,Z)) \\ &= -4C_2((d\omega \wedge \omega)(X,Y,Z)) - \frac{4}{3}C_2((\omega \wedge [\omega,\omega])(X,Y,Z)) \\ &= -4TC_2(\omega)(X,Y,Z) \end{aligned}$$

The Lemma follows immediately:

$$\begin{aligned} \mathfrak{cs}_M(\omega', \delta') &= \int_M (\delta')^* TP_1(\omega') = \int_M \delta^* f^* TP_1(\omega') \\ &= -4\int_M \delta^* TC_2(\omega) = -4\mathfrak{cs}_M(\omega, \delta). \end{aligned}$$

□

Now consider the following commutative diagram

$$\begin{array}{ccc} SL(2;\mathbb{C}) & \longrightarrow & PSL(2;\mathbb{C}) \\ \uparrow & & \uparrow \\ SU(2;\mathbb{C}) & \longrightarrow & SO(3;\mathbb{R}) \end{array}$$

and recall that the Lie algebra of $SL(2;\mathbb{C})$ is the complexification of the Lie algebra of $SU(2;\mathbb{C})$. The same holds for the right side and thus the Chern-Simons classes of $SL(2;\mathbb{C})$ and $PSL(2;\mathbb{C})$ are also based on C_2 and P_1. Hence the Lemma also holds for P a $SL(2;\mathbb{C})$-bundle over M and P' a $PSL(2;\mathbb{C})$-bundle over M.

2.5.3. Volume and Chern-Simons classes in Seifert geometry. This section is devoted to the proof of the following proposition:

PROPOSITION 2.30. *Let $\rho : \pi_1 M \to G = Iso_e(\widetilde{SL_2(\mathbb{R})})$ be a representation and A the induced flat connection on the associated principal bundle $P = M \times_\rho G$. Assume that P admits a section $\delta : M \to P$. Then:*

$$\mathfrak{cs}_M(A, \delta) = \int_M \delta^* \mathbf{R}\left(dA \wedge A + \frac{1}{3}A \wedge [A, A]\right) = 2vol_G(M, \rho),$$

where the Chern-Simons class is based on the polynomial $\mathbf{R}(A \otimes A) = Tr(X^2) + t^2$ *for* $A = X + t \in Lie(\widetilde{SL_2(\mathbb{R})}) \oplus \mathbb{R} \equiv Lie(G)$.

PROOF. The proof is inspired from [**36**],p. 532 and follows their representation. The matrices

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} Z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

together with the generator T of $\mathbb{R}$ form a basis of the Lie algebra $\mathfrak{g}$ of G. We get a new basis (X, Y, Z, W) by setting $W = Z - Y - T$ which satisfies the following commutator relations:

$$[X, Y] = -2Y, [X, Z] = 2Z$$

$$[Y, Z] = [Y, W] = [Z, W] = -X, [X, W] = 2Y + 2Z.$$

Denoting by ϕ_X, ϕ_Y, ϕ_Z, ϕ_W the dual basis of $\mathfrak{g}^*$ the Maurer Cartan form of G is given by

$$\omega_{M.C:} = \phi_X \otimes X + \phi_Y \otimes Y + \phi_Z \otimes Z + \phi_W \otimes W$$

It is immediate from the commutator relations that

$$\begin{aligned}[\omega_{M.C.}, \omega_{M.C.}] = & -4(\phi_X \wedge \phi_Y) \otimes Y + 4(\phi_X \wedge \phi_Z) \otimes Z \\ & + 4(\phi_X \wedge \phi_W) \otimes Y + 4(\phi_X \wedge \phi_W) \otimes Z \\ & - 2(\phi_Y \wedge \phi_Z) \otimes X - 2(\phi_Y \wedge \phi_W) \otimes X - 2(\phi_Z \wedge \phi_W) \otimes X\end{aligned}$$

Let $\widetilde{M}$ be the universal covering of M, $q_G : \widetilde{M} \times G \to G$ the projection onto the second component and $^-: \widetilde{M} \times G \to \widetilde{M} \times_\rho G$ the canonical projection onto the associated bundle. Then the corresponding flat connection A on $\widetilde{M} \times_\rho G$ corresponds to the form $\overline{q_G^*(\omega_{M.C.})}$, where the push-forward under $^-$ is well defined, since $q_G^*(\omega_{M.C.})$ is $\pi_1 M$-invariant.

According to Section 2.5.2 we obtain

$$T\mathbf{R}(\omega_{M.C.}) = -\frac{1}{6}\mathbf{R}\left(\omega_{M.C.} \wedge [\omega_{M.C.}, \omega_{M.C.}]\right).$$

Since

$$\begin{aligned}&\omega_{M.C.} \wedge [\omega_{M.C.}, \omega_{M.C.}] = \\ &-2(\phi_X \wedge \phi_Y \wedge \phi_Z) \otimes (X \otimes X) - 2(\phi_X \wedge \phi_Y \wedge \phi_W) \otimes (X \otimes X) \\ &-2(\phi_X \wedge \phi_Z \wedge \phi_W) \otimes (X \otimes X) - 4(\phi_X \wedge \phi_Y \wedge \phi_Z) \otimes (Y \otimes Z) \\ &-4(\phi_X \wedge \phi_Y \wedge \phi_W) \otimes (Y \otimes Y) - 4(\phi_X \wedge \phi_Y \wedge \phi_W) \otimes (Y \otimes Z) \\ &-2(\phi_Y \wedge \phi_Z \wedge \phi_W) \otimes (Y \otimes X) - 4(\phi_X \wedge \phi_Y \wedge \phi_Z) \otimes (Z \otimes Y) \\ &-4(\phi_X \wedge \phi_Z \wedge \phi_W) \otimes (Z \otimes Y) - 4(\phi_X \wedge \phi_Z \wedge \phi_W) \otimes (Z \otimes Z) \\ &+2(\phi_Y \wedge \phi_Z \wedge \phi_W) \otimes (Z \otimes X) - 4(\phi_X \wedge \phi_Y \wedge \phi_W) \otimes (W \otimes Y) \\ &+4(\phi_X \wedge \phi_Z \wedge \phi_W) \otimes (W \otimes Z) - 2(\phi_Y \wedge \phi_Z \wedge \phi_W) \otimes (W \otimes X)\end{aligned}$$

Recall that $\mathbf{R}$ is the symmetric polynomial defined by $\mathbf{R}((A+t) \otimes (A+t)) = Tr(A^2) + t^2$ for $A + t \in \mathfrak{sl}_2(\mathbb{R}) \oplus \mathbb{R} = Lie(Iso_e\widetilde{SL_2(\mathbb{R})})$ and thus by linearity of the Trace

$$\mathbf{R}((A_1 + t_1) \otimes (A_2 + t_2)) = \frac{1}{2}(A_1 A_2 + A_2 A_1) + t_1 t_2,$$

for $A_1 + t_1, A_2 + t_2 \in \mathfrak{sl}_2(\mathbb{R}) \oplus \mathbb{R}$ Hence we get the following non-zero values for $\mathbf{R}$, when evaluating on the base elements of $Lie(Iso_e\widetilde{SL_2(\mathbb{R})}) \otimes Lie(Iso_e\widetilde{SL_2(\mathbb{R})})$

$$\mathbf{R}(X \otimes X) = 2, \mathbf{R}(Y \otimes Z) = \mathbf{R}(Z \otimes Y) = 1, \mathbf{R}(W \otimes W) = 1$$

Thus

$$T\mathbf{R}(\omega_{M.C.}) = 2\phi_X \wedge \phi_Y \wedge \phi_Z + \frac{4}{3}(\phi_X \wedge \phi_Y + \phi_X \wedge \phi_Z) \wedge \phi_W.$$

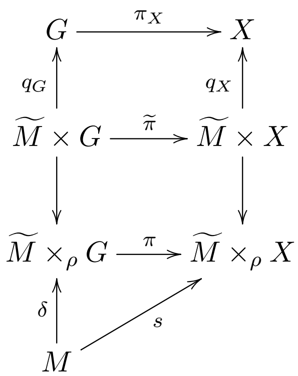
It is easy to see that

$$d(\phi_Y - \phi_Z) = 2(\phi_X \wedge \phi_Y + \phi_X \wedge \phi_Z),$$
$$d(\phi_W) = 0.$$

Thus we obtain

$$T\mathbf{R}(\omega_{M.C.}) = 2\phi_X \wedge \phi_Y \wedge \phi_Z + \frac{2}{3}d(\phi_Y \wedge \phi_W - \phi_Z \wedge \phi_W).$$

Now consider the following commutative diagram:



Since $\phi_X \wedge \phi_Y \wedge \phi_Z$ represents the volume form on $X = \widetilde{SL_2(\mathbb{R})}$ the proposition follows from the diagram and Stokes formula:

$$\begin{aligned}\int_M \delta^* T\mathbf{R}(A) &= \int_M \overline{q_G^* T\mathbf{R}(\omega_{M.C.})} \\ &= 2\int_M \delta^* \overline{q_G^* \pi_G^*(\phi_X \wedge \phi_Y \wedge \phi_Z)} + \int_M d\alpha \\ &= 2\int_M s^* \overline{q_X^*(\phi_X \wedge \phi_Y \wedge \phi_Z)} \\ &= 2vol_G(M, \rho)\end{aligned}$$

where an appropriate 2-form α exists due to G-invariance of $(\phi_Y - \phi_Z) \wedge \phi_W$.

□

2.5.4. Volume and Chern-Simons classes in Hyperbolic Geometry. This section is devoted to the proof of the following proposition:

PROPOSITION 2.31. *Let M be a 3-dimensional manifold, $\widetilde{M}$ its universal covering and $\rho : \pi_1 M \to PSL(2;\mathbb{C}) = G$ a representation. Denote by A*

the induced flat connection on the associated principal $PSL(2;\mathbb{C})$*-bundle* $P = \widetilde{M} \times_\rho G \to M$*. Assume that* P *admits a section* $\delta : M \to P$*. Then*

$$\mathcal{I}(\mathfrak{cs}_M(A,\delta)) = -\frac{1}{\pi^2} vol_G(M,\rho)$$

PROOF. Recall that $p : PSL(2;\mathbb{C}) \cong Iso_+\mathbb{H}^3 \to \mathbb{H}^3$ is a principal $SO(3;\mathbb{R})$-bundle over $\mathbb{H}^3$, where p denotes the natural projection, and in particular $K = SO(3;\mathbb{R})$ is a maximal compact subgroup of $PSL(2;\mathbb{C})$ with $X = G/K \cong \mathbb{H}^3$. Denote by $\omega_{\mathbb{H}^3}$ the G-invariant volume form on $\mathbb{H}^3$ corresponding to the hyperbolic metric.

Note that the principal G-bundle $\widetilde{M} \times_\rho G$ is trivial if and only if ρ admits a lift into $SL(2;\mathbb{C})$. That holds, since we know from obstruction theory that any principal bundle over a 3-manifold M with simply connected group is trivial.

Consider the complex basis

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of $\mathfrak{sl}(2;\mathbb{C})$ which satisfies the commutator relations

$$[X,Y] = 2Y, [X,Z] = -2Z, [Y,Z] = X$$

and denote by ϕ_X, ϕ_Y, ϕ_Z the dual basis of $\mathfrak{sl}^*(2;\mathbb{C})$. With respect to this basis the Maurer Cartan form of G is

$$\omega_{M.C} = \phi_X \otimes X + \phi_Y \otimes Y + \phi_Z \otimes Z.$$

Thus we obtain

$$[\omega_{M.C.}, \omega_{M.C.}] = 4(\phi_X \wedge \phi_Y) \otimes Y - 4(\phi_X \wedge \phi_Z) \otimes Z + 2(\phi Y \wedge \phi_Z) \otimes X$$

and consequently

$$\omega_{M.C.} \wedge [\omega_{M.C.}, \omega_{M.C.}] = \tag{2.5.1}$$
$$4(\phi_X \wedge \phi_Y \wedge \phi_Z) \otimes (Z \otimes Y) + 4(\phi_X \wedge \phi_Y \wedge \phi_Z) \otimes (Y \otimes Z) \tag{2.5.2}$$
$$+ 2(\phi_X \wedge \phi_Y \wedge \phi_Z) \otimes (X \otimes X). \tag{2.5.3}$$

Thus we obtain that

$$TC_2(\omega_{M.C.}) = -\frac{1}{6} C_2(\omega_{M.C.} \wedge [\omega_{M.C.}, \omega_{M.C.}]) = -\frac{1}{4\pi^2} \phi_X \wedge \phi_Y \wedge \phi_Z.$$

It is immediate from Lemma 2.29 and its proof that

$$TP_1(\omega_{M.C.}) = \frac{1}{\pi^2} \phi_X \wedge \phi_Y \wedge \phi_Z,$$

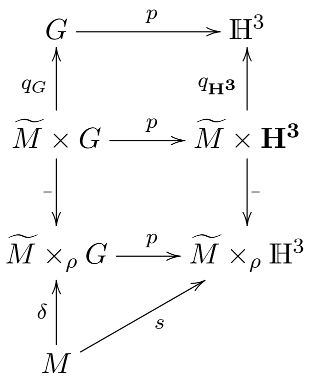
where $\omega_{M.C.}$ now denotes the Maurer Cartan form on $PSL(2;\mathbb{C})$ by abuse of notation.

According to Lemma 3.1 by Yoshida [**49**] we thus have

$$iTP_1(\omega_{M.C.}) = \frac{1}{\pi^2} p^* \omega_{\mathbb{H}^3} + i\mathfrak{cs}_{L.C.}(\mathbb{H}^3) + d\gamma \tag{2.5.4}$$

where $\mathfrak{cs}_{L.C.}(\mathbb{H}^3)$ is the Chern-Simons 3-form of the Levi-Civita connection over $\mathbb{H}^3$ and $d\gamma$ is an exact real 3-form. Note in particular that γ, as given in [**49**], is invariant under the G left-action.

To conclude the proposition we evaluate the following commutative diagram



where $s = p \circ \delta$. In the following computations we will drop the index of q_G and $q_{\mathbb{H}^3}$ and simply write q for both, since the maps will be clear from the context.

The forms $q^*TP_1(\omega_{M.C.})$, $q^*p^*\omega_{\mathbb{H}^3}$, and $q^*\mathfrak{cs}_{L.C.}(\mathbb{H}^3)$ and $q^*d\gamma$ are invariant under the left G-action. Consequently we can take the push-forward of equation (2.5.4) under $\bar{}$ and obtain the equation

$$\overline{iq^*TP_1(\omega_{M.C.})} = \frac{1}{\pi^2}\overline{q^*p^*\omega_{\mathbb{H}^3}} + \overline{iq^*\mathfrak{cs}_{L.C.}(\mathbb{H}^3)} + \overline{q^*d\gamma}$$

of 3-forms on $\widetilde{M} \times_\rho \mathbb{H}^3$.

Taking the pull-back under δ and integrating over M, equation (2.5.4) becomes

$$i\mathfrak{cs}_M(A,\delta) = \frac{1}{\pi^2}\int_M \delta^*\overline{q^*p^*\omega_{\mathbb{H}^3}} + i\int_M \delta^*\overline{q^*\mathfrak{cs}_{L.C.}(\mathbb{H}^3)} + \int_M \delta^*\overline{q^*d\gamma}.$$

By commutativity of the diagram $\delta^*\overline{q^*p^*\omega_{\mathbb{H}^3}} = \delta^*p^*\overline{q^*\omega_{\mathbb{H}^3}} = s^*\overline{q^*\omega_{\mathbb{H}^3}}$. By left-invariance of $q^*\gamma$ under G the push-forward $\overline{q^*\gamma}$ exists and we obtain $\delta^*\overline{q^*d\gamma} = \delta^*\overline{d(q^*\gamma)} = \delta^*d(\overline{q^*\gamma}) = d(\delta^*(\overline{q^*\gamma}))$.

By Stokes formula and closedness of M we obtain $\int_M d(\delta^*(\overline{q^*\gamma})) = 0$ and by Section 2.3.2

$$vol(M,\rho) = \int_M s^*\overline{q^*\omega_{\mathbb{H}^3}}.$$

Alltogether that implies

$$\mathfrak{cs}_M(A,\delta) = -\frac{i}{\pi^2}vol(M,\rho) + \mathfrak{cs}(M_\rho;\delta), \tag{2.5.5}$$

where $\mathfrak{cs}(M_\rho;\delta) = \int_M \delta^*\overline{q^*\mathfrak{cs}(\mathbb{H}^3)}$. □

2.5.5. Normal forms. To compute volumes of representations explicitly we want to recall from [**29**] and [**26**] a few facts on normal forms of connections near toral boundaries of 3-manifolds and their relation to the Chern-Simons invariant. To simplify the situation we consider a compact oriented 3-manifold M with toral boundary $\partial M = T$ and we choos a basis (s,h) of $H_1(\partial M;\mathbb{Z})$. Note that analogous statements hold in the case of more than one toral boundary component.

Let $\rho : \pi_1 M \to G$ be a representation where $G = PSL(2;\mathbb{C})$ or $G = \widetilde{SL_2(\mathbb{R})}$. Here it is important to note that there is no reason for the following statements to hold when $G = Iso_e\widetilde{SL_2(\mathbb{R})}$ which is one of the central obstructions to a generalization of Proposition 2.4.2 to non-geometric prime manifolds. Consider the space of flat connections on the trivialized principal bundle $P = M \times G$ and recall that every principal bundle with structure group $\widetilde{SL_2(\mathbb{R})}$ is trivial whereas for $G = PSL(2;\mathbb{C})$ triviality is equivalent to ρ admitting a lift into $SL(2;\mathbb{C})$.

It follows from [**49**] and [**26**] that, for both choices of G, after conjugation the representation attains one of three possible normal forms on ∂M.

For $A \in G = \widetilde{SL_2(\mathbb{R})}$ we distinguish the following three normal forms after taking the projection of A onto $PSL(2;\mathbb{R})$.

A is said to be in *elliptic* normal form if

$$p(A) \in \overline{\Gamma}_{ell} = \left\{ \left(\begin{array}{cc} cos\phi & sin\phi \\ -sin\phi & cos\phi \end{array} \right) \middle| \phi \in [0, \pi[\right\}$$

in *parabolic* normal form if

$$p(A) \in \overline{\Gamma}_{par} = \left\{ \left(\begin{array}{cc} 1 & c \\ 0 & 1 \end{array} \right) \middle| c \in \mathbb{R} \right\}$$

and in *hyperbolic* normal form if

$$p(A) \in \overline{\Gamma}_{hyp} = \left\{ \left(\begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right) \middle| 0 < a \in \mathbb{R} \right\}$$

Remark 2.32.

(1) Every element in $\widetilde{SL_2(\mathbb{R})}$ is conjugated to an element in normal form.
(2) It is known (and not hard to prove) that whenever two non-central elements $A, B \in \widetilde{SL_2(\mathbb{R})}$ commute and A is in elliptic/parabolic/hyperbolic normal form, then B is also in elliptic/parabolic/hyperbolic normal form.

In the following we will mainly consider the elliptic case.

In a similar way we distinguish the following normal forms for $A \in PSL(2;\mathbb{C})$. If $trA \neq 2$ and A is in normal form then

$$A = \left(\begin{array}{cc} e^{2i\pi\alpha} & 0 \\ 0 & e^{-2i\pi\alpha} \end{array} \right), \ \alpha \in \mathbb{C}.$$

We call A *elliptic* if $|trA| < 2$ and *hyperbolic* if $|trA| > 2$. If $trA = 2$ and in A is in normal form then

$$A = \left(\begin{array}{cc} 1 & a \\ 0 & 1 \end{array} \right), \ a \in \mathbb{C}.$$

In the following we will only consider the first case. Thus assume that both $\rho(h)$ and $\rho(s)$ attain the elliptic normal form for $G = \widetilde{SL_2(\mathbb{R})}$ and the hyperbolic or elliptic normal form for $G = PSL(2;\mathbb{C})$. Let A denote the corresponding connection on P. Then there exists a Gauge transformation g such that the induced connection $g * A$ satisfies

$$g * A|_{T\times[0,1]} = (i\alpha dx + i\beta dy) \otimes X.$$

The behaviour of the Chern-Simons invariant under Gauge transformations that fix a given connection near the boundary is investigated in [**29**], Lemma 3.3, and in [**49**], Theorem 4.2 and can be summarized as follows

PROPOSITION 2.33. *Let A, B be two flat connections in normal form over an oriented 3-manifold with toral boundary. If A and B are equal near the boundary and if they are Gauge equivalent then*

(1) $\mathfrak{cs}_M(A,\delta) = \mathfrak{cs}_M(B,\delta')$ *for* $G = \widetilde{SL_2(\mathbb{R})}$,
(2) $\mathfrak{cs}_M(A,\delta) - \mathfrak{cs}_M(B,\delta') \in \mathbb{Z}$ *for* $G = PSL(2;\mathbb{C})$.

Denote by $\varepsilon\mathcal{R}^0_M(PSL(2;\mathbb{C}))$ the image, up to conjugation, of the elliptic/hyperbolic presentation into $SL(2;\mathbb{C})$ induced by the projection $SL(2;\mathbb{C}) \to PSL(2;\mathbb{C})$.

Clearly we obtain a natural map $t : \mathbb{C}^2 \to \varepsilon\mathcal{R}^0_{\partial M}(PSL(2;\mathbb{C}))$ by sending $(\alpha,\beta) \in \mathbb{C}^2$ to the conjugation class of the representation defined by

$$\begin{array}{ccc} \mathbb{Z}\oplus\mathbb{Z} & \to & PSL(2;\mathbb{C}) \\ s & \mapsto & \begin{pmatrix} e^{2i\pi\alpha} & 0 \\ 0 & e^{-2i\pi\alpha} \end{pmatrix} \\ h & \mapsto & \begin{pmatrix} e^{2i\pi\beta} & 0 \\ 0 & e^{-2i\pi\beta} \end{pmatrix} \end{array}$$

Consider the restriction map $r : \varepsilon\mathcal{R}^0_M(PSL(2;\mathbb{C})) \to \varepsilon\mathcal{R}^0_{\partial M}(PSL(2;\mathbb{C}))$. The map $\rho \mapsto (\alpha,\beta)$ defines a lift L of r and thus the following commutative diagram

$$\begin{array}{ccc} & & \mathbb{C}^2 \\ & \overset{L}{\nearrow} & \downarrow t \\ \varepsilon\mathcal{R}^0_M(PSL(2;\mathbb{C})) & \xrightarrow{r} & \varepsilon\mathcal{R}^0_{\partial M}(PSL(2;\mathbb{C})) \end{array}$$

Fix a lift L of r and associate to any ρ in $\varepsilon\mathcal{R}^0_M(PSL(2;\mathbb{C}))$ the triple

$$(\alpha,\beta,\mathfrak{cs}^*_M(A)) \in \mathbb{C}^2 \times \mathbb{C}^*$$

where A is the connection over M corresponding to $L(\rho)$ in normal form

$$A|_{T\times[0,1]} = (i\alpha dx + i\beta dy) \otimes X$$

near the boundary.

As mentioned in the beginning of Chapter 3 of [**29**] all results of [**29**], including Theorem 2.5, carry over to $SL(2;\mathbb{C})$ and thus we obtain that replacing A by B with a lift $(\alpha+1/2,\beta,\mathfrak{cs}^*_M(B))$ or by C with a lift $(\alpha,\beta+1/2,\mathfrak{cs}^*_M(C))$ gives us:

$$\mathfrak{cs}^*_M(B) = \mathfrak{cs}^*_M(A)e^{-4i\pi\beta} \text{ and } \mathfrak{cs}^*_M(C) = \mathfrak{cs}^*_M(A)e^{4i\pi\alpha} \tag{2.5.6}$$

whereas $t(\alpha,\beta) = t(\alpha+1/2,\beta) = t(\alpha,\beta+1/2)$.

The following two results can be found in [**29**], Theorem 2.7 and end of p.55:

PROPOSITION 2.34. *Let M be an oriented compact 3-manifold with toral boundary $\partial M = T$ and $\rho_t : \pi_1 M \to PSL(2;\mathbb{C})$ a path of elliptic/hyperbolic representations in $\varepsilon\mathfrak{R}^0_M(PSL(2;\mathbb{C}))$. Fix a lifting $L(\rho_t) = (\alpha(t),\beta(t))$ of the* *restriction* $r : \varepsilon^0_M(PSL(2;\mathbb{C})) \to \varepsilon\mathfrak{R}^0_{\partial M}(PSL(2;\mathbb{C}))$ *and denote by* A_t *the corresponding path of flat connections. Then*

$$\mathfrak{cs}^*_M(A_1) = \mathfrak{cs}^*_M(A_0)exp\left(-8i\pi\int_0^1(\alpha(t)\beta'(t)-\beta(t)\alpha'(t))dt\right)$$

and as a Corollary

COROLLARY 2.35. *Let* $V = \mathbf{D}^2 \times \mathbf{S}^1$ *denote the solid torus and let* $\rho : \pi_1 V \to PSL(2;\mathbb{C})$ *denote a representation. If* $L(\rho) = (1/2, \beta)$ *with respect to the meridian longitude basis on* $\partial V = \partial\mathbf{D}^2 \times \mathbf{S}^1$ *then*

$$\mathfrak{cs}^*_V(B) = exp(-4i\pi\beta)$$

where B *is the flat connection over* V *corresponding to the lifting* $L(\rho)$*.*

PROOF. First compute $\mathfrak{cs}^*_V(A)$ where A is a connection in normal form corresponding to the lift $(0,\beta) \in \mathbb{C}^2$. For that consider the path of connections $A_t = (0, t\beta)$. Triviality of A_0 implies $\mathfrak{cs}^*_V(A_0) = 1$ and thus by Proposition 2.34 $\mathfrak{cs}^*_V(A_1) = 1$. The formula is proven by applying equation (2.5.6) once. □

2.5.6. Additivity principle. In this section a very important tool in the computation of volumes of representations of non-geometric manifolds is derived. Let M be a closed 3-manifold M and denote by $[M]$ its orientation class. Let T be a separating torus cutting M into two pieces M_1 and M_2 and denote by $[M_1, \partial M_1]$ and $[M_2, \partial M_2]$ the induced orientation classes so that the orientations on ∂M_1 and ∂M_2 are opposite on T. In particular $[M] = [M_1, \partial M_1] + [M_2, \partial M_2]$.

Let $W(T) = [0,1] \times T$ be a regular neighbourhood of T such that $T = \{1/2\} \times T$, $M_1 \cap W(T) = [0, 1/2] \times T$, and $M_2 \cap W(T) = [1/2, 1] \times T$. Let G be either $PSL(2;\mathbb{C})$ or $\widetilde{SL_2(\mathbb{R})}$. Then by applying the same arguments as in [**29**], respectively [**26**], we may assume that A is in normal form. Then by linearity of integration

$$\mathfrak{cs}^*_M(A) = \mathfrak{cs}^*_{M_1}(A|M_1)\mathfrak{cs}^*_{M_2}(A|M_2).$$

Now denote by V the solid torus $\mathbf{D}^2 \times \mathbf{S}^1$ with meridian m and fix a slope c in T. For $i = 1, 2$ perform a Dehn filling identifying m with the slope c and let $\widehat{M_i} = M_i \cup V = M_i(c)$ be the resulting closed oriented 3-manifold. Suppose that $A|M_1$ and $A|M_2$ smoothly extend to flat connections $\widehat{A_1}$ and $\widehat{A_2}$ on $\widehat{M_1}$ and $\widehat{M_2}$. That is equivalent to: Some/Any representation ρ corresponding to A satisfies $[c] \in ker\rho$. Again using linearity of integration we obtain

$$\mathfrak{cs}^*_{\widehat{M_1}}(\widehat{A_1})\mathfrak{cs}^*_{\widehat{M_2}}(\widehat{A_2}) = \mathfrak{cs}^*_{M_1}(A|M_1)\mathfrak{cs}^*_V(\widehat{A_1}|V)\mathfrak{cs}^*_{M_2}(A|M_2)\mathfrak{cs}^*_V(\widehat{A_2}|V).$$

Since the chosen extensions of $A|M_i$ to $\widehat{A_i}$ on V coincide on the T direction and are opposite on the $[0,1]$ direction, then Proposition 2.33 and implies

$$\mathfrak{cs}^*_V(\widehat{A_1}|V)\mathfrak{cs}^*_V(\widehat{A_2}|V) = 1.$$

Thus we obtain from Proposition 2.31 and Proposition 2.30 together with the above equations that

$$vol_G(M,\rho) = vol_G(M_1(c), \hat{\rho}_1) + vol_G(M_2(c), \hat{\rho}_2),$$

where $\hat{\rho}_i$ is the extension of $\rho|\pi_1 M_i$ to $\pi_1 \widehat{M_i}$, $i = 1, 2$.

2.6. Manifolds with virtually positive hyperbolic volume

In this section we prove Proposition 2.5. Given a compact oriented 3-manifold Q with connected, toral boundary, denote by $i : \partial Q \to Q$ the natural inclusion. Then, according to Lemma 3.5 in [**22**], we obtain that

$$Rank(H_1(\partial Q;\mathbb{R}) \stackrel{i_\#}{\to} H_1(Q;\mathbb{R})) = 1, \tag{2.6.1}$$

since $dim_{\mathbb{R}} H_1(\partial Q) = 2$. Then we obtain by the universal coefficient theorem in homology, that we can choose a *meridian-longitude* basis (μ, λ) of $H_1(\partial Q;\mathbb{Z})$, such that $i_\#(\lambda)$ has infinite order, whereas $i_\#(\mu)$ is a torsion element in $H_1(Q;\mathbb{Z})$. Remember for the sequel that we can make this choice, since we will frequently use it when choosing representations. When Q has non-connected toral boundary we write $\partial Q = T_1 \cup \cdots \cup T_r$ and we fix a basis (λ_i, μ_i) on each component of T_i. Before going into the proof of the Proposition we recall the following Lemma (cf. [**24**],[**32**]):

Lemma 2.36. *Let Q be a compact, oriented and irreducible 3-manifold with toral boundary. Then there exists a prime number q_0, depending only on Q, such that for any prime number $q \geq q_0$ there exists a finite covering $p : \widetilde{Q} \to Q$ inducing a $q \times q$-characteristic covering over ∂Q such that each Seifert piece of $\widetilde{Q}$ is a product of a surface with positive genus by the circle.*

Remark 2.37. In particular the Lemma applies to all compact oriented 3-manifolds Q with toral boundary whose interior admits a complete (finite volume) hyperbolic metric, since Q is irreducible. For this it is obviously enough to check it for the interior of Q. But this is indeed true, because of Theorem 2.3 in [**44**] together with the fact that every orientable prime 3-manifold is either irreducible or $\mathbf{S}^2 \times \mathbf{S}^1$ (cf. [**22**]) and the latter obviously possesses a geometric structure modelled on $\mathbf{S}^2 \times \mathbb{R}$. Furthermore the Lemma also holds for all Seifert fibered spaces S over an orientable surface with nonempty boundary and $\chi(S) < 0$. Additionally in that case the preimage of a boundary component of S under the q^2-characteristic covering consists of at least two components. For the latter see Theorem 2.4 in [**32**].

From this lemma we derive:

Lemma 2.38. *Let Q be a compact oriented 3-manifold with toral boundary whose interior admits a complete (finite volume) hyperbolic metric. Then there exists a prime number p_0, depending only on Q, such that for any family of slopes $(m_1, \ldots, m_r)$ in ∂Q with $m_i \subset T_i$ for $i = 1, \ldots, r$ and for any prime number $q \geq p_0$ there exists a finite covering $p : \widetilde{Q} \to Q$ inducing the $q \times q$-characteristic covering over ∂Q and a representation $\rho : \pi_1 \widetilde{Q}(p^{-1}(m_1 \cup \cdots \cup m_r)) \to PSL(2;\mathbb{C})$ of positive volume, where $\widetilde{Q}(p^{-1}(m_1 \cup \cdots \cup m_r))$ denotes the closed surgered manifold obtained from $\widetilde{Q}$ after performing a Dehn filling on each component U_i^j of $p^{-1}(T_i)$ identifying the meridian of a solid torus with a component of $p^{-1}(m_i) \cap U_i^j$ for any $i = 1, \ldots, r$.*

Proof. Now choose a prime number p_0 such that $p_0 > max\,\{C, q_0\}$, where C is chosen as in Theorem 2.16 and q_0 is the prime number given in Lemma 2.36, when applied to Q. Thus given a slope $m_i = a_i\lambda_i + b_i\mu_i$ in T_i, with (a_i, b_i) co-prime, then for any prime number $q \geq p_0$ we have $\sqrt{qa_i^2 + qb_i^2} > C$ for $i = 1, \ldots, r$. Therefore we can apply Theorem 2.16 implying that $Q((qa_1, qb_1), \ldots, (qa_r, qb_r))$ is a hyperbolic orbifold. In particular there exists a representation $\rho : \pi_1 Q((qa_1, qb_1), \ldots, (qa_r, qb_r)) \to PSL(2; \mathbb{C})$ with positive volume and such that for each $i = 1, \ldots, r$ the element $\rho(m_i)$ is conjugated to $\begin{pmatrix} e^{i\pi/q} & 0 \\ 0 & e^{-i\pi/q} \end{pmatrix}$ and $\rho(l_i)$ is conjugated to $\begin{pmatrix} x_i & 0 \\ 0 & x_i^{-1} \end{pmatrix}$ for some $x_i \in \mathbb{C}^*$, where l_i is the slope $c_i\lambda_i + d_i\mu_i$ in T_i with $a_id_i - b_ic_i = 1$. The latter condition makes sure that l_i and m_i form a basis for the toral boundary component. This choice of representation is possible by 2.6.1 and the corresponding conclusions, since these choices lead to a discrete and faithful representation and therefore by Theorem 2.1 we obtain that the volume of this representation is identical with the hyperbolic volume and consequently especially positive. On the other hand, Lemma 2.36 applies to Q for any $q \geq p_0$ which gives rise to a finite covering $p : \widetilde{Q} \to Q$. Since it induces the $q \times q$-characteristic covering on the boundary then it induces an orbifold finite covering $\widehat{p} : \widetilde{Q}(p^{-1}(m_1 \cup \cdots \cup m_r)) \to Q((qa_1, qb_1), \ldots, (qa_r, qb_r))$. Accordingly $\rho \circ \widetilde{p_*}$ is a representation of the (non-singular) manifold $\widetilde{Q}(p^{-1}(m_1 \cup \cdots \cup m_r))$ whose volume is positive, where for the last step we use Theorem 2.1 (4). □

Now we can prove the Proposition:

Proof. Let N have a hyperbolic piece Q such that each non-separating component of ∂Q is shared by a Seifert piece. Let $N = Q \cup (Q_1 \cup \cdots \cup Q_l)$, where each Q_i is a component of $N \backslash Q$. For each of the components Q_i with connected boundary fix its meridian-longitude basis (μ_i, λ_i); and for each component Q_i with non-connected boundary $T_j^1, \ldots T_j^{l_j}$, denote by h_j^l, $l = 1, \ldots, l_j$ the regular fiber of the Seifert piece adjacent to Q that is represented in T_j^l. The latter is possible due to the conditions posed on N.

For each component Q_i denote by q_i the prime number such that for any prime number $q \geq q_i$ there exists a $q \times q$-characteristic covering map $\widetilde{Q}_i \to Q$ by Lemma 2.36 and Remark 2.37. When ∂Q_i is connected denote by $T_i^k \subset \partial \widetilde{Q}_i$, $k = 1, \ldots, r_i$ the connected components over ∂Q_i. Then the pair (μ_i^k, λ_i^k) with $\mu_i^k = p^{-1}(\mu_i) \cap T_i^k$ and $\lambda_i^k = p^{-1}(\lambda_i) \cap T_i^k$ is a basis of $H_1(T_i^k; \mathbb{Z})$. When ∂Q_j is non-connected then we fix a trivialization of the Seifert pieces adjacent to $\partial \widetilde{Q}_j$ providing a section-fiber basis of $H_1(T_j^l; \mathbb{Z})$ for each component of $\partial \widetilde{Q}_j$.

Apply Lemma 2.38 to Q with the family of slopes $\{\mu_i\}_i$ $\{h_j^l\}_{l,j}$ and let p_0 be as in the Lemma. Let $q > max\,\{p_0, q_1, \cdots, q_l\}$ be prime and $\widetilde{Q} \to Q$ the corresponding $q \times q$-characteristic covering. Denote by $p : M \to N$ a finite covering such that each component of $p^{-1}(Q_i)$ is homeomorphic to $\widetilde{Q_i}$ and each component of $p^{-1}(Q)$ to $\widetilde{Q}$. Such a covering exists and a description of its construction is given in [**32**].

For each component $\widetilde{Q}$ of $p^{-1}(Q)$, let $\widehat{Q} = \widetilde{Q}(\cup_i p^{-1}(\mu_i) \cup_{l,j} p^{-1}(h_j^l))$. By Lemma 2.38, there is a representation

$$\rho : \pi_1 \widehat{Q} \to PSL(2; \mathbb{C})$$

with

$$vol(\widehat{Q}, \rho) > 0.$$

ρ satisfies the following conditions:

(1) when ∂Q_i is connected $\rho(\mu_i^k)$ is trivial and $\rho(\lambda_i^k)$ is conjugated to an element of type $\begin{pmatrix} x_i & 0 \\ 0 & x_i^{-1} \end{pmatrix}$ where $x_i \in \mathbb{C}^*$

(2) when ∂Q_i is not connected then the fibers of the Seifert pieces adjacent to $\partial \widetilde{Q}$, over ∂Q_i, are sent to the trivial element under ρ.

REMARK 2.39. Notice that we do not index ρ by the components of $p^{-1}(Q)$ to keep notation simple.

When ∂Q_i is connected we choose a basis $e_i^1, \dots e_i^{n_i}$ of the torsion-free submodulus of $H_1(Q_i;\mathbb{Z})$ so that $\lambda_i \in \langle e_i^1 \rangle$. Denote by k_i the non-trivial integer such that $\lambda_i = k_i \dot{e}_i^1$ and choose a complex number $\xi_i = a_i e^{i\theta_i} \in \mathbb{C}^*$ such that $\xi_i^{qk_i} = x_i$ and consider the homomorphism $\eta_i : \langle e_i^1 \rangle \to PSL(2;\mathbb{C})$ sending e_i^1 to $\begin{pmatrix} \xi_i & 0 \\ 0 & \xi_i \end{pmatrix}$. For each component $\widetilde{Q}_i$ of $p^{-1}(Q_i)$, consider the representation given by the composition

$$\pi_1 \widetilde{Q}_i \to H_1(\widetilde{Q}_i;\mathbb{Z}) \stackrel{p_\#}{\to} H_1(Q_i;\mathbb{Z}) \to \langle e_i^1 \rangle \stackrel{\eta_i}{\to} PSL(2;\mathbb{C})$$

where $H_1(Q_i;\mathbb{Z}) \to \langle e_i^1 \rangle$ denotes the natural projection onto the $\mathbb{Z}$-factor spanned by e_i^1. Note the composition of the above homomorphisms sends μ_i^k to the unit of $PSL(2;\mathbb{C})$, since μ_i^k was sent to the torsion part of $H_1(Q_i;\mathbb{Z})$ first, and then was sent to the unit under the natural projection. Therefore the composition of the above homomorphisms gives rise to a cyclic representation

$$\eta : \pi_1 \widehat{Q}_i \to PSL(2;\mathbb{C})$$

where $\widehat{Q}_i = \widetilde{Q}_i \left(\cup_{k=1}^{r_i} \mu_i^k \right)$, such that

$$\eta(\lambda_i^k) = \begin{pmatrix} x_i & 0 \\ 0 & x_i^{-1} \end{pmatrix}$$

and

$$\mathrm{vol}(\widehat{Q}_i, \eta) = 0$$

For the latter we need to work a little bit. First observe that we obtain a path

$$(\eta_i)_t(e_i^1) = \begin{pmatrix} \omega_t e^{it\theta_i} & 0 \\ 0 & \frac{1}{\omega_t} e^{-it\theta_i} \end{pmatrix}$$

of representations $\langle e_i^1 \rangle \to PSL(2;\mathbb{C})$ where $\omega_t = ta_i + (1-t)$. It induces a path of representations

$$\eta_t : \pi_1 \widehat{Q}_i \to PSL(2;\mathbb{C})$$

such that $\eta_1 = \eta$ and η_0 is the trivial representation. Observe that by definition the path of representations η_t lifts to a path of representations $\widetilde{\eta}_t : \pi_1 \widehat{Q} \to SL(2;\mathbb{C})$. Since from the obstruction theory it follows that any principal bundle with simply connected structure group is trivial and $SL(2;\mathbb{C})$ is simply connected we obtain that the induced bundle $M \times_{\widetilde{\eta}_t} SL(2;\mathbb{C})$ is trivial for all t. Consequently also the bundle $M \times_{\eta_t} PSL(2;\mathbb{C})$

is trivial for all t. From that we obtain that the path of representations η_t induces a path of flat connections A_t on the trivial $PSL(2;\mathbb{C})$-bundle over M. This defines a connection $\mathbb{A}$ on the product $\widehat{Q}_i \times [0,1]$ (actually on the corresponding trivial $PSL(2;\mathbb{C})$-bundle, but by triviality we can identify it with a unique 1-form on the base) that is not flat anymore but whose curvature $F^{\mathbb{A}}$ satisfies $F^{\mathbb{A}} \wedge F^{\mathbb{A}} = 0$. Where the latter is due to the fact that $F^{A_t} = 0$ for all t and hence $F^{\mathbb{A}} = \delta \wedge dt$. Hence the same argument as in [**28**], page 351, proves that $\mathfrak{cs}^*_{\widehat{Q}_i}(A_0) = \mathfrak{cs}^*_{\widehat{Q}_i}(A_1) = 1$. Therefore $\mathrm{vol}(\widehat{Q}_i, \eta) = 0$.

Let now ∂Q_j be not connected and denote by $\widetilde{Q_j}$ a component of $p^{-1}(Q_j)$. Denote by $\{S_j^l\}$ the Seifert pieces of $\widetilde{Q_j}$ adjacent to $\partial\widetilde{Q_j}$ with a fixed trivialization $F_j^l \times \mathbf{S}^1$. Denote by $\{s_{k,l}\}_{k,l}$ the components of $\partial F_j^l \cap \partial\widetilde{Q_j}$, by $\{s'_{k,l}\}_{k,l}$ the components of $\partial F_j^l \setminus \partial F_j^l \cap \partial\widetilde{Q_j}$ and by $\widetilde{h_l}$ its $\mathbf{S}^1$-fiber. Let $\widehat{F}_j^l$ be the surface obtained from F_j^l by collapsing each boundary component $s'_{k,l}$ to a point and let A be the set of "'singular points"' obtained by collapsing each component of $\cup_l(\partial F_j^l \setminus \partial F_j^l \cap \partial\widetilde{Q}_j)$ to a point.

Define a relation $\mathcal{R}$ on $\widetilde{Q}_j$: $x\mathcal{R}y$ iff either x and y are in the closure $\overline{\widetilde{Q}_j \setminus \cup_l S_j^l}$ or x and y lie on the same $\mathbf{S}^1$-fiber of a Seifert piece $F_j^l \times \mathbf{S}^1$ for some l. This is an equivalence relation and we can consider the *crunching* map $\xi : \widetilde{Q}_j \to \widetilde{Q}_j/\mathcal{R}$. Clearly the quotient space $\widetilde{Q}_j/\mathcal{R}$ is homeomorphic to $\cup_l \widehat{F}_j^l/A$. The fundamental group of $\widetilde{Q}_j/\mathcal{R}$ is an extension of $*_l \pi_1 \widehat{F}_j^l$ by some cycles $\gamma_1, \cdots, \gamma_\nu$ obtained due to the identification of all points of A.

Fix A/A as base point in $\widetilde{Q}_j/\mathcal{R}$. There is a representation $\phi : \pi_1(\widetilde{Q}_j/\mathcal{R}) \to PSL(2;\mathbb{C})$ with

$$\phi(\widetilde{h_l}) = 1 \forall l;\ \phi(s_{k,l}) = \rho(s_{k,l}) \forall k,l;\ \phi(s'_{k,l}) = 1 \forall k,l \text{ and } \phi(\gamma_i) = 1,$$

since $\widehat{F}_j^l$ has positive genus and each element of $PSL(2;\mathbb{C})$ is a commutator by [**40**].

Let $\widehat{Q}_j$ be the closed manifold $\widetilde{Q}_j(\cup_l p^{-1}(h_j^l))$. By continuity ξ induces a map $\xi_* \pi_1 \widetilde{Q}_j \to \pi_1(\widetilde{Q}_j/\mathcal{R})$. It clearly factors through $\widehat{Q}_j$ and thus gives rise to a representation $\pi_1 \widehat{Q}_j \to \pi_1(\widetilde{Q}_j/\mathcal{R}) \to PSL(2;\mathbb{C})$, which we also denote by ϕ.

Note that

$$vol(\widehat{Q}_j, \phi) = 0 :$$

Consider the long exact sequence in cohomology that is induced by $\cup_l \widehat{F}_j^l \to \left(\cup_l \widehat{F}_j^l\right)/A$:

$$\cdots \to \widetilde{H}^{n-1}(A) \to \widetilde{H}^n((\cup_l \widehat{F}_j^l)/A) \to \widetilde{H}^n(\cup_l \widehat{F}_j^l) \to \widetilde{H}^n(A) \to \cdots$$

Since A is a set of points $\widetilde{H}^n = \{0\}$ for all $n > 0$ and since $\widehat{F}_j^l$ is a surface $\widetilde{H}^n(\cup_l \widehat{F}_j^l) = 0$ for all $n > 2$. Thus it leads to an isomorphism $H^3(\widetilde{Q}_j/\mathcal{R}) \simeq \widetilde{H}^3(\widetilde{Q}_j/\mathcal{R}) \simeq \widetilde{H}^3(\cup_l \widehat{F}_j^l) \simeq \{0\}$.

We use the definition of the volume through continuous cohomology from Section 2.3.2. The induced homomorphism $\phi^* : H^3_{cont}(PSL(2;\mathbb{C})) \to$

$H^3(\widehat{Q}_j)$ factors through $H^3(\widetilde{Q}_j/\mathcal{R}) = 0$. Thus $\phi^*(\omega_{\mathbb{H}^3}) = 0$ and

$$vol_{PSL(2;\mathbb{C})}(\widehat{Q}_j, \phi) = \left| \int_M \phi^*(\omega_{\mathbb{H}^3}) \right| = 0.$$

Now we exploit additivity in order to complete the proof. Choose flat $\mathfrak{sl}_2(\mathbb{C})$-connections A,B_i (when ∂Q_i is connected) and B_j (when ∂Q_j is not connected) in normal form and corresponding to the representations $\rho|\pi_1\widetilde{Q}$, $\eta|\pi_1\widetilde{Q}_i$ and $\phi|\pi_1\widetilde{Q}_j$ over each component $\widetilde{Q}$, $\widetilde{Q}_i$ and $\widetilde{Q}_j$ of $p^{-1}(Q)$, $p^{-1}(Q_i)$ and $p^{-1}(Q_j)$. By construction they glue together in a smooth and flat way and give rise to a global representation $\psi : \pi_1 M \to PSL(2;\mathbb{C})$ and thus by the additivity principle

$$vol(M, \psi) = d\,vol(\widehat{Q}, \rho) > 0$$

where d denotes the number of connected components of $p^{-1}(Q)$. This completes the proof. □

2.7. Manifolds with positive simplicial, but zero hyperbolic volume

PROPOSITION 2.40. *There are infinitely many 1-edged 3-manifolds N with non-vanishing simplicial volume but $vol(N, PSL(2;\mathbb{C})) = \{0\}$.*

PROOF OF PROPOSITION 2.40. To obtain a non-vanishing simplicial volume it is sufficient that one of the geometric pieces is hyperbolic. We construct the 3-manifolds N, by gluing a hyperbolic piece to a Seifert piece.

Let $M_1 = F \times \mathbf{S}^1$ where F is a surface with positive genus and connected boundary and let $(s, h) \subset \partial M_1$ be a canonical section-fiber basis. It is proved in [**37**] that there are infinitely many one cusped, complete, finite volume hyperbolic manifolds M_2 endowed with a basis $(\mu, \lambda) \subset \partial M_2$ such that both $M_2(\lambda)$ and $M_2(\mu)$ are connected sums of lense spaces and thus have zero hyperbolic volume.

Glue M_1 and M_2 by the homeomorphism $\phi : \partial M_1 \to \partial M_2$ defined by $\phi(s) = \mu$ and $\phi(h) = \lambda^{-1}$. $M_\phi = M_1 \cup_\phi M_2$. M_ϕ is a one-edged Haken manifold with positive simplicial volume. Let T be the image of ∂M_1 in M_ϕ.

Let $\rho : \pi_1 M_\phi \to PSL(2;\mathbb{C})$ be a representation and A the corresponding connection over M_ϕ. Then either $\rho(s)$ or $\rho(h)$ is trivial, since if $\rho(h) \neq 1$, then its centralizer $Z(\rho(h))$ must be abelian in $PSL(2;\mathbb{C})$. h is central in $\pi_1 M_1$ which implies that $\rho(\pi_1 M_1)$ is abelian Furthermore $s$is homologically zero in M_1, because it is a product of commutators in $\pi_1 M_1$ and thus $\rho(s) = 1$.

Let now ζ be either s or h so that $\rho(\zeta) = 1$ then after conjugation we may assume that A is in normal form with respect to T. Conjugation doesn't change the volume of ρ, since M_ϕ is closed. Denote by A_1 and A_2 the flat connections over M_1 and M_2. Due to triviality of $\rho(\zeta)$ there are extensions of A_1 and A_2 to flat connections $\widehat{A_1}$ and $\widehat{A_2}$ on $M_1(\zeta)$ and $M_2(\zeta)$. By additivity

$$\mathfrak{cs}^*_{M_\phi}(A) = \mathbf{cs}^*_{M_1(\zeta)}(\widehat{A_1}) \times \mathfrak{cs}^*_{M_2(\zeta)}(\widehat{A_2}).$$

Proposition 2.31 implies

$$vol(M_\phi, \rho) = vol(M_1(\zeta), \widehat{\rho_1}) + vol(M_2(\zeta), \widehat{\rho_2}) \tag{2.7.1}$$

where $\widehat{\rho_i}$ denotes the extension of $\rho_{\pi_1 M_i}$ to $\pi_1 M_i(\zeta)$. Thus $vol(M_\phi, \rho) = 0$ by choice of M_1 and M_2. □

REMARK 2.41. Note that equation (2.7.1) still holds replacing M_2 by any one cusped complete hyperbolic manifold. In this case it follows from Section 2.2 that $vol(M_2(\zeta), \widehat{\rho_2}) < vol M_2$ and therefore, the volume of the hyperbolic piece is not reached by the volume of any representation of $\pi_1 M_\phi$ into $PSL(2;\mathbb{C})$.

2.8. Volumes of representations of 1-edged manifolds

This section is devoted to the proofs of Propositions 2.13 and 2.14.

Consider an one-edged 3-manifold $N = Q_- \cup_\tau Q_+$, where the gluing map $\tau : \partial Q_- = T_- \to \partial Q_+ = T_+$ is an orientation reversing homeomorphism. On each of the T_ϵ, $\epsilon = \pm$ we fix a basis $T_\epsilon(s_\epsilon, h_\epsilon)$ with the properties given in Section 2.1.4. Then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ denotes the integral matrix of τ under the basis (s_-, h_-) and (s_+, h_+), which satisfies $detA = -1$, and

$$\tau(s_-) = as_+ + bh_+, \tau(h_-) = bs_+ + dh_+. \tag{2.8.1}$$

It follows from 2.36 and [**24**] that N admits a n-sheeted covering $\widetilde{N}$, where n depends only on the pieces Q_- and Q_+, that induces a $q \times q$-characteristic covering over $T = \mathrm{T}_N$, such that for $p = n/q^2$:

(1) $\widetilde{N} = \widetilde{Q_-} \cup_\tau \widetilde{Q_+}$, where $\widetilde{Q_\epsilon}$ covers Q_ϵ for $\epsilon = \pm$ and $\widetilde{Q_\epsilon}$ is a product of a surface of genus $\geq p+2$ with $\mathbf{S}^1$ if Q_ϵ is a Seifert piece.
(2) when Q_- and Q_+ are both Seifert manifolds then $\widetilde{Q_-}$ and $\widetilde{Q_+}$ can be chosen connected so that $\widetilde{N}$ is a p-edged manifold with $p \geq 2$ (A p-edged 3-manifold is a manifold whose dual graph consists of 2 vertices and p edges and each edge is shared by the two vertices),
(3) The basis $T_\epsilon(s_\epsilon, h_\epsilon)$ can be lifted and denoted by $T_\epsilon^j(s_\epsilon^j, h_\epsilon^j)$ for $j = 1, \cdots, p$, and the matrix of the gluing $\widetilde{\tau} : T_-^j \to T_+^j$ is the same than A for all j under the lifted basis, where the T_ϵ^j's denote the components of $\partial\widetilde{Q_\epsilon}$.

The following Remark gives an explanation for these assumptions.

Remark 2.42. In the case where both pieces are Seifert, i.e. $N = Q_+ \cup_\tau Q_-$ with Q_+ and Q_- Seifert we proceed as follows:

Step 1: Apply Lemma 3.5 (ii) of [**9**]. That implies that we may, after possibly switching to a covering, assume that we have two Seifert pieces that are products of a surface of genus ≥ 2 by a circle.

Step 2: Now increase the number of edges to $d \geq 2$, by performing a 1-characteristic covering "‘along one of the tori of the surface"’ of sufficiently high degree on both Seifert pieces (See figure 4 in [**9**]). Using a covering of the same degree on both pieces makes sure that we can glue them in order to obtain a closed manifold $\widetilde{N}$ and a covering $\widetilde{p} : \widetilde{N} = \widetilde{Q}_+ \cup \widetilde{Q}_- \to N$.

Step 3: Finally apply Lemma 4.4 which gives us a d-characteristic covering $\widehat{p} : \widehat{N} = \widehat{Q}_+ \cup \widehat{Q}_- \to \widetilde{N}$. For this covering we obtain, according to its proof, the following formulas for the genus of the underlying surfaces $\widehat{F}_\pm$ and $\widetilde{F}_\pm$:

$$g(\widehat{F}_\pm) = d(g(\widetilde{F}_\pm) - 1) + \left(\frac{d(d-1)}{2} + 1\right) \geq d + 2.$$

The last inequality holds, since $g(\widetilde{F}_\pm) \geq 2$ and $d \geq 2$. This shows that we may assume properties (1)-(3) when both pieces are Seifert.

In the case where $N = Q_+ \cup_\tau Q_-$ with Q_+ hyperbolic and Q_- Seifert we neither need that the genus of the surface is $\geq p+2$, nor that $\widetilde{Q}_-$ or $\widetilde{Q}_+$ is connected. Hence an application of Lemma 6.1 immediately gives us the required covering.

The notations used in Section 2.8.2 don't pay attention to the fact that $\widetilde{Q}_-$ and $\widetilde{Q}_+$ might not be connected, but, as far as I see, it is really only a matter of some notational modifications. The reason that these two properties are not required is basically due to the fact that we cannot write every element of $\widetilde{SL_2(\mathbb{R})}$ as a product of commutators and these conditions make sure it is possible, whereas we can write every element of $PSL(2;\mathbb{C})$ as a product of commutators.

2.8.1. Graph manifolds. In this section Proposition 2.13 is proved. Obviously for that it is sufficient to prove the following Lemma:

LEMMA 2.43. *Let N denote a closed oriented graph manifold satisfying conditions (1)-(3) and denote by G the group $\mathbb{R} \times_{\mathbb{Z}} \widetilde{SL_2(\mathbb{R})}$. Then:*

(i) if $a = d = 0$, then $8\pi^2 p \in vol(N, G)$
(ii) if $ac \neq 0$ then $4\pi^2 p/|ac| \in vol(N, G)$
(iii) if $cd \neq 0$ then $4\pi^2 p/|cd| \in vol(N, G)$
(iv) if $c = 0$ then $4\pi^2 p/|b| \in vol(N, G)$

PROOF. $b \neq 0$, since otherwise N would be a Seifert manifold (cf. Section 2.1.4).

Denote by $Q_- = F_- \times \mathbf{S}^1$ and by $Q_+ = F_+ \times \mathbf{S}^1$ the two (connected) Seifert pieces and recall that (s^i_-, h^i_-) and (s^i_+, h^i_+) are section fiber bases of ∂Q_- and ∂Q_+ respectively, where $i = 1, \cdots, p$. Choose base points $x_- \in intQ_-$ and $x_+ \in intQ_+$ for the fundamental groups and p arcs connecting x_- with $h^i_- \cap s^i_-$, resp. x_+ with $h_+ \cap s_+$ to see these elements in $\pi_1 Q_-$, resp. $\pi_1 Q_+$.

(i) Suppose first $a = d = 0$. Then $b = c = \pm 1$. First suppose $b = c = 1$. Since the genus of both F_- and F_+ is greater equal than p+2 we can choose a representation $\rho : \pi_1 N \to \widetilde{SL_2(\mathbb{R})}$ with $\rho(s^i_\epsilon) = sh(1)$ and $\rho(h_{\epsilon^i}) = sh(1)$, because of Proposition 2.3 in [**32**].

We can apply Proposition 2.4.2 to the $\widetilde{SL_2(\mathbb{R})}$-manifolds $Q_-(1,-1)$ and $Q_+(-1,1)$, obtained by attaching p solid tori along the p boundary tori $T^i_\pm$, identifying the meridians of the solid tori with $a^i_- s^i_- + b^i_- h^i_-$ for Q_-, where $a^i_- = 1 = -b^i_-$, respectively $a^i_+ s^i_+ + b^i_- h^i_-$ for Q_+, where $a^i_+ = -1 = -b^i_+$. Then an easy computation shows that in both cases (+ and -) choosing $n^i_\pm = b^i_\pm$ and $z^i_\pm = 0$ ρ restricts to a well-defined representation on $Q_-(1,-1)$, resp. $Q_+(-1,1)$, that satisfies all conditions in Proposition 2.4.2 and hence $vol_G(Q_-(1,-1), \rho) = vol_G(Q_+(-1,1)) = 4\pi^2 p$. Then the claim is a direct consequence of the additivity principle for the volume $vol_G(N, \rho)$, which applies since $\tau(s^i_- - h^i_-) = -s^i_+ + h^i_+$ (cf. Section 2.5.6).

In case $b = c = -1$ proceed analogously, taking $\rho(s^i_+) = sh(-1)$, $\rho(s^i_-) = sh(-1)$, $a^i_+ = 1 = -b^i_+$ and $n^i_+ = a^i_+$ instead.

(ii) Now assume $ac \neq 0$. Then the closed manifold $Q_+((a,c),\cdots,(a,c))$ has Euler characteristic $\pm pc/a$.

Since $g(F_+) > p$ we can clearly apply Proposition 2.4.2 with $z_1 = \cdots = z_p = n = 0$ and $n_1 = \cdots = n_p = 1$, yielding a representation $\rho : \pi_1 Q_+ \to \mathbb{R} \times_{\mathbb{Z}} \widetilde{SL_2(\mathbb{R})}$ such that

$$\rho_+(s^i_+) = \overline{\left(0, sh\left(-\frac{1}{a}\right)\right)}, \rho_+(h^i_+) = \overline{\left(\frac{1}{c}, 1\right)}. \tag{2.8.2}$$

Using equation (2.1.1) we obtain:

$$\rho_+(s^i_-) = \overline{(0, sh(0))} = \overline{(0,1)}, \rho_+(h^i_-) = \overline{\left(\frac{d}{c}, sh\left(-\frac{b}{a}\right)\right)}. \tag{2.8.3}$$

Then we can obviously extend ρ_+ to $\rho : \pi_1 N \to \mathbb{R} \times_{\mathbb{Z}} \widetilde{SL_2(\mathbb{R})}$ by sending the subgroup $\pi_1 F_-$ of $\pi_1 Q_-$ to the unit of G. To apply the additivity principle we need that the representation has image in $\widetilde{SL_2(\mathbb{R})}$. For that it suffices to construct a c-fold cyclic covering $q : \widetilde{N_c} \to N$, so that the induced representation $\widetilde{\rho} = \rho \circ q_* : \pi_1 \widetilde{N}_c \to \mathbb{R} \times_{\mathbb{Z}} \widetilde{SL_2(\mathbb{R})}$ has image in $\widetilde{SL_2(\mathbb{R})}$. This covering $p : \widetilde{N_c} \to N$ can be obtained as follows: Consider the covering $p_+ : \widetilde{Q_+} \to Q_+$ defined by $\phi_+ : \pi_1 Q_+ \to \mathbb{Z}/c\mathbb{Z}$ with $\phi_+(s^i_+) = \overline{0}$, $\phi_+(h^i_+) = \overline{1}$; and the covering $p_- : \widetilde{Q_-} \to Q_-$ defined by $\phi_- : \pi_1 Q_- \to \mathbb{Z}/c\mathbb{Z}$ with $\phi_-(s^i_-) = \overline{0}$, $\phi_-(h^i_-) = \overline{d}$. Then by 2.8.1 there is a map $\phi : \pi_1 N \to \mathbb{Z}/c\mathbb{Z}$ which restricts to ϕ_+ on Q_+ and ϕ_- on Q_-. This defines the covering p. It is straightforward to check that

$$\begin{aligned} \widetilde{\rho}(\widetilde{s^i_+}) &= \overline{\left(0, sh\left(-\tfrac{1}{a}\right)\right)}, \quad \widetilde{\rho}(\widetilde{h^i_+}) = \overline{(0, sh(1))}, \\ \widetilde{\rho}(\widetilde{s^i_-}) &= \overline{(0, sh(0))}, \quad \widetilde{\rho}(\widetilde{h^i_-}) = \overline{\left(0, sh\left(\tfrac{1}{a}\right)\right)}, \end{aligned}$$

where the $(\widetilde{s^i_\epsilon}, \widetilde{h^i_\epsilon})$'s are the lifts of the elements s^i_+, s^i_-, $(h^i_+)^c$, $(h^i_-)^{c/gcd(c,d)} \in ker\phi$ to $\widetilde{N_c}$. Hence indeed $\widetilde{\rho}$ takes values in $SL_2(\mathbb{R})$ and hence we can apply the additivity principle.

Since $s^i_- = as^i_+ + ch^i_+$, we obtain $\widetilde{s^i_-} = a\widetilde{s^i_+} + \widetilde{h^i_+}$. Thus applying that $e(\widetilde{Q_+}((a,1),\cdots,(a,1))) = p/a$, Proposition 2.4.2 implies:

$$vol(\widetilde{Q_+}((a,1),\cdots,(a,1),\widetilde{\rho})) = 4\pi^2 e^2/|e| = 4\pi^2|p/a| \tag{2.8.4}$$

and furthermore

$$vol(\widetilde{Q_-}((1,0),\cdots,(1,0))) = 0. \tag{2.8.5}$$

The latter equation is obtained as follows: First observe that $\widetilde{Q}_-((1,0),\cdots,(1,0)) = F \times \mathbf{S}^1$ for F a closed surface of genus equal to the genus of F_- and therefore especially greater equal than $p+2 \geq 2$. But surfaces of genus ≥ 2 are hyperbolic and hence $\widetilde{Q}_-((1,0),\cdots,(1,0))$ is modelled on the $\mathbb{H}^2 \times \mathbf{S}^1$-geometry. But then it is immediate from Remark 2.2, that $vol(\widetilde{Q}_-((1,0),\cdots,(1,0))) = 0$. Hence by the additivity principle:

$$vol(\widetilde{N_c}, \widetilde{\rho}) = 4\pi^2|p/a| \tag{2.8.6}$$

Since by Theorem 2.1 $vol(\widetilde{N_c}, \widetilde{\rho}) = |c| vol(N,\rho)$, we obtain $vol(N,\rho) = 4\pi^2 p/|ac|$. This completes the proof of (ii).

(iii) This is a consequence of (ii) by interchanging the roles of Q_+ and Q_- and therefore replacing A by A^{-1}.

(iv) Observe that for $c = 0$ we obtain $a = -d = \pm 1$, since $-1 = ad - bc = ad$. Define a representation $\rho : \pi_1 Q_-((b,-a), \cdots, (b,-a)) \to \widetilde{SL_2(\mathbb{R})}$ by $\rho(s_-^i) = sh(a/b)$, $\rho(h_-^i) = sh(1)$, for $i = 1, \cdots, p$. Such a representation exists, since $|p/b| \leq p < 2g - 2$ and hence we can find elements of $SL_2(\mathbb{R})$ satisfying the commutator relation. For that representation Proposition 2.4.2 with $n_1 = \cdots = n_p = -a$ and $z_1 = \cdots = z_p = n = 0$ can be applied. Hence we obtain that:

$$vol_G(Q_-((b,-a), \cdots, (b,-a)), \rho) = 4\pi^2 \frac{1}{|e|} e^2 = 4\pi^2 |e| = 4\pi^2 |p/b|$$

On the other hand in order to apply the additivity principle we choose the corresponding representation $\psi : \pi_1 Q_- \to \widetilde{SL_2(\mathbb{R})}$, defined by $\psi(s_+^i) = sh(1/b)$, $\psi(h_+^i) = sh(1)$. Then the manifold $Q_+((0,1), \cdots, (0,1))$ is a connected sum of copies of $\mathbf{S}^2 \times \mathbf{S}^1$. This is proved in part 6.2.4 of Theorem 6.3.3 in [**46**]. Then, under these conditions, Theorem 1 in [**30**] says that a connected sum of copies of $S^2 \times S^1$ is dominated by a product $\sigma \times S^1$ for σ some surface. Since such a product has zero Seifert volume it is immediate that also a connected sum of copies of $S^2 \times S^1$ has zero Seifert volume. Thus $vol_G(Q_+((0,1), \cdots, (0,1)), \rho) = 0$ and therefore

$$vol_G(N, \rho) = vol_G(Q_+((b,-a), \cdots, (b,-a)), \rho)$$

where by slight abuse of notation we denoted the extension of ρ to N also by ρ. This completes the proof. □

2.8.2. Seifert and hyperbolic pieces do appear. In this section the proof of Proposition 2.14 is carried out. The proof is divided in two cases, which involve quite different arguments in certain stages. As in [**29**] denote by D the space of deformations of hyperbolic structures on $intQ_+$ near the complete one $d_0 \in D$. Since Q_+ has only one cusp there are functions

$$(\alpha, \beta) : D^* = D \backslash d_0 \to \mathbb{C}^2$$

such that for each $d \in D^*$ there exists a representation $\rho_d^+ : \pi_1 Q_+ \to PSL(2;\mathbb{C})$ induced on the boundary by the representation

$$s_+ \mapsto \begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\alpha} \end{pmatrix} \text{ and } h_+ \mapsto \begin{pmatrix} e^{2\pi i\beta} & 0 \\ 0 & e^{-2\pi i\beta} \end{pmatrix} \tag{2.8.7}$$

In this situation the map $D^* \ni d \mapsto (\alpha, \beta) \in \mathbb{C}^2$ is a lift of the composition map

$$D^* \to \epsilon\mathfrak{R}_M^0(PSL(2;\mathbb{C})) \to \epsilon\mathfrak{R}_{\partial M}^0(PSL(2;\mathbb{C})).$$

By Thurston Hyperbolic Dehn filling Theorem there is a constant $C > 0$ such that if $\sqrt{a^2 + c^2} > C$ then there exists $d \in D^*$ such that

$$a\alpha + c\beta = \frac{1}{2} \tag{2.8.8}$$

as is mentioned in [**29**] on page 555. Let V= $\mathbf{D}^2 \times \mathbf{S}^1$ be a solid torus endowed with the standard meridian-parallel basis (m, l) on the boundary.

The representation ρ_d^+ extends to a discrete and faithful representation

$$\widehat{\rho_d^+} : \pi_1 Q_+(a,c) \to PSL(2;\mathbb{C})$$

where $Q_+(a,c)$ is obtained by gluing ∂V to ∂Q_+ identifying the meridian m of V with the slope $as_+ + ch_+$. Let then $\widehat{A_d^+}$ denote the flat connection over $Q_+(a,c)$ in normal hyperbolic form over ∂Q_+, corresponding to this representation. It decomposes into $A_d^+ \cup A_d^0$ over $Q_+ \cup V$. Then we can choose a lift $L(\rho_d^+)$ such that

$$A_d^+ | \partial Q_+ = (i\alpha dx + i\beta dy) \otimes X$$

in the basis (s_+, h_+). By (2.8.7) and (2.8.8) this means that

$$A_d^0 | \partial V = \left(i\frac{1}{2}dx + i(b\alpha + d\beta)dy \right) \otimes X$$

in the basis (m, l). A similar construction can be done replacing (a,c) by (b,d).

1. Gluing along the section $s_- = \partial F_-$. Denote by $\rho_+ : \pi_1 \widetilde{Q_+} \to PSL(2;\mathbb{C})$ the representation defined by the composition $\rho_d^+ \circ p_*$, where $p : \widetilde{Q_+} \to Q_+$ is the $q \times q$-characteristic covering map defined at the beginning of the section. Evaluating this representation on s_-^j and h_-^j we obtain that $\rho_+(s_-^j)$ is the trivial element and $\rho_+(h_-^j)$ is given by

$$\begin{pmatrix} e^{2i\pi q(b\alpha + d\beta)} & 0 \\ 0 & e^{-2i\pi q(b\alpha + d\beta)} \end{pmatrix}$$

in $PSL(2;\mathbb{C})$. Consequently there exists a global representation $\rho : \pi_1 \widetilde{N} \to PSL(2;\mathbb{C})$ such that $\rho | \pi_1 \widetilde{Q_+} = \rho_+$, since h_-^j is an infinite cyclic element in the center of $\pi_1 \widetilde{Q_-} = \pi_1(F_- \times \mathbf{S}^1)$. Denote $\rho_- = \rho | \pi_1 \widetilde{Q_-}$. Let A be a flat connection in hyperbolic normal form with respect to $\mathrm{T}_{\widetilde{N}}$ such that $A = A_- \cup A_+$ where A_-, resp. A_+, is the restriction of A over $\widetilde{Q_-}$, resp. $\widetilde{Q_+}$. Note that $A_+ = (p|\widetilde{Q_+})^*(A_d^+)$. FOr each $j = 1, \cdots, p$ we identify the meridian of a solid torus $V_\pm^j = \mathbf{D}^2 \times \mathbf{S}^1$ with s_-^j, resp. $as_+^j + ch_+^j$, and then we get closed manifolds $\widetilde{Q_-}(1,0)$ and $\widetilde{Q_+}(a,c)$ where A_- and A_+, extend to flat connections $\widehat{A_-}$ and $\widehat{A_+}$ such that

$$\mathfrak{cs}^*_{\widetilde{N}}(A) = \mathfrak{cs}^*_{\widetilde{Q_+(a,c)}}(\widehat{A_+}) \times \mathfrak{cs}^*_{\widetilde{Q_-(1,0)}}(\widehat{A_-})$$

by the additivity principle.

Remark 2.44. Again $\widehat{A_+}|V_+^j = (p|V_+^j)^*(A_d^0)$ but mind that $p|V_+^j : V_+^j \to V$ is a $q \times q$-characteristic covering branched along the core of the solid torus. Therefore the lifting of the representations induced on the V_j^+'s is $(1/2, q(b\alpha + d\beta))$.

Denote by $\widehat{\rho_+}$, $\widehat{\rho_-}$ the extension of ρ_+ and ρ_- to $\pi_1 \widetilde{Q_+}(a,c)$ and $\pi_1 \widetilde{Q_-}(1,0)$. Splitting the former equality into real and imaginary parts we obtain:

$$vol(\widetilde{N}, \rho) = vol(\widetilde{Q_+}(a,c), \widehat{\rho_+}) + vol(\widetilde{Q_-}(1,0), \widehat{\rho_-})$$

Since $\widetilde{Q_-}(1,0)$ is by construction a product of a surface $\widetilde{F}_{(1,0)}$ with the circle, the simplicial volume satisfies:

$$0 \leq ||\widetilde{Q_-}(1,0)|| = ||\widetilde{F}_{(1,0)} \times \mathbf{S}^1|| \leq K \cdot ||\widetilde{F}_{(1,0)}|| \cdot ||\mathbf{S}^1|| = 0$$

where the inequality is true for some $K > 0$, since $\widetilde{F}_{(1,0)}$ and $\mathbf{S}^1$ are both oriented closed connected manifolds and the last equality holds since $\pi_1 \mathbf{S}^1$ is abelian (cf. [**20**], Chapters 0.2, 3.0 & 3.1). It is immediate from Theorem 2.1 (3) that

$$vol(\widetilde{Q_-}(1,0), \rho_-) = 0.$$

On the other hand the representation ρ_+ does, a priori, not lie in any deformation space of hyperbolic structures of $\widetilde{Q_+}$ and hence we have to simplify first for getting a geometric interpretation of $vol(\widetilde{N}, \rho) = vol(\widetilde{Q_+}(a,c), \widehat{\rho_+})$. As consequence of Remark 2.44, Corollary 2.35 and additivity of integration we obtain that

$$\mathfrak{cs}^*_{\widetilde{Q_+}(a,c)}(\widehat{A}) = \mathfrak{cs}^*_{\widetilde{Q_+}}(A_+) exp(-4i\pi pq(b\alpha + d\beta)).$$

Since ρ_+ is induced by restriction from ρ_d^+ then $\mathfrak{cs}^*_{\widetilde{Q_+}}(A_+) = (\mathfrak{cs}^*_{Q_+}(A_d^+))^{pq^2}$ and therefore

$$\mathfrak{cs}^*_{\widetilde{Q_+}(a,c)}(\widehat{A}) = (\mathfrak{cs}^*_{\widetilde{Q_+}}(A_d^+))^{pq^2} exp(-4i\pi pq(b\alpha + d\beta))$$

Applying Corollary 2.35 once more we obtain

$$\mathfrak{cs}^*_{Q_+(a,c)}(\widehat{A_d^+}) = \mathfrak{cs}_{Q_+}(A_d^+) exp(-4i\pi(b\alpha + d\beta))$$

leading to

$$\mathfrak{cs}^*_{\widetilde{Q_+}(a,c)}(\widehat{A_+}) = (\mathfrak{cs}^*_{Q_+(a,c)}(\widehat{A_d^+}))^{pq^2} exp(4i\pi pq(q-1)(b\alpha + d\beta)).$$

Applying equation (2.5.5) and Remark 2.12, where the latter can be applied to $\mathfrak{cs}_{Q_+(a,c)}(\widehat{A_d^+})$ since $\widehat{\rho_d^+}$ is discrete and faithful, we obtain:

$$\mathfrak{cs}^*_{\widetilde{Q_+}(a,c)}(\widehat{A_+}) = exp(2i\pi \mathfrak{cs}_{L.C.}(Q_+(a,c)))^{pq^2} \times \\ exp\left(\frac{2pq^2}{\pi} vol Q_+(a,c) + 4i\pi pq(q-1)(b\alpha + d\beta)\right)$$

Using equation (2.5.5), we obtain:

$$exp(2i\pi \mathfrak{cs}(M_\rho;\delta)) \times exp\left(\frac{2}{\pi}\,\mathrm{vol}\left(\widetilde{Q}_+(a,c), \widehat{\rho}_+\right)\right) = \\ exp(2i\pi \mathfrak{cs}_{L.C.}(Q_+(a,c)))^{pq^2} \times exp\left(\frac{2pq^2}{\pi} vol Q_+(a,c) + 4i\pi pq(q-1)(b\alpha + d\beta)\right)$$

Separating real and imaginary terms and multiplying by $\pi/2$ we then obtain that the following equality must hold:

$$\mathrm{vol}\left(\widetilde{Q}_+(a,c), \widehat{\rho}_+\right) = pq^2\,\mathrm{vol}\,Q_+(a,c) + \mathfrak{Re}\left(2i\pi^2 pq(q-1)(b\alpha + d\beta)\right)$$

Hence it is only left to prove that

$$\mathfrak{Re}\left(2i\pi^2 pq(q-1)(b\alpha + d\beta)\right) = \frac{\pi pq(q-1)}{2}\,\mathrm{length}(\gamma)$$

and simplifying further:

$$\mathfrak{Re}\,(2\pi i(b\alpha + d\beta)) = \frac{1}{2}\,\text{length}(\gamma)$$

It is an immediate consequence of Lemma 4.2 in [**37**] that

$$\text{length}(\gamma) = \mathfrak{Re}(2\pi i(b\alpha + d\beta))$$

Therefore completing the proof up to a factor of 1/2 which I couldn't figure out yet.

2. Gluing along the fiber h_-. Finally the verification for (b, d) is carried out. As above, by the Thurston Hyperbolic Dehn filling Theorem there is a constant $C > 0$ such that if $\sqrt{b^2 + d^2} > C$ then there exists $d \in D^*$ such that

$$b\alpha + d\beta = \frac{1}{2}. \tag{2.8.9}$$

Let $V = \mathbf{D}^2 \times \mathbf{S}^1$ be a solid torus endowed with the standard meridian parallel basis (m, l). The representation ρ_d extends to a discrete and faithful representation $\widehat{\rho}_+ : \pi_1 Q_+(b, d) \to PSL(2; \mathbb{C})$, where $Q_+(b, d)$ is obtained by gluing ∂V to ∂Q_+ identifying the meridian of V with the curve $bs_+ + dh_+$. Let $\widehat{A}_d^+$ denote the connection over $Q_+(b, d)$ in normal hyperbolic form over ∂Q_+ which decomposes into $A_d^+ \cup A_d^0$ over $Q_+ \cup V$ We choose a lifting $L(\rho_d^+)$ such that

$$A_d^+|\partial Q_+ = (i\alpha dx + i\beta dy) \otimes X$$

in the basis (s_+, h_+). By equation (2.8.9) this means that

$$A_d^0|\partial V = \left(i\frac{1}{2}dx + i(a\alpha + c\beta)dy\right) \otimes X$$

in the basis (m, l). Denote by $\rho_+ : \pi_1 \widetilde{Q}_+ \to PSL(2; \mathbb{C})$ the representation defined by the composition $\rho_d^+ \circ p_*$, where $p : \widetilde{Q}_+ \to Q_+$ is the $q \times q$-characteristic covering map defined above. This representation induces the following relations: $\rho_+(h_-^j)$ is the trivial element and $\rho_+(s_-^j)$ is sent to

$$\begin{pmatrix} e^{2i\pi q(a\alpha + c\beta)} & 0 \\ 0 & e^{-2i\pi q(a\alpha + c\beta)} \end{pmatrix}$$

in $PSL(2; \mathbb{C})$. Again since by [**40**] each element of $PSL(2; \mathbb{C})$ is a commutator, there exists a global representation $\rho : \pi_1 \widetilde{N} \to PSL(2; \mathbb{C})$ such that $\rho|\pi_1 \widetilde{Q}_+ = \rho_+$, because the only further restriction for its existence is that $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = s_-^1 \cdots s_-^p$. Denote $\rho_- = \rho|\pi_1 \widetilde{Q}_-$. Let A be a flat connection in hyperbolic normal form with respect to $\mathfrak{T}_{\widetilde{N}}$ such that $A = A_- \cup A_+$ where A_-, resp. A_+, is the restriction of A over $\widetilde{Q}_-$, resp. $\widetilde{Q}_+$. For each $j = 1, \cdots, p$ we identify the meridian of a solid torus $\mathbf{D}^2 \times \mathbf{S}^1$ with h_-^j and with $bs_+^j + dh_+^j$ then we get closed manifolds $\widetilde{Q}_-(0, 1)$ and $\widetilde{Q}_+(b, d)$ where A_+ and A_- extend to flat connections $\widehat{A}_+$ and $\widehat{A}_-$ such that

$$\mathfrak{cs}^*_{\widetilde{N}}(A) = \mathfrak{cs}^*_{\widetilde{Q}_+(b,d)}(\widehat{A}_+) \times \mathfrak{cs}^*_{\widetilde{Q}_-(0,1)}(\widehat{A}_-).$$

Keeping the same notation as in the previous section we get, by splitting the former equality into real and imaginary parts according to the additivity principle, we get

$$\mathrm{vol}(\widetilde{N},\rho) = \mathrm{vol}(\widetilde{Q}_+(b,d),\widehat{\rho}_+) + \mathrm{vol}(\widetilde{Q}_-(0,1),\widehat{\rho}_-)$$

Since now $\widetilde{Q}_-(0,1)$ is by construction a connected sum of $\mathbf{S}^2 \times \mathbf{S}^1$-factors, then $\mathrm{vol}(\widetilde{Q}_-(0,1),\widehat{\rho}_-) = 0$. Now applying the same arguments as in the former section we get

$$\mathrm{vol}(\widetilde{N},\rho) = pq^2 \,\mathrm{vol}\, Q_+(b,d) + \pi \frac{pq(q-1)}{2} length(\gamma)$$

where γ is now the geodesic added to Q_+ to complete the cusp with respect to the (b,d)-Dehn filling. This completes the proof of 2.14.

CHAPTER 3

New results

3.1. Introduction

In this chapter the new results contained in this thesis are proved. They are split into two parts. In Section 3.2 we derive an additivity result for the volume of a representation of a non-geometric 3-manifold when decomposing along tori. Namely we prove the following proposition:

PROPOSITION 3.1. *Let* $M = M_1 \cup_\tau M_2$ *be a 1-edged manifold and* $\rho : M \to G = Iso_e\widetilde{SL_2(\mathbb{R})}$ *a representation and* $X = G/K$ *for* K *a maximal compact subgroup. Let* (s_1, h_1) *be a basis of* $H_1(\partial M_1;\mathbb{Z}) = H_1(T^2;\mathbb{Z})$ *and* $(a,b) \in \mathbb{Z}^2$ *with* $gcd(a,b) = 1$ *and* $\rho(as_1 + bh_1) = \overline{(0,1)}$.

Then for $\widehat{M_i} = M_i(a,b)$, $i = 1,2$, ρ *extends to* $\widehat{\rho}_1 : \pi_1\widehat{M_1} \to G$ *and* $\widehat{\rho}_2 : \pi_1\widehat{M_2} \to G$ *and*

$$vol(M,\rho) = vol(\widehat{M_1}, \widehat{\rho}_1) + vol(\widehat{M_2}, \widehat{\rho}_2).$$

For a non-geometric prime 3-manifold M and a representation $\rho : \pi_1 M \to Iso_e\widetilde{SL_2(\mathbb{R})}$, the main difficulty in generalizing the additivity result in 2.5.6 is that we can not use Chern-Simons theory, since the associated principal G-bundle of ρ over M does not admit a section in general.

Instead of using Chern-Simons theory we directly consider the associated principal X-bundle of ρ over M. We prove that it is trivial over a neighbourhood $T \times (-1,1)$ of each essential torus $T \subset M$ and derive from this that we can normalize every section $s : M \to \widetilde{M} \times_\rho X$ in a neighbourhood of T.

In Section 3.3 we focus on the question whether there is a representation of non-zero volume for a given graph manifold M with a particular focus on one-edged graph manifolds.

Given a one-edged graph manifold M, we are able to compute the volume for most representations (see Proposition 3.14) and we derive from these results that for a large class of one-edged graph manifolds there is a representation of non-zero volume (see Proposition 3.15).

3.2. Additivity for representations into $Iso_e\widetilde{SL_2(\mathbb{R})}$

This section generalizes the additivity results in Section 2.5.6 to representations into $Iso_e\widetilde{SL_2(\mathbb{R})}$.

3.2.1. The centralizer of an element in $Iso_e\widetilde{SL_2(\mathbb{R})}$. In this section we prove the following proposition

PROPOSITION 3.2. *Let $\overline{(\zeta, sh(n))} \neq \overline{(\zeta, A)} \in Iso_e\widetilde{SL_2(\mathbb{R})}$.*
Then the centralizer $\Gamma_{\overline{(\zeta,A)}}$ of $\overline{(\zeta, A)}$ in $Iso_e\widetilde{SL_2(\mathbb{R})}$ is the group $\mathbb{R}\times_{\mathbb{Z}}\Gamma_A$, where Γ_A is the 1-dimensional abelian centralizer of A in $\widetilde{SL_2(\mathbb{R})}$.
In particular $\Gamma_{\overline{(\zeta,A)}}$ is connected, abelian and 2-dimensional.

Consider the universal covering $p : \widetilde{SL_2(\mathbb{R})} \to PSL_2(\mathbb{R})$ and a non-central element $A \in \widetilde{SL_2(\mathbb{R})}$, i.e. $A \notin \{sh(n)|n \in \mathbb{Z}\}$.

LEMMA 3.3. *Let $A \in \widetilde{SL_2(\mathbb{R})}$ be a non-central element. The centralizer Γ_A of A is the preimage of the centralizer $\overline{\Gamma}_{p(A)}$ of $p(A)$ in $PSL_2(\mathbb{R})$.*

PROOF. Let $B \in \widetilde{SL_2(\mathbb{R})}$ with $BA = AB$. Then $p(B)$ is in the centralizer of $p(A)$ and thus $p(\Gamma_A) \subset \overline{\Gamma}_{p(A)}$. Conversely let $B \in \widetilde{SL_2(\mathbb{R})}$ with $p(B) \in \overline{\Gamma}_{p(A)}$, that is $p(B)p(A) = p(A)p(B)$. Then $p([A, B]) = \mathbb{1}$ and thus $[A, B] = sh(n)$ $n \in \mathbb{Z}$.

By Lemma 2.24, we can write $sh(n)$, $n \in \mathbb{Z}$, as commutator if and only if $n < 2 - 1 = 1$ and $n > 1 - 2 = -1$ and thus $n = 0$. It follows that $[A, B] = sh(0) = \mathbb{1}$ and consequently $B \in \Gamma_A$. Hence we obtain that $p(\Gamma_A) \supset \overline{\Gamma}_{p(A)}$. □

COROLLARY 3.4. *Let $A \in \widetilde{SL_2(\mathbb{R})}$ be a non-central elliptic/parabolic/ hyperbolic element then the centralizer $\overline{\Gamma}_{p(A)}$ of $p(A)$ is isomorphic to $\overline{\Gamma}_{ell}$/ $\overline{\Gamma}_{par}$/$\overline{\Gamma}_{hyp}$ as Lie group and thus in particular abelian, connected, and 1-dimensional.*

PROOF. We consider only the elliptic case, the other cases are analogous. By Remark 2.32 there is $g \in PSL_2(\mathbb{R})$, such that $gp(A)g^{-1}$ is in elliptic normal form. Then clearly $\overline{\Gamma}_{p(A)} = g^{-1}\Gamma_{ell}g$ and the isomorphism is given by conjugation by g: $h \mapsto ghg^{-1}$. Thus $\Gamma_{p(A)}$ is connected and abelian of dimension 1. □

Consider $A \in \widetilde{SL_2(\mathbb{R})}$ with its centralizer $\Gamma_A = p^{-1}(\overline{\Gamma}_{p(A)})$. Corollary 3.4 implies that Γ_A is a 1-dimensional abelian Lie group, but it is not connected in general.

Let $\Gamma_{A,e}$ be the identity component of Γ_A. Then $p(\Gamma_{A,e}) = \overline{\Gamma}_{p(A)}$ and since the kernel of p is $\{sh(n)|n \in \mathbb{Z}\}$, we obtain that

$$\Gamma_A = p^{-1}(\overline{\Gamma}_{p(A)}) = \{sh(n)B|B \in \Gamma_{A,e}\}\,. \tag{3.2.1}$$

Denote by Γ_{ell}, Γ_{par}, Γ_{hyp} the preimages of the groups $\overline{\Gamma}_{ell}$, $\overline{\Gamma}_{par}$, $\overline{\Gamma}_{hyp}$ under p.

PROOF OF PROPOSITION 3.2. First observe that $\Gamma_{\overline{(\zeta,A)}} = \mathbb{R} \times_{\mathbb{Z}} \Gamma_A$:

Let $\overline{(z,B)} \in \Gamma_{\overline{(\zeta,A)}}$, that is $\overline{(z+\zeta, BA)} = \overline{(z,B)} \cdot \overline{(\zeta,A)} = \overline{(\zeta,A)} \cdot \overline{(z,B)} = \overline{(\zeta+z, AB)}$. Thus $BA = AB$ and therefore $B \in \Gamma_A$ which implies that $\overline{(z,B)} \in \mathbb{R} \times_{\mathbb{Z}} \Gamma_A$. Clearly all implications can be reversed. Hence we obtain that $\Gamma_{\overline{(\zeta,A)}} = \mathbb{R} \times_{\mathbb{Z}} \Gamma_A$.

$\Gamma_{\overline{(\zeta,A)}}$ is clearly abelian and 2-dimensional.

Let $\overline{(\zeta,A)}, \overline{(z,B)} \in \mathbb{R} \times_{\mathbb{Z}} \Gamma_A$. Then by equality (3.2.1) there is $n \in \mathbb{Z}$ such that $sh(n)B$ and A are in the same connected component of Γ_A.

Let $\gamma_2 : [0,1] \to \Gamma_A$ be a path with $\gamma_2(0) = A$ and $\gamma_2(1) = sh(n)B$ and define $\gamma_1 : [0,1] \to \mathbb{R}$ by $\gamma_1(t) = (1-t)\zeta + t(z-n)$.

We obtain a path

$$\gamma = \overline{(\gamma_1, \gamma_2)} : [0,1] \to \mathbb{R} \times_{\mathbb{Z}} \Gamma_A$$

with $\gamma(0) = \overline{(\zeta,A)}$ and $\gamma(1) = \overline{(z-n, sh(n)B)} = \overline{(z,B)}$. Thus $\Gamma_{\overline{(\zeta,A)}}$ is connected. □

3.2.2. Additivity of the volume of a representation. Let $M = M_1 \cup_\tau M_2$ be a closed 3-manifold, where M_1 and M_2 are glued along a torus $T^2 = M_1 \cap M_2$.

LEMMA 3.5. *Let M, X be smooth manifolds with X contractible, G a Lie group, and assume G acts on X from the left. Let $\rho : \pi_1 M \to G$ be a representation and $M_1 \subset M$ a submanifold. Assume that ρ restricts to $\rho_1 : \pi_1 M_1 \to G$. Then there is a natural diffeomorphism*

$$\widetilde{M_1} \times_{\rho_1} X \cong p^{-1}(M_1),$$

where $p : M \times_\rho X \to M$ is the projection.

PROOF. Choose a connected subset $\widehat{M_1}$ of $\widetilde{M}$ which covers M_1. Then there is a covering $\phi : \widetilde{M_1} \to \widehat{M_1}$ and as such ϕ is in particular open and continuous. Consider the map

$$\overline{\psi} : \widetilde{M_1} \times_{\rho_1} X \to p^{-1}(M_1)$$
$$[(m,x)]_{M_1} \mapsto [(\phi(m), x)]_M .$$

$\overline{\psi}$ is well-defined:

Since ϕ is a covering $\phi(\gamma \cdot m) = \gamma \cdot \phi(m)$. For $\gamma \in \pi_1 M_1$:

$$\begin{aligned}\overline{\psi}(\left[(\gamma \cdot m, \rho_1(\gamma)^{-1}x)\right]_{M_1}) &= \left[(\phi(\gamma \cdot m), \rho(\gamma)^{-1}x)\right]_M \\ &= \left[(\gamma \cdot \phi(m), \rho(\gamma)^{-1}x)\right]_M \\ &= [(\phi(m), x)]_M\end{aligned}$$

It is not hard to see that $\overline{\psi}$ is continuous and open and in fact it is a homeomorphism. For that it is left to check that $\overline{\psi}$ is bijective:

Injectivity:

Let $[(\phi(m_1), x_1)]_M = [(\phi(m_2), x_2)]_M$. Then there exists $\gamma \in \pi_1 M$ such that $(\gamma.\phi(m_1), \rho(\gamma)^{-1}x_1) = (\phi(m_2), x_2)$. In particular $\gamma.\phi(m_1) = \phi(m_2)$, and $\rho(\gamma)^{-1}x_1 = x_2$.

Since $\phi(m_1), \phi(m_2) \in \widehat{M_1}$, we obtain that γ can be represented by a loop in M_1 and thus be interpreted as element of $\pi_1 M_1$. It follows that $[(m_1, x_1)]_{M_1} = [(m_2, x_2)]_{M_1}$.

Surjectivity:

Let $[(m, x)]_M \in p^{-1}(M_1)$. Then there is $\gamma \in \pi_1 M$ such that $\gamma.m \in \widehat{M_1}$.

Let $m_1 \in \widetilde{M_1}$ with $\phi(m_1) = \gamma.m$. Then clearly $\left[(m_1, \rho(\gamma)^{-1}x)\right]_{M_1} \in \widetilde{M_1} \times_{\rho_1} X$ is mapped to $[(m, x)]_M$.

$\overline{\psi}$ is a diffeomorphism, since all maps are smooth. □

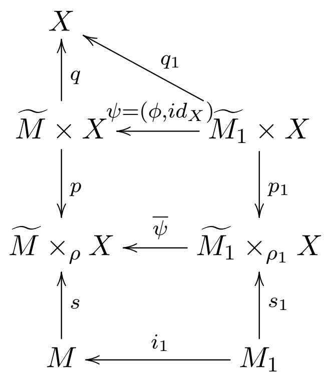
The following Lemma will be used, when proving additivity:

Lemma 3.6. *Let M be a 3-manifold, X a contractible 3-manifold, $\rho : \pi_1 M \to G$ a representation, $s : M \to M \times_\rho X$ a section, and ω a G-invariant closed form on X. Let $i_1 : M_1 \hookrightarrow M$ be a submanifold, $s_1 = s \circ i_1 : M_1 \to M_1 \times_{\rho_1} X$ the induced section, and $\rho_1 : \pi_1 M_1 \to G$ the induced representation. Then*

$$s_1^* \omega_1' = s^* \omega' |_{M_1},$$

where ω_1', resp. ω', are the closed forms that ω induces on $M_1 \times_{\rho_1} X$, resp. $M \times_\rho X$.

Proof. Let $p : \widetilde{M} \times X \to \widetilde{M} \times_\rho X$, $p_1 : \widetilde{M_1} \times X \to_{\rho_1} X$ the natural projections. Consider the following diagram:



By definition of the maps and Lemma 3.5 the diagram is clearly commutative. Let ω be a G-invariant closed k-form on X.

By commutativity of the upper triangle we clearly obtain

$$\psi^* q^* \omega = q_1^* \omega.$$

Let ω' and ω_1' be the k-forms on $\widetilde{M} \times_\rho X$ and $\widetilde{M_1} \times_\rho X$ induced by $q^*\omega$, resp. $q_1^*\omega$. Commutativity of the upper square implies

$$\overline{\psi}^* \omega' = \omega_1' :$$

Let $Z_1, \cdots, Z_k \in T(\widetilde{M_1} \times_{\rho_1} X)$ and let $\widetilde{Z_1}, \cdots, \widetilde{Z_k} \in T(\widetilde{M_1} \times X)$ with $p_{1*}\widetilde{Z}_i = Z_i$, $i = 1, \cdots, k$. By commutativity:

$$p_* \psi_* \widetilde{Z}_i = \overline{\psi} p_{1*} \widetilde{Z}_i, i = 1, \cdots, k$$

Thus

$$\begin{aligned}(\overline{\psi}^*\omega')(Z_1,\cdots,Z_k) &= \omega'(\overline{\psi}_*Z_1,\cdots,\overline{\psi}_*Z_k) = \omega'(\overline{\psi}_*p_{1*}\widetilde{Z_1},\cdots,\overline{\psi}_*p_{1*}\widetilde{Z_k})\\ &= \omega'(p_*\psi_*\widetilde{Z_1},\cdots,p_*\psi_*\widetilde{Z_k}) = (q^*\omega)(\psi_*\widetilde{Z_1},\cdots,\psi_*\widetilde{Z_k})\\ &= (q_1^*\omega)(\widetilde{Z_1},\cdots,\widetilde{Z_k}) = \omega_1'(Z_1,\cdots,Z_k).\end{aligned}$$

Commutativity of the lower square then clearly implies

$$s^*\omega'|_{M_1} = i_1^*s^*\omega' = s_1^*\overline{\psi}^*\omega' = s_1^*\omega_1'$$

□

REMARK 3.7. As a consequence of Lemma 3.6 from now we will not distinguish between ω_1' and ω' and simply write ω' always.

Fix a section $s: M \to \widetilde{M}\times_\rho X$. The inclusions $i_1: M_1 \to M$, $i_2: M_2 \to M$ induce sections

$$s_1 = i_1\circ s : M_1 \to \widetilde{M_1}\times_\rho X,$$
$$s_2 = i_2\circ s : M_2 \to \widetilde{M_2}\times_\rho X.$$

We derive a normal form for s in a neighbourhood $T\times(-1,1)$ of T.

Let $\rho_T : \pi_1 T \to G$ be the restriction of ρ to $\pi_1 T$. As a consequence of Lemma 3.5 we obtain that $p^{-1}(T\times(-1,1)) \cong (T\times(-1,1))\times_{\rho_T} X$ as X-bundle over $T\times(-1,1)$.

Since $\pi_1 T = \mathbb{Z}\oplus\mathbb{Z}$ is abelian, Proposition 3.2 implies that $\rho_T(\pi_1 T)$ is contained in a 2-dimensional connected, abelian subgroup $\Gamma < G$. Hence there is a homotopy of representations $\rho^t : \pi_1 T \to \Gamma$ such that $\rho^0 = \rho_e$ and $\rho^1 = \rho_T$, where $\rho_e \equiv e$ denotes the trivial representation.

LEMMA 3.8. $(T\times(-1,1))\times_{\rho_T} X \to T\times(-1,1)$ *is trivial.*

PROOF. Recall that $T = \mathbb{R}^2/\mathbb{Z}^2$, where $\mathbb{Z}^2 = \pi_1 T$ acts by $(n,m)\cdot(\alpha,\beta) = (\alpha+n,\beta+m)$ on $\mathbb{R}^2$ and consider the map

$$(T\times(-1,1))\times X \to (T\times(-1,1))\times_{\rho_T} X$$
$$\left((\overline{(\alpha,\beta)},z),x\right) \mapsto \left((\overline{(\alpha,\beta)},z),\rho^{\alpha-\lfloor\alpha\rfloor}(-1,0)\rho^{\beta-\lfloor\beta\rfloor}(0,-1)\rho_T(-\lfloor\alpha\rfloor,-\lfloor\beta\rfloor)x\right)$$

It is not hard to see that the map is a well-defined, continuous isomorphism of X-bundles over $T\times(-1,1)$ and consequently $(T\times(-1,1))\times_{\rho_T} X$ is trivial. □

PROOF OF PROPOSITION 3.1. Let (s_1,h_1) be a basis of $H_1(\partial M_1;\mathbb{Z})$ and as_1+bh_1 a slope in T, with $gcd(a,b)=1$. Clearly, after a diffeomorphism of the base $T\times(-1,1)$ of the restricted bundle $(T\times(-1,1))\times_{\rho_T} X$, we may assume that as_1+bh_1 is homotopic to the generator $(1,0)\in\pi_1 T$ of the meridian $\mathbf{S}^1\times pt\times 0 \subset T\times(-1,1)$.

The restriction $s: T\times(-1,1) \to T\times(-1,1)\times X$ of the section s is of the form $s\left((\overline{(\alpha,\beta)},z)\right) = \left((\overline{(\alpha,\beta)},z), s_T\left((\overline{(\alpha,\beta)},z)\right)\right)$ for a smooth $s_T : T\times(-1,1)\to X$.

Let $\phi : X\times[0,1]\to X$ be a deformation retraction of X to $x_0\in X$ with $\phi(x,0)=x$, $\phi(x,1)\equiv x_0$ and let $f:(-1,1)\to[0,1]$ be a smooth cutoff function with $f(z)\equiv 1$, for $|z|<1/4$, and $f(z)\equiv 0$, for $|z|>1/2$.

Define $\widetilde{s}_T : T\times(-1,1) \to X$ by $\widetilde{s}_T(\overline{(\alpha,\beta)}, z) = \phi(s(\overline{(\alpha,\beta)}, z), \underline{f(z)})$. $\widetilde{s}_T$ is a smooth section, since ϕ, s, and f are smooth, and satisfies $\widetilde{s}_T(\overline{(\alpha,\beta)}, z) = x_0$, for $|z| < 1/4$, and $\widetilde{s}_T(\overline{(\alpha,\beta)}, z) = s_T(\overline{(\alpha,\beta)}, z)$, for $|z| > 1/2$.Then by construction $\widetilde{s}_T$ extends to a smooth section $\widetilde{s} : M \to M \times_\rho X$.

Assume that $\rho_T(1,0) = \overline{(0,1)} \in G$ and recall that $(1,0)$ corresponds to the slope $as_1 + bh_1$ in M_1. Then the representation ρ extends to a representation $\widehat{\rho}_1$ on $\widehat{M_1} = M_1(a,b) = M_1 \cup V_1$, where V_1 is a solid torus attached to M_1 by identifying the slope $as_1 + bh_1$ in ∂M_1 with the meridian of V_1. Obviously the trivialization in Lemma 3.8 induces a trivialization over $(T \times (-1,0]) \cup V_1$. In this trivialization the section $\widetilde{s} : T \times (-1,0] \to X$ extends to a smooth section $\widehat{s}_1 : T \times (-1,0] \cup V_1 \to X$ with $\widehat{s}_T|_{V_1} \equiv x_0$, which extends $\widetilde{s}|_{M_1}$ to $\widehat{s} : \widehat{M_1} \to \widehat{M_1} \times_{\widehat{\rho}_1} X$.

Similarly the representation ρ extends to a representation $\widehat{\rho}_2$ on $\widehat{M_2} = M_2(a,b) = M_2 \cup V_2$, the trivialization in Lemma 3.8 induces a trivialization over $V_2 \cup (T \times [0,1))$ and $\widetilde{s} : T \times [0,1) \to X$ extends to a section $\widehat{s}_2 : V_2 \cup [0,1) \to X$ with $\widehat{s}_2(p) \equiv x_0$ on V_2, which extends $\widetilde{s}|_{M_2}$ to $\widehat{s}_2 : \widehat{M_2} \to \widehat{M_2} \times_{\widehat{\rho}_2} X$.

Observe that since the gluing between ∂M_1 and ∂M_2 is orientation reversing and the sections $\widehat{s}_1$ and $\widehat{s}_2$, as well as the representations $\widehat{\rho}_1$ and $\widehat{\rho}_2$ coincide on the solid torus $V_1 = V_2$, we obtain

$$\int_{V_1} \widehat{s}_1^* \omega'_X = - \int_{V_2} \widehat{s}_2^* \omega'_X$$

Thus using Lemma 3.6, the volume $vol(M,\rho)$ (without absolute values) can be computed as follows:

$$\begin{aligned} vol(M,\rho) &= \int_M \widetilde{s}^* \omega'_X = \int_{M_1} \widetilde{s}|_{M_1}^* \omega'_X + \int_{M_2} \widetilde{s}|_{M_2}^* \omega'_X \\ &= \int_{M_1} \widehat{s}|_{M_1}^* \omega'_X + \int_{V_1} \widehat{s}_1^* \omega'_X + \int_{V_2} \widehat{s}_2^* \omega'_X + \int_{M_2} \widehat{s}|_{M_2}^* \omega'_X \\ &= \int_{\widehat{M_1}} \widehat{s}_1^* \omega'_X + \int_{\widehat{M_2}} \widehat{s}_2^* \omega'_X \\ &= vol(\widehat{M_1}, \widehat{\rho}_1) + vol(\widehat{M_2}, \widehat{\rho}_2) \end{aligned}$$

Thus we proved additivity directly from the definition of the volume of a representation without using Chern-Simons theory. □

REMARK 3.9. Additivity can be generalized in an obvious way to non-geometric manifolds that contain more than one essential torus $\{T_1, \cdots T_k\}$, since all arguments in the proof only depend on neighbourhoods $T_i \times (-1,1)$ of the tori.

3.3. Representations on Graph manifolds

Let M be a connected graph manifold. That is, the JSJ decomposition of M consists of Seifert pieces only which we denote $M_1, \cdots, M_l$. Denote further by $T_1, \cdots, T_k$ the tori along which the JSJ-decomposition is performed and by $A_1, \cdots, A_k \in GL(2;\mathbb{Z})$ the gluing maps. For the moment assume that every two Seifert pieces are connected by at least one torus.

We want to give an explicit description of the set

$$\left\{ Vol_G(M,\rho) \middle| \rho : \pi_1 M \to Iso_e\widetilde{SL_2(\mathbb{R})} \text{ a representation} \right\},$$

where $Vol_G(M,\rho)$ is the volume of the representation.

For the classification the following Lemma is useful.

Lemma 3.10. *Let M be as above and let $\rho : \pi_1 M \to Iso_e\widetilde{SL_2(\mathbb{R})} = G$ be a representation. Denote by $h_1, \cdots, h_l$ the fibers of the Seifert pieces. Then there exists a connected 2-dimensional Lie subgroup $C < G$ such that for each $i = 1, \cdots, l$ either $\rho(h_i) = \overline{(\zeta_i, 1)}$ or $\rho(\pi_1 M_i) \subset C$ where $\pi_1 M_i$ is the subgroup of the fundamental group generated by all loops in M_i.*

Proof. Assume that $\rho(h_i) = \overline{(x, A)} \neq \overline{(\zeta_i, 1)}$ for some i, that is, $\rho(h_i)$ is not central in G. Without loss of generality we may assume that $i = 1$. Since h_1 commutes with every $\delta \in \pi_1 M_1$, Proposition 3.2 implies that $\rho(\pi_1 M_1)$ is contained in the 2-dimensional connected abelian subgroup $C = \mathbb{R} \times_{\mathbb{Z}} \Gamma_A$ of G, where Γ_A is the centralizer of A in $\widetilde{SL_2(\mathbb{R})}$. Now consider the boundary tori of M_1. For simplicity we assume that there is only one boundary torus T_1 which connects M_1 to M_2. If there is more than one boundary torus then the proof generalizes in an obvious way.

Choose section fiber bases on the boundaries of M_1 and M_2 that belong to T_1 and denote them by (s_1, h_1) and (s_2, h_2). Let $A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the gluing map, satisfying

$$(s_2 h_2) = (s_1 h_1) A_1.$$

Then $\rho(s_2) = \rho(s_1)^a \rho(h_1)^c$ and $\rho(h_2) = \rho(s_1)^b \rho(h_1)^d$ and thus $\rho(h_2) = \overline{(\zeta_2, 1)}$ or $\rho(h_2) \in C$. The claim follows, since by assumption M_i and M_j, $i \neq j$ are connected by some torus for all i, j. □

3.3.1. One-edged Graph manifolds.

We now consider a one-edged Graph manifold M. That is M consists of two Seifert pieces M_1 and M_2 glued along one torus $T = \partial M_1 = \partial M_2$.

Choose section fiber bases $(s^{(1)}, h^{(1)})$, $(s^{(2)}, h^{(2)})$ for the boundaries ∂M_1, resp. ∂M_2. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2;\mathbb{Z})$ be the gluing matrix with $\left(s^{(2)}, h^{(2)}\right) = \left(s^{(1)}, h^{(1)}\right) A$. In particular $b \neq 0$ since M is not a Seifert fibered manifold and after possibly changing the orientation of the section fiber basis $detA = -1$.

We fix the notation for the generators of the fundamental group of M_j, $j = 1, 2$ as follows:

$$\pi_1 M_j = \left\langle \begin{array}{l} s^{(j)}, h^{(j)} \\ \alpha_1^{(j)}, \beta_1^{(j)}, \cdots, \alpha_{g^{(j)}}^{(j)}, \beta_{g^{(j)}}^{(j)}, \\ \gamma_1^{(j)}, \cdots, \gamma_{d^{(j)}}^{(j)} \end{array} \middle| \begin{array}{l} \left[\alpha_1^{(j)}, \beta_1^{(j)}\right] \cdots \left[\alpha_{g^{(j)}}^{(j)}, \beta_{g^{(j)}}^{(j)}\right] = \gamma_1^{(j)} \cdots \gamma_{d^{(j)}}^{(j)} s^{(j)}, \\ (\gamma_i^{(j)})^{a_i} (h^{(j)})^{b_i} = 1, i = 1, \cdots, d^{(j)}, \\ \left[h^{(j)}, *\right] = 1 \end{array} \right\rangle$$

The Seifert van Kampen Theorem (see for instance [**23**, p.43]) implies that the fundamental group $\pi_1 M$ of M is generated by

$$\begin{array}{l} s^{(1)}, h^{(1)}, \alpha_1^{(1)}, \beta_1^{(1)}, \cdots, \alpha_{g^{(1)}}^{(1)}, \beta_{g^{(1)}}^{(1)}, \gamma_1^{(1)}, \cdots, \gamma_{d^{(1)}}^{(1)}, \\ s^{(2)}, h^{(2)}, \alpha_1^{(2)}, \beta_1^{(2)}, \cdots, \alpha_{g^{(2)}}^{(2)}, \beta_{g^{(2)}}^{(2)}, \gamma_1^{(2)}, \cdots, \gamma_{d^{(2)}}^{(2)}, \end{array}$$

which are subject to the following relations

$$\begin{array}{l} \left[\alpha_1^{(j)}, \beta_1^{(j)}\right] \cdots \left[\alpha_{g^{(j)}}^{(j)}, \beta_{g^{(j)}}^{(j)}\right] = \gamma_1^{(j)} \cdots \gamma_{d^{(j)}}^{(j)} s^{(j)}, \\ (\gamma_i^{(j)})^{a_i} (h^{(j)})^{b_i} = 1, i = 1, \cdots, d^{(j)}, \\ \left[h^{(j)}, \pi_1 M_j\right] = 1, j = 1, 2, \\ s^{(2)} = (s^{(1)})^a (h^{(1)})^c, h^{(2)} = (s^{(1)})^b (h^{(1)})^d \end{array}$$

Assume we are given a representation $\rho : \pi_1 M \to G = Iso_e \widetilde{SL_2(\mathbb{R})} = \mathbb{R} \times_{\mathbb{Z}} \widetilde{SL_2(\mathbb{R})}$ and we want to compute $vol(M, \rho)$. By Lemma 3.10 it is sufficient to distinguish 3 cases:

(1) $\rho(\pi_1 M) \subset C$ for C some two-dimensional Lie subgroup of G.
(2) $\rho(h^{(1)}) = \overline{(\zeta_1, 1)}$ and $\rho(h^{(2)}) = \overline{(\zeta_2, 1)}$ for some $\zeta_1, \zeta_2 \in \mathbb{R}$.
(3) $\rho(h^{(1)}) = \overline{(\zeta_1, 1)}$ and $\rho(\pi_1 M_2) \subset C$ for some two-dimensional Lie subgroup of G.

.

Case 1: A dimension argument in cohomology directly implies $vol(M, \rho) = 0$. It can be found in the proof of Proposition 2.4.2.

Case 2: Since $A^{-1} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$ we obtain:

$$(*) \begin{cases} h^{(1)} & = bs^{(2)} - ah^{(2)} \\ h^{(2)} & = bs^{(1)} + dh^{(1)} \end{cases}$$

Let $\rho(s^{(1)}) = (z^{(1)}, x^{(1)})$, $\rho(s^{(2)}) = (z^{(2)}, x^{(2)})$. Then the relations $(*)$ imply

$$(**) \begin{cases} \overline{(\zeta_1, 1)} & = \overline{(bz^{(2)} - a\zeta_2, (x^{(2)})^b)} \\ \overline{(\zeta_2, 1)} & = \overline{(bz^{(1)} + d\zeta_1, (x^{(1)})^b)}. \end{cases}$$

Since $b \neq 0$ it is immediate that there exist $n^{(1)}$, $n^{(2)}$ such that $x^{(1)}$ is a conjugate of $sh\left(\frac{n^{(1)}}{b}\right)$ and $x^{(2)}$ is a conjugate of $sh\left(\frac{n^{(2)}}{b}\right)$.

LEMMA 3.11. *Given a representation $\rho : \pi_1 M \to G$ with $\rho(h^{(j)}) = \overline{(\zeta_j, 1)}$, $\zeta_j \in \mathbb{R}$, $j = 1, 2$, then ρ is homotopic to a representation with $\zeta_j \in \mathbb{Q}$, $j = 1, 2$.*

PROOF. Fix the following notations for $j = 1, 2$, $i = 1, \cdots, d^{(j)}$:

$$\rho(h^{(j)}) = \overline{(\zeta_j, 1)},$$
$$\rho(s^{(j)}) = \overline{(z^{(j)}, x^{(j)})},$$
$$\rho(\gamma_i^{(j)}) = \overline{(z_i^{(j)}, x_i^{(j)})}.$$

and let $e_j = \sum_{i=1}^{d^{(j)}} \frac{b_i^{(j)}}{a_i^{(j)}}$, j=1,2.

From the relations in the fundamental group we obtain that the $x_i^{(j)}$ are conjugated to elements of the form $sh(-n_i^{(j)}/a_i)$ for integers $n_i^{(j)}$. Let $r_j = \sum_{i=1}^{d^{(j)}} \frac{n_i^{(j)}}{a_i^{(j)}}$, $j = 1, 2$. Then

$$z_i^{(j)} = \frac{n_i^{(j)}}{a_i^{(j)}} - \frac{b_i^{(j)}}{a_i^{(j)}}\zeta_j \tag{3.3.1}$$

$$z^{(j)} = e_j\zeta_j - r_j. \tag{3.3.2}$$

From the gluing we obtain that $\zeta_1 = bz^{(2)} - a\zeta_2 + n^{(2)}$. Plugging in equation (3.3.2), we obtain

$$\zeta_1 = b\,(e_2\zeta_2 - r_2) - a\zeta_2 + n^{(2)} \tag{3.3.3}$$

Again from the gluing along the torus it follows that for some $n_{z^{(2)}} \in \mathbb{Z}$

$$z^{(2)} = az^{(1)} + c\zeta_1 + n_{z^{(2)}}.$$

One easily checks that $n_{z^{(2)}} = \frac{an^{(1)}-n^{(2)}}{b}$ and as a direct consequence we observe that necessarily $an^{(1)} - n^{(2)} \equiv 0 \bmod b$. Plugging in equation (3.3.2) for $z^{(1)}$, we obtain

$$z^{(2)} = a\,(e_1\zeta_1 - r_1) + c\zeta_1 + n_{z^{(2)}}$$

and equation (3.3.3) then implies

$$\begin{aligned} z^{(2)} =& a\left(e_1\left(b\,(e_2\zeta_2 - r_2) - a\zeta_2 + n^{(2)}\right) - r_1\right) \\ &+ c\left(b\,(e_2\zeta_2 - r_2) - a\zeta_2 + n^{(2)}\right) + n_{z^{(2)}} \\ =&(ae_1 + c)(be_2 - a)\zeta_2 + (ae_1 + c)(n^{(2)} - br_2) + n_{z^{(2)}} - ar_1 \end{aligned}$$

With equation (3.3.2) that implies

$$(ae_1 + c)(be_2 - a)\zeta_2 + (ae_1 + c)(n^{(2)} - br_2) + n_{z^{(2)}} - ar_1 = e_2\zeta_2 - r_2$$

$$\overset{ad-bc=-1}{\Leftrightarrow} a((be_1+d)e_2-(ae_1+c))\zeta_2 = (ae_1+c)b\left(r_2 - \frac{n^{(2)}}{b}\right)+a\left(r_1 - \frac{n^{(1)}}{b}\right)-\left(r_2 - \frac{n^{(2)}}{b}\right)$$

$$\overset{ad-bc=-1}{\Leftrightarrow} a((be_1 + d)e_2 - (ae_1 + c))\zeta_2 = a\left((be_1 + d)\left(r_2 - \frac{n^{(2)}}{b}\right) + \left(r_1 - \frac{n^{(1)}}{b}\right)\right)$$

Hence we obtain that

$$\zeta_2 = \frac{(be_1 + d)\left(r_2 - \frac{n^{(2)}}{b}\right) + \left(r_1 - \frac{n^{(1)}}{b}\right)}{(be_1 + d)e_2 - (ae_1 + c)} \tag{3.3.4}$$

Hence ζ_2 must be rational whenever $(be_1 + d)e_2 - (ae_1 + c) \neq 0$. If the left side is zero, then we see from the fundamental group that we can homotope the representation in such a way that ζ_2 becomes rational. Indeed, we see that we can easily homotope ρ in such a way that ζ_2 becomes rational, if we ignore the obstruction given by $s^{(2)} = (s^{(1)})^a(h^{(1)})^b$. But whenever $((ae_1 + c)(be_2 - a) - e_2) = 0$, then this obstruction clearly does not affect ζ_2 due to equation (3.3.4). Thus we may assume that ζ_2 is rational and as a direct consequence also ζ_1 must be rational. □

Remark 3.12. Note that the proof of Lemma 3.11 in particular shows that if $(be_1 + d)e_2 - (ae_1 + c) \neq 0$, then ζ_j, $j = 1, 2$ depend only on r_1, r_2, $n^{(1)}$, $n^{(2)}$ and M.

Thus we can write $\zeta_1 = \frac{p_h}{q_h}$ and it is easy to see that the obstructions on the fundamental group imply that then also $z^{(1)} = \frac{p_s}{q_s}$ is rational.

For gluing in a solid torus along the slope $\alpha s^{(1)} + \beta h^{(1)}$ so that the representation extends to this torus it is necessary and sufficient that

$$\overline{(\alpha z^{(1)} + \beta\zeta_1, (x^{(1)})^\alpha)} = \rho(s^{(1)})^\alpha \rho(h^{(1)})^\beta = \overline{(0,1)}.$$

It suffices to find $r, s \in \mathbb{Z}$ such that $b|r$ (since then $r\frac{n^{(1)}}{b} \in \mathbb{Z}$) and

$$rz^{(1)} + s\zeta_1 + r\frac{n^{(1)}}{b} = r\frac{(bz^{(1)} + n^{(1)})}{b} + s\zeta_1 = 0,$$

since that implies

$$\rho(rs^{(1)} + s\zeta_1) = \overline{(rz^{(1)} + s\zeta_1, (x^{(1)})^r)} = \overline{\left(-r\frac{n^{(1)}}{b}, sh(r\frac{n^{(1)}}{b})\right)} = \overline{(0,1)}.$$

We see that such $r, s \in \mathbb{Z}$ exist, and a possible choice is $r = \frac{bp_h}{gcd(b,p_h)}$, $s = -\frac{bz^{(1)}+n^{(1)}}{p_h}q_h r$. Furthermore at least one of them can be assumed to be non-zero.

Thus it is possible to perform a Dehn filling on M_1 so that the representation extends to $M_1(\alpha, \beta)$, where $\alpha = r$ and $\beta = s$ and perform the same Dehn filling on M_2, denoting the slope by $\widetilde{\alpha}s^{(2)} + \widetilde{\beta}h^{(2)}$.

Remark 3.13. Note that $M_1(\alpha, \beta)$ and $M_2(\widetilde{\alpha}, \widetilde{\beta})$ might be orbifolds, since there is no reason for which we can always assume $gcd(\alpha, \beta) = 1$, because we need that $b|\alpha$.

If $gcd(\alpha, \beta) = 1$ we can apply Proposition 3.1 and obtain

$$vol(M, \rho) = vol(M_1(\alpha, \beta)) + vol(M_2(\widetilde{\alpha}, \widetilde{\beta})).$$

Case 3: After possibly taking a conjugate representation we may assume that C is one of the following three 2-dimensional Lie groups $\mathbb{R} \times_{\mathbb{Z}} \Gamma_{ell}$, $\mathbb{R} \times_{\mathbb{Z}} \Gamma_{par}$, $\mathbb{R} \times_{\mathbb{Z}} \Gamma_{hyp}$.

Note that in any of the three cases the restriction of ρ to $\pi_1 M_2$ is completely determined for a fixed $\rho(h^{(1)})$ up to homotopies of the $\rho(\alpha_i^{(1)})$ and $\rho(\beta_i^{(1)})$.

The elliptic case is very similar to case 2 above and thus will not be discussed here. For the other two cases note that $C \cong \mathbb{R}^2$, where identification is as follows:

By Proposition 3.2, the centralizer of $\overline{(x, A)}$ is the group $\mathbb{R} \times_{\mathbb{Z}} \Gamma_A$, where Γ_A is either Γ_{par} or Γ_{hyp}. We know from Section 3.2.1 that $\Gamma_A = \{sh(n)B | B \in \Gamma_{A,e}\}$.

Thus we can uniquely write $\overline{(x,B)} \in \Gamma_{par}/\Gamma_{hyp}$ as $\overline{(x',B')}$ with $B' \in \Gamma_{par,e}/\Gamma_{hyp,e}$. Since $\Gamma_{par,e}/\Gamma_{hyp,e} \cong \mathbb{R}$ this gives us the desired identification of C with $\mathbb{R}^2$.

We claim that in those two cases $\rho(h^{(2)})^{-a}\rho(s^{(2)})^b = \rho(h^{(1)}) = \overline{(\zeta_1,1)} = \overline{(0,1)}$. To see this consider the obstructions coming from $\pi_1 M_2$. As above let

$$\begin{aligned}\rho(h^{(2)}) &= \overline{(\zeta_2, y^{(2)})}\\ \rho(s^{(2)}) &= \overline{(z^{(2)}, x^{(2)})}\\ \rho(\gamma_i^{(2)}) &= \overline{(z_i^{(2)}, x_i^{(2)})}\end{aligned}$$

The obstructions we get are

(1) $\rho(h^{(2)})^{-a}\rho(s^{(2)})^b = \overline{(\zeta_1,1)} \Rightarrow x^{(2)} = \frac{a}{b}y^{(2)}$

(2) $\rho(\gamma_i^{(2)})^{a_i^{(2)}}\rho(h^{(2)})^{b_i^{(2)}} = \overline{(0,1)} \Rightarrow x_i^{(2)} = -\frac{b_i^{(2)}}{a_i^{(2)}}y^{(2)}, z_i^{(2)} = -\frac{b_i^{(2)}}{a_i^{(2)}}\zeta_2$

(3) $\overline{(0,1)} = \rho\left(\left[\alpha_1^{(2)},\beta_1^{(2)}\right]\cdots\left[\alpha_{g^{(2)}}^{(2)},\beta_{g^{(2)}}^{(2)}\right]\right) = \rho(\gamma_1^{(2)})\cdots\rho(\gamma_{d^{(2)}}^{(2)})\rho(s^{(2)})$
$\overset{(1),(2)}{\Rightarrow}, \frac{a}{b}y^{(2)} - \left(\sum_{i=1}^{d^{(2)}}\frac{b_i^{(2)}}{a_i^{(2)}}\right)y^{(2)} = 0,\ z^{(2)} = \left(\sum_{i=1}^{d^{(2)}}\frac{b_i^{(2)}}{a_i^{(2)}}\right)\zeta_2$

But then since $y^{(2)} \neq 0$ the first equation in (3) implies

$$-\sum_{i=1}^{d^{(2)}}\frac{b_i}{a_i} + \frac{a}{b} = 0.$$

Plugging that into the second equation that we derived in (3) implies that

$$z^{(2)} = \frac{a}{b}\zeta_2 \Rightarrow bz^{(2)} - a\zeta_2 = 0.$$

Hence $\rho(h^{(1)}) = \overline{(0,1)}$. But then we cut M open along T and perform a Dehn filling on M_2 with slope $bs^{(2)} - ah^{(2)}$ and on M_1 with slope $h^{(1)}$. Additivity implies that

$$vol(M,\rho) = vol(M_2(b,-a),\rho) + vol(M_1(0,1),\rho) = vol(M_1(0,1),\rho),$$

where the last equality follows since C is 2-dimensional.

It follows from [**46**, part 6.2.4 of Thm. 6.3.3] that $M_1(0,1)$ is a connected sum of copies of $\mathbf{S}^2 \times \mathbf{S}^1$. By [**30**, Theorem 1] a connected sum of copies of $\mathbf{S}^2 \times \mathbf{S}^1$ is dominated by a product $\sigma \times \mathbf{S}^1$ for a surface σ and thus has zero Seifert volume. Hence we obtain

$$vol(M,\rho) = 0$$

The discussion in particular shows that Case 2 and Case 3 with C elliptic are the most interesting cases, since in all other Cases the volume of the representation is zero. We now concentrate on Case 2 and give a description of all representations that occur in this case. From this result we will conclude that for a large class of one-edged Graph manifolds there is a representation of non-zero volume.

PROPOSITION 3.14. *Let $M = M_1 \cup_T M_2$ be a one-edged graph manifold, where $M_j = \left(g^{(j)}, 1; \frac{b_1^{(j)}}{a_1^{(j)}}, \cdots, \frac{b_{d^{(j)}}}{a_{d^{(j)}}}\right)$, $j = 1, 2$ with section-fiber bases $(h^{(j)}, s^{(j)})$, with respect to which the gluing is given by the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, that is $(s^{(1)}, h^{(1)}) = (s^{(2)}, h^{(2)})A$.*

Assume that $(be_1 + d)e_2 - (ae_1 + c) \neq 0$. Let $n^{(j)}, n_1^{(j)}, \cdots, n_{d^{(j)}}^{(j)} \in \mathbb{Z}$, $j = 1, 2$, satisfying the conditions

$$\sum_{i=1}^{d^{(j)}} \left\lfloor \frac{n_i^{(j)}}{a_i^{(j)}} \right\rfloor + \left\lfloor -\frac{n^{(j)}}{b} \right\rfloor \leq 2g^{(j)} - 2 \text{ and } \sum_{i=1}^{d^{(j)}} \left\lceil \frac{n_i^{(j)}}{a_i^{(j)}} \right\rceil + \left\lceil -\frac{n^{(j)}}{b} \right\rceil \geq 2 - 2g^{(j)},\ j = 1, 2, \tag{3.3.5}$$

$$an^{(1)} - n^{(2)} \equiv 0 \text{ mod } b. \tag{3.3.6}$$

Let furthermore $e_j = \sum_{i=1}^{d^{(j)}} \frac{b_i^{(j)}}{a_i^{(j)}}$, $r_j = \sum_{i=1}^{d^{(j)}} \frac{n_i^{(j)}}{a_i^{(j)}}$, and $\widehat{r}_j = r_j - \frac{n^{(j)}}{b}$, $j = 1, 2$. Then there is a representation $\rho : \pi_1 M \to Iso_e\widetilde{SL_2(\mathbb{R})}$ with

$$\rho(h^{(j)}) = \overline{(\zeta_j, 1)},$$
$$\rho(\gamma_i^{(j)}) = \overline{(z_i^{(j)}, x_i^{(j)})},$$
$$\rho(s^{(j)}) = \overline{(z^{(j)}, x^{(j)})},$$

where $z_i^{(j)} = \frac{n_i^{(j)}}{a_i^{(j)}} - \frac{b_i^{(j)}}{a_i^{(j)}}\zeta_j$, $z^{(j)} = e_j\zeta_j - r_j$, $x_i^{(j)}$ is conjugated to $sh(-n_i^{(j)}/a_i^{(j)})$, $x^{(j)} = sh(n^{(j)}/b)$, for $j = 1, 2$, and

$$\zeta_1 = \frac{(be_2 - a)\widehat{r}_1 + \widehat{r}_2}{(be_1 + d)e_2 - (ae_1 + c)}$$

$$\zeta_2 = \frac{(be_1 + d)\widehat{r}_2 + \widehat{r}_1}{(be_1 + d)e_2 - (ae_1 + c)}$$

are uniquely determined by $\widehat{r}_1$, $\widehat{r}_2$, and M.

Conversely given a representation $\rho : \pi_1 M \to Iso_e\widetilde{SL_2(\mathbb{R})}$ with $\rho(h^{(j)}) = \overline{(\zeta_j, 1)}$, $j = 1, 2$, then there are integers $n_i^{(j)}$, $n^{(j)}$ satisfying the conditions (3.3.5) and (3.3.6), and up to conjugation the representation also satisfies all the other properties.

PROOF. Assume we are given $n_1^{(j)}, \cdots, n_{d^{(j)}}^{(j)}, n^{(j)} \in \mathbb{Z}$, $j = 1, 2$, satisfying conditions (3.3.5) and (3.3.6). Set $x^{(j)} = sh(n^{(j)}/b)$. Condition (3.3.5) and Lemma 2.24 imply that there are conjugates $x_i^{(j)}$ of $sh(-n_i^{(j)}/a_i^{(j)})$, $i = 1, \cdots d_i^{(j)}$, $j = 1, 2$ such that

$$x_1^{(j)} \cdots x_{d^{(j)}}^{(j)} x^{(j)} = \prod_{i=1}^{g^{(j)}} \left[v_i^{(j)}, w_i^{(j)}\right],\ j = 1, 2,$$

where $v_1^{(j)}, w_1^{(j)}, \cdots, v_{g^{(j)}}^{(j)}, w_{g^{(j)}}^{(j)} \in \widetilde{SL_2(\mathbb{R})}$.

Setting $\rho(\alpha_i^{(j)}) = \overline{(0, v_i^{(j)})}$, $\rho(\beta_i^{(j)}) = \overline{(0, w_i^{(j)})}$, $i = 1, \cdots g^{(j)}$, $z_i^{(j)} = \frac{n_i^{(j)}}{a_i^{(j)}} - \frac{b_i^{(j)}}{a_i^{(j)}}\zeta_j$,

$$z^{(j)} = e_j\zeta_j - r_j$$

, $x_i^{(j)}$, $h^{(j)} = \zeta_j \in \mathbb{R}$, $j = 1, 2$, it is straightforward to check that ρ satisfies all relations in $\pi_1 M$ except for $\rho(s^{(2)}) = \rho(s^{(1)})^a \rho(h^{(1)})^c$, $\rho(h^{(2)}) = \rho(s^{(1)})^b \rho(h^{(1)})^d$.

Using condition (3.3.6), these two relations are equivalent to

$$z^{(2)} = az^{(1)} + c\zeta_1 + \frac{an^{(1)} - n^{(2)}}{b}$$

$$\zeta_2 = bz^{(1)} + d\zeta_1 + n^{(1)}$$

Thus we obtain the system of linear equations

$$\begin{pmatrix} 1 & -e_1 & 0 & 0 \\ -a & -c & 1 & 0 \\ 0 & 0 & 1 & -e_2 \\ -b & -d & 0 & 1 \end{pmatrix} \begin{pmatrix} z^{(1)} \\ \zeta_1 \\ z^{(2)} \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} -r_1 \\ \frac{an^{(1)} - n^{(2)}}{b} \\ -r_2 \\ n^{(1)} \end{pmatrix} \tag{3.3.7}$$

and the representation is well-defined if and only if equation (3.3.7) is fulfilled. Computing the determinant of the 4×4 real matrix implies that there is a unique solution if and only if $(be_1 + d)e_2 - (ae_1 + c) \neq 0$, which is satisfied by assumption.

Thus there are unique $\zeta_j, z^{(j)} \in \mathbb{R}$ solving the linear equation and we saw in the proof of Lemma (3.3.4) that

$$\zeta_2 = \frac{\widehat{r}_1 + (be_1 + d)\widehat{r}_2}{(be_1 + d)e_2 - (ae_1 + c)}$$

and since the situation is symmetric in M_1 and M_2 up to exchanging A by A^{-1}, we obtain

$$\zeta_1 = \frac{\widehat{r}_2 + (be_2 - a)\widehat{r}_1}{(be_1 + d)e_2 - (ae_1 + c)}.$$

This completes the first part of the Proposition.

The second part is an easy consequence of the relations in the fundamental group and Lemma 2.24. □

We apply this result to prove that there is a representation with non-trivial volume for a large class of one-edged graph manifolds.

Proposition 3.15. *Let $M = M_1 \cup_\tau M_2$ be a one-edged graph manifold, where $M_j = \left(g^{(j)}, 1; \frac{b_1^{(j)}}{a_1^{(j)}}, \cdots, \frac{b_{d^{(j)}}}{a_{d^{(j)}}}\right)$, $j = 1, 2$ with section-fiber bases $(h^{(j)}, s^{(j)})$, with respect to which the gluing is given by the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, that is $(s^{(1)}, h^{(1)}) = (s^{(2)}, h^{(2)})A$.*

Assume that

$$gcd(aq_{e_2}p_{e_1} + cq_{e_2}q_{e_1} - dp_{e_2}q_{e_1}, b) = 1, \tag{3.3.8}$$

with $e_1 = \frac{p_{e_1}}{q_{e_1}}$, $e_2 = \frac{p_{e_2}}{q_{e_2}}$, $gcd(p_{e_1}, q_{e_1}) = gcd(p_{e_2}, q_{e_2}) = 1$ *and that* $g^{(1)}, g^{(2)} \geq 2$.

Then for every $n^{(2)}$ *with* $\widetilde{gcd(n^{(2)}}, b) = 1$ *and* $-2b \leq n^{(2)} \leq 2b$ *there is a representation* $\rho : \pi_1 M \to Iso_e \widetilde{SL_2}(\mathbb{R})$ *of the form in Proposition 3.14 with volume*

$$vol(M, \rho) = 4\pi^2 \left(\widehat{r}_1 \zeta_1 + \widehat{r}_2 \zeta_2 \right).$$

In particular choosing $n^{(2)} = 1$, *there is a representation with non-zero volume.*

Proof. The above results in Case 2 imply that we obtain manifolds $\widehat{M_1} = M_1(\lambda_1, \mu_1)$, $\widehat{M_2} = M_2(\lambda_2, \mu_2)$ whenever we can choose $\lambda_2, \mu_2 \in \mathbb{Z}$ such that $gcd(\lambda_2, \mu_2) = 1$, $b|\lambda_2$, and

$$\lambda_j z^{(j)} + \lambda_j \frac{n^{(j)}}{b} + \mu_j \zeta_j = 0,\ j = 1, 2.$$

Using that $z^{(2)} = e_2 \zeta_2 - r_2$ and solving for $\frac{\mu_2}{\lambda_2}$, we obtain

$$\frac{\mu_2}{\lambda_2} = -e_2 + \left(r_2 - \frac{n^{(2)}}{b} \right) \frac{1}{\zeta_2} = -e_2 + \frac{\widehat{r}_2}{\zeta_2}. \tag{3.3.9}$$

Observe that $\widehat{e}_j = e_j + \frac{\mu_j}{\lambda_j}$, is the Euler number of $\widehat{M_j}$, $j = 1, 2$ and thus

$$\widehat{e}_j = \frac{\widehat{r}_2}{\zeta_2}. \tag{3.3.10}$$

Since condition (3.3.8) immediately implies condition (3.3.5), we can use equation (3.3.4) in equation (3.3.9), and obtain

$$\begin{aligned}
\frac{\mu_2}{\lambda_2} &= -e_2 + \left(r_2 - \frac{n^{(2)}}{b} \right) \frac{(be_1 + d)e_2 - (ae_1 + c)}{(be_1 + d)\left(r_2 - \frac{n^{(2)}}{b} \right) + \left(r_1 - \frac{n^{(1)}}{b} \right)} \\
&= - \frac{e_2 \left(r_1 - \frac{n^{(1)}}{b} \right) + (ae_1 + c)\left(r_2 - \frac{n^{(2)}}{b} \right)}{(be_1 + d)\left(r_2 - \frac{n^{(2)}}{b} \right) + \left(r_1 - \frac{n^{(1)}}{b} \right)} \\
&= - \frac{\frac{p_{e_2}}{q_{e_2}} \left(\frac{p_{r_1}}{q_{r_1}} - \frac{n^{(1)}}{b} \right) + \left(a\frac{p_{e_1}}{q_{e_1}} + c \right)\left(\frac{p_{r_2}}{q_{r_2}} - \frac{n^{(2)}}{b} \right)}{\left(b\frac{p_{e_1}}{q_{e_1}} + d \right)\left(\frac{p_{r_2}}{q_{r_2}} - \frac{n^{(2)}}{b} \right) + \left(\frac{p_{r_1}}{q_{r_1}} - \frac{n^{(1)}}{b} \right)}
\end{aligned}$$

Let $r_j = \frac{p_{r_j}}{q_{r_j}}$, $gcd(p_{r_j}, q_{r_j}) = 1$, $j = 1, 2$. Then

$$\frac{\mu_2}{\lambda_2} = - \frac{b\left(p_{e_2} p_{r_1} q_{e_1} q_{r_2} + (ap_{e_1} + cq_{e_1}) p_{r_2} q_{r_1} q_{e_2} \right) - \left(n^{(1)} p_{e_2} q_{e_1} + n^{(2)} (aq_{e_2} p_{e_1} + cq_{e_2} q_{e_1}) \right) q_{r_1} q_{r_2}}{\left[b\left((bp_{e_1} + d) p_{r_2} q_{r_1} - n^{(2)} p_{e_1} q_{r_1} q_{r_2} \right) - (n^{(2)} d + n^{(1)}) q_{e_1} q_{r_1} q_{r_2} \right] q_{e_2}}.$$

Assuming that $gcd(q_{r_1} q_{r_2}, b) = 1$, we obtain that for $b|\lambda_2$ it is sufficient to choose $n^{(1)}, n^{(2)} \in \mathbb{Z}$ such that

$$b| \left(n^{(2)} d + n^{(1)} \right)$$

and

$$gcd(n^{(1)} p_{e_2} q_{e_1} + n^{(2)} \left(aq_{e_2} p_{e_1} + q_{e_2} q_{e_1} c \right), b) = 1.$$

Multiplying $n^{(2)}d+n^{(1)}$ by a and remembering that $gcd(a,b)=1$ we see that

$$b|n^{(2)}d+n^{(1)} \Leftrightarrow b|n^{(2)}(bc-1)+an^{(1)} \Leftrightarrow b|an^{(1)}-n^{(2)}.$$

Hence this conditions is equivalent to condition (3.3.6) and is thus satisfied for every representation ρ of the form in Proposition 3.14 and conversely we do not have to take care of condition (3.3.6), when choosing $n^{(1)}$ and $n^{(2)}$. In particular $n^{(1)} \equiv -dn^{(2)} \bmod b$ and $n^{(2)} \equiv an^{(1)} \bmod b$. That implies that

$$n^{(1)}p_{e_2}q_{e_1}+n^{(2)}\left(aq_{e_2}p_{e_1}+q_{e_2}q_{e_1}c\right) \equiv n^{(2)}\left(aq_{e_2}p_{e_1}+q_{e_2}q_{e_1}c-dp_{e_2}q_{e_1}\right) \bmod b.$$

Condition (3.3.8) implies that $gcd(n^{(2)}\left(aq_{e_2}p_{e_1}+q_{e_2}q_{e_1}c-dp_{e_2}q_{e_1}\right),b)=1$ if and only if $gcd(n^{(2)},b)=1$ and in particular every choice of $n^{(2)}$ with $gcd(n^{(2)},b)=1$ guarantees that $\widehat{M}_1$ and $\widehat{M}_2$ are manifolds.

Remember that the proof of Proposition 2.25 showed that for a $\widetilde{SL_2(\mathbb{R})}$-manifold N, and a representation f mapping the fiber to $\overline{(\zeta,1)}$, we obtain

$$vol(N,f)=4\pi^2 e(N)\zeta^2.$$

Since by assumption $g^{(1)},g^{(2)}\geq 2$, we know that $\widehat{M}_j$ is an $\widetilde{SL_2(\mathbb{R})}$-manifold if and only if $\widehat{e}_j\neq 0$ and else it has geometry $\mathbb{H}^2\times\mathbb{R}$ in which case the volume of any representation vanishes, thus the Volume formula always holds and using equation (3.3.10) we obtain

$$\begin{aligned} vol(M,\rho) &= vol(\widehat{M}_1,\rho|_{\widehat{M}_1})+vol(\widehat{M}_2,\rho|_{\widehat{M}_2}) \\ &= 4\pi^2(\widehat{e}_1\zeta_1^2+\widehat{e}_2\zeta_2^2) \\ &= 4\pi^2\left(\frac{\widehat{r}_1}{\zeta_1}\zeta_1^2+\frac{\widehat{r}_2}{\zeta_2}\zeta_2^2\right) \\ &= 4\pi^2\left(\widehat{r}_1\zeta_1+\widehat{r}_2\zeta_2\right). \end{aligned}$$

Now we prove that there exists a choice for $n_i^{(j)},n^{(j)}$, $i=1,\cdots,d^{(j)}$, $j=1,2$ whenever $gcd(n^{(2)},b)=1$ and $0\leq n^{(2)}\leq b$.

Choose $n_i^{(j)}=0$, for all i and j and $n^{(1)}\equiv -dn^{(2)} \bmod b$ with $-2b\leq n^{(1)}\leq 2b$.

Then we obtain $\widehat{r}_1=-\frac{n^{(1)}}{b}$, $n^{(1)}=kb-dn^{(2)}$, $\widehat{r}_2=-\frac{n^{(2)}}{b}$ and, since $-2b\leq n^{(1)},n^{(2)}\leq 2b$, and $g^{(1)},g^{(2)}\geq 2$, it is immediate that condition (3.3.5) is satisfied. Thus there is indeed a representation for every such $n^{(2)}$.

Finally we prove that for $n^{(2)}=1$ we can choose $n^{(1)}$ such that we obtain a representation with non-zero volume. Define $u=4\pi^2/((be_1+d)e_2-(ae_1+$

$c)) \neq 0$. Then we obtain the volume

$$\begin{aligned}
vol(M,\rho) &= 4\pi^2\left(\widehat{r}_1\zeta_1 + \widehat{r}_2\zeta_2\right) \\
&= u\left(2\widehat{r}_1\widehat{r}_2 + (be_2 - a)\,\widehat{r}_1^2 + (be_1 + d)\,\widehat{r}_2^2\right) \\
&= \frac{u}{b^2}\left(2(kb-d) + \left(b\frac{p_{e_2}}{q_{e_2}} - a\right)(n^{(1)})^2 + \left(b\frac{p_{e_1}}{q_{e_1}} + d\right)\right) \\
&= \frac{u}{b^2 q_{e_1} q_{e_2}}\left[(2kb-2d)q_{e_1}q_{e_2} + (bp_{e_2}q_{e_1} - aq_{e_1}q_{e_2})(kb-d)^2\right. \\
&\quad \left. + bp_{e_1}q_{e_2} + dq_{e_1}q_{e_2}\right] \\
&= \frac{u}{b^2 q_{e_1} q_{e_2}}\left[k^2 b^2 (bp_{e_2}q_{e_1} - aq_{e_1}q_{e_2})\right. \\
&\quad + k\left(2bq_{e_1}q_{e_2} - 2bd\left(bp_{e_2}q_{e_1} - aq_{e_1}q_{e_2}\right)\right) \\
&\quad \left. + d^2\left(bp_{e_2}q_{e_1} - aq_{e_1}q_{e_2}\right) + bp_{e_1}q_{e_2} + dq_{e_1}q_{e_2}\right]
\end{aligned}$$

Recall that $q_{e_1}, q_{e_2}, b \neq 0$. Hence $b^2(bp_{e_2}q_{e_1} - aq_{e_1}q_{e_2})$ and $(2bq_{e_1}q_{e_2} - 2bd\,(bp_{e_2}q_{e_1} - aq_{e_1}q_{e_2}))$ cannot vanish simultaneously. Thus $vol(M,\rho)$ is a non-constant polynomial of degree 2 in k. Since we have at least three different choices for k with $-2b \leq n^{(1)} \leq 2b$ it is immediate that there is a choice for k with $vol(M,\rho) \neq 0$. □

Remark 3.16. In this section we restricted ourselves to a closer analysis of one-edged graph manifolds. The main tools in our analysis were additivity and Lemma 3.10. As mentioned in Remark 3.9, one can apply the additivity formulas also to other non-geometric manifolds and thus both results also hold for a much larger class of graph manifolds. This suggests that using similar methods one might be able to derive non-vanishing results for volumes of representations for a larger class of graph manifolds.

Bibliography

[1] S. Baseilhac, *Chern Simons theory in Dimension Three*, www.math.univ-montp2.fr/~baseilhac/CS.pdf

[2] G. Besson, G. Courtois, S. Gallot, *Inegalites de Milnor Wood geometriques*, Comment. Math. Helv. **82** (2007), 753-803.

[3] S. Bleiler, C. Hodgson, *Spherical space forms and Dehn filling*, Topology **35** (1996), no. 3, 809C833.

[4] M. Boileau, S. Maillot, J. Porti, *Three-dimensional orbifolds and their geometric structures*, Panoramas et Synthèses, **15**. Société Mathématique de France, Paris 2003, viii+167 pp.

[5] R.Brooks and W.Goldman, *The Godbillon-Vey invariant of a transversely homogeneous foliation*, Transactions of the American Mathematical Society **286,2**, 651-664 (1984).

[6] R. Brooks, W. Goldman, *Volumes in Seifert space*, Duke Math J., **51** (1984), no.3, 529-545.

[7] P. Buser, *Geometry and spectra of compact Riemann surfaces*, Progress in Mathematics vol. **106**, Birkhäuser (1992).

[8] S. Chern, J. Simons, *Characteristic forms and geometric invariants*, Ann. of Math. (2), **99**, (1974), 4869.

[9] P.Derbez, S.Wang, *Finiteness of mapping degrees and $PSL(2,\mathbb{R})$-volume on graph manifolds*, Algebraic and Geometric Topology, **9**, (2009), 1727-1749.

[10] P.Derbez, S.Wang, *Graph manifolds have virtually positive Seifert volume*, J. London Math. Soc. (2), **86**, (2012), 17-35.

[11] P.Derbez and S.Wang, *Chern Simons theory and the volume of 3-manifolds*, Preprint, ArXiv math/1111.6153.

[12] B.A. Dubrovin, A.T. Fermenko, S.P. Novikov, *Modern Geometry - Methods and Applications, Part II*, Springer series in Soviet mathematics. Graduate texts in mathematics; 93, Springerverlag New York Inc., (1985).

[13] W. Dunbar, G. Robert Meyerhoff, *Volumes of hyperbolic 3-orbifolds*, Indiana Univ. Math. J. **43** (1994), no. 2, 611-637.

[14] J.L. Dupont, *Curvature and Characteristic classes*, Vol. **640**. Springer-Verlag, Berlin-New York, 1978. viii+175 pp.

[15] J.L. Dupont, *Simplicial de Rham cohomology and characteristic classes of flat bundles*, Topology **15** (1976), no. 2, 233245.

[16] D.Eisenbud, U.Hirsch, W.Neumann, *Transverse foliations of Seifert bundles and self homeomorphism of the circle*, Comment. Math. Helvetici **56**, 638-660 (1981).

[17] S. Francaviglia, *Hyperbolic volume of representations of fundamental groups of cusped 3-manifolds.*, Int. Math. Res. Not., **9**, 425-459 (2004).

[18] G. Godbillon, J. Vey, *Un invariant des feuilletages de codimension 1.* (French) C.R. Acad. Sci. Paris Ser. A-B **273** (1971) A92-A95.

[19] M. Gromov, *Hyperbolic manifolds according to Thurston and Jørgensen*, Seminaire N. Bourbaki, 1979-1980, exp. no. **546**, 40-53.

[20] M. Gromov, *Volume and bounded cohomology*, Publications mathématiques de l'I.H.É.S., **56**, (1982), 5-99.

[21] J. Hass, H. Rubinstein, S. Wang, *Boundary slopes of immersed surfaces in 3-manifolds*, J. Diff. Geom., **52**, (1999), 303-325.

[22] A.Hatcher, *Notes on Basic 3-Manifold Topology*, www.math.cornell.edu/~hatcher/3M/3M.pdf

[23] A. Hatcher, *Algebraic Topology* www.math.cornell.edu/~hatcher/AT/AT.pdf

[24] J.Hempel *Residual finiteness for 3-manifolds. Combinatorial group theory and topology*(Alta, Utah, 1984), 379C396, Ann. of Math Stud., **111**,Princeton Univ. 1987.

[25] J.A. Hoffman, D. Matignon, *Examples of bireducible Dehn fillings*, Pacific Journal of Math., Vol. **209**, No. 1 (2003), 67-83.

[26] V.T. Khoi, *A cut-and-paste method for computing the Seifert volumes*, Math. Annalen, **326**, (2003), 759-801.

[27] R. Kirby, *Problems in low-dimensional topology*, Geometric topology, Edited by H. Kazez, AMS/IP, Vol.2, International Press, 1997.

[28] P.A. Kirk and E.P. Klaasen, *Chern-Simons invariants of 3-manifolds and representation spaces of knot groups*, Math. Ann., **287**, (1990) 343-367.

[29] P.A. Kirk, E.P. Klaasen, *Chern-Simons invariants of 3-manifolds decomposed along tori and the circle bundle over the representation space of* T^2, Comm. Math. Phys. **153**, (1993), no.3, 521-557.

[30] D. Kotschick, C. Neofytidis, *On three-manifolds dominated by circle bundles* Math. Z. **247** (2013), 21-32.

[31] T. Kuessner, *Relative Simplicial Volume*, Tübingen, Univ., Diss., (2001).

[32] J.Luecke, *Finite covers of 3-manifolds containing essential tori*, Trans. Amer. Soc. **310**,(1988), no.1, 381-191.

[33] J.Luecke, Y.-Q. Wu, *Relative Euler number and finite covers of Graph manifolds*, from: "'Geometric topology (Athens, GA, 1993)"', (W.H. Kazez, editor), AMS/IP Stud. Adv. Math. 2, Amer. Math. Soc. (1997) 80-103.

[34] J. Milnor, *On the existence of a connection with curvature zero.* Comment. Math. Helv. **32** (1958) 215-223.

[35] J. Milnor, W. Thurston, *Characteristic numbers of 3-manifolds*, Enseign. Math. **23** (1977), 249-254.

[36] W.Neumann and F.Raymond *Seifert manifolds, plumbing μ-invariant and orientation reversing maps*, Algebraic and Geometric Topology, Lecture Notes in Mathematics **664**, (1978), 163-196.

[37] W. Neumann, D.Zagier, *Volumes of hyperbolic three-manifolds*, Topology Vol.24, no.3, (1985), 307-332.

[38] A. Reznikov, *Volumes of discrete groups and topological complexity of homology spheres*, Math. Ann. **306** (1996), no.3, 547-554.

[39] A. Reznikov, *Rationality of secondary classes*, J. Differential Geom. **43** (1996), 674-692.

[40] P. Samuel, H.-C. Wang, *Commutators in a semi-simple Lie group*, Proc. Amer. Math. Soc., **13**, (1962), 907-913.

[41] P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Society (1983), **15(5)**, 401-487.

[42] H. Sun, S. Wang, J. Wu, H. Zheng, *Self-mapping degrees of 3-manifolds*, Osaka J. Math., **49** (2012), 247 -269.

[43] W.P. Thurston, *The Geometry and Topology of 3-manifolds*, Lecture Notes, Princeton (1977).

[44] W.P. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc., **6**, (1982), no.3, 357-381.

[45] W. Van Est, *Une application d'une methode de Cartan-Leray.* (French) Indag. Math., **17** (1955), 542-544.

[46] F. Waldhausen, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten. II* Inventiones Math. **4** (1967), 87-117.

[47] S.C. Wang, Y.Q. Wu, *Covering invariants and co-Hopficity of 3-manifold groups*, Proc. London Math. Soc.(3) **68** (1994), no.1, 203C224.

[48] J. Wood, *Bundles with totally disconnected structure group.* Comment. Math. Helv. **46** (1971), 257 - 273.

[49] T. Yoshida, *The η-invariant of hyperbolic 3-manifolds*, Invent. Math., **81**, no.3, 473 - 514.

[50] `www.map.mpim-bonn.mpg.de/Simplicial_volume`

Selbstständigkeitserklärung

Ich versichere, die Arbeit selbstständig angefertigt und dazu nur die im Literaturverzeichnis angegebenen Quellen benutzt zu haben.

München, den 25. September 2013