SPECTRAL RADIi OF GENERALIZED INVERSES OF SIMPlEy POlar MATRiCES

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Abstract. In this note we study the spectral radii of generalized inverses of square matrices $A$ such that $\text{rank}(A) = \text{rank}(A^2)$.

1. General and introductory material

For positive integers $n$ and $m$, $\mathbb{C}^{n \times m}$ denotes the vector space of all complex $n \times m$ matrices. Let $A \in \mathbb{C}^{n \times n}$ be a square matrix. A matrix $C \in \mathbb{C}^{n \times n}$ is called a $g_1$-inverse of $A$ if

$$ACA = A.$$ 

If $B \in \mathbb{C}^{n \times n}$ and

$$ABA = A \quad \text{and} \quad BAB = B,$$

then $B$ is called a $g_2$-inverse of $A$. By $\mathcal{G}_1(A)$ we denote the set of all $g_1$-inverses of $A$. $\mathcal{G}_2(A)$ is the set of all $g_2$-inverses of $A$. It is well-known that $\mathcal{G}_1(A) \neq \emptyset$ (see [1]). Furthermore it is easy to see that if $C \in \mathcal{G}_1(A)$, then $B = CAC \in \mathcal{G}_2(A)$, hence

$$\phi \neq \mathcal{G}_2(A) \subseteq \mathcal{G}_1(A).$$

If $A$ is non-singular, then $\mathcal{G}_2(A) = \mathcal{G}_1(A) = \{A^{-1}\}$.

For $A \in \mathbb{C}^{n \times n}$ we denote the set of eigenvalues of $A$ by $\sigma(A)$ and the spectral radius $r(A)$ of $A$ is defined by

$$r(A) = \max_{\lambda \in \sigma(A)} |\lambda|.$$ 

Let $A \in \mathbb{C}^{n \times m}$, $A^T$ denotes the transpose of $A$ and $A^*$ denotes the conjugate transpose of $A$. The range of $A$ is given by

$$\mathcal{R}(A) = \{Ax : x \in \mathbb{C}^n\}$$

and the kernel of $A$ is the set

$$\mathcal{N}(A) = \{x \in \mathbb{C}^n : Ax = 0\}$$

(we follow the convention $\mathbb{C}^n = \mathbb{C}^{n \times 1}$).

In this note we study the set

$$R_A = \{r(C) : C \in \mathcal{G}_1(A)\}$$

for $A \in \mathbb{C}^{n \times n}$ such that $\text{rank}(A) = \text{rank}(A^2)$, where $\text{rank}(A) = \dim \mathcal{R}(A)$. Such matrices are called simply polar.

Examples. If $A$ is non-singular, then $R_A = \{r(A)^{-1}\}$. If $A = 0$, then $ACA = A$ for each $C \in \mathbb{C}^{n \times n}$, hence $R_A = [0, \infty)$.
Throughout this paper we will assume that \( n \geq 2 \). The identity on \( \mathbb{C}^n \) is denoted by \( I_n \).

1.1. Proposition. If \( A \in \mathbb{C}^{n \times n} \) and \( B \in \mathcal{G}_2(A) \), then
\[
\mathcal{G}_1(A) = \{ B + T - BATAB : T \in \mathbb{C}^{n \times n} \}
\]

Proof. [1, Theorem 2 in Chapter 2.3].

It follows from Proposition 1.1, that if \( A \) is singular, then \( \mathcal{G}_1(A) \) is an infinite set. In [6], the following result is shown:

1.2. Proposition. Suppose that \( A \in \mathbb{C}^{n \times n} \) is singular. We have:
(1) for each \( z \in \mathbb{C} \), there is \( B \in \mathcal{G}_1(A) \) with \( z \in \sigma(B) \);
(2) if \( B \in \mathcal{G}_2(A) \), then
\[
B + z(I_n - BA), B + z(I_n - AB) \in \mathcal{G}_1(A)
\]
for all \( z \in \mathbb{C} \) and
\[
r(B + z(I_n - BA)) = r(B + z(I_n - AB)) = \begin{cases} r(B), & \text{if } |z| \leq r(B) \\ |z|, & \text{if } |z| > r(B) \end{cases}
\]
(3) \( [r(B), \infty) \subseteq R_A \) for each \( B \in \mathcal{G}_2(A) \).

1.3. Proposition. Suppose that \( A \in \mathbb{C}^{n \times n} \), \( r = \text{rank}(A) > 0 \) and that \( A \) has a decomposition
\[
A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V^{-1}
\]
with \( U, V \in \mathbb{C}^{n \times n} \) non-singular and \( D \in \mathbb{C}^{r \times r} \) non-singular. Then
\[
B = V \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^{-1} \in \mathcal{G}_2(A)
\]
and
\[
\mathcal{G}_1(A) = \left\{ V \begin{bmatrix} D^{-1} & A_1 \\ A_2 & A_3 \end{bmatrix} U^{-1} : A_1 \in \mathbb{C}^{r \times (n-r)}, A_2 \in \mathbb{C}^{(n-r) \times r}, A_3 \in \mathbb{C}^{(n-r) \times (n-r)} \right\}.
\]

Proof. It is easy to verify that \( B \in \mathcal{G}_2(A) \). Let \( T \in \mathbb{C}^{n \times n} \), let \( \varphi(T) = V^{-1}TU \) and set \( B_0 := B + T - BATAB \). Then
\[
B_0 = V \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^{-1} + T - V^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \varphi(T) \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^{-1}
\]
\[
= V \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \varphi(T) - \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \varphi(T) \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^{-1}
\]
\[
= V \begin{bmatrix} D^{-1} & A_1 \\ A_2 & A_3 \end{bmatrix} U^{-1}.
\]
Since the mapping \( \varphi : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n} \) is bijective, the result follow from Proposition 1.1. □

Recall that a matrix \( A \in \mathbb{C}^{n \times n} \) is called simply polar if \( \text{rank}(A) = \text{rank}(A^2) \).
1.4. Proposition. Let $A \in \mathbb{C}^{n \times n}$ be singular. The following assertions are equivalent:
(1) $A$ is simply polar;
(2) $0$ is a simple pole of the resolvent $(\lambda I_n - A)^{-1}$;
(3) $\mathbb{C}^n = \mathcal{R}(A) \oplus \mathcal{N}(A)$;
(4) there is $B \in \mathcal{D}_2(A)$ such that $AB = BA$.

Proof. [3, Satz 72.4], [3, Satz 101.2] and [1, Theorem 5.2].

If $A \in \mathbb{C}^{n \times n}$ is simply polar, then, by Proposition 1.4, there is $B \in \mathbb{C}^{n \times n}$ such that $ABA = A, BAB = B$ and $AB = BA$. It is shown in [1, Theorem 5.1], that there is no other $g_2$-inverse of $A$ which commutes with $A$. $B$ is called the Drazin-inverse of $A$. The following result is shown in [1, p. 53].

1.5. Proposition. If $A \in \mathbb{C}^{n \times n}$, $A \neq 0$ and if $A$ is simply polar, then the Drazin-inverse $B$ of $A$ satisfies

$$\sigma(B) \setminus \{0\} = \left\{ \frac{1}{\lambda} : \lambda \in \sigma(A) \setminus \{0\} \right\}$$

and hence $r(B) = r(A)^{-1}$.

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2. Generalized inverses of simply polar matrices

Throughout this section we assume that $A \in \mathbb{C}^{n \times n}$ is simply polar and that $\text{rank}(A) > 0$.

By [5, 4.3.2 (4)] (see also [4]), $A$ has a decomposition

$$A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^{-1},$$

where $U \in \mathbb{C}^{n \times n}$ and $D \in \mathbb{C}^{r \times r}$ are non-singular. From Proposition 1.3 we know that

$$B = U \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^{-1} \in \mathcal{D}_2(A).$$

It is easy to see that the matrix $B$ in (2.2) is the Drazin-inverse of $A$.

2.1. Theorem. The following assertions are equivalent:
(1) $\dim \mathcal{N}(A) \geq \text{rank}(A)$.
(2) there is $B \in \mathcal{D}_2(A)$ with $B^2 = 0$.

A consequence of Theorem 2.1 is:

2.2. Corollary. If $\dim \mathcal{N}(A) \geq \text{rank}(A)$, then there is an entire function $F : \mathbb{C} \to \mathbb{C}^{n \times n}$ such that

$$F(z) \in \mathcal{D}_2(A), \sigma(F(z)) = \{z, 0\} \text{ and } r(F(z)) = |z| \text{ for all } z \in \mathbb{C}.$$ 

Furthermore we have $R_A = [0, \infty)$.

Proof. By Theorem 2.1, there is $B \in \mathcal{D}_2(A)$ with $B^2 = 0$. Define $F$ by $F(z) = B + z(I_n - AB)$. Then $F(z) \in \mathcal{D}_2(A)$ for each $z \in \mathbb{C}$ (Proposition 1.2). [6, Theorem 3] gives

$$\{z\} \subseteq \sigma(F(z)) \subseteq \{z, 0\} \quad (z \in \mathbb{C}).$$

Assume that $F(z)$ is non-singular for some $z \in \mathbb{C}$. Thus there is $C \in \mathbb{C}^{n \times n}$ with $F(z)C = I_n$. Since $BF(z) = 0$, we get $0 = BF(z)C = B$, thus $A = AB = 0$, a contradiction. \(\square\)
Proof of Theorem 2.1. Let \( r = \text{rank}(A) \).

(1) \( \Rightarrow \) (2): Proposition 1.4 (3) shows that \( n - r = \dim \mathcal{N}(A) \geq r \).

Case 1: \( n - r = r \). Let \( D \) be as in (2.1) and let

\[
S = \begin{bmatrix} D^{-1} & D^{-1} \\ -D^{-1} & -D^{-1} \end{bmatrix} \quad \text{and} \quad B = \text{USU}^{-1}.
\]

Then it is easy to see \( B \in \mathcal{G}_2(A) \) and \( B^2 = 0 \).

Case 2: \( n - r > r \). Then \( r < n/2 \).

Case 2.1: \( n = 2m \) for some \( m \in \mathbb{N} \). Let

\[
T = \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} T & T \\ -T & -T \end{bmatrix} \quad \in \mathbb{C}^{n \times n},
\]

and \( B = \text{USU}^{-1} \). Then \( B \in \mathcal{G}_2(A) \) and \( B^2 = 0 \).

Case 2.2: \( n = 2m + 1 \) for some \( m \in \mathbb{N} \). Then \( r < m \). Set

\[
T = \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} T & T & 0 \\ -T & -T & 0 \\ 0 & \cdots & 0 \end{bmatrix} \quad \in \mathbb{C}^{n \times n},
\]

and \( B = \text{USU}^{-1} \). As above, \( B \in \mathcal{G}_2(A) \) and \( B^2 = 0 \).

(2) \( \Rightarrow \) (1): Since \( B^2 = 0 \), \( A \) is singular. We have \((BA)^2 = BA, \mathcal{R}(BA) = \mathcal{R}(B), \mathcal{N}(A) = \mathcal{R}(I - BA), \mathcal{R}(AB) = \mathcal{R}(A), (AB)^2 = AB \) and

\[
\mathbb{C}^n = \mathcal{R}(B) \oplus \mathcal{N}(A),
\]

thus, by Proposition 1.4 (3), \( \text{rank}(B) = r = \text{rank}(A) \). Now let \( z \in \mathcal{R}(A) \cap \mathcal{R}(B) \). Then \( z = ABz = BAz \), therefore \( z = AB^2z = 0 \). This gives \( \mathcal{R}(A) \cap \mathcal{R}(B) = \{0\} \). Since

\[
\mathcal{R}(A) \oplus \mathcal{R}(B) \subseteq \mathbb{C}^n,
\]

we derive \( 2r \leq n \), hence \( \text{rank}(A) = r \leq n - r = \dim \mathcal{N}(A) \).

A square matrix \( D \) is said to be non-derogatory if its characteristic polynomial is also its minimal polynomial.

2.3. Theorem. Suppose that \( \text{rank}(A) = \text{rank}(A^2) = n - 1 \), let \( D \) be as in (2.1) and suppose that \( D \) is non-derogatory. Then \( A \) has a nilpotent \( g_1 \)-inverse and hence \( \min R_A = 0 \).

Proof. Since \( D^{-1} \) is also non-derogatory, it follows from [2, Theorem 3.4] that there are \( a_1 \in \mathbb{C}^{n-1}, a_2 \in \mathbb{C}^{n-1} \) and \( a_3 \in \mathbb{C} \) such that

\[
S = \begin{bmatrix} D^{-1} & a_1 \\ 0 & a_2 \\ a_3 \end{bmatrix}
\]

is nilpotent,

hence \( S^q = 0 \) for some positive integer \( q \). Let \( B = \text{USU}^{-1} \). Then \( B^q = 0 \). By Proposition 1.3, \( B \in \mathcal{G}_1(A) \).

A matrix \( N \in \mathbb{C}^{n \times n} \) is called normal if \( NN^* = N^*N \). The spectral theorem for normal matrices implies that

\[
N = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*,
\]

with \( U \in \mathbb{C}^{n \times n} \) unitary (that is \( UU^* = U^*U = I_n \)) and \( D = \text{diag} (\lambda_1, \ldots, \lambda_r) \), where \( \lambda_1, \ldots, \lambda_r \) are the non-zero eigenvalues of \( N \). It follows (see [5, 4.3.2 (4)]) that \( N \) is simply polar.

Now suppose that \( \text{rank}(N) = n - 1 \). If \( \lambda_i \neq \lambda_j \) \( (i \neq j; i, j = 1, \ldots, n - 1) \) then the matrix \( D \) in (2.3) is
non-derogatory.

Thus we have proved:

**2.4. Corollary.** If \( N \in \mathbb{C}^{n \times n} \) is normal, \( \text{rank}(N) = n - 1 \) and if \( \lambda_i \neq \lambda_j \) \((i \neq j; i, j = 1, \ldots, n - 1)\) for the non-zero eigenvalues of \( N \), then there is a nilpotent \( g_1 \)-inverse of \( A \).

### 3. The case \( n = 2 \)

**3.1. Proposition.** If \( A \in \mathbb{C}^{2 \times 2} \) and \( A^2 = 0 \), then there is \( B \in \mathcal{H}_2(A) \) such that \( B^2 = 0 \).

**Proof.** The Schur decomposition of \( A \) is

\[
A = U \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} U^*,
\]

where \( U \in \mathbb{C}^{2 \times 2} \) is unitary and \( \alpha \in \mathbb{C} \) (see [5, 5.2.3 (1)]). If \( \alpha = 0 \), we are done. So assume that \( \alpha \neq 0 \). Let

\[
B = U \begin{bmatrix} 0 & 0 \\ \alpha^{-1} & 0 \end{bmatrix} U^*.
\]

then it is easy to see that \( B \in \mathcal{H}_2(A) \) and \( B^2 = 0 \). \( \square \)

**3.2. Theorem.** Suppose that \( A \in \mathbb{C}^{2 \times 2} \) is singular. Then there is \( B \in \mathcal{H}_2(A) \) with \( B^2 = 0 \) and hence \( R_A = [0, \infty) \).

**Proof.** Because of Proposition 3.1, we assume that \( A^2 \neq 0 \). Since \( A \) is singular, we have \( \text{rank}(A) = \text{rank}(A^2) = 1 \), \( A \) is simply polar and \( \dim \mathcal{N}(A) = \text{rank}(A) \). Theorem 2.1 gives the result. \( \square \)

### 4. Generalized inverses of projections

In this section we assume that \( P \in \mathbb{C}^{n \times n}, \ 0 \neq P \neq I_n \) and \( P^2 = P \). Hence \( P \) is simply polar.

Since \( \mathcal{A}(P) = \{ x \in \mathbb{C}^n : Px = x \} \), it follows that \( \sigma(P) = \{0, 1\} \) and that there is a non-singular \( U \in \mathbb{C}^{n \times n} \) such that

\[
(4.1) \quad P = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^{-1},
\]

([5, 9.8 (3)]), where \( r = \text{rank}(P) \).

From Theorem 2.1 we know that

\[
\dim \mathcal{N}(P) \geq \text{rank}(P) \Leftrightarrow \text{there is } B \in \mathcal{H}_2(P) \text{ such that } B^2 = 0.
\]

So it remains to investigate the case where \( \dim \mathcal{N}(P) < \text{rank}(P) \):

**4.1. Theorem.** If \( \dim \mathcal{N}(P) < \text{rank}(P) \) and if \( B \in \mathcal{H}_1(P) \), then \( 1 \in \sigma(B) \) and hence \( r(B) \geq 1 \).

**Proof.** Proposition 1.3 and (4.1) show that there are \( A_1 \in \mathbb{C}^{r \times (n - 1)}, A_2 \in \mathbb{C}^{(n - r) \times r} \) and \( A_3 \in \mathbb{C}^{(n - r) \times (n - r)} \) such that

\[
B = U \begin{bmatrix} I_r & A_1 \\ A_2 & A_3 \end{bmatrix} U^{-1}.
\]
Denote by $a^{(1)}, \ldots, a^{(r)}$ the columns of $A_2$. Since
\[
\text{rank}(A_2) \leq n - r = \dim \mathcal{N}(A) < \text{rank}(P) = r,
\]
there is $(\alpha_1, \ldots, \alpha_r)^T \in \mathbb{C}^r$ such that $(\alpha_1, \ldots, \alpha_r) \neq 0$ and
\[
\alpha_1 a^{(1)} + \cdots + \alpha_r a^{(r)} = 0.
\]
Set $x = (\alpha_1, \ldots, \alpha_r, 0, \ldots, 0)^T \in \mathbb{C}^n$ and $z = Ux$, then $z \neq 0$ and
\[
Bz = U \begin{bmatrix} I_r & A_1 \\ A_2 & A_3 \end{bmatrix} x = Ux = z,
\]
thus $1 \in \sigma(B)$.

4.2. Corollary.

(1) $R_P = [0, \infty) \iff \dim \mathcal{N}(P) \geq \text{rank}(P)$.

(2) $R_P = [1, \infty) \iff \dim \mathcal{N}(P) < \text{rank}(P)$.

Proof. (1) Theorem 4.1 and Corollary 2.2. (2) Theorem 4.1 and Proposition 1.2 (3). Observe that $P \in \mathcal{G}_1(P)$ and $r(P) = 1$.

\section*{References}


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