Universität Karlsruhe (TH)
Research University • founded 1825
Institute for Applied and Numerical Mathematics
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Advanced Topics in Numerical Analysis I
Homework Assignment No. 3

(WS 2006/2007) November 2, 2006

Problem 9 (to be handed in)

Consider the space $C[-1,1]$ with the scalar product

$$\langle f,g\rangle := \int_{-1}^{1} f(x)\,g(x)\,\sqrt{1-x^2}\,dx\ .$$

We define the functions $U_n(x) := \sqrt{\frac{2}{\pi}}\frac{\sin((n+1)\arccos x)}{\sqrt{1-x^2}}$. Show:

a) $\langle U_n, U_m\rangle = \delta_{nm}$ for $n,m \in \mathbb{N}_0$.

b) $U_n(x)$ is a polynomial of degree n and for $n \geq 1$, we have $T_n'(x) = \sqrt{\frac{\pi}{2}}\,n\,U_{n-1}(x)$, with $T_n(x) = \cos(n \arccos x)$ the n^{th} Tschebyscheff polynomial of the first kind.

c) The polynomials $U_n(x)$ satisfy the recursion formula $2xU_n(x) = U_{n+1}(x) + U_{n-1}(x)$ for $n \geq 1$.

d) The estimate $|U_n(x)| \leq \sqrt{\frac{2}{\pi}}\,(n+1)$, $x \in [-1,1]$ holds.

e) Between two zeros of $T_n(x)$ there is exactly one zero of $U_{n-1}(x)$.

Problem 10 (no correction)

Let $f \in C^\infty([a,b])$. There exists a constant $C < \infty$, such that $f^{(n)}(x) \leq C^n$ for $x \in [a,b]$ and all $n \in \mathbb{N}$. p_n denotes the algebraic interpolation polynomial of f with respect to $a \leq x_0^{(n)} < x_1^{(n)} < \cdots < x_n^{(n)} \leq b$. Show that p_n converges uniformly to f on the interval $[a,b]$.

Problem 11 (to be handed in)

Let $m \in \mathbb{N}$. Π_m denotes the space of trigonometric polynomials

$$t(x) := \frac{a_0}{2} + \sum_{\nu=1}^{m} (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

with real coefficients $a_0, \ldots, a_m$ and $b_1, \ldots, b_m$. Show:

a) A trigonometric polynomial that has more than $2m$ distinct zeros in the interval $[0, 2\pi)$ must vanish identically; i.e. $a_0 = \ldots = a_m = b_1 = \ldots = b_m = 0$.

b) Let $p \in \Pi_m$ and $q \in \Pi_M$. Then, $pq \in \Pi_{m+M}$ holds.

c) For given nodes $0 \leq x_0 < x_1 \cdots < x_{2m} < 2\pi$ and data $y_0, y_1, \ldots, y_{2m}$ there exists a unique trigonometric polynomial $t \in \Pi_m$ with $t(x_j) = y_j$ for $j = 0,1,\ldots,2m$. It can be represented by

$$t(x) = \sum_{k=0}^{2m} y_k t_k(x) \text{ with } t_k(x) = \prod_{\substack{j=0\\ j\neq k}}^{2m} \frac{\sin\frac{x-x_j}{2}}{\sin\frac{x_k-x_j}{2}}\ .$$

Instruction:
For a) choose the representation $t(x) = \sum_{\nu=-m}^{m} c_\nu \exp(i\nu x)$ with $c_\nu \in \mathbb{C}$.

Problem 12 (no correction)

Consider a circulant matrix $A \in \mathbb{C}^{n\times n}$, i.e.

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ a_n & a_1 & a_2 & \ldots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \ldots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \ldots & a_1 \end{pmatrix}.$$

a) Show that for $f_k = (\omega_n^{0k}, \omega_n^{k}, \ldots, \omega_n^{(n-1)k})^T$ with $\omega_n = e^{2\pi i/n}$ the equation

$$Af_k = \lambda_k f_k, \quad k = 1,2,\ldots,n$$

holds and identify λ_k, $k = 1,2,\ldots,n$.

b) Show that with the results of part a) the matrix vector multiplication $y = Ax$ can be done using two fast Fourier transforms. What is the computational cost?

Please hand in your homework problems (No. **9** and **11**) due **Thursday**, **November 9, 2006**, **13:00h**. Put them in the slot marked „Numerische Mathematik I/II/III“ in the Math-Building (20.30), 2nd floor opposite room 112. Please print your name and registration number on your problems.

On **Thursday**, **November 9, 2006**, **14:00-15:30 h** the problems will be discussed in the Neuer Hörsaal (Building 20.40).

Each Thursday a homework assignment will be handed out in the tutorial. The homework assignments are also available for download in the WWW:

`http://www.mathematik.uni-karlsruhe.de/ianm3/lehre/numana12006w` .