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Institute for Applied and Numerical Mathematics

Prof. Dr. Rudolf Scherer, Dipl.-Math. Wolfgang Müller

Advanced Topics in Numerical Analysis I
Homework Assignment No. 4

(WS 2006/2007) November 9, 2006

Problem 13 (to be handed in)

Let $a \in \mathbb{C}^n$. Then, the discrete Fourier transformation $\hat{a} = \textbf{FFT}(a)$ is defined by

$$\hat{a}_k = \sum_{\ell=0}^{n-1} a_\ell z_n^{\ell k} \text{ with } z_n = \exp(2\pi i/n) \text{ for } k = 0, \ldots, n-1 \ .$$

a) Show that the inverse discrete Fourier transformation $a = \textbf{IFFT}(\hat{a}) := \textbf{FFT}^{-1}(\hat{a})$ with the identities

$$a_j = \frac{1}{n} \sum_{k=0}^{n-1} \hat{a}_k z_n^{-jk} = \frac{1}{n}\Big(\hat{a}_0 z_n^{0j} + \sum_{k=1}^{n-1} \hat{a}_{n-k} z_n^{kj}\Big) \text{ for } j = 0, \ldots, n-1$$

is well defined.

b) Let $b \in \mathbb{C}^{n-1}$. Use the **FFT**-algorithm to compute the transformation $\tilde{b} = \textbf{FST}(b)$ with

$$\tilde{b}_k = \sum_{j=1}^{n-1} b_j \sin(\pi jk/n) \text{ for } k = 1, \ldots, n-1 \ .$$

Show that $b = \frac{2}{n}\textbf{FST}(\textbf{FST}(b))$ holds.

Problem 14 (no correction)

A function $f \in C[0,1]$ is interpolated by a linear spline s with respect to the equidistant partition $\Delta_n = \{0, h, 2h, \ldots, 1\}$, $h = \frac{1}{n}$.

Show that for $f \in C^2[0,1]$ there exist constants C_1, C_2 independent of h such that

$$\|f - s\|_\infty \le C_1 h^2 \text{ and } \|f - s\|_2 \le C_2 h^2 \ .$$

Problem 15 (to be handed in)

Consider the nodes $x_\nu = \nu$ and the data y_ν, $\nu = 0, \ldots, k$. Show that the interpolating cubic spline s with respect to the given nodes, data, and the boundary conditions $s''(0) = s''(k) = 0$ satisfies the relation

$$\int_0^k (s''(x))^2\,dx = \min_{f \in \mathcal{M}} \{ \int_0^k (f''(x))^2\,dx \} \ ,$$

$\mathcal{M} := \{f \in C^2[0,k] : f(\nu) = y_\nu \text{ for } \nu = 0, \ldots, k\}$.

Hint:

Use the identity

$$\int_0^k (f''(x))^2\,dx = \int_0^k (s''(x))^2\,dx + 2\int_0^k s''(x)(f''(x) - s''(x))\,dx + \int_0^k (f''(x) - s''(x))^2\,dx.$$

Problem 16 (no correction)

Consider the partition $\Delta = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and $f \in C^2[a,b]$. Let s be the interpolating cubic spline of f with respect to Δ, satisfying the boundary conditions

$$s'(a) = f'(a), \quad s'(b) = f'(b) \ .$$

Show:

a) $\|f'' - s''\|_2 \le \|f'' - t''\|_2$ for all $t \in S_3(\Delta)$.

b) If $f \in C^4[a,b]$ then $\|f'' - s''\|_2^2 = \int_a^b (f(x) - s(x)) f^{(4)}(x)\,dx$.

c) If $\bar{\Delta} = \{a = \bar{x}_0 < \bar{x}_1 < \cdots < \bar{x}_m = b\}$ is a refinement of the partition Δ (i.e. $\Delta \subseteq \bar{\Delta}$) and if $\bar{s}$ is the interpolating cubic spline of f with respect to $\bar{\Delta}$, satisfying $\bar{s}'(a) = f'(a), \bar{s}'(b) = f'(b)$ then $\|s''\|_2 \le \|\bar{s}''\|_2 \le \|f''\|_2$.

Please hand in your homework problems (No. **13** and **15**) due **Thursday**, **November 16, 2006**, **13:00h**. Put them in the slot marked „Numerische Mathematik I/II/III" in the Math-Building (20.30), 2nd floor opposite room 112. Please print your name and registration number on your problems.

On **Thursday**, **November 16, 2006**, **14:00-15:30 h** the problems will be discussed in the Neuer Hörsaal (Building 20.40).

Each Thursday a homework assignment will be handed out in the tutorial. The homework assignments are also available for download in the WWW:

http://www.mathematik.uni-karlsruhe.de/ianm3/lehre/numana12006w .