Universität Karlsruhe (TH)
Research University • founded 1825

Institute for Applied and Numerical Mathematics

Prof. Dr. Rudolf Scherer, Dipl.-Math. Wolfgang Müller

Advanced Topics in Numerical Analysis I
Homework Assignment No. 5

(WS 2006/2007) November 16, 2006

Problem 17 (to be handed in)

a) Let the partition $\Delta = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and $f \in C^4[a, b]$. By using the result of problem 5a show that the quadrature formula

$$\int_a^b f(x)\,dx = \frac{1}{2}\sum_{i=1}^n h_i(f(x_{i-1}) + f(x_i)) + \frac{1}{12}\sum_{i=1}^n h_i^2(f'(x_{i-1}) - f'(x_i)) + Rf$$

$h_i := x_i - x_{i-1}$, $i = 1, \ldots, n$ is exact (i.e. $Rf = 0$) for all spline functions $s \in S_3(\Delta)$.

b) How does the formula read for equidistant nodes ($h_i = h, i = 1, \ldots, n$)? By using the result of problem 5d show that the error representation

$$Rf = \frac{b-a}{720}\, h^4 f^{(4)}(\xi)\ , \xi \in [a, b] \text{ holds.}$$

Problem 18 (no correction)

Proof the existence of the improper integral $I = \displaystyle\int_0^1 \frac{\cos x}{\sqrt{x}}\,dx$ and compute an approximation to I in the following ways:

a) After applying a suitable substitution approximate the resulting integral using the Gauss-Legendre formula with two nodes and Kepler's formula.

b) Subtract the singularity of the integrand, i.e. split the function

$$\frac{\cos x}{\sqrt{x}} = \frac{1}{\sqrt{x}} + \frac{\cos x - 1}{\sqrt{x}}$$

and the apply Kepler's formula to the non-singular part.

c) Identify a series expansion of the integrand and integrate the first four nonzero terms.

Problem 19 (no correction)

Consider the Gauss-Tschebyscheff quadrature formula of the first kind with s nodes in $(-1, 1)$ with respect to $\omega(x) = \frac{1}{\sqrt{1-x^2}}$. Show:

a) The weights $\omega_\nu^{(s)}$ are all equal, i.e. $\omega_\nu^{(s)} = \frac{\pi}{s}$, $\nu = 1, \ldots, s$.

b) If $f \in C^{2s}[-1, 1]$, the the quadrature error

$$R_s f = \frac{\pi}{2^{2s-1}(2s)!} f^{(2s)}(\xi) \quad \xi \in (-1, 1).$$

Problem 20 (to be handed in)

Let $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$ and consider the scalar product

$$\langle f, g\rangle := \int_a^b f(x)g(x)\omega(x)\,dx$$

with $\omega(x) \geq 0$ in $[a, b]$. Where ω has a countable number of zeros at most.

a) Proof that the recursion

$$p_{n+1}(x) = xp_n(x) - \sum_{k=0}^n \frac{\langle xp_n, p_k\rangle}{\langle p_k, p_k\rangle}\, p_k(x) \text{ for } n = 0, 1, 2, \ldots \text{ with } p_0 \equiv 1$$

defines a series of orthogonal polynomials with the property $\langle p_i, p_j\rangle = 0$ for $i \neq j$.

b) Show: If $j \leq n - 2$, then $\langle p_{n+1}, p_j\rangle = 0$ holds.

c) Compute the nodes and weights of the Gaussian quadrature formula with two nodes with respect to the weight function $\omega(x) = \cos x$ in the interval $[-\pi/2, \pi/2]$.

Please hand in your homework problems (No. **17** and **20**) due **Thursday**, **November 23, 2006, 13:00h**. Put them in the slot marked „Numerische Mathematik I/II/III" in the Math-Building (20.30), 2nd floor opposite room 112. Please print your name and registration number on your problems.

On **Thursday**, **November 23, 2006**, **14:00-15:30 h** the problems will be discussed in the Neuer Hörsaal (Building 20.40).

Each Thursday a homework assignment will be handed out in the tutorial. The homework assignments are also available for download in the WWW:

http://www.mathematik.uni-karlsruhe.de/ianm3/lehre/numana12006w .