Universität Karlsruhe (TH)
Research University • founded 1825

Institute for Applied and Numerical Mathematics

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Advanced Topics in Numerical Analysis I
Homework Assignment No. 6

(WS 2006/2007) November 23, 2006

Problem 21 (to be handed in)

One wants to compute an integral over the interval $[-1,1]$. Find a quadrature formula

$$\int_{-1}^{1} f(x)\,dx = w_1 f(-1) + w_s f(1) + \sum_{i=2}^{s-1} w_i f(x_i)$$

which is exact for polynomials of degree $2s-3$ (Lobatto-Quadrature).

Hint: Use the orthogonal polynomials with respect to $\omega(x) = (1-x)(1+x)$.

Problem 22 (no correction)

$\{p_k\}_{k\geq 0}$ is a series of orthognoal normal polynomials with leading coefficient $a_k > 0$. Show:

a) For $x \neq y$ the formula of Christoffel-Darboux is valid:

$$\sum_{k=0}^{n} p_k(x)p_k(y) = \frac{a_n}{a_{n+1}} \frac{p_n(y)p_{n+1}(x) - p_n(x)p_{n+1}(y)}{x-y} \; .$$

b) The weights w_j, $j = 1,\ldots,s$ can be computed using $w_j = \left[\sum_{k=0}^{s-1} p_k^2(x_j)\right]^{-1}$.

Problem 23 (no correction)

$\{\tilde{p}_k\}_{k\geq 0}$ is a series of real orthogonal normal polynomials with the three term recurrence relation $\tilde{p}_{k+1}(x) = (x-\beta_k)\tilde{p}_k(x) - \gamma_k^2 \tilde{p}_{k-1}(x)$, $k = 0,1,\ldots,s-1$ with $\tilde{p}_{-1} = 0, \tilde{p}_0 = 1$. Show that the zeros of p_s are exactly the solutions of an eigenvalue problem $B_s\,\tilde{p} = x\tilde{p}$. Transform the problem with the diagonal matrix

$$D = \mathrm{diag}(1, \gamma_1, \gamma_1\gamma_2, \ldots, \gamma_1\gamma_2\cdots\gamma_{s-1})$$

to a symmetric problem.

How can you compute the weights of the quadrature formula?

Problem 24 (to be handed in)

The number e is to be approximated using $T(h) := (1+h)^{1/h}, h > 0$ with an extrapolation method.

a) Proof the existence of an asymptotic expansion of the form

$$e - T(h) = \sum_{n=1}^{\infty} \alpha_n h^n$$

and its convergence for $|h| < 1$.

b) Show that the approximation $\tilde{T}(h) := \frac{1}{2}(T(h) + T(-h))$ has an asymptotic expansions in powers of h^2.

c) Use Richardson extrapolation for $h \to 0$ on $T(h)$ and on $\tilde{T}(h)$ with respect to the step size series

$$h_0 = \frac{1}{2},\ h_j = \frac{h_{j-1}}{2}, \quad j = 1,2,\ldots$$

Give the formulas to build the scheme and compute the values in the scheme for h_0, h_1, h_2, and h_3.

d) How has $T(h)$ to be changed in order to approximate the number e^x, x fixed?

Please hand in your homework problems (No. **21** and **24**) due **Thursday**, **November 30, 2006**, **13:00h**. Put them in the slot marked „Numerische Mathematik I/II/III“ in the Math-Building (20.30), 2nd floor opposite room 112. Please print your name and registration number on your problems.

On **Thursday**, **November 30, 2006**, **14:00-15:30 h** the problems will be discussed in the Neuer Hörsaal (Building 20.40).

Each Thursday a homework assignment will be handed out in the tutorial. The homework assignments are also available for download in the WWW:

http://www.mathematik.uni-karlsruhe.de/ianm3/lehre/numana12006w .