Universität Karlsruhe (TH)
Research University • founded 1825

**Institute for Applied and Numerical Mathematics**

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# Advanced Topics in Numerical Analysis I
# Homework Assignment No. 14

(WS 2006/2007) February 1, 2007

**Problem 53** (to be handed in)

Show:

(a) The one-step method defined by

$$(c, A, b) := \left( \begin{array}{c|ccc|c} 0 & 0 & 0 & 0 & 1/6 \\ 1/2 & 1/2 & 0 & 0 & 2/3 \\ 1 & -1 & 2 & 0 & 1/6 \end{array} \right)$$

(method of Kutta) has order of consistency $p = 3$.

(b) The method of Kutta is exact for the initial value problem

$$y' = g(x), \quad y(x_0) = y_0, \quad g \in \mathcal{P}_3.$$

**Problem 54** (to be handed in)

A function $f \in C([0, T] \times \mathbb{R}^m, \mathbb{R}^m)$ is called *strictly dissipative*, if there exist a constant $\rho < 0$ such that

$$\langle f(x, y) - f(x, z), y - z \rangle \le \rho \, \|y - z\|^2, \qquad y, z \in \mathbb{R}^m$$

a) Show: The function $f(x, y) = \exp(-y) - \exp(y)$ with $m = 1$ is strictly dissipative.

b) Show: For $y' = f(x, y)$ and $z' = f(x, z)$ holds the estimate

$$\|y(x) - z(x)\| \le \exp(\rho x) \, \|y(0) - z(0)\|.$$

Hint: Use the functions

$$w(x) \;=\; \frac{1}{2}\|y(x) - z(x)\|^2 \;, \quad \varphi(x) = \int_0^x -2\rho w(\xi)\, d\xi \;, \quad \psi(x) = w(x) + \varphi(x) \;.$$

First, show that $\psi(x) \le w(0)$. Then, represent $\varphi(x)$ as a solution of a suitable ODE.

**Problem 55** (no correction)

(a) Find the multi-step method of the form

$$y_{n+3} + \alpha_2 y_{n+2} + \alpha_1 y_{n+1} + \alpha_0 y_n = h(\beta_2 f_{n+2} + \beta_0 f_n)$$

of the highest possible order.

(b) Apply the method from (a) to the initial value problem

$$y' = y, \quad y(0) = 1.$$

Compute an approximation to $y(2)$ with the step-size $h = \frac{1}{2}$. Use the Euler polygon method to determine the missing starting values.

**Problem 56** (no correction)

Consider the boundary value problem

$$y'' - \alpha y' - \beta y = r(t), \quad y(a) = y_0, \quad y(b) = y_{N+1},$$

$\alpha \in \mathbb{R}$, $\beta > 0$ and $r$ continuous.

We seek approximations $y_i$ to the exact values $y(t_i)$, $i = 1, 2, \ldots, N$, $t_i = a + ih$, $h = \frac{b-a}{N+1}$.

By approximation of the derivatives using symmetric (centered) differences we obtain the system of linear equations

$$Ax = c, \quad x = (y_1, y_2, \ldots, y_N)^T.$$

Compute the matrix $A$, the vector $c$ and give a sufficient condition on $h$ such that the system of linear equations can be solved uniquely.

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Please hand in your homework problems (No. **53** and **54**) due **Thursday**, **February 8, 2007**, **13:00h**. Put them in the slot marked „Numerische Mathematik I/II/III“ in the Math-Building (20.30), 2nd floor opposite room 112. Please print your name and registration number on your problems.

On **Thursday**, **February 8, 2007**, **14:00-15:30 h** the problems will be discussed in the Neuer Hörsaal (Building 20.40).

Each Thursday a homework assignment will be handed out in the tutorial. The homework assignments are also available for download in the WWW:

http://www.mathematik.uni-karlsruhe.de/ianm3/lehre/numana12006w .