Institute for Applied and Numerical Mathematics

Prof. Dr. Christian Wieners, Dr. Johannes Ernesti   2018/09/20

Summer School on Full Waveform Inversion – Practice Sheet

Exercise 1 (Getting and Compiling the Code)

The code can be installed (with user m++ and password m++) and the examples can be started by

```
svn co -r 963 https://svn.math.kit.edu/svn/M++/FWI-SummerSchool
cd FWI-SummerSchool/
make -j FWI
mpirun -n 4 fwi forward
paraview &
python3 tools/seismogram_lib.py data/FWI/seis_1.1_2.1
```

Compiling FWI generates the executable file fwi, it is started (in parallel) by mpirun. The output is written to the log-files in the directory log, vtk-files are in data/vtk and can be viewed with paraview.
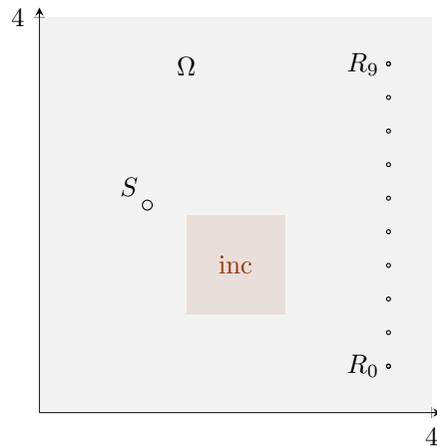

Figure 1: The wave that is exited at $S \in \Omega$ propagates through the domain and is recorded at the receiver positions $R_0, \ldots, R_9 \in \Omega$.

Exercise 2 (Solving the Forward Problem)

In this simple example, we consider the 2D acoustic wave equation $\rho_0 \partial_t \mathbf{v} = \nabla p$ and $\partial_t p = \kappa \operatorname{div} \mathbf{v} + b$ with constant density ρ_0 and variable permeability κ. The first-order system is discretized with a discontinuous Galerkin method of degree deg in space and by the implicit midpoint rule in time.

We start with an approximate point source b located at

```
source_x = 1.1;
source_y = 2.1;
```

Parameters of the configuration can be changed in FWI/conf/forward.conf.

(a) Run the forward simulation with procs processes

```
mpirun -n <procs> fwi forward
```

This creates snapshot plots of the wavefields in the directory data/vtk and seismograms in data/FWI (with make xxx all previous results are removed).

Use the program paraview to visualize the pressure P and the material distribution Kappa and Rho in data/vtk.

(b) Now test different source locations and compare the seismograms. Therefore, set different sname in the config file and compare

```
python3 tools/seismogram_lib.py <sname1> <sname2> ...
```

By adding -n to the call of seismogram_lib.py each trace gets scaled by its individual amplitide (making differences more visible).

Using -t when calling seismogram_lib.py, the program calculates seismogram differences.

How sensitive is this meassurement? What is the maximal distance of the sources, so that the relative difference of the seismograms is smaller than 0.8?

(c) In our test, we fix the density ρ and the background permeability κ_{bg}, and in a small region the permeability is changed to

```
kappa_inc = 1.5;
```

How sensitive is the meassurement? Which difference of κ_{bg} and κ_{inc} is required, so that the relative difference of the seimograms for the source position (1.1/2.1) is smaller than 0.5?

Exercise 3 (Convergence of the Forward Problem)

Solving partial differential equations (PDEs) numerically yields an approximation of the true solution of the PDE. Here, we consider how the approximation behaves under mesh refinement in space and time.

We use a uniform mesh of mesh width $\triangle x$, fixed time increments $\triangle t$, and a fixed polynomial degree in the ansatz spaces.

In `FWI/conf/forward.conf`, the discretization parameters are defined by

```
level = 4;
deg = 1;
dt = 0.01;
```

Use `sname = automatic` in the config file to let the program pick a name according to the discretization. Further, reset the source position to $(1.1/2.1)$ and the inclusion to `kappa_inc = 1.5`.

Make a few tests with different discretization parameters and compare the seismograms.

By adding `-t` to the call of `seismogram_lib.py` only the relative differences of the seismograms are computed.

(a) Run simulations for `deg = 1, 2` and `dt = 0.01` with `level = 3, 4` and compare the resulting seismograms to the reference solution provided by `FWI/s_ref_1.1_2.1__kappa_inc_1.5`.

   Which convergence order do you observe by refining the `level`?

(b) Consider the convergence order in time by computing the solution for `deg = 1, 2` on `level = 4` using different timestep sizes.

   Compare the relative differences of the seismograms for the timestep sizes `dt = 0.08, 0.04, 0.02, 0.01`.

(c) Find a configuration which is accurate up to a relative difference in the seimograms smaller than 0.5 compared to the reference solution.

Exercise 4 (The Inverse Problem)

We use the forward simulation to create synthetic data for the inverse problems. In order to reduce the inverse crime, one can use a finer discretization for the data generation than for the inversion. In this example, we use the same discretization for both, generation of data and inversion.

The data for three shots are computed with

```
mpirun -n 4 fwi generate
```

and, starting with $\kappa \equiv \kappa_0$, we recover the material distribution approximately by

```
mpirun -n 4 fwi invert
```

using CG-REGINN combining Newton's method and conjugate gradients for the normal equation in every Newton step.

Here, $m = (\rho, \kappa) \in \mathcal{P} \subset \mathrm{L}_\infty(\Omega; \mathbb{R}^2)$ are the material parameters,

$$\mathbf{y} = (p, \mathbf{v}) \in Y \subset \mathrm{C}^1\big([0, T]; \mathrm{H}^1(\Omega) \times \mathrm{H}(\mathrm{div}, \Omega)\big) \subset \mathrm{L}_2(Q; \mathbb{R} \times \mathbb{R}^2)$$

are the state variables in the space-time cylinder $Q = (0, T) \times \Omega$, and

$$L_m \mathbf{y} = (\rho \partial_t \mathbf{v} - \nabla p, \partial_t p - \kappa \, \mathrm{div} \, \mathbf{v})$$

is the first-order system corresponding to the acoustic wave equation for the parameter distribution $m = (\rho, \kappa)$. This defines the parameter-to-wavefield map

$$\mathcal{F} \colon \mathcal{P} \longrightarrow Y, \qquad m \longmapsto \mathbf{y} \quad \text{with} \quad L_m \mathbf{y} = (\mathbf{0}, b).$$

Given a set of space-time receivers $\mathcal{R} \subset Q$, every wave solution defines a seismogram in $\mathcal{S} = \mathbb{R}^{\mathcal{R}}$. The corresponding measument operator

$$\Psi \colon \mathrm{L}_2(Q; \mathbb{R} \times \mathbb{R}^2) \longrightarrow \mathcal{S}, \qquad \mathbf{y} \longmapsto \sum_{r \in \mathcal{R}} (\boldsymbol{\varphi}_r, \mathbf{y})_Q$$

is defined by measurement kernels $\boldsymbol{\varphi}_r$ approximating point evaluations at $r \in \mathcal{R}$. Together, this defines the parameter-to-seismogram map $\Phi = \Psi \circ \mathcal{F} \colon \mathcal{P} \longrightarrow \mathcal{S}$.

In this setting, the problem of Full-Waveform-Inversion reads as follows:

Given $s_{\mathrm{obs}} \in \mathcal{S}$, find $m \in \mathcal{P}$ with $\quad \Phi(m) \approx s_{\mathrm{obs}}^\delta$.

This problem is solved by Newton's method.

Therefore, let $\Phi'(m)\colon \mathcal{P} \longrightarrow \mathcal{S}$ by the linearized parameter-to-seismogram map.

Algorithm 1 Newton's method with approximate updates

1: Choose $m^0 \in \mathcal{P}$, $k \leftarrow 0$
2: **while** not converged **do**
3:     $r^k \leftarrow s_{\text{obs}} - \Phi(m^k) \in \mathcal{S}$
4:     Select $\vartheta_k > 0$.
5:     Find $\triangle m^k \in \mathcal{P}$ with $\|\Phi'(m^k)[\triangle m^k] - r^k\| \leq \vartheta_k \|r^k\|$.
6:     $m^{k+1} \leftarrow m^k + \triangle m^k$
7:     $k \leftarrow k + 1$

Since the linearization in this application in general is ill-conditioned, the approximate update $\triangle m^k \in \mathcal{P}$ is computed by the CG iteration for the corresponding normal equation.

Algorithm 2 Conjugate gradient algorithm for $\Phi'(m)^*\Phi'(m)[\triangle m] = \Phi'(m)^* r$

1: $j \leftarrow 0$, $\beta \leftarrow 0$, $r^0 \leftarrow r \in \mathcal{S}$
2: $p^0, \triangle m^0 \leftarrow 0 \in \mathcal{P}$
3: **while** not converged **do**
4:     $j \leftarrow j + 1$
5:     $d \leftarrow \Phi'(m)^*[r^{j-1}] \in \mathcal{P}$
6:     $p^j \leftarrow d + \beta \|d\|^2 p^{j-1}$
7:     $q \leftarrow \Phi'(m)[p^j] \in \mathcal{S}$
8:     $\alpha \leftarrow \|d\|^2 / \|q\|^2$
9:     $\triangle m^j \leftarrow \triangle m^{j-1} + \alpha \, p^j$
10:     $r^j \leftarrow r^{j-1} - \alpha \, q$
11:     $\beta \leftarrow 1/\|d\|^2$

In our example we fix the density $\rho \equiv \rho_0$ and the a small number source functions b_n $(n = 0, 1, ..., N_{\text{shots}} - 1)$ so that the inverse algorithm only tries to recover κ. In the Newton step k, we use the source function b_n and seismogram $s_{n,\text{obs}}$ alternating with $n = k \mod N_{\text{shots}}$.

(a) How large is the relative difference in the seismograms at the beginning?

(b) How many steps in the inverse method are required to obtain a relative difference in the seismograms smaller than 0.1?

(c) Which accuracy can be obtain with the method? Why is the accuracy limited and what is required to obtain better results?