Lecture Notes for the MFO seminar on wave phenomena

Space-time approximations for linear acoustic, elastic, and electro-magnetic wave equations
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1. Modeling of acoustic, elastic, and electro-magnetic waves

The goal of mathematical modeling of physical processes is to derive a partial differential equation that describes the behavior of a physical system correctly and allows for analytical and numerical predictions of the system behavior.

Here we start by shortly summarizing modeling principles which are then specified for different wave equations.

1.1. Modeling in continuum mechanics

Describing a model in continuum mechanics is a complex process combining physical principles, parameters and data. For a mathematical framework, we introduce the following terminology:

- **Geometric configuration**
  We select a domain in space \( \Omega \subset \mathbb{R}^d \) (\( d \in \{1, 2, 3 \} \)) and a time interval \( I \subset \mathbb{R} \), and for the specification of boundary conditions we select \( \Gamma_j \subset \partial \Omega \), \( j = 1, \ldots, m \), where \( m \) is the number of components of the variables which describe the current state of the physical system.

- **Constituents**
  Which physical quantities determine the model?
  Which quantities directly depend on these quantities?
  For the mathematical formulation it is required to select a set of primary variables.

- **Parameters**
  Which material data are required for the model?
  Which properties do these parameters have in order to be physically meaningful.

- **Balance relations**
  This collects relations between the physical quantities (and external sources) which are derived from basic energetic or kinematic principles. These relations are independent of specific materials and applications.

- **Material laws**
  This collects relations between the physical quantities that have to be determined by measurements and depend on the specific material and application.

- **Boundary and initial data**
  Additional conditions and data on the boundary on \( I \times \Omega \) are required to determine a unique solution.
1.2. The wave equation in 1d

This formalism is now specified for the most simple wave model in 1d with homogeneous coefficients. Therefore, we assume that all quantities are sufficiently smooth, so that all derivatives and integrals are well-defined.

Configuration. We consider an interval \( \Omega = (0, X) \subset \mathbb{R} \) in space and a time interval \( I = (0, T) \subset \mathbb{R} \).

Constituents. Here, we consider the simplified situation that material points in \( \mathbb{R}^2 \) move up and down vertically. Then, the state of this physical system is determined by the vertical *displacement*

\[
u: [0, T] \times \Omega \rightarrow \mathbb{R}
\]

describing the position of the material point \((x, u(t,x)) \in \mathbb{R}^2 \) at time \(t\), and the *tension*

\[
\sigma: [0, T] \times \Omega \rightarrow \mathbb{R}
\]

describing the forces between the points \(x \in \Omega\). In this simplified 1d setting with vertical displacements the tension corresponds to the *shear stress* in higher dimensions.

Depending on the primal variable \(u\), we can define the *velocity* \(v = \partial_t u\), the *acceleration* \(a = \partial_t v = \partial_t^2 u\), the *strain* \(\varepsilon = \partial_x u\), and the *strain rate* \(\partial_t \varepsilon = \partial_t v = \partial_x \partial_t u\).

Material parameters. This simple model only depends on the *mass density* \(\rho > 0\) and the *stiffness* \(\kappa > 0\); together, this defines \(c = \sqrt{\kappa/\rho}\). We will see that \(c\) is the wave speed which characterizes this model.

Balance of momentum. Depending on the velocity \(v\) and the mass density \(\rho\) we define the *momentum* \(\rho v\). Newton’s law states that the temporal change of the momentum in time equals the sum of all driving forces. Here, without any external forces, this balance relation reads as follows: for all \(0 < x_1 < x_2 < X\) and \(0 < t_1 < t_2 < T\) we have

\[
\int_{x_1}^{x_2} \rho(x)(v(t_2,x) - v(t_1,x)) \, dx = \int_{t_1}^{t_2} (\sigma(t,x_2) - \sigma(t,x_1)) \, dt.
\]

For smooth functions this yields

\[
\int_{x_1}^{x_2} \int_{t_1}^{t_2} \rho(x) \partial_t v(t,x) \, dt \, dx = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \partial_x \sigma(t,x) \, dx \, dt,
\]

and since this holds for all \(0 < x_1 < x_2 < X\) and \(0 < t_1 < t_2 < T\), this holds point-wise, i.e.,

\[
\rho(x) \partial_t v(t,x) = \partial_x \sigma(t,x), \quad (t,x) \in (0,T) \times (0,X).
\]

Material law. One observes that the tension \(\sigma(t,x)\) only depends on the strain \(\varepsilon(t,x) = \partial_x u(t,x)\). This is formulated as a material law: a material is by definition *elastic*, if a function \(\Sigma\) exists such that \(\sigma = \Sigma(\partial_x u)\), and it is *linear elastic*, if \(\sigma = \kappa \varepsilon\) with stiffness \(\kappa > 0\). In a homogeneous material, the stiffness \(\kappa\) is independent of \(x \in (0,X)\).

Boundary and initial data. The actual physical state at time \(t\) of the system depends on its state at the beginning \(t = 0\) and on constraints at the boundary. Here, assume that at \(t = 0\) the system is given by the initial displacement \(u(0,x) = u_0(x)\) and velocity \(v(0,x) = v_0(x)\) for \(x \in \Omega\), and we use homogeneous boundary conditions \(u(t,0) = u(t,X) = 0\) for \(t \in [0,T]\) corresponding to a string with is fixed at the endpoints.

Together, inserting \(v = \partial_t u\) and \(\varepsilon = \partial_x u\) we obtain the second-order formulation of the wave equation

\[
\partial_t^2 u(t,x) - c^2 \partial_x^2 u(t,x) = 0 \quad \text{for } (t,x) \in (0,T) \times (0,X), \quad (1a)
\]
\[
u(0,x) = u_0(x) \quad \text{for } x \in (0,X) \text{ at } t = 0, \quad (1b)
\]
\[
\partial_t u(0,x) = v_0(x) \quad \text{for } x \in (0,X) \text{ at } t = 0, \quad (1c)
\]
\[
u(t,x) = 0 \quad \text{for } x \in \{0,X\} \text{ and } t \in (0,T). \quad (1d)
\]

Note that the same equation can be derived for a 1d wave with horizontal displacement, corresponding to an actual position of the material point \(x + u(x) \in \mathbb{R}\).
The solution of the linear wave equation in 1D in homogeneous media. The equation (1) with constant wave speed \( c > 0 \) can be solved explicitly. For given initial values (1b) and (1c) the solution is given by the d’Alembert formula

\[
u(t,x) = \frac{1}{2} \left( u_0(x - ct) + u_0(x + ct) + \frac{1}{c} \int_{x-ct}^{x+ct} v_0(\xi) \, d\xi \right), \quad (x - ct, x + ct) \subset (0, X).
\]

Now we consider the solution in the bounded interval \( \Omega = (0, X) \) of length \( X = \pi \) with homogeneous Dirichlet boundary conditions (1d). The solution can be expanded into eigenmodes of the operator \(-\partial_x^2 u\) in \( H^1_0(\Omega) \cap H^2(\Omega)\), so that we obtain

\[
u(t,x) = \sum_{k=1}^{\infty} \left( \alpha_k \cos(ckt) + \beta_k \sin(ckt) \right) \sin(kx),
\]

where the coefficients are determined by the initial values (1b) and (1c). For the special example \( u_0(x) = 1, \ v_0(x) = 0 \) for \( x \in (0, \pi) \), and \( c = 1 \), we obtain the explicit Fourier representation

\[
u(t,x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \cos \left( (2k+1)t \right) \sin \left( (2k+1)x \right) = \frac{1}{2} \left( u_0(x + t) + u_0(x - t) \right), \quad (2)
\]

where the initial function \( u_0 \) is extended to the periodic function

\[
u_0(x) = \begin{cases} 1 & x \in (0, \pi) + 2\pi \mathbb{Z}, \\ 0 & x \in \pi \mathbb{Z}, \\ -1 & x \in (-\pi, 0) + 2\pi \mathbb{Z}, \end{cases}
\]

cf. Fig. 1. We observe that this solution solves the wave equation only in a weak sense since it is discontinuous along linear characteristics \( x \pm ct = \text{const.} \).

**Figure 1.** Weak solution \( u \in L^2((0, 8) \times (0, \pi)) \) with initial values for \( u(0, \cdot) = 1, \ \partial_t u(0, \cdot) = 0 \), and homogeneous Dirichlet boundary values \( u(\cdot, 0) = u(\cdot, \pi) = 0 \).
1.3. Harmonic, anharmonic and viscous waves

Special solutions of the linear wave equation (1) can be derived by the ansatz

\[ u(t, x) = \exp(-i\omega t)a(x) \]

with a fixed frequency \( \omega \in \mathbb{R} \). This yields in case of constant wave speed \( c = \sqrt{\kappa/\rho} \)

\[ \partial_t^2 u(t, x) - c^2 \partial_x^2 u(t, x) = -\left(\omega^2 a(x) + c^2 \partial_x^2 a(x)\right) \exp(-i\omega t). \]

The equation \( \omega^2 a(x) + c^2 \partial_x^2 a(x) = 0 \) is solved by \( a(x) = a_0 \exp(ikx) \) with \( k = \omega/c \) and \( a_0 \in \mathbb{R} \), cf. Tab. 1.

<table>
<thead>
<tr>
<th>wave number ( k )</th>
<th>angular frequency ( \omega )</th>
<th>frequency ( \nu = \omega/2\pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>wave speed ( c = \omega/k )</td>
<td>wave length ( \lambda = c/\nu )</td>
<td>amplitude ( a_0 )</td>
</tr>
</tbody>
</table>

Table 1. Characterizing quantities for harmonic waves \( u(t, x) = a_0 \exp(i(kx - \omega t)) \).

Interaction with material: anharmonic waves. The harmonic wave with constant amplitude is an idealistic model. This contradicts to observations: a wave traveling through material interacts with the particles in some sense, so that the amplitude is decreasing in time. A simple ansatz are waves of the form

\[ u(t, x) = a(t) \exp \left( i(kx - \omega t) \right), \quad a(t) = a_0 \exp(-\tau t) \quad (3) \]

depending on wave number \( k \), angular frequency \( \omega \), and relaxation time \( \tau > 0 \). Then, we observe for (3) in case of constant \( \rho \) and \( \kappa \)

\[
(\rho \partial_t^2 - \kappa \partial_x^2)u(x, t) = \left(\rho(\tau + i\omega)^2 + \kappa k^2\right)u(x, t), \quad \partial_t u(x, t) = -i(\tau + i\omega)u(x, t)
\]

which yields for \( \tau < \kappa k^2/\rho \) and \( \omega = \sqrt{\kappa^2/\rho - \tau^2} \in \mathbb{R} \) a solution of the wave equation with attenuation

\[ \rho \partial_t^2 u(t, x) - \kappa \partial_x^2 u(t, x) + 2\tau \rho \partial_t u(t, x) = 0. \quad (4) \]

In general, one observes that the wave speed depends on the frequency of the wave, i.e., the wave is dispersive. For the case of constant parameters this is characterized by the dispersion relation \( \omega = \omega(k) \). In this example, we find the dispersion relation for the wave equation with attenuation (4). For the general description of real media this approach is too simple and describes the wave propagation only within a limited frequency range, in particular since the relaxation time also depends on the frequency. For viscous waves suitable material laws are constructed where the parameters can be determined from measurements of the dispersion relation at sample frequencies which are relevant for the application. This is now demonstrated for a specific example.

A model for viscous waves. One approach to characterize waves with dispersion is to use a linear superposition of the constitutive law for a harmonic wave with several relations for anharmonic waves. In this ansatz the material law for the stress is based on a decomposition \( \sigma = \sigma_0 + \sigma_1 + \cdots + \sigma_r \) with Hooke’s law for \( \sigma_0 \), i.e.,

\[ \sigma_0 = \kappa_0 \varepsilon, \quad (5a) \]

and several Maxwell bodies for \( \sigma_1, \ldots, \sigma_r \) described by the relation

\[ \partial_t \sigma_j + \tau_j^{-1} \sigma_j = \kappa_j \partial_x \varepsilon, \quad j = 1, \ldots, r. \quad (5b) \]

This model depends on the stiffness of the components \( \kappa_0, \ldots, \kappa_r \) and relaxation times \( \tau_1, \ldots, \tau_r \). Solving the linear ODE (5b) with initial value \( \sigma_j(0) = 0 \) and inserting \( \partial_t \varepsilon = \partial_x \nu \) yields

\[ \sigma_j(t) = \int_0^t \kappa_j \exp \left(-\frac{1}{\tau_j}(t-s)\right) \partial_x v(s) \, ds, \]

and together with (5a) we obtain the retarded material law

\[ \sigma(t) = \kappa_0 \partial_x u(t) + \int_0^t \sum_{j=1}^r \kappa_j \exp \left(-\frac{1}{\tau_j}(t-s)\right) \partial_x v(s) \, ds. \]
This yields

$$\partial_t \sigma(t) = \kappa_0 \partial_x v(t) + \sum_{j=1}^r \kappa_j \partial_x v(t) - \int_0^t \sum_{j=1}^r \frac{\kappa_j}{\tau_j} \exp \left( - \frac{1}{\tau_j} (t - s) \right) \partial_x v(s) \, ds$$

$$= \kappa \partial_x v(t) + \int_0^t \dot{\kappa}(t - s) \partial_x v(s) \, ds$$

with stiffness $\kappa = \kappa_0 + \kappa_1 + \cdots + \kappa_r$ and retardation kernel

$$\dot{\kappa}(s) = - \sum_{j=1}^r \frac{\kappa_j}{\tau_j} \exp \left( - \frac{s}{\tau_j} \right).$$

Together with the balance relation $\rho \partial_t v = \partial_x \sigma$ this is a model for viscous waves.
1.4. Elastic waves

In the next step we derive equations for waves in solids. We consider heterogeneous media where the material parameters depend on the position, and we assume that the wave energy is sufficiently small, so that the material law can be approximated by a linear relation.

Configuration. We consider an elastic body in the spatial domain \( \Omega \subset \mathbb{R}^3 \) and we fix a time interval \([0, T]\). The boundary \( \partial \Omega = \Gamma_D \cup \Gamma_S \) is decomposed into a parts corresponding to dynamic or static boundary conditions.

Constituents. The current state of the body is described by the deformation or by the *displacement*

\[
\varphi = \text{id} + \mathbf{u} : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad \mathbf{u} : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}^3,
\]

i.e., \( \varphi(t, x) = x + \mathbf{u}(t, x) \) is the actual position of the point \( x \in \Omega \) at time \( t \).

The internal forces in the material are described by the *stress tensor*

\[
\sigma : [0, T] \times \Omega \rightarrow \mathbb{R}^{3 \times 3}.
\]

In the balance relation (6) only the normal stress \( \sigma(t, x)n \) for all directions \( n \in S^2 \) on the boundary of a subvolume \( K \subset \Omega \) is included. This described the force between material points left and right from \( x \in \partial K \) with respect to the direction \( n \). The existence of such a vector for all directions and all points is postulated by the Cauchy axiom, and by the Cauchy theorem a tensor representing this force exists; moreover, the symmetric of this tensor is a consequence of the balance of angular momentum.

Depending on the displacement, we define the *velocity* \( \mathbf{v} = \partial_t \mathbf{u} \), the *strain* \( \varepsilon(\mathbf{u}) = \text{sym}(D\mathbf{u}) \), the *acceleration* \( \mathbf{a} = \partial_t \mathbf{v} = \partial_t^2 \mathbf{u} \), and the *strain rate* \( \varepsilon(\mathbf{u}) = \text{sym}(D\mathbf{v}) = \partial_t \varepsilon \).

Material parameters. Measurements are required to determine the distribution of the mass density

\[
\rho : \Omega \rightarrow (0, \infty)
\]

and to determine the material stiffness in all directions which are collected in Hooke’s tensor

\[
C : \Omega \rightarrow \mathcal{L}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3}_{\text{sym}}).
\]

Balance of momentum. Newton’s law postulates equality of the temporal change of the momentum \( \rho \mathbf{v} \) in any time interval \((t_1, t_2) \in (0, T)\) within any subvolume \( K \subset \Omega \) with the driving forces on the boundary \( \partial K \) described by the stress in direction of the outer normal vector \( \mathbf{n} \) on \( \partial K \). This results in the balance relation (without external loads)

\[
\int_K \rho(x)(\mathbf{v}(t_2, x) - \mathbf{v}(t_1, x)) \, dx = \int_{t_1}^{t_2} \int_{\partial K} \sigma(t, x)\mathbf{n}(x) \, da \, dt. \quad (6)
\]

For smooth functions we obtain by the Gauß theorem

\[
\int_K \int_{t_1}^{t_2} \rho(x)\partial_t \mathbf{v}(t_2, x) \, dx \, dt = \int_{t_1}^{t_2} \int_K \text{div} \sigma(t, x) \, dx \, dt,
\]

and since this holds for all time intervals and subvolumes, we get

\[
\rho \partial_t \mathbf{v} = \text{div} \sigma \quad \text{in} \ (0, T) \times \Omega. \quad (7)
\]

Material law. Since the forces between the material points \( x_1 \) and \( x_2 \) only depend on the difference of the actual positions \( \mathbf{u}(t, x_2) - \mathbf{u}(t, x_2) \), the stress \( \sigma(t, x) \) only depends on the deformation gradient \( D\varphi \).

By definition, a material is elastic, if a function \( \Sigma \) exists such that \( \sigma = \Sigma(D\varphi) \). Then, \( \partial_t \sigma = D\Sigma(D\varphi)[D\mathbf{v}] \). A material response is linear in the limit of small strains, i.e., we assume \( D\varphi \approx I \), and we use the linear relation \( \partial_t \sigma = D\Sigma(I)[D\mathbf{v}] \). In addition, we assume that the stress response is *objective*, i.e., it is independent of the observer's position; then it can be shown that it only depends on the symmetric strain \( \varepsilon(\mathbf{u}) = \text{sym}(D\mathbf{u}) \).

Together, we obtain Hooke’s law

\[
\partial_t \sigma = C \varepsilon(\mathbf{v}).
\]
Introducing the corresponding stress decomposition $\sigma$ observable quantities. The stress components $\sigma_v$, and the total stress $\sigma$ are obtained by substituting $\sigma_v$ into the momentum balance equation together with the initial and boundary conditions for the velocity $v$. Including external body forces $f$, we obtain the second-order formulation of the linear wave equation

$$ \rho \partial_t^2 u - \text{div } C \varepsilon(u) = f $$

in $(0, T) \times \Omega$, \hspace{1cm} (8a)

$$ u(0) = u_0 $$

in $\Omega$ at $t = 0$, \hspace{1cm} (8b)

$$ \partial_t u_0 = v_0 $$

in $\Omega$ at $t = 0$, \hspace{1cm} (8c)

$$ u(t) = u_D(t) $$

on $\Gamma_D$ for $t \in (0, T)$, \hspace{1cm} (8d)

$$ \sigma(u) n = g_S(t) $$

on $\Gamma_S$ for $t \in (0, T)$. \hspace{1cm} (8e)

and, equivalently, the first-order formulation

$$ \rho \partial_t v - \text{div } \sigma = f $$

in $(0, T) \times \Omega$, \hspace{1cm} (9a)

$$ \partial_t \sigma - C \varepsilon(u) = 0 $$

in $(0, T) \times \Omega$, \hspace{1cm} (9b)

$$ v(0) = v_0 $$

in $\Omega$ at $t = 0$, \hspace{1cm} (9c)

$$ \sigma(0) = C \varepsilon(u_D) $$

in $\Omega$ at $t = 0$, \hspace{1cm} (9d)

$$ v(t) = \partial_t u_D(t) $$

on $\Gamma_D$ for $t \in (0, T)$, \hspace{1cm} (9e)

$$ \sigma(t) n = g_S(t) $$

on $\Gamma_S$ for $t \in (0, T)$. \hspace{1cm} (9f)

1.5. Visco-elastic waves

The balance of momentum (7) together with Hooke’s law $\sigma = C \varepsilon(v)$ describes linear elastic waves. We observe

$$ \sigma(t) = \sigma(0) + \int_0^t \partial_t \sigma(s) \, ds = \sigma(0) + \int_0^t C \varepsilon(v(s)) \, ds. $$

General linear visco-elastic waves are described by a \textit{retarded material law}

$$ \sigma(t) = \sigma(0) + \int_0^t C(t-s) \varepsilon(v(s)) \, ds \quad \Rightarrow \quad \partial_t \sigma(t) = C(0) \varepsilon(v(t)) + \int_0^t C(t-s) \varepsilon(v(s)) \, ds, $$

with a time-dependent extension of the elasticity tensor $C$.

In analogy to the 1D model (5), for Generalized Standard Linear Solids the relaxation tensor is chosen as

$$ \tilde{C}(s) = -\sum_{j=1}^r \frac{1}{\tau_j} \exp\left(-\frac{s}{\tau_j}\right) C_j, \quad C(0) = C_0 + C_1 + \cdots + C_r. $$

Introducing the corresponding stress decomposition $\sigma = \sigma_0 + \cdots + \sigma_r$ with

$$ \sigma_j(t) = \int_0^t \exp\left(-\frac{s-t}{\tau_j}\right) C_j \varepsilon(v(s)) \, ds, \quad j = 1, \ldots, r, $$

results in the first-order system for visco-elastic waves

$$ \rho \partial_t v - \nabla \cdot (\sigma_0 + \cdots + \sigma_r) = f, $$

$$ \partial_t \sigma_0 - C_0 \varepsilon(v) = 0, $$

$$ \partial_t \sigma_j - C_j \varepsilon(v) + \tau_j^{-1} \sigma_j = 0, \quad j = 1, \ldots, r. $$

This is complemented by initial and boundary conditions for the velocity $v$ and the total stress $\sigma$, which are the observable quantities. The stress components $\sigma_1, \ldots, \sigma_r$ are inner variables describing the retarded material law; they can be replaced, e.g., by memory variables encoding the material history.
1.6. Acoustic waves in solids

In isotropic media, Hooke’s tensor only depends on two parameters, e.g., the Lamé parameters $\mu, \lambda$

$$C \varepsilon = 2\mu \varepsilon + \lambda \text{tr}(\varepsilon) I$$

$$= 2\mu \text{dev}(\varepsilon) + \kappa \text{tr}(\varepsilon) I, \quad \text{dev}(\varepsilon) = \varepsilon - \frac{1}{3} \text{tr}(\varepsilon) I.$$ 

For the wave dynamics, one uses a decomposition into components corresponding to shear waves depending on the shear modulus $\mu$, and compressional waves depending on the compression modulus $\kappa = \frac{2}{3} \mu + \lambda$.

Then, the linear second order elastic wave equation (8a) takes the form

$$\partial_t^2 \mathbf{u} + \mu \nabla \times \nabla \times \mathbf{u} - 3\kappa \nabla (\nabla \cdot \mathbf{u}) = \mathbf{f}.$$ 

Vanishing shear modulus $\mu \to 0$ gives the linear acoustic wave equation for the hydrostatic pressure $p = \frac{1}{3} \text{tr}(\sigma)$ and the velocity, described by the first-order system

$$\rho \partial_t \mathbf{v} - \nabla p = \mathbf{f} \quad \text{in } (0, T) \times \Omega,$$

$$\partial_t p - \kappa \nabla \cdot \mathbf{v} = 0 \quad \text{in } (0, T) \times \Omega,$$

$$\mathbf{v}(0) = \mathbf{v}_0 \quad \text{in } \Omega \text{ at } t = 0,$$

$$p(0) = p_0 \quad \text{in } \Omega \text{ at } t = 0,$$

$$\mathbf{n} \cdot \mathbf{v}(t) = g_D(t) \quad \text{on } \Gamma_D \text{ for } t \in (0, T),$$

$$p(t) = p_S(t) \quad \text{on } \Gamma_S \text{ for } t \in (0, T),$$

where we set $p_S = \mathbf{n} \cdot \mathbf{g}_S$ and $g_D = \mathbf{n} \cdot \mathbf{v}_D$.

In homogeneous media and for $\mathbf{f} = \mathbf{0}$, (11a) and (11b) combine to the linear second-order acoustic wave equation

$$\partial_t^2 p - c^2 \Delta p = 0, \quad c = \sqrt{\kappa / \rho}.$$ 

**Remark 1.** Simply neglecting the shear component is only an approximation and not fully realistic for waves in solids, in particular since by reflections compressional waves split in compressional and shear components. Nevertheless, in applications the acoustic wave equation is used also in solids since the system is much smaller so that computations are much faster.

**Remark 2.** One obtains the same acoustic wave equations describing compression waves in a fluid or a gas. Note that, historically, the sign conventions for pressure and stress are different in fluid and solid mechanics.

**Visco-acoustic waves.** Generalized Standard Linear Solids also reduce to acoustics. The corresponding retarded material law for the hydrostatic pressure takes the form

$$\partial_t p(t) = \kappa \nabla \cdot \mathbf{v}(t) + \int_0^t \dot{\kappa}(t - s) \nabla \cdot \mathbf{v}(s) \, ds, \quad \dot{\kappa}(s) = -\sum_{j=1}^r \frac{\kappa_j}{\tau_j} \exp \left( -\frac{s}{\tau_j} \right).$$

Defining $\kappa = \kappa_0 + \kappa_1 + \cdots + \kappa_r$ and

$$p_j(t) = \int_0^t \exp \left( \frac{s - t}{\tau_j} \right) \kappa_j \nabla \cdot \mathbf{v}(s) \, ds, \quad j = 1, \ldots, r, \quad p_0 = p - (p_1 + \cdots + p_r)$$

results into the first-order system for linear visco-acoustic waves

$$\rho \partial_t \mathbf{v} - \nabla (p_0 + \cdots + p_r) = \mathbf{f},$$

$$\partial_t p_0 - \kappa_0 \nabla \cdot \mathbf{v} = 0,$$

$$\partial_t p_j - \kappa_j \nabla \cdot \mathbf{v} + \tau_j^{-1} p_j = 0, \quad j = 1, \ldots, r.$$ 

This is complemented by initial and boundary conditions (11c)–(11f).
1.7. Electro-magnetic waves

Electric fields induce a magnetic field, and vice versa. This is formulated by Maxwell’s equations describing the propagation of electro-magnetic waves.

**Configuration.** We consider a spatial domain $\Omega \subset \mathbb{R}^3$, a time interval $I = (0, T)$, and a boundary decomposition $\partial \Omega = \Gamma_E \cup \Gamma_I$ corresponding to perfect conducting or transmission boundaries.

**Constituents.** Electro-magnetic waves are determined by the **electric field** and the **magnetic field intensity**

$$E: \mathbb{T} \times \Omega \to \mathbb{R}^3, \quad H: \mathbb{T} \times \Omega \to \mathbb{R}^3,$$

and by the **electric flux density** and **magnetic induction**

$$D: \mathbb{T} \times \Omega \to \mathbb{R}^3, \quad B: \mathbb{T} \times \Omega \to \mathbb{R}^3.$$

Further quantities and the **electric current density** and the **electric charge density**

$$J: I \times \Omega \to \mathbb{R}^3, \quad \rho: I \times \Omega \to \mathbb{R}.$$

**Balance relations.** Faraday’s law states that the temporal change of the magnetic induction through a two-dimensional subset $A \subset \Omega$ induces an electric field along the boundary $\partial A$, so that for all $0 < t_1 < t_2 < T$

$$\int_A \left( B(t_2) - B(t_1) \right) \cdot da = - \int_{t_1}^{t_2} \int_{\partial A} E \cdot d\ell \, dt.$$

Ampere’s law states that the temporal change of the electric flux density together with the electric current density through a two-dimensional manifold $A \subset \Omega$ induces a magnetic field intensity along the boundary $\partial A$, i.e.,

$$\int_A \left( D(t_2) - D(t_1) \right) \cdot da + \int_{t_1}^{t_2} \int_A J \cdot da \, dt = \int_{t_1}^{t_2} \int_{\partial A} H \cdot d\ell \, dt.$$

Here, we use $u \cdot da = u \cdot n \, da$ and $u \cdot d\ell = u \cdot \tau \, d\ell$, the normal vector field $n: A \to \mathbb{R}^3$ and the tangential vector field $\tau: \partial A \to \mathbb{R}^3$ (where the orientation of $\partial A$ is given by $n$).

The Gauß laws state for all subvolumes $K \subset \Omega$ the conservation of the magnetic induction

$$\int_{\partial K} B \cdot da = 0$$

and the equilibrium of electric charge density in the volume with electric flux density across the boundary $\partial K$

$$\int_{\partial K} D \cdot da = \int_K \rho \, dx.$$

Together, by the theorems from Stokes and Gauss we obtain

$$\int_A \int_{t_1}^{t_2} \partial_t B \cdot da \, dt = - \int_{t_1}^{t_2} \int_A \nabla \times E \cdot da \, dt, \quad \int_K \nabla \cdot B \, dx = 0,$$

$$\int_A \int_{t_1}^{t_2} \partial_t D \cdot da \, dt + \int_{t_1}^{t_2} \int_A J \cdot da \, dt = \int_{t_1}^{t_2} \int_A \nabla \times H \cdot da \, dt, \quad \int_K \nabla \cdot D \, dx = \int_K \rho \, dx,$$

and since this holds for all $(t_1, t_2) \subset I$ and all $A, K \subset \Omega$, it results into the Maxwell system

$$\partial_t B + \nabla \times E = 0, \quad \partial_t D - \nabla \times H = -J, \quad \nabla \cdot B = 0, \quad \nabla \cdot D = \rho. \quad (12)$$

Note that this implies the conservation of charge $\partial_t \rho + \nabla \cdot J = 0$. 
Material laws in vacuum. Without the interaction with material electric field and the electric flux density with \(D = \varepsilon_0 E\), and magnetic induction and magnetic field intensity with \(B = \mu_0 H\) are proportional up to multiplication with the constant permittivity \(\varepsilon_0\) and permeability \(\mu_0\), which together results in the linear second-order Maxwell equation

\[ \partial_t^2 E - c^2 \nabla \times \nabla \times E = 0 \]

with speed of light \(c = 1/\sqrt{\varepsilon_0 \mu_0}\).

In vacuum, there is no electric current and no electric charge, i.e., \(J = 0\) and \(\rho = 0\).

Effective material laws for electro-magnetic waves in matter. The interaction of electro-magnetic with the atoms in materials are described the polarization \(P\) and the magnetization \(M\) depending on electric field \(E\) and the magnetic induction \(B\). For the electric flux density holds

\[ D = \varepsilon_0 E + P(E, B) \]

and the magnetic field intensity is given by

\[ \mu_0 H = B - M(E, B) \]

For the electric current density we have

\[ J = \sigma(E, H)E + J_0 \]

depending on the conductivity \(\sigma\) and the external current \(J_0\).

In case of linear materials with instantaneous response, the polarization is proportional to the electric field

\[ P = \varepsilon_0 \chi E \]

with the susceptibility \(\chi\), that yields \(D = \varepsilon_r E\) with relative permittivity \(\varepsilon_r = \varepsilon_0(1 + \chi)\).

Linear materials with retarded response are given by

\[ P(t) = \varepsilon_0 \int_{-\infty}^{t} \chi(t - s)E(s) \, ds \tag{13} \]

A special case is the Debye model with \(\chi(t) = \exp \left( \frac{t}{\tau} \frac{\varepsilon_s - \varepsilon_\infty}{\varepsilon_\infty} \right)\), so that the polarization is determined by

\[ \tau \partial_t P + P = \varepsilon_0 (\varepsilon_s - \varepsilon_\infty) E \]

This model is dispersive with a dispersion relation similar to the model for viscous elastic waves.

The relation (13) extends to nonlinear materials by, e.g.,

\[ P(t) = \varepsilon_0 \int_{-\infty}^{t} \chi_1(t - s)E(s) \, ds + \int_{-\infty}^{t} \int_{-\infty}^{t} \int_{-\infty}^{t} \chi_3(t - s_1, t - s_2, t - s_3)E(s_1)E(s_2)E(s_3) \, ds_1ds_2ds_3 \]

For materials of Kerr-type this response is instantaneous, i.e.,

\[ P = \chi_1 E + \chi_3 |E|^2 E \]

In more complex material models, the Maxwell system (12) is coupled to evolution equations for polarization or magnetization. E.g., in the Maxwell–Lorentz system the evolution of the polarization is determined by the ODE

\[ \partial_t^2 P = \frac{1}{\varepsilon_0^2} (E - P) + |P|^2 P \]

In the Landau–Lifshitz–Gilbert (LLG) equation the magnetization \(M\) is determined by

\[ \partial_t M - \alpha M \times \partial_t M = -M \times H_{\text{eff}} \]
Boundary conditions. The Maxwell system is complemented by conditions on $\partial \Omega$.

On a perfectly conducting boundary $\Gamma_E$, we have

$$E \times n = 0 \text{ and } B \cdot n = 0,$$

and on the impedance (or Silver–Müller) boundary $\Gamma_I$, we have

$$H \times n + (\zeta (E \times n) E \times n) \times n = 0$$

depending on the conductivity $\zeta$.

Together, we obtain for general nonlinear instantaneous material laws $D(E, H)$ and $B(E, H)$ the quasilinear first-order system

$$\frac{\partial}{\partial t} D(E, H) - \nabla \times H + \sigma(E, H) E = -J_0,$$
$$\frac{\partial}{\partial t} B(E, H) + \nabla \times E = 0$$

in $(0, T) \times \Omega$, \quad (14a)

$$E(0) = E_0 \quad \text{in } \Omega \text{ at } t = 0,$$
$$H(0) = H_0 \quad \text{in } \Omega \text{ at } t = 0,$$

$$E \times n = 0 \quad \text{on } \Gamma_E \text{ for } t \in (0, T),$$
$$H \times n + (\zeta (E \times n) E \times n) \times n = g \quad \text{on } \Gamma_I \text{ for } t \in (0, T).$$

A second-order quasi-linear wave model. In the special case of a nonlinear instantaneous nonmagnetic material law $D(E) = \varepsilon_0 E + \mathbf{P}(E)$ and $M \equiv 0$ the Maxwell system reduces to the second-order equation

$$\frac{\partial^2}{\partial t^2} D(E) + \mu_0^{-1} \nabla \times \nabla \times E + \sigma(E) E = -\partial_t J_0$$

complemented by initial and boundary conditions.

Bibliographic comments

The mathematical foundations of modeling elastic solids (including a detailed discussion and a proof of the Cauchy theorem) is given in [Ciarlet, 1988], more physical background in [Davis, 2012]. For generalized standard linear solids we refer to [Fichtner, 2011]. An overview on modeling of electro-magnetic waves in given in [Jackson, 1999], the mathematical aspects of photonics are considered in [Dörfler et al., 2011]. The example (2) is taken from [Leis, 2013, Example 3.4]. Dispersion relations and the analogy in the modeling of elastic and electromagnetic waves are collected in [Carcione, 2014, Chap. 2 and Chap. 8].
2. Space-time solutions for linear hyperbolic systems

The linear wave equation can be analyzed in the framework of symmetric Friedrichs systems as a special case of linear hyperbolic conservation laws. Here, we introduce a general framework for the existence and uniqueness of strong and weak solutions in space and time which applies to general linear wave equations.

We start with operators in space and time of the form \( L = M \partial_t + A \), where \( A \) is an hyperbolic operator in space. All results transfer to operators of the form \( L = M \partial_t + A + D \) with an additional dissipative operator \( D \). This applies to visco-acoustic and visco-elastic models, to mixed boundary conditions of Robin type and impedance boundary conditions.

2.1. Linear hyperbolic first-order systems

Let \( \Omega \subset \mathbb{R}^d \) be a domain in space with Lipschitz boundary, \((0, T)\) a time interval, and together we denote the space-time cylinder by \( Q = (0, T) \times \Omega \). Boundary conditions will be imposed on \( \Gamma_k \subset \partial \Omega \) for \( k = 1, \ldots, m \) depending on the model, so that the corresponding equations are well-posed.

We consider a linear operator in space and time of the form \( L = M \partial_t + A \) with a uniformly positive definite operator \( M \) defined by \( My(x) = M(x)y(x) \) with a matrix function \( M \in L_\infty(\Omega; \mathbb{R}^{m \times m}) \), and a differential operator \( Ay = \sum_{j=1}^d B_j \partial_j y \) with matrices \( B_j \in \mathbb{R}^{m \times m} \). Moreover, we define the matrix \( B_n = \sum_j n_j B_j \in \mathbb{R}^{m \times m} \) for \( n \in \mathbb{R}^d \) and the corresponding boundary operator \( (B_n y)(x) = B_n(x)y(x) \).

In the first step, we consider the properties of the operators \( A \) and \( L \) for smooth functions. Then the operators are extended to Hilbert spaces and, by specifying boundary conditions, we define maximal domains for the operators.

Example 3. This applies to the linear acoustic wave equation (11) with \( m = d + 1 \) and the operators

\[
\begin{align*}
y &= \begin{pmatrix} v \\ p \end{pmatrix}, & My &= \begin{pmatrix} \rho v \\ \kappa^{-1} p \end{pmatrix}, & Ay &= \begin{pmatrix} -\nabla p \\ -\nabla \cdot v \end{pmatrix}, & B_n y &= \begin{pmatrix} -\rho n \\ -n \cdot v \end{pmatrix}.
\end{align*}
\]  

(15)

In two space dimensions, this corresponds to the boundary parts \( \Gamma_1 = \Gamma_S \) and \( \Gamma_2 = \Gamma_3 = \Gamma_D \), and the matrices

\[
M = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \kappa^{-1} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.
\]

For linear elastic waves we have

\[
\begin{align*}
y &= \begin{pmatrix} v \\ \sigma \end{pmatrix}, & My &= \begin{pmatrix} \rho v \\ C^{-1} \sigma \end{pmatrix}, & Ay &= \begin{pmatrix} -\text{div} \sigma \\ -\varepsilon(v) \end{pmatrix}, & B_n y &= \begin{pmatrix} -\sigma n \\ -(nv^\top + vn^\top) \end{pmatrix}.
\end{align*}
\]  

(16)

Note that \( \frac{1}{2} My \cdot y = \frac{1}{2} (\rho|v|^2 + \sigma \cdot C^{-1} \sigma) = \frac{1}{2} (\rho|\partial_t u|^2 + \varepsilon(u) \cdot C \varepsilon(u)) \) is the kinetic and potential energy.

For linear electro-magnetic waves we have

\[
\begin{align*}
y &= \begin{pmatrix} E \\ H \end{pmatrix}, & My &= \begin{pmatrix} \varepsilon_0 E \\ \mu_0 H \end{pmatrix}, & Ay &= \begin{pmatrix} -\nabla \times H \\ \nabla \times E \end{pmatrix}, & B_n y &= \begin{pmatrix} -n \times H \\ n \times E \end{pmatrix}.
\end{align*}
\]  

(17)

and \( \frac{1}{2} My \cdot y = \frac{1}{2} (\varepsilon_0 |E|^2 + \mu_0 |H|^2) \) is the electro-magnetic energy.
Linear conservation laws. Defining \( \mathbf{B} = (\mathbf{B}_j)_{j=1}^d \), we observe
\( \mathbf{Ay} = \sum_{j=1}^d \partial_j \mathbf{B}_j \mathbf{y} = \text{div}(\mathbf{B} \mathbf{y}) \) and \( \mathbf{B}_n = \mathbf{n} \cdot \mathbf{B} = \sum_{j=1}^d n_j \mathbf{B}_j \), so that this system takes the form of a linear conservation law
\[
\mathbf{M} \partial_t \mathbf{y} + \text{div}(\mathbf{B} \mathbf{y}) = \mathbf{f}.
\]
Integration by parts yields for differentiable functions with compact support in \( \Omega \)
\[
(A \mathbf{y}, \mathbf{z}) = \sum_{j=1}^d \int_\Omega \mathbf{B}_j \partial_j \mathbf{y} \cdot \mathbf{z} \, dx = \sum_{j=1}^d \sum_{k,l=1}^m \int_\Omega B_{jkl}(\partial_j y_l) z_k \, dx
\]
so that \( A^* = -A \) on \( C^1_c(\Omega; \mathbb{R}^m) \). On the boundary \( \partial \Omega \) with outer unit normal \( \mathbf{n} \), integration by parts yields
\[
(A \mathbf{y}, \mathbf{z}) + (\mathbf{y}, A \mathbf{z}) = \sum_{j=1}^d \sum_{k,l=1}^m \int_\Omega (B_{jkl}(\partial_j y_l) z_k + y_l B_{jlk}(\partial_j z_k)) \, dx
\]
Together, we obtain in space and time for \( L = \mathbf{M} \partial_t + A \) and its adjoint \( L^* = -L \)
\[
(L \mathbf{v}, \mathbf{w})_Q - (\mathbf{v}, L^* \mathbf{w})_Q = (\mathbf{M} \mathbf{v}(T), \mathbf{w}(T))_\Omega - (\mathbf{M} \mathbf{v}(0), \mathbf{w}(0))_\Omega + (\mathbf{B}_n \mathbf{v}, \mathbf{w})_{(0,T) \times \partial \Omega}
\]
for \( \mathbf{v}, \mathbf{w} \in C^1(\Omega; \mathbb{R}^m) \cap C^0(\overline{\Omega}; \mathbb{R}^m) \).

**Example 4.** For linear acoustic waves (15) we have
\[
(L(\mathbf{v}, p), (\mathbf{w}, q))_Q + ((\mathbf{v}, p), L(\mathbf{w}, q))_Q = (\rho \mathbf{v}(T), \mathbf{w}(T))_\Omega + (\kappa^{-1} p(T), p(T))_\Omega
\]
For linear elastic waves (16) we have
\[
(L(\mathbf{v}, \sigma), (\mathbf{w}, \tau))_Q + ((\mathbf{v}, \sigma), L(\mathbf{w}, \tau))_Q = (\rho \mathbf{v}(T), \mathbf{w}(T))_\Omega + (C^{-1} \sigma(T), \tau(T))_\Omega
\]
For linear electro-magnetic waves (17) we have
\[
(L(\mathbf{E}, \mathbf{H}), (\mathbf{e}, \mathbf{h}))_Q + ((\mathbf{E}, \mathbf{H}), L(\mathbf{e}, \mathbf{h}))_Q = (\varepsilon_0 \mathbf{E}(T), \mathbf{e}(T))_\Omega + (\mu_0 \mathbf{H}(T), \mathbf{h}(T))_\Omega
\]
Here we use the following calculus: for vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3 \) we have \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} \), and for vector fields \( \mathbf{u}, \mathbf{v} : \Omega \to \mathbb{R}^3 \) we have \( \nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v}) \). Thus, the Gauß theorem gives
\[
\int_{\Omega} \mathbf{v} \cdot (\nabla \times \mathbf{u}) \, dx - \int_{\Omega} \mathbf{u} \cdot (\nabla \times \mathbf{v}) \, dx = \int_{\partial \Omega} \nabla \cdot (\mathbf{u} \times \mathbf{v}) \, da - \int_{\partial \Omega} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{n} \, da = \int_{\partial \Omega} \mathbf{u} \cdot (\mathbf{v} \times \mathbf{n}) \, da.
\]
2.2. Solution spaces

We define the Hilbert spaces

\[ H(A, \Omega) = \{ y \in L_2(\Omega; \mathbb{R}^m) : z \in L_2(\Omega; \mathbb{R}^m) \text{ exists with } (z, w)_{\Omega} = (y, A^* w)_{\Omega} \text{ for all } w \in C^1(\Omega; \mathbb{R}^m) \} \]

\[ = \{ y \in L_2(\Omega; \mathbb{R}^m) : Ay \in L_2(\Omega; \mathbb{R}^m) \}, \quad \| y \|_{H(A, \Omega)} = \sqrt{\| y \|^2_{\Omega} + \| Ay \|^2_{\Omega}}, \]

\[ H(L, Q) = \{ v \in L_2(Q; \mathbb{R}^m) : z \in L_2(Q; \mathbb{R}^m) \text{ exists with } (z, w)_Q = (v, L^* w)_Q \text{ for all } w \in C^1(Q; \mathbb{R}^m) \} \]

\[ = \{ v \in L_2(Q; \mathbb{R}^m) : Lv \in L_2(Q; \mathbb{R}^m) \}, \quad \| v \|_{H(L, Q)} = \sqrt{\| v \|^2_Q + \| Lv \|^2_Q}. \]

Depending on homogeneous boundary conditions on \( \Gamma_k \subset \partial \Omega, \ k = 1, \ldots, m \), we define

\[ V = \{ w \in C^1(\Omega; \mathbb{R}^m) \cap C^0(\overline{\Omega}; \mathbb{R}^m) : w(0) = 0, \ (B_n w)_k = 0 \text{ on } (0, T) \times \Gamma_k, \ k = 1, \ldots, m \}, \]

\[ V^* = \{ z \in C^1(\Omega; \mathbb{R}^m) \cap C^0(\overline{\Omega}; \mathbb{R}^m) : z(T) = 0, \ (B_n z)_k = 0 \text{ on } (0, T) \times \Gamma^*_k, \ k = 1, \ldots, m \}, \]

where \( \Gamma^*_k \subset \partial \Omega \) is minimal such that \((B_n w, z)_{(0, T) \times \partial \Omega} = 0\) for \( w \in V \) and \( z \in V^* \).

Let \( V \subset H(L, Q) \) be the closure of \( V \), and \( V^* \subset H(L^*, Q) \) be the closure of \( V^* \). Then, we obtain from (18)

\[ (Lv, w)_Q - (v, L^* w)_Q = 0, \quad v \in V, \ w \in V^*. \]

Example 5. For linear acoustic waves (15) we have \( H(A, \Omega) = H(\text{div}, \Omega) \times H^1(\Omega) \), and for the boundary parts \( \Gamma_1 = \Gamma_S \) and \( \Gamma_2 = \Gamma_3 = \Gamma_D \) we obtain with \( \Gamma_k = \Gamma^*_k \) and thus

\[ V \subset \{(v, p) \in L_2(0, T; H(\text{div}, \Omega) \times H^1(\Omega)) : v(0) = 0, \ p(0) = 0, \]

\[ v \cdot n = 0 \text{ on } (0, T) \times \Gamma_D, \ p = 0 \text{ on } (0, T) \times \Gamma_S \}, \]

\[ V^* \subset \{(w, q) \in L_2(0, T; H(\text{div}, \Omega) \times H^1(\Omega)) : w(T) = 0, \ q(T) = 0, \]

\[ \quad w \cdot n = 0 \text{ on } (0, T) \times \Gamma_D, \ q = 0 \text{ on } (0, T) \times \Gamma_S \}. \]

In \( Y = L_2(\Omega; \mathbb{R}^m) \) and \( W = L_2(Q; \mathbb{R}^m) \) we use the energy norms

\[ \| y \|_Y = \sqrt{(My, y)_\Omega}, \quad \| w \|_W = \sqrt{(Mw, w)_Q} \]

and for the \( L_2 \) adjoints

\[ \| y \|_{Y^*} = \sup_{z \in Y \setminus \{0\}} \frac{(y, z)_\Omega}{\| z \|_Y} = \sqrt{(M^{-1} y, y)_\Omega}, \quad \| w \|_{W^*} = \sqrt{(M^{-1} w, w)_Q}. \]

In \( V \) and \( V^* \) we use the weighted norms

\[ \| v \|_V = \sqrt{\| v \|^2_W + \| Lv \|^2_{W^*}}, \quad \| z \|_{V^*} = \sqrt{\| z \|^2_W + \| L^* z \|^2_{V^*}}, \quad v \in V, \ z \in V^*. \]

Remark 6. For the extension to visco-acoustic and visco-elastic models the same solution spaces can be used. For mixed boundary conditions of Robin type or impedance-elastic boundary conditions a modification is required by including by additional conditions on the boundary; see Sect. 2.7.
2.3. Solution concepts

We consider different solutions spaces of the equation $Lu = f$ with initial and boundary conditions.

**Definition 7.** Depending on regularity of the data, we define:

a) $u \in C^1(Q; \mathbb{R}^m) \cap C^0(\overline{Q}; \mathbb{R}^m)$ is a classical solution, if

\[
Lu = f \\
u(0) = u_0 \\
(B_u u)_k = g_k
\]

for $f \in C^0(Q; \mathbb{R}^m)$, $u_0 \in C^0(\Omega; \mathbb{R}^m)$, $g_k \in C^0((0, T) \times \Gamma_k)$.

b) $u \in H(L, Q)$ is a strong solution, if

\[
Lu = f \\
u(0) = u_0 \\
(B_u u)_k = g_k
\]

for $f \in L^2(Q; \mathbb{R}^m)$, $u_0 \in L^2(\Omega; \mathbb{R}^m)$, $g_k \in L^2((0, T) \times \Gamma_k)$.

c) $u \in L^2(Q; \mathbb{R}^m)$ is a weak solution, if

\[
(u, L^* z)_Q = (f, z)_Q + (Mu_0, z(0))_\Omega - (g, z)_{(0,T) \times \partial \Omega}, \quad z \in V^*
\]

for $f \in L^2(Q; \mathbb{R}^m)$, $u_0 \in L^2(\Omega; \mathbb{R}^m)$, $g_k \in L^2((0, T) \times \Gamma_k)$, and $g = (g_k)_{k=1,...,m} \in L^2((0, T) \times \partial \Omega; \mathbb{R}^m)$ with $g_k = 0$ on $\partial \Omega \setminus \Gamma_k$.

**Example 8.** A weak solution $(\nu, \sigma) \in L^2((0, T) \times (0, X); \mathbb{R}^2)$ of the linear wave equation (1) in 1d with wave speed $c = \sqrt{\kappa/\rho}$ and homogeneous Dirichlet boundary conditions satisfies

\[
(\nu, -\rho \partial_t w + \partial_x \tau)_{0,(0,T) \times (0,X)} + (\sigma, -\kappa^{-1} \partial_t \tau + \partial_x w)_{0,(0,T) \times (0,X)} = (\nu_0, w(0))_{(0,X)} + (\sigma_0, \tau(0))_{(0,X)}
\]

for all test functions $w, \tau \in C^1([0, T] \times [0, X])$ with $w(T, x) = \tau(T, x) = 0$, $x \in (0, X)$, and $w(t, 0) = w(t, X) = 0$, $t \in (0, T)$. This allows for discontinuities along the characteristics

\[
\left\{ \left( \frac{t}{x_0 \pm ct} \right) \in (0, T) \times \mathbb{R} : x_0 \pm ct \in \Omega \right\} = \left\{ \left( \frac{t}{x} \right) \in (0, T) \times \Omega : \left( \frac{t}{x-x_0} \right) \cdot (\pm c) = 0 \right\}.
\]

Here we illustrate this for a simple example: consider a piecewise constant function

\[
\begin{pmatrix}
\nu(t, x) \\
\sigma(t, x)
\end{pmatrix} = \begin{cases}
\begin{pmatrix}
v_L \\
\sigma_L
\end{pmatrix} & x < x_0 + ct, \\
\begin{pmatrix}
v_R \\
\sigma_R
\end{pmatrix} & x > x_0 + ct,
\end{cases}
\]

\[
[v] = v_R - v_L, \\
[\sigma] = \sigma_R - \sigma_L.
\]
Then, we have for all \((w, \tau) \in C_c([0, T] \times [0, X], \mathbb{R}^2)\)

\[
\int_0^T \int_0^X \left( \frac{v}{\sigma} \right) \cdot \left( \begin{array}{c}
-\rho \partial_t w + \partial_x \tau \\
-\kappa \sigma \partial_t \tau + \partial_x w
\end{array} \right) \, dx \, dt
\]

\[
= \int_{x < x_0 + ct} \left( \frac{\partial v}{\partial x} \right) \cdot \left( \begin{array}{c}
-\rho \sigma v - \kappa \sigma w \\
v v + \sigma w
\end{array} \right) \, dx \, dt + \int_{x > x_0 + ct} \left( \frac{\partial v}{\partial x} \right) \cdot \left( \begin{array}{c}
-\rho \sigma v - \kappa \sigma w \\
v \sigma v + \sigma w
\end{array} \right) \, dx \, dt
\]

\[
= -\frac{1}{\sqrt{1+c^2}} \int_{x=x_0 + ct} \left( c \right) \cdot \left( -\rho [v] - \kappa [\sigma] \right) \, da
\]

\[
= \frac{1}{\sqrt{1+c^2}} \int_{x=x_0 + ct} \left( c \rho 0 \kappa - \left( \begin{array}{c}
0 \\
1
\end{array} \right) \left( \begin{array}{c}
[v] \\
[\sigma]
\end{array} \right) \right) \cdot \left( \begin{array}{c}
\sigma \tau \\
\tau
\end{array} \right) \, da
\]

and we observe, that \((v, \sigma)\) is a weak solution if the jump \([v]/[\sigma]\) is an eigenvector of

\[
A \left( \begin{array}{c}
[v] \\
[\sigma]
\end{array} \right) = cM \left( \begin{array}{c}
[v] \\
[\sigma]
\end{array} \right).
\]

This is equivalent to the jump conditions \(|\sigma| - \rho |v| = 0\) and \(|v| - c \kappa^{-1} |\sigma| = 0\).

Based on the the jump conditions we construct a weak solution \((v, \sigma) \in L_2((0, T) \times (0, L), \mathbb{R}^2)\) with \(L = ct\) which is discontinuous along the characteristics \((t, j \Delta x \pm ct)\) on a special mesh in space and time depending of the wave speed \(c\) with \(\Delta x = c \Delta t\) and \(\Delta t = T/N, N \in \mathbb{N}\), cf. Fig. 2. Starting with \(v(0, x) = v_{j-\frac{1}{2}}^0\) and \(\sigma(0, x) = \sigma_{j-\frac{1}{2}}^0\) for \((j-1)\Delta x < x < j\Delta x\), we obtain from the jump condition recursively for \(n = 1, 2, \ldots, N\)

\[
v_{j-\frac{1}{2}}^n = \frac{1}{2} \left( v_{j+1}^{n-\frac{1}{2}} + v_{j-\frac{1}{2}}^{n-\frac{1}{2}} + \sigma_{j+\frac{1}{2}}^{n-\frac{1}{2}} - \sigma_{j-\frac{1}{2}}^{n-\frac{1}{2}} \right), \quad v_{j-\frac{1}{2}} = -v_{j+\frac{1}{2}},
\]

\[
\sigma_{j-\frac{1}{2}}^n = \frac{1}{2} \left( v_{j+1}^{n-\frac{1}{2}} - v_{j-\frac{1}{2}}^{n-\frac{1}{2}} + \sigma_{j+\frac{1}{2}}^{n-\frac{1}{2}} + \sigma_{j-\frac{1}{2}}^{n-\frac{1}{2}} \right), \quad j = 0, \ldots, N, \quad v_{N+\frac{1}{2}} = -v_{N-\frac{1}{2}},
\]

\[
v_{j-\frac{1}{2}}^n = \frac{1}{2} \left( v_{j+1}^{n-\frac{1}{2}} + v_{j-\frac{1}{2}}^{n-\frac{1}{2}} + \sigma_{j+\frac{1}{2}}^{n-\frac{1}{2}} - \sigma_{j-\frac{1}{2}}^{n-\frac{1}{2}} \right), \quad \sigma_{j+\frac{1}{2}} = \sigma_{j-\frac{1}{2}},
\]

\[
\sigma_{j-\frac{1}{2}}^n = \frac{1}{2} \left( v_{j+1}^{n-\frac{1}{2}} - v_{j-\frac{1}{2}}^{n-\frac{1}{2}} + \sigma_{j+\frac{1}{2}}^{n-\frac{1}{2}} + \sigma_{j-\frac{1}{2}}^{n-\frac{1}{2}} \right), \quad j = 1, \ldots, N, \quad \sigma_{N+\frac{1}{2}} = \sigma_{N-\frac{1}{2}},
\]

with suitable extensions for homogeneous Dirichlet boundary conditions for \(v\).

**Figure 2.** Illustration of a piecewise constant weak solution in 1d of the wave equation in space and time with jumps along the characteristics. The solution is computed by the explicit time stepping scheme in Ex. 8.
2.4. Existence and uniqueness of space-time solutions

Now we construct strong and weak solutions by a least squares approach. Therefore, we define the linear functional $\ell$ by $\langle \ell, z \rangle = (f, z)_Q + \langle M u_0, z(0) \rangle - \langle g, z \rangle_{(0,T)\times\partial\Omega}$ depending on $f \in L^2(Q;\mathbb{R}^m)$, $u_0 \in L^2(\Omega;\mathbb{R}^m)$, and $g \in L^2((0,T) \times \partial\Omega;\mathbb{R}^m)$, and we define and the quadratic functionals

$$J(w) = \frac{1}{2} \| Lw - f \|^2_{W^*}, \quad w \in H(L, Q),$$

$$J^*(z) = \frac{1}{2} \| L^*z \|^2_{W^{**}} - \langle \ell, z \rangle, \quad z \in V^*.$$

**Theorem 9.** Depending on regularity of the data, we obtain:

a) Assume that $C_L > 0$ exists with

$$\| w \|_V \leq C_L \| Lw \|_{W^*}, \quad w \in V. \quad \text{(19)}$$

Then, a unique minimizer $u \in V$ of $J(\cdot)$ exists, and if $L(V) = W$, the minimizer $u \in V$ is the unique strong solution of

$$(Lu, w)_Q = (f, w)_Q, \quad w \in W. \quad \text{(20)}$$

b) Assume that $C_{L^*} > 0$ and $C_\ell > 0$ exists with

$$\| z \|_W \leq C_{L^*} \| L^*z \|_{W^*}, \quad \| \ell, z \| \leq C_\ell \| z \|_{V^*}, \quad z \in V^*. \quad \text{(21)}$$

Then, $J^*(\cdot)$ extends to $V^*$, a unique minimizer $z^* \in V^*$ of $J^*(\cdot)$ exists, and if $L^*(V^*) \subset W$ is dense, $u = L^*z^* \in L^2(Q;\mathbb{R}^m)$ is the unique weak solution of

$$(u, L^*z)_Q = \langle \ell, z \rangle, \quad z \in V^*. \quad \text{(22)}$$

Moreover, strong solutions are also weak solutions, and if a weak solution $u$ is in $H(L, Q)$, it is a strong solution.

**Proof.** a) The functional $J(\cdot) > 0$ is bounded from below, and any minimizing sequence $\{u_n\}_{n \in \mathcal{N}} \subset V$ with

$$\lim_{n \to \infty} J(u_n) = \inf_{v \in V} J(v) := J_{\inf}$$

satisfies

$$\frac{1}{2} \| Lu_n + Lu_k \|^2_{W^*} = \frac{1}{2} \| Lu_n - f \|^2_{W^*} + \frac{1}{2} \| Lu_k - f \|^2_{W^*} - \| Lu_n + Lu_k - f \|^2_{W^*} = J(u_n) + J(u_k) - 2J(u_n + u_k) \leq J(u_n) + J(u_k) - 2J_{\inf} \to 0 \text{ for } n, k \to \infty.$$

The condition (19) implies norm equivalence

$$\| Lw \|_{W^*} \leq \| w \|_V = \sqrt{\| w \|^2_{V^*} + \| Lw \|^2_{W^*}} \leq \sqrt{1 + C_L^2} \| Lw \|_{W^*}, \quad w \in V,$$

so that the minimizing sequence is a Cauchy sequence converging to $u \in V$. Since $J(\cdot)$ is strictly convex, the minimizer is unique. Moreover, since $J(\cdot)$ is differentiable, $u$ is a critical point, i.e.,

$$0 = \partial J(u)[v] = (Lu - f, Lv)_{W^*} = (Lu - f, M^{-1}Lv)_{0,Q}.$$

If $L$ is surjective, this implies (20) by inserting $w = M^{-1}Lv \in M^{-1}L(V) = W$.

b) By assumption (21), $J^*$ and $\ell$ are continuous in $V^*$ with respect to the norm in $V^*$, so they extend to $V^*$. By the same arguments as above a unique minimizer $z^* \in V^*$ exists; it is characterized by

$$0 = \partial J^*(z^*)[z] = (L^*z^*, L^*z)_{W^{**}} - \langle \ell, z \rangle.$$

Inserting $u = L^*z^*$ implies (22), and if $L^*$ is surjective, the weak solution is unique. \qed

**Remark 10.** Strong solutions with inhomogeneous initial and boundary data exists, if the initial function $u_0$ in $\Omega$ can be extended to a function $u_0 \in H(L, Q)$ satisfying the boundary conditions.
2.5. Mapping properties of the space-time operator

Lemma 11. \( \|w\|_W \leq C_L \|Lw\|_{W^*} \) for \( w \in V \) holds with \( C_L = 2T \).

Proof. For \( w \in V \) we have \( w(0) = 0 \) and thus

\[
\|w\|_W^2 = \int_0^T (Mw(t), w(t))_\Omega dt
\]

\[
= \int_0^T \left( (Mw(t), w(t))_\Omega - (Mw(0), w(0))_\Omega \right) dt
\]

\[
= \int_0^T \int_0^t \partial_t (Mw(t), w(t))_\Omega dt
\]

\[
= 2 \int_0^T \int_0^t \partial_t (M\partial_t w(t), w(t))_\Omega dt
\]

\[
= 2 \int_0^T \int_0^t \left( (M\partial_t w(t), w(t))_\Omega + (Aw(t), w(t))_\Omega \right) dt
\]

\[
= 2 \int_0^T \int_0^t (Lw(t), w(t))_\Omega dt = 2 \int_0^T (T - t)(Lw(t), w(t))_\Omega dt
\]

\[
\leq 2T \|Lw\|_{W^*} \|w\|_W .
\]

Since \( V \) is dense in \( V \), this extends to \( V \).

\( \square \)

Remark 12. The same arguments hold for \( L^* \), i.e.,

\[
\|w\|_W \leq C_L \|L^*w\|_{W^*}, \quad w \in V^* .
\]

As a consequence of Lemma 11, the operator \( L: V \rightarrow L_2(Q; \mathbb{R}^m) \) is injective and continuous, i.e., \( L \in \mathcal{L}(V,W) \).

Corollary 13. \( L(V) \subset L_2(Q; \mathbb{R}^m) \) is closed.

Proof. For any sequence \( (w_n)_{n \in \mathbb{N}} \subset V \) with \( \lim_{n \to \infty} LW_n = f \in W \) we have

\[
\|w_n - w_k\|_W + \|Lw_n - LW_k\|_{W^*} \leq \sqrt{1 + C_L^2} \|Ly_n - Ly_k\|_{W^*} \rightarrow 0, \quad n,k \rightarrow \infty ,
\]

so that \( (w_n) \) is a Cauchy sequence in \( V \); since \( V \subset H(L,Q) \) is closed, \( w = \lim w_n \in V \) and \( LW = f \) exists. \( \square \)

Let the domain \( \mathcal{D}(A) = Z \subset H(A,\Omega) \) of the operator \( A \) be the closure of

\[
Z = \{ z \in C^1(\Omega; \mathbb{R}^m) \cap C^0(\overline{\Omega}; \mathbb{R}^m) : (B_nz)_k = 0 \text{ on } \Gamma_k \text{ for } k = 1,\ldots,m \},
\]

and select \( \Gamma_k \subset \partial\Omega \) such that \( (Az,z)_\Omega = \frac{1}{2} (B_nz,z)_{\partial\Omega} = 0 \) for \( z \in Z \).

Then, \( (M + A)z,z)_\Omega = (Mz,z)_\Omega > 0 \) for \( z \neq 0 \), i.e., \( M + A \) is injective on \( Z \). Moreover, we require that \( M + A \) is surjective on \( Z \), which is achieved in our applications be a suitable balanced selection of \( \Gamma_k \subset \partial\Omega \).

Lemma 14. Assume that \( M + A: Z \rightarrow L_2(\Omega; \mathbb{R}^m) \) is surjective. Then, \( L(V) \subset L_2(Q; \mathbb{R}^m) \) is dense.

Together with Corollary 13, \( L(V) = L_2(Q; \mathbb{R}^m) \), i.e., \( L: V \rightarrow L_2(Q; \mathbb{R}^m) \) is surjective.

Proof. For \( f \in L_2(Q; \mathbb{R}^m) \), \( N \in \mathbb{N} \), \( t_n = n \frac{T}{N} \), and \( f_N = f \big|_{(t_n, t_{n+1}, t_{n,N})} \) so that \( \lim_{N \rightarrow \infty} f_N = f \). Since \( M + A: Z \rightarrow L_2(\Omega; \mathbb{R}^m) \) is surjective and \( M \) is positive definite, also \( M + \frac{T}{N}A: Z \rightarrow L_2(\Omega; \mathbb{R}^m) \) is surjective. Thus, \( u_{N,n} \in Z \) exists with \( (M + \frac{T}{N}A)u_{N,n} = f_{N,n} \), and set \( u_{N,0} = 0 \). Let \( u_N \in H^1(0,T; Z) \subset V \) be the piecewise linear interpolation, i.e., \( u_N(t_{N,n}) = u_{N,n} \) for \( n = 0,\ldots,N \), and \( u_N \) linear in \( (t_{N,n-1}, t_{N,n}) \). Then, we observe \( \lim_{N \rightarrow \infty} \|Lu_N - f\|_Q = 0 \). \( \square \)
2.6. Inf-sup stability

From the previous section we directly obtain the following results.

**Lemma 15.** We have $C_L = C_{L^*}$ and

$$V = \{ v \in H(L, Q) : (Lv, z)_Q = (v, L^*z)_Q \text{ for } z \in V^\ast \},$$

$$V^* = \{ z \in H(L^*, Q) : (L^*z, v)_Q = (z, Lv)_Q \text{ for } v \in V \},$$

i.e., $V^*$ is the Hilbert adjoint space of $V$, and $V$ is the Hilbert adjoint space of $V^*$.

**Theorem 16.** The bilinear form $b : V \times W \to \mathbb{R}$, $b(v, w) = (Lv, w)_Q$ is inf-sup stable:

$$\inf_{v \in V \setminus \{0\}} \sup_{w \in W \setminus \{0\}} \frac{b(v, w)}{\|v\| V \|w\| W} = \inf_{w \in W \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{b(v, w)}{\|v\| V \|w\| W} = \beta, \quad \beta \geq \frac{1}{\sqrt{C_L^2 + 1}}.$$

Thus, for all $f \in L_2(Q, \mathbb{R}^m)$ a unique Petrov–Galerkin solution $u \in V$ of

$$b(u, w) = (f, w)_Q, \quad w \in W,$$

exists, and the solution is bounded by $\|u\| V \leq \beta^{-1} \|f\| W^*$. 

**Proof.** For $v \in V \setminus \{0\}$ we test with $w = M^{-1}Lv$, so that

$$\sup_{w \in W \setminus \{0\}} \frac{b(v, w)}{\|w\| W} \geq \frac{b(v, M^{-1}Lv)}{\|M^{-1}Lv\| W} = \|M^{-1}Lv\| W \geq (1 + 4T^2)^{-1/2} \|v\| V.$$

**Corollary 17.** $f \in H^1(0, T; L_2(\Omega; \mathbb{R}^m))$ implies $u \in H^1(0, T; L_2(\Omega; \mathbb{R}^m))$ and $\|\partial_t u\| W \leq C_L \|\partial_t f\| W^*$. 

**Proof.** This simply follows from $Lu = f$, which gives formally $L\partial_t u = \partial_t f$. If $\partial_t f \in W$, a solution $v \in V$ solving $Lv = \partial_t f$ exists, and since the solution is unique, $v = \partial_t u$. \qed
2.7. Applications to acoustics and visco-elasticity

Acoustic waves. In this case we have \( A(v, p) = -(\nabla p, \nabla v) \) and

\[
(A(v, p), (w, q))_Q + ((v, p), A(w, q))_Q = -(p, n \cdot w)_{(0,T) \times \partial \Omega} - (n \cdot v, q)_{(0,T) \times \partial \Omega}.
\]

Depending on the choice of boundary parts \( \partial \Omega = \Gamma_D \cap \Gamma_D \) we define

\[
Z = \{(v, p) \in H(\text{div}, \Omega) \times H^1(\Omega) : n \cdot v = 0 \text{ on } \Gamma_D, \ p = 0 \text{ on } \Gamma_S \}.
\]

For all \((f, g) \in L^2(\Omega; \mathbb{R}^{d+1})\) we define in the first step \( p \in H^1(\Omega) \) with \( p = 0 \) on \( \Gamma_S \) by solving the elliptic equation

\[
(\rho^{-1} \nabla p, \nabla \phi)_\Omega + (\kappa^{-1} p, \phi)_\Omega = (g, \phi)_\Omega - (\rho^{-1} f, \nabla \phi)_\Omega, \quad \phi \in H^1(\Omega) \text{ with } \phi = 0 \text{ on } \Gamma_S.
\]

(23)

Then, we define \( v = \rho^{-1}(\nabla p - f) \in L^2(\Omega; \mathbb{R}^d) \), and inserting (23), we observe

\[
(v, \nabla \phi)_\Omega = (g, \phi)_\Omega - (\kappa^{-1} p, \phi)_\Omega, \quad \phi \in C^c(\Omega),
\]

i.e., \( \nabla \cdot v = g - \kappa^{-1} p \in L^2(\Omega) \), and thus

\[
0 = (v, \nabla \phi)_\Omega + (\nabla \cdot v, \phi)_\Omega = \langle n \cdot v, \phi \rangle_{\partial \Omega}, \quad \phi \in C(\overline{\Omega}), \ \phi = 0 \text{ on } \Gamma_S,
\]

so that \( n \cdot v = 0 \) on \( \partial \Omega \setminus \Gamma_S = \Gamma_D \). Together, \((v, p) \in Z \) and \((M + A)(v, p) = (f, g)\). Moreover, the solution is unique, so that \( M + A \) is injective and surjective. This shows that the assumption in Lem. 14 is satisfied.

Visco-elastic waves. The space-time setting extends to the operator \( L = M \partial_t + A + D \), where the operator \( Dy(x) = D(x)y(x) \) is defined by a positive semi-definite matrix \( D \in L^\infty(\Omega; \mathbb{R}^{m \times m}) \), i.e., \( (Dy, y)_\Omega \geq 0 \) for all \( y \in L^2(\Omega; \mathbb{R}^m) \).

For the visco-elastic system (10) we set \( y = (v, \sigma_0, \ldots, \sigma_r)^\top \) and

\[
M = \begin{pmatrix}
\rho & 0 & \cdots & 0 \\
0 & C_0^{-1} & & \\
\vdots & & \ddots & \\
0 & & & C_r^{-1}
\end{pmatrix}, \quad A = -\begin{pmatrix}
0 & \text{div} & \cdots & \text{div} \\
\varepsilon & 0 & & \\
\vdots & & \ddots & \\
\varepsilon & 0 & & 0
\end{pmatrix}, \quad D = M \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \ddots & \\
0 & 0 & \cdots & \tau_r^{-1}
\end{pmatrix}
\]

with \( m = 2 + 3(1 + r) \) components for \( d = 2 \) and \( m = 3 + 6(1 + r) \) for \( d = 3 \). The formal adjoint operator is \( L^* = -M \partial_t - A + D \).

Remark 18. The extension to mixed boundary conditions on \( \Gamma_R \subset \partial \Omega \) requires \( L^2(\Gamma_R) \) regularity of the traces on the mixed boundary parts. Then, extending the norm \( \| \cdot \|_V \) by a corresponding boundary term again defines \( V \) as closure of \( V \) with respect to this stronger norm, and the space-time operator \( L \) is extended by a dissipative boundary operator \( D \).

Bibliographic comments

Least squares for first-order systems for finite elements are considered in [Cai et al., 1994, Cai et al., 2001]. Here this is applied to the space-time setting, see [Dörfler et al., 2016, Dörfler et al., 2018, Ernesti and Wieners, 2019b, Ernesti and Wieners, 2019a]. The extension to mixed boundary condition is considered in [Dörfler et al., 2020].
3. Discontinuous Galerkin methods for linear hyperbolic systems

Step by step we derive a discretization in space which is derived by solving Riemann problems, i.e., by construction of piecewise constant solutions in space and time.

3.1. Traveling wave solutions in homogeneous media

We consider linear hyperbolic first-order systems \( L = M \partial_t + A \) introduced in Sect. 2.1, and we start start with the case of homogeneous material parameters, so that the operator \( M \) is defined by a symmetric positive definite matrix \( M \in \mathbb{R}^{m \times m} \) which is constant in \( \Omega \).

Let \((\lambda, w) \in \mathbb{R} \times \mathbb{R}^m\) be an eigensystem of \( B_n w = \lambda M w \), and let \( a \in C^1(\mathbb{R}) \) be an amplitude function. Then, we observe that \( y(t, x) = a(n \cdot x - \lambda t) \) solves \( Ly = 0 \) in \( \Omega = \mathbb{R}^d \).

Example 19. For acoustic waves with wave speed \( c = \sqrt{\kappa/\rho} \) we have

\[
y = \begin{pmatrix} v_p \\ p \end{pmatrix}, \quad M_y = \begin{pmatrix} \rho v \\ \kappa^{-1} p \end{pmatrix}, \quad B_n y = -\begin{pmatrix} \rho n \\ v \cdot n \end{pmatrix}, \quad \lambda \in \{0, \pm c\}, \quad w = \begin{pmatrix} \mp cn \\ \kappa \end{pmatrix}.
\]

For elastic waves with wave speeds \( c_p = \sqrt{(2\mu + \lambda)/\rho} \) for compressional waves and \( c_s = \sqrt{\mu/\rho} \) for shear waves, we have

\[
y = \begin{pmatrix} v \\ \sigma \end{pmatrix}, \quad M_y = \begin{pmatrix} \rho v \\ C^{-1} \sigma \end{pmatrix}, \quad B_n y = -\begin{pmatrix} \sigma n \\ \frac{1}{2}(nv^\top + vn^\top) \end{pmatrix}, \quad \lambda \in \{0, \pm c_p, \pm c_s\}, \quad w_c = \begin{pmatrix} \mp c_p n \\ 2\mu nn^\top + \lambda I \end{pmatrix}, \quad w_s = \begin{pmatrix} \mp c_s \tau \\ \mu (n\tau^\top + \tau n^\top) \end{pmatrix},
\]

where \( \tau \in \mathbb{R}^d \) is a tangential unit vector, i.e., \( \tau \cdot n = 0 \) and \( |\tau| = 1 \).
Figure 3. Reflection and transmission of traveling waves.
3.2. Reflection of traveling acoustic waves at boundaries

In the next step we consider weak solutions in $L_{2, loc}(\Omega_R; \mathbb{R}^{d+1})$ of the acoustic wave equation in the half space

$$
\left( \rho \partial_t^2 \mathbf{v} - \nabla p \right)_{\kappa^{-1}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } \Omega_R = \{ \mathbf{x} \in \mathbb{R}^d : \mathbf{n} \cdot \mathbf{x} > 0 \}
$$

with initial value

$$
\begin{pmatrix} \mathbf{v}(0, \mathbf{x}) \\ p(0, \mathbf{x}) \end{pmatrix} = a(n \cdot \mathbf{x}) \begin{pmatrix} cn \\ \kappa \end{pmatrix}, \quad a(n \cdot \mathbf{x}) = 0 \text{ for } n \cdot \mathbf{x} < c t_0, \ t_0 > 0.
$$

The wave starts traveling from right to left, and at time $t = t_0$ is reaches the boundary. Then, in case of homogeneous Neumann boundary condition $\mathbf{v} \cdot \mathbf{n} = 0$ it is reflected, i.e.,

$$
\begin{pmatrix} \mathbf{v}(t, \mathbf{x}) \\ p(t, \mathbf{x}) \end{pmatrix} = \begin{cases} \frac{a(ct + n \cdot \mathbf{x})}{\kappa} \begin{pmatrix} cn \\ \kappa \end{pmatrix} & 0 < c(t_0 - t) < n \cdot \mathbf{x}, \\ \frac{a(ct + n \cdot \mathbf{x})}{\kappa} + a(ct - n \cdot \mathbf{x}) \begin{pmatrix} -cn \\ \kappa \end{pmatrix} & 0 < n \cdot \mathbf{x} < c(t - t_0). \end{cases}
$$

Otherwise, with homogeneous Dirichlet boundary conditions $p = 0$ the reflection also changes sign, i.e.,

$$
\begin{pmatrix} \mathbf{v}(t, \mathbf{x}) \\ p(t, \mathbf{x}) \end{pmatrix} = \begin{cases} \frac{a(ct + n \cdot \mathbf{x})}{\kappa} \begin{pmatrix} cn \\ \kappa \end{pmatrix} & 0 < c(t_0 - t) < n \cdot \mathbf{x}, \\ \frac{a(ct + n \cdot \mathbf{x})}{\kappa} - a(ct - n \cdot \mathbf{x}) \begin{pmatrix} -cn \\ \kappa \end{pmatrix} & 0 < n \cdot \mathbf{x} < c(t - t_0). \end{cases}
$$

3.3. Transmission and reflection of waves at interfaces

Now we consider weak solutions in $L_{2, loc}(\mathbb{R}^d; \mathbb{R}^{d+1})$ of the acoustic wave equation

$$
\left( \rho \partial_t^2 \mathbf{v} - \nabla p \right)_{\kappa^{-1}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } \Omega_L \cup \Omega_R, \quad \begin{cases} \Omega_L = \{ \mathbf{x} \in \mathbb{R}^d : \mathbf{n} \cdot \mathbf{x} < 0 \} \\ \Omega_R = \{ \mathbf{x} \in \mathbb{R}^d : \mathbf{n} \cdot \mathbf{x} > 0 \} \end{cases}
$$

with constant coefficients ($\rho_L, \kappa_L$) in $\Omega_L$ and ($\rho_R, \kappa_R$) in $\Omega_R$ defining $M_L$ and $M_R$, starting with

$$
\begin{pmatrix} \mathbf{v}(0, \mathbf{x}) \\ p(0, \mathbf{x}) \end{pmatrix} = a(n \cdot \mathbf{x}) \begin{pmatrix} n \\ Z_R \end{pmatrix}, \quad a(n \cdot \mathbf{x}) = 0 \text{ for } n \cdot \mathbf{x} < c_R t_0, \ t_0 > 0,
$$

where $Z_L = \sqrt{\kappa_L \rho_L}$, $Z_R = \sqrt{\kappa_R \rho_R}$ is the impedance. The weak solution is given by

$$
\begin{pmatrix} \mathbf{v}(t, \mathbf{x}) \\ p(t, \mathbf{x}) \end{pmatrix} = \begin{cases} \frac{a(t + n \cdot \mathbf{x} / c_L)}{Z_L} \begin{pmatrix} n \\ Z_R \end{pmatrix} & 0 < c_L(t_0 - t) < n \cdot \mathbf{x} \\ \frac{a(t + n \cdot \mathbf{x} / c_R)}{Z_R} + \beta_R a(t - n \cdot \mathbf{x} / c_R) \begin{pmatrix} -n \\ Z_R \end{pmatrix} & 0 < n \cdot \mathbf{x} < c_R(t - t_0) \\ \beta_L a(t + n \cdot \mathbf{x} / c_L) \begin{pmatrix} n \\ Z_L \end{pmatrix} & c_L(t_0 - t) < n \cdot \mathbf{x} < 0 \end{cases}
$$

with transmission and reflection coefficients derived from the interface condition $B_n[y] = 0$ are given by

$$
\beta_L = \frac{2Z_R}{Z_R + Z_L}, \quad \beta_R = \frac{Z_L - Z_R}{Z_R + Z_L}, \quad Z_L = \sqrt{\kappa_L \rho_L}, \quad Z_R = \sqrt{\kappa_R \rho_R}.
$$
3.4. The Riemann problem for acoustic waves

Next we consider weak solutions in $L_{2,\text{loc}}(\mathbb{R}^d;\mathbb{R}^{d+1})$ of the acoustic wave equation

$$
\begin{align*}
\left( \rho \partial_t v - \nabla p - \nabla \cdot \mathbf{v} \right) &= (0)_t \quad \text{in } \Omega_L \cup \Omega_R, \\
\left( \kappa^{-1} \partial_t p - \nabla \cdot \mathbf{v} \right) &= (0)_t \\
\end{align*}
$$

with constant coefficients $(\rho_L, \kappa_L)$ in $\Omega_L$ and $(\rho_R, \kappa_R)$ in $\Omega_R$, and with piecewise constant initial values

$$
\begin{align*}
(v(0,x), p(0,x)) &= (v_L, p_L), \quad x \in \Omega_L, \\
(v(0,x), p(0,x)) &= (v_R, p_R), \quad x \in \Omega_R.
\end{align*}
$$

The weak solution is of the form

$$
\begin{align*}
(v(t,x), p(t,x)) &= \begin{cases}
(v_L, p_L) & x \cdot n < -c_L t \\
(v_L + \beta_L (n \cdot Z_L), p_L) & -c_L t < x \cdot n < 0 \\
(v_R, p_R + \beta_R (n \cdot Z_R)) & 0 < x \cdot n < c_R t \\
(v_R, p_R) & c_R t < x \cdot n
\end{cases}
\end{align*}
$$

depending on $\beta_L, \beta_R \in \mathbb{R}$ determined by the flux condition

$$
B_n \left( \begin{array}{c} v_L \\ p_L \end{array} \right) + \beta_L \left( \begin{array}{c} n \\ Z_L \end{array} \right) = B_n \left( \begin{array}{c} v_R \\ p_R \end{array} \right) + \beta_R \left( \begin{array}{c} n \\ Z_R \end{array} \right),
$$

which yields $\beta_L = \frac{[p] + Z_R n \cdot [v]}{Z_L + Z_R}, \beta_R = \frac{[p] - Z_L n \cdot [v]}{Z_L + Z_R}$ depending on $[p] = p_R - p_L; [v] = v_R - v_L$.

3.5. The Riemann problem for linear conservation laws

We construct a weak solution of the Riemann problem for general linear conservation laws, i.e., a piecewise constant weak solution of $Ly = 0$ in $L_{2,\text{loc}}(\mathbb{R}^d;\mathbb{R}^m)$ with discontinuous initial values

$$
y_0(x) = \begin{cases}
y_L & x \in \Omega_L = \{ x \in \mathbb{R}^d; n \cdot x < 0 \}, \\
y_R & x \in \Omega_R = \{ x \in \mathbb{R}^d; n \cdot x > 0 \},
\end{cases} \quad y_L, y_R \in \mathbb{R}^m, \quad M_L, M_R \in \mathbb{R}^{m \times m}.
$$

Let $\{ (\lambda_j^L, w_j^L) \}_{j=1,...,m}$ and $\{ (\lambda_j^R, w_j^R) \}_{j=1,...,m}$ be eigenpairs, i.e.,

$$
B_n w_j^L = \lambda_j^L M_L w_j^L, \quad B_n w_j^R = \lambda_j^R M_R w_j^R, \quad w_k^L \cdot M_L w_j^L = w_k^R \cdot M_R w_j^R = 0 \quad \text{for } j \neq k,
$$

The weak solution is a superposition of traveling waves

$$
y(t,x) = \begin{cases}
y_L + \sum_{\lambda_j^L < 0} \beta_j^L w_j^L & x \in \Omega_L \\
y_R + \sum_{\lambda_j^R > 0} \beta_j^R w_j^R & x \in \Omega_R
\end{cases}
$$

and is obtained by solving the equation for $\beta_j^L, \beta_j^R$ (only depending on $[y_0] = y_R - y_L$)

$$
B_n \left( y_L + \sum_{\lambda_j^L < 0} \beta_j^L w_j^L \right) = B_n \left( y_R + \sum_{\lambda_j^R > 0} \beta_j^R w_j^R \right) \quad \text{on } \partial \Omega_L \cap \partial \Omega_R. \tag{24}
$$

Together, the solution of the Riemann problem defines the upwind flux

$$
B_n^{\text{up}} y_0 = B_n \left( y_L + \sum_{\lambda_j^L < 0} \beta_j^L w_j^L \right). \tag{25}
$$
By construction, we have \( B_n^\text{num} y_0 = B_n^\text{num} y_0 \) on inner faces. On the boundary, depending on the boundary conditions a system corresponding to (24) is solved defining an operator \( B_n^\text{bnd} \) with

\[
B_n^\text{num} y_0 = B_n y_L + B_n^\text{bnd} g,
\]

where \( g \) are the boundary data. This is specified in the following sections for our examples.

### 3.6. The DG discretization with full upwind

Let \( \Omega_h = \bigcup_{K \in \mathcal{K}} K \subset \mathbb{R}^d \) be a decomposition in open cells \( K \subset \Omega \) with \( \partial \Omega_h = \overline{\Omega} \setminus \Omega_h \).

Let \( \mathcal{F}_K \) be the set of faces \( F \subset K \), and define \( \mathcal{F} = \bigcup_{K \in \mathcal{K}} \mathcal{F}_K \). For inner faces \( F \in \mathcal{F} \setminus \Omega \), let \( K_F \) be the neighboring cell such that \( \mathcal{F} = \partial K \cap \partial K_F \). For boundary conditions on \( \Gamma_k \subset \partial \Omega \) we assume compatibility with the mesh so that \( \Gamma_k = \bigcup_{F \in \mathcal{F}_{\Gamma_k}} \mathcal{F} \).

Let \( Y_h \subset \mathbb{P}(\Omega_h; \mathbb{R}^m) = \prod_{K \in \mathcal{K}} \mathbb{P}(K; \mathbb{R}^m) \) be a discontinuous finite element space.

For \( y_h \in Y_h \), let \( y_{h,K} \in \mathbb{P}(K; \mathbb{R}^m) \) be the continuous extension of \( y_{h,K} = y_h|_K \). We define \( [y_h]_{K,F} = y_{h,K_F} - y_{h,K} \) on inner faces \( F \in \mathcal{F} \setminus \Omega \).

**Lemma 20.** We have for \( y_h \in Y_h \):

\[
y_h \in H(A, \mathbb{R}^m) \iff B_{nK}[y_h]_{K,F} = 0 \text{ for all } F \in \mathcal{F}_K \cap \partial \Omega, \ K \in \mathcal{K}.
\]

**Proof.** We define \( f_K = A y_K \) in \( K \), and since \( |\Omega \setminus \Omega_h|_d = 0 \), this defines a function in \( f_h \in L_2(\Omega; \mathbb{R}^m) \) with \( f_h|_K = f_K \). Now we observe for test functions \( z \in C^1_0(\Omega, \mathbb{R}^m) \)

\[
(f_h, z)_\Omega + (y_h, Az)_\Omega = \sum_K \left( (f_h - A y_h, z)_K + (B_{nK} y_K, z)_{\partial K} \right)
\]

\[
= -\frac{1}{2} \sum_{F \in \mathcal{F}_K \cap \partial \Omega} \sum_{F \in \mathcal{F}_K \cap \partial \Omega} (B_{nK}[y_K]_{K,F}, z)_F
\]

using \( B_{nK,F} = -B_{nK} \). Thus, \( y_h \in H(A, \mathbb{R}^m) \) and \( A y_h = f_h \in L_2(\Omega; \mathbb{R}^m) \) if and only if \( B_{nK}[y_K]_{K,F} \) vanishes on all inner faces. \( \square \)

For \( y_h, z_h \in Y_h \) we observe

\[
(A y_h, z_h)_\Omega = \sum_K \left( \text{div} B y_h, K, z_h, K \right)_K = \sum_K \left( (B_{nK} y_h, K, z_h, K)_{\partial K} - (y_h, K, A z_h, K)_{\partial K} \right).
\]

Inserting the upwind flux (25) defines the DG approximation \( A_h \), where \( B_{nK} \) is replaced by \( B_{nK}^\text{num} y_h \), i.e.,

\[
(A_h y_h, z_h)_\Omega = \sum_K \left( (B_{nK}^\text{num} y_h, z_h, K)_{\partial K} - (y_h, K, A z_h, K)_{\partial K} \right)
\]

\[
= \sum_K \left( (A y_h, K, z_h, K)_K + \sum_{F \in \mathcal{F}_K} (B_{nK}^\text{num} y_h - B_{nK} y_h, K, z_h, K)_F \right).
\]

(27)

For inhomogeneous boundary conditions the corresponding right-hand side is defined by

\[
(f_{bnd}^g, z_h)_\Omega = (f, z_h)_\Omega - \sum_{F \in \mathcal{F}_{\Gamma_k} \cap \partial \Omega} (B_{n}^\text{bnd} g, z_h)_F.
\]

(28)

As we see in our examples, the boundary term is consistent with

\[
(B_{n}^\text{bnd} g_h, z)_{(0,T) \times \Gamma_k} = \sum_{k=1}^m (g_k, z_k)_{(0,T) \times \Gamma_k}
\]

(29)

for all test functions \( z \in \mathcal{D}(A) \) with homogeneous boundary conditions \( z_k = 0 \) on \( \partial \Omega \setminus \Gamma_k, k = 1, \ldots, m \).
3.7. Consistent extension of the discrete operator $A_h$

For sufficiently smooth functions $y \in H^1(\Omega_h, \mathbb{R}^m)$ traces on the skeleton exists, so that the discrete operator $A_h$ extends to $A_h \in L(H^1(\Omega_h, \mathbb{R}^m), Y_h)$ by

$$
(A_h y, z_h)_\Omega = \sum_K \left( (A y, z_h)_K + \sum_{F \in \mathcal{F}_K} (B_{n_K}^{\text{num}} y - B_{n_K} y_K, z_{h,K})_F \right), \quad y \in H^1(\Omega_h, \mathbb{R}^m), z_h \in Y_h.
$$

(30)

Moreover, since in the conforming case by construction

$$B_{n_K}^{\text{num}} y = B_{n_K} y_K, \quad K \in \mathcal{K}, \quad F \in \mathcal{F}_K, \quad y \in H^1(\Omega, \mathbb{R}^m),$$

we obtain consistency for sufficiently smooth functions, i.e.,

$$
(A_h y, z_h)_\Omega = (A y, z_h)_\Omega, \quad y \in H^1_0(\Omega, \mathbb{R}^m), \quad z_h \in Y_h.
$$

(31)

In case of homogeneous boundary conditions, this extends to $y \in Y_h \cap \mathcal{D}(A)$; this is proved for acoustics in Lem. 21.
3.8. The full upwind discretization for visco-acoustic and visco-elastic waves

For acoustics, the continuity of the flux requires on inner faces \( F \in \mathcal{F} \cap \Omega \)
\[
B_{nK} \left( \begin{pmatrix} v_{h,K} \\ p_{h,K} \end{pmatrix} + \beta_K \begin{pmatrix} n_K \\ Z_K \end{pmatrix} \right) = B_{nK} \left( \begin{pmatrix} v_{h,K_F} \\ p_{h,K_F} \end{pmatrix} + \beta_{K_F} \begin{pmatrix} n_K \\ -Z_K \end{pmatrix} \right)
\]
\[
\implies 0 = \begin{pmatrix} n_K \\ Z_K \end{pmatrix} \cdot B_{nK} \left( \begin{pmatrix} v_{h,F} \\ p_{h,F} \end{pmatrix} - \beta_K \begin{pmatrix} n_K \\ Z_K \end{pmatrix} \right)
\]
\[
\implies B_{nK}^{num} \left( \begin{pmatrix} v_{h} \\ p_{h} \end{pmatrix} \right) = B_{nK} \left( \begin{pmatrix} v_{h,K} \\ p_{h,K} \end{pmatrix} - \frac{[p_h]_{K,F} + Z_{K_F} n_K \cdot [v_h]_{K,F}}{Z_K + Z_{K_F}} \begin{pmatrix} Z_K n_K \\ 1 \end{pmatrix} \right).
\]

This extends to the boundary by defining the jump terms depending on the boundary conditions. On boundary faces \( F \in \mathcal{F} \cap \partial \Omega \), we obtain from
\[
B_{nK} \left( \begin{pmatrix} v_K \\ p_K \end{pmatrix} + \beta_K \begin{pmatrix} n_K \\ Z_K \end{pmatrix} \right) = \begin{pmatrix} p_K n_K \\ n_K \cdot v_K \end{pmatrix} - \beta_K \begin{pmatrix} Z_K n_K \\ 1 \end{pmatrix}
\]
in case of Dirichlet boundary conditions \( \beta_K = \frac{1}{Z_K} \), which corresponds for the numerical flux to \([p_h]_{K,F} = -2p_h\) and \( n_K \cdot [v_h]_{K,F} = 0 \).

In case of Neumann boundary conditions we obtain \( \beta_K = 1 \) corresponding to \( n_K \cdot [v_h]_{K,F} = -2n_K \cdot v_h \) and \([p_h]_{K,F} = 0\). In both cases we extend the impedance on boundary faces \( F \) by \( Z_{K_F} = Z_K \).

**The DG operator for acoustics and visco-acoustics.** The operator \( A_h \in \mathcal{L}(Y_h, Y_h) \), \( A_h = \sum_{K \in \mathcal{K}} A_{h,K} \) for acoustics (with \( r = 0 \)) and visco-acoustics (\( r \geq 1 \)) with full upwind is defined by

\[
(A_{h,K} y_h, z_h)_K = - (\nabla \cdot v_{h,K}, q_{h,K})_K - (\nabla p_{h,K}, w_{h,K})_K
\]
\[- \sum_{F \in \mathcal{F}_K} \frac{1}{Z_K + Z_{K_F}} ([p_h]_{K,F} + Z_{K_F} n_K \cdot [v_h]_{K,F}, q_{h,K} + Z_K n_K \cdot w_{h,K})_F
\]
\[- \sum_{F \in \mathcal{F}_K \cap \Omega} \frac{1}{Z_K + Z_{K_F}} ([p_h]_{K,F} + Z_{K_F} n_K \cdot [v_h]_{K,F}, q_{h,K} + Z_K n_K \cdot w_{h,K})_F
\]
\[+ \sum_{F \in \mathcal{F}_K \cap \Gamma_D} \frac{1}{Z_K} (p_{h,K}, q_{h,K} + Z_K n_K \cdot w_{h,K})_F
\]
\[+ \sum_{F \in \mathcal{F}_K \cap \Gamma_S} (n_K \cdot v_{h,K}, q_{h,K} + Z_K n_K \cdot w_{h,K})_F
\]

for \( y_h = (v_h, p_{h,0}, \ldots, p_{r,h}), \ z_h = (w_h, q_{h,0}, \ldots, q_{r,h}) \in Y_h \) with \( p_h = p_{0,h} + \cdots + p_{r,h}, \ q_h = q_{0,h} + \cdots + q_{r,h} \).

For inhomogeneous boundary conditions we obtain the right-hand side (28)
\[
\left( \ell_{h}^{up}, z_h \right)_\Omega = \left( f_h, z_h \right)_\Omega - \sum_{F \in \mathcal{F}_K \cap \Gamma_S} \frac{1}{Z_K} (p_{S,h}, q_{h,K} + Z_K n_K \cdot w_{h,K})_F - \sum_{F \in \mathcal{F}_K \cap \Gamma_D} (g_D, q_{h,K} + Z_K n_K \cdot w_{h,K})_F.
\]

**Lemma 21.** The DG discretization is

a) consistent, i.e.,
\[
(A_h y_h, z_h)_\Omega = (A y_h, z_h)_\Omega, \quad y_h \in Y_h \cap D(A), \ z_h \in Y_h,
\]
\[
(A_h y_h, z_h)_\Omega = -(y_h, A z_h)_\Omega, \quad y_h \in Y_h, \ z \in Y_h \cap D(A);
\]

b) monotone / dissipative satisfying
\[
(A_h y_h, y_h)_\Omega = \frac{1}{2} \sum_{K \in \mathcal{K}} \sum_{F \in \mathcal{F}_K} \frac{1}{Z_K + Z_{K_F}} \left( \| [p_h]_{K,F} \|_F^2 + Z_K Z_{K_F} \| n_K \cdot [v_h]_{K,F} \|_F^2 \right) \geq 0, \quad y_h \in Y_h.
\]
Proof. It is sufficient to consider \( r = 0 \). For \( y = (v, p) \in Y_h \cap D(A) \) we obtain \( n_K \cdot [v_h]_{K,F} = [p_h]_{K,F} = 0 \) for \( F \in F_K \cap \Omega, p_h = 0 \) on \( F \in F_K \cap \Gamma_D \), and \( n_K \cdot v_h = 0 \) on \( F \in F_K \cap \Gamma_S \), so that consistency is obtained by

\[
(A_h y, z_h)_\Omega = \sum_K (A_h y_K, z_h)_K = \sum_K (Ay, z_h)_K = (Ay, z_h)_\Omega, \quad z_h \in Y_h
\]
since all flux terms on the faces are vanishing.

Integration by parts yields for \( y_h = (v_h, p_h) \in Y_h \) and \( z = (w, q) \in Y_h \in D(A) \) dual consistency by

\[
(A_h y_h, z)_\Omega = \sum_K \left( - (\nabla \cdot v_h, q)_K - (\nabla p_h, w)_K \right)
- \sum_{F \in F_K \cap \Omega} \frac{1}{Z_K + Z_{KF}} ([p_h]_{K,F} + Z_{KF} n_K \cdot [v_h]_{K,F}, q_K + Z_K n_K \cdot w_K)_F
+ \sum_{F \in F_K \cap \Gamma_D} (p_h, n_K \cdot w_K)_F + \sum_{F \in F_K \cap \Gamma_S} (n_K \cdot v_h, q_K)_F
\]

\[
= \sum_K \left( (v_h, \nabla \cdot q_h)_K \right) (p_h, \nabla \cdot w)_K
- \sum_{F \in F_K \cap \Omega} \left( \frac{1}{Z_K + Z_{KF}} ([p_h]_{K,F} + Z_{KF} n_K \cdot [v_h]_{K,F}, p_h, K) + Z_K n_K \cdot v_h)_F
+ (p_h, n_K \cdot w_K)_F + (n_K \cdot v_h, q_K)_F \right)
\]

\[
= \sum_K \left( -(y_h, A z_h, K) \right)_K
- \sum_{F \in F_K \cap \Omega} \frac{1}{Z_K + Z_{KF}} \left( ([p_h]_{K,F}, q_K)_F + Z_K Z_{KF} (n_K \cdot [v_h]_{K,F}, n_K \cdot w_K)_F \right)
\]

For \( y_h \in Y_h \) we obtain the identity

\[
(A_h y_h, y_h)_\Omega = \sum_K (A_h y_h, y_h)_K
\]

\[
= \sum_K \left( - (\nabla \cdot v_h, p_h, K) - (\nabla p_h, v_h, K) \right)_K
- \sum_{F \in F_K \cap \Omega} \frac{1}{Z_K + Z_{KF}} ([p_h]_{K,F} + Z_{KF} n_K \cdot [v_h]_{K,F}, p_h, K) + Z_K n_K \cdot v_h)_F
+ \sum_{F \in F_K \cap \Gamma_D} (p_h, n_K \cdot v_h, K) + Z_K n_K \cdot v_h, K)_F
\]

\[
= \frac{1}{2} \sum_{K \in F_K} \sum_{F \in F_K} \frac{1}{Z_K + Z_{KF}} \left( \| [p_h]_{K,F} \|_F^2 + Z_K Z_{KF} \| n_K \cdot [v_h]_{K,F} \|_F^2 \right)
\]

using \( (\nabla \cdot v_h, p_h, K) + (\nabla p_h, v_h, K) = (n_K \cdot v_h, K, p_h, K) \) and

\[
\sum_K \left( - (n_K \cdot v_h, K, p_h, K) \right)_\Omega
- \sum_{F \in F_K \cap \Omega} \frac{Z_K}{Z_K + Z_{KF}} ([p_h]_{K,F}, n_K \cdot v_h, K)_F - \sum_{F \in F_K \cap \Gamma_D} \frac{Z_{KF}}{Z_K + Z_{KF}} (n_K \cdot [v_h]_{K,F}, p_h, K)_F
+ \sum_{F \in F_K \cap \Gamma_S} (p_h, n_K \cdot v_h, K)_F + \sum_{F \in F_K \cap \Gamma_S} (n_K \cdot v_h, K, p_h, K)_F = 0.
\]
The DG operator for visco-elasticity. The operator $A_h \in \mathcal{L}(Y_h, Y_h)$, $A_h = \sum_{K \in \mathcal{K}} A_{h,K}$ with full upwind is defined by

$$
(A_{h,K} y_h, z_h)_K = -((\nabla \cdot \sigma, \psi, \eta)_{h,K} - ((v, \sigma, \eta)_{h,K})_{K} - \sum_{F \in \mathcal{F}_K} \frac{1}{Z_{p}^K} (n_K \cdot (\sigma_{h,F} n_K + Z_{K,F}^{p} v_h[F]), n_K \cdot (\eta_{h,F} n_K + Z_{K,F}^{p} w_h[F]))_{F} - \sum_{F \in \mathcal{F}_K} \frac{1}{Z_{s}^K} (n_K \times (\sigma_{h,F} n_K + Z_{K,F}^{s} v_h[F]), n_K \times (\eta_{h,F} n_K + Z_{K,F}^{s} w_h[F]))_{F}
$$

for $y_h = (v_h, \sigma_0, \ldots, \sigma_r, h), z_h = (w_h, \eta_0, \ldots, \eta_r, h) \in Y_h, \sigma_h = \sum \sigma_{j,h}, \eta_h = \sum \eta_{j,h}$. The coefficients $Z_{p}^K = \sqrt{(2\mu + \lambda)\rho |K}$ and $Z_{s}^K = \sqrt{\mu\rho |K}$ are the impedance of compressional waves and shear waves, respectively.

On boundary faces $F \in \mathcal{F} \cap \Gamma_D$, we set $[v_h]_{K,F} = 0$ and $[\sigma_{h,F}]_{K,F} n_K = -2 \sigma_{h,F} n_K$, and on $F \in \mathcal{F} \cap \Gamma_S$ we set $[v_h]_{K,F} = -2v_h$ and $[\sigma_{h,F}]_{K,F} n_K = 0$. We have

$$
(A_h y_h, y_h)_{\Omega} = \frac{1}{2} \sum_{K \in \mathcal{K}} \sum_{F \in \mathcal{F}_K} \frac{1}{Z_{p}^K + Z_{K,F}^{p}} \left( \|n_K \cdot (\sigma_{h,K,F} n_K)\|_F^2 + Z_{p}^K Z_{K,F}^{p} \|n_K \cdot [v_h]_{K,F}\|_F^2 \right)
$$

$$
+ \frac{1}{2} \sum_{K \in \mathcal{K}} \sum_{F \in \mathcal{F}_K} \frac{1}{Z_{s}^K + Z_{K,F}^{s}} \left( \|n_K \times (\sigma_{h,K,F} n_K)\|_F^2 + Z_{s}^K Z_{K,F}^{s} \|n_K \times [v_h]_{K,F}\|_F^2 \right) \geq 0.
$$

Bibliographic comments

An introduction to discontinuous Galerkin methods for hyperbolic conservation laws is given, e.g., in [Hesthaven and Warburton, 2008,Hesthaven, 2017]. The numerical flux for wave equations is evaluated in [Hochbruck et al., 2015] and extended to visco-acoustic and visco-elastic waves in [Ziegler, 2019]. For the explicit evaluation of the numerical flux for inhomogeneous boundary conditions we refer to [Dörfler et al., 2018], and the case of electro-magnetic impedance boundary conditions is evaluated in [Schulz, 2015].
4. A Petrov–Galerkin space-time approximation for linear hyperbolic systems

4.1. Decomposition of the space-time cylinder

For the discretization, we use tensor product space-time cells combining mesh in Sect. 3.6 of \( \Omega \) with a decomposition in time. For \( 0 = t_0 < t_1 < \cdots < t_N = T \), we define

\[
I_h = (t_0, t_1) \cup \cdots \cup (t_{N-1}, t_N) \subset I = (0, T)
\]

This is combined with the decomposition in space \( \Omega_h = \bigcup_{K \in K_h} K \) into open cells \( K \subset \Omega \subset \mathbb{R}^d \) with \( \partial \Omega_h = \overline{\Omega} \setminus \Omega_h \).

Together, we obtain a decomposition \( Q_h = \bigcup_{R \in \mathcal{R}_h} R = I_h \times \Omega_h \) of the space-time cylinder \( Q = I \times \Omega \).

For every space-time cell \( R = (t_{n-1}, t_n) \times K \) we select polynomial degrees \( p_R = p_{n, K} \geq 1 \) in time and \( q_R = q_{n, K} \geq 0 \) in space. This defines the discontinuous test space in the space-time cylinder

\[
W_h = \prod_{R \in \mathcal{R}_h} P_{p_R - 1} \otimes P_{q_R} (K; \mathbb{R}^m) \subset P(I_h \times \Omega_h; \mathbb{R}^m) \subset L_2(Q; \mathbb{R}^m).
\]

Defining the discontinuous spaces

\[
Y_{n,h} = \prod_{K \in K_h} P_{q_{n, K}} (K; \mathbb{R}^m) \subset P(\Omega_h; \mathbb{R}^m) \subset L_2(\Omega; \mathbb{R}^m), \quad Y_h = Y_{1,h} + \cdots + Y_{N,h},
\]

we observe \( W_h \subset L_2(0, T; Y_h) \), and in every time slice \( w_h(t) \in Y_{n,h} \subset Y_h \) for all \( t \in (t_{n-1}, t_n) \times \Omega \) and \( w_h \in W_h \).

4.2. The Petrov–Galerkin setting

Let \( L_h = M_h \partial_t + D_h + A_h \) be an approximation of \( L = M \partial_t + D + A \) with the following properties:

a) \( M_h \in \mathcal{L}(Y_h, Y_h) \) is uniformly positive definite, i.e., \( c_M > 0 \) exists with

\[
(M_h y_h, y_h)_\Omega \geq c_M \| y_h \|^2_W, \quad y_h \in Y_h;
\]

b) \( D_h \in \mathcal{L}(Y_h, Y_h) \) is monotone, i.e.,

\[
(D_h y_h, y_h)_Q \geq 0, \quad y_h \in Y_h;
\]

c) \( A_h \in \mathcal{L}(Y_h, Y_h) \) is monotone and consistent, i.e.,

\[
(A_h y_h, y_h)_Q \geq 0, \quad (A_h z, y_h)_Q = (A_z, y_h)_Q \quad \text{and} \quad (A_h y_h, z)_Q = -(y_h, A z)_Q, \quad y_h \in Y_h, \quad z \in Y_h \cap D(A).
\]

In the next step we construct a suitable ansatz space \( V_h \subset P(Q_h; \mathbb{R}^m) \). Therefore, in every time slice \( (t_{n-1}, t_n) \), let

\[
\Pi_{n,h} : L_2(\Omega; \mathbb{R}^m) \rightarrow Y_{n,h}
\]

be the projection defined by

\[
(M_h \Pi_{n,h} y, z_h)_\Omega = (M_h y, z_h)_\Omega, \quad y \in L_2(\Omega; \mathbb{R}^m), \quad z_h \in Y_{n,h}.
\]

For \( v_h \in P(I_h \times \Omega_h; \mathbb{R}^m) \) let \( v_{n,h} \in P([t_{n-1}, t_n] \times \Omega_h; \mathbb{R}^m) \) be the extension of \( v_h|_{(t_{n-1}, t_n) \times \Omega_h} \) to \([t_{n-1}, t_n]\).

Then, we define

\[
V_h = \left\{ v_h \in \prod_{R \in \mathcal{R}_h} P_{p_R} \otimes P_{q_R} (K; \mathbb{R}^m) \subset P(I_h \times \Omega_h; \mathbb{R}^m) : \right. \left. v_h(0) = 0 \text{ for } t = 0, \ v_{n,h}(t_{n-1}) = \Pi_{n,h} v_{n-1,h}(t_{n-1}) \text{ for } n = 2, \ldots, N \right\} \subset H^1(0, T; Y_h)
\]

By construction, we have \( \partial_t V_h = W_h \) in \( I_h \) and \( \dim V_h = \dim W_h \).
Note that $V_h$ includes homogeneous initial data. For inhomogeneous initial data $u_0$ we define the affine space

$$V_h(u_0) = \left\{ v_h \in \prod_{R \in R_h} \mathbb{P}_{p_R} \otimes \mathbb{P}_{q_R}(K; \mathbb{R}^m) \subset \mathbb{P}(I_h \times \Omega_h; \mathbb{R}^m) : v_h(0) = \Pi_{1,h}u_0 \text{ for } t = 0, \ v_{n,h}(t_{n-1}) = \Pi_{n,h}v_{n-1,h}(t_{n-1}) \text{ for } n = 2, \ldots, N \right\} \subset H^1(0, T; Y_h).$$

4.3. Inf-sup stability

Let

$$\Pi_h : L_2(Q; \mathbb{R}^m) \rightarrow W_h$$

be the projection defined by

$$(M_h \Pi_h v, w_h)_Q = (M_h v, w_h)_Q, \quad v \in L_2(Q; \mathbb{R}^m), \ w_h \in W_h.$$  

Note that $\Pi_h M_h = M_h \Pi_h$.

The analysis of the discrete problem is based on the norms

$$\|w_h\|_{W_h} = \sqrt{(M_h w_h, w_h)_Q}, \quad w_h \in W_h$$

and

$$\|v_h\|_{V_h} = \sqrt{\|v_h\|_{W_h}^2 + \|\Pi_h M_h^{-1} L_h v_h\|_{W_h}^2}, \quad v_h \in V_h. \quad (32)$$

**Theorem 22.** The bilinear form $b_h : V_h \times W_h \rightarrow \mathbb{R}, \ b_h(v_h, w_h) = (L_h v_h, w_h)_Q$ is inf-sup stable:

$$\sup_{w_h \in W_h \setminus \{0\}} \frac{b_h(v_h, w_h)}{\|w_h\|_{W_h}} \geq \beta \|v_h\|_{V_h}, \quad v_h \in V_h \quad \text{with} \ \beta \geq \frac{1}{4T^2 + 1}. \quad (33)$$

**Corollary 23.** For given $f \in L_2(Q; \mathbb{R}^m)$ there exists a unique solution $u_h \in V_h$ of

$$(L_h u_h, w_h)_Q = (f, w_h)_Q, \quad w_h \in W_h$$

satisfying the a priori bound $\|u_h\|_{V_h} \leq \beta^{-1}\|f\|_{W_h}$.

The proof of the inf-sup stability is based on the following estimates.

**Lemma 24.** Let $\lambda_{n,k} \in \mathbb{P}_k, \ k = 0, 1, 2, \ldots$, be the orthonormal Legendre polynomials in $L_2(t_{n-1}, t_n)$. Then, we have $(t \partial_t \lambda_{n,k}, \lambda_{n,k})_{(t_{n-1}, t_n)} \geq 0$.

**Proof.** The orthonormal Legendre polynomials with respect to $(\cdot, \cdot)_{(t_{n-1}, t_n)}$ are given by

$$\lambda_{n,k}(t) = c_{n,k} \lambda_{n,k}^{og}(t), \quad \lambda_{n,k}^{og}(t) = \partial_t^k (t - t_{n-1})(t - t_n)^k, \quad c_{n,k} = \|\lambda_{n,k}^{og}\|_{(t_{n-1}, t_n)}^{-1}.$$

For $k = 0$ we have $\partial_t \lambda_{n,0} = 0$ and thus $(t \partial_t \lambda_{n,0}, \lambda_{n,0})_{(t_{n-1}, t_n)} = 0$. For $k \geq 1$ we have

$$(t \partial_t \lambda_{n,k}, \lambda_{n,k})_{(t_{n-1}, t_n)} = (tc_{n,k} \partial_t^{k+1} ((t - t_{n-1})(t - t_n))^k, \lambda_{n,k})_{(t_{n-1}, t_n)}$$

$$= (tc_{n,k} \partial_t^{k+1} t^{2k}, \lambda_{n,k})_{(t_{n-1}, t_n)}$$

$$= (c_{n,k} k \partial_t^{k+1} t^{2k}, \lambda_{n,k})_{(t_{n-1}, t_n)} = k(\lambda_{n,k}, \lambda_{n,k})_{(t_{n-1}, t_n)} = k > 0$$

using $t \partial_t^{k+1} t^{2k} = t \cdot 2k \cdot (2k - 1) \cdots k \cdot t^{k-1} = k \cdot \partial_t^k t^{2k}$.

□
Lemma 25. We have for $v_h \in V_h$

\[ \|v_h\|_{W_h} \leq 2T \|\Pi_h M^{-1}_h L_h v_h\|_{W_h}, \quad v_h \in V_h. \] (34)

Proof. Set $p = \max_{R \in \mathcal{R}_h} p_R$. For $v_h \in V_h$ in every time slice $(t_{n-1}, t_n)$ a representation

\[ v_{n,h}(x, t) = \sum_{k=0}^{p} \lambda_{n,k}(t)v_{n,k,h}(x), \quad v_{n,k,h} \in Y_{n,h}, \quad (t, x) \in Q_h \]

exists with $v_{n,k,h}(x) = 0$ for $(t, x) \in R = (t_{n-1}, t_n) \times K$ and $k > p_R$, so that

\[ \Pi_h v_{n,h}(t, x) = \sum_{k=0}^{p_{R-1}} \lambda_{n,k}(t)v_{n,k,h}(x), \quad (t, x) \in R = (t_{n-1}, t_n) \times K. \]

The proof of (34) relies on estimates with respect to the weighting function in time $d_T(t) = T - t$, and on the application of Fubini's theorem

\[ \int_0^T \int_0^t \phi(s) ds \, dt = \int_0^{T} d_T(t) \phi(t) \, dt, \quad \phi \in L_1(0, T). \] (35)

In the first step, we prove

\[ (M_h \partial_t v_h, d_T v_h)_Q \leq (M_h \partial_t v_h, d_T \Pi_h v_h)_Q, \]

\[ 0 \leq (\Pi_h A_h v_h, d_T \Pi_h v_h)_Q, \]

\[ 0 \leq (\Pi_h D_h v_h, d_T \Pi_h v_h)_Q. \] (36) (37) (38)

Since $A_h$ and $D_h$ are monotone, we obtain (37) and (38) from

\[ (\Pi_h A_h v_h, d_T \Pi_h v_h)_Q = \sum_{n=1}^{N} (\Pi_h A_h v_{n,h}, d_T \Pi_h v_{n,h})_{(t_{n-1}, t_n) \times \Omega} \]

\[ = \sum_{n=1}^{N} \sum_{R = (t_{n-1}, t_n) \times K} \sum_{k=0}^{p_{R-1}} \sum_{l=0}^{p_{R-1}} (\lambda_{n,k}, d_T \lambda_{n,l})_{(t_{n-1}, t_n)} (A_h v_{n,k,h}, v_{n,l,h})_K \geq 0, \]

\[ (\Pi_h D_h v_h, d_T \Pi_h v_h)_Q = \sum_{n=1}^{N} \sum_{R = (t_{n-1}, t_n) \times K} \sum_{k=0}^{p_{R-1}} \sum_{l=0}^{p_{R-1}} (\lambda_{n,k}, d_T \lambda_{n,l})_{(t_{n-1}, t_n)} (D_h v_{n,k,h}, v_{n,l,h})_K \geq 0. \]

For $k \geq 1$ we have $(d_T \partial_t \lambda_{n,k}, \lambda_{n,k})_{(t_{n-1}, t_n)} = -(t \partial_t \lambda_{n,k}, \lambda_{n,k})_{(t_{n-1}, t_n)} < 0$ by Lem. 24, which gives

\[ (d_T M_h \partial_t v_h, v_h - \Pi_h v_h)_Q = \sum_{n=1}^{N} (d_T M_h \partial_t v_h, v_h - \Pi_h v_h)_{(t_{n-1}, t_n) \times \Omega} \]

\[ = \sum_{n=1}^{N} \sum_{R = (t_{n-1}, t_n) \times K} \sum_{k=0}^{p_{R}} (d_T \partial_t \lambda_{n,k}, \lambda_{n,p_{R}})_{(t_{n-1}, t_n)} (M_h v_{n,k,h}, v_{n,p_{R},h})_K \]

\[ = \sum_{n=1}^{N} \sum_{R = (t_{n-1}, t_n) \times K} (d_T \partial_t \lambda_{n,p_{R}}, \lambda_{n,p_{R}})_{(t_{n-1}, t_n)} (M_h v_{n,p_{R},h}, v_{n,p_{R},h})_K \leq 0. \]

Thus we obtain (36) by

\[ (M_h \partial_t v_h, d_T v_h)_Q = (d_T M_h \partial_t v_h, v_h)_Q \leq (d_T M_h \partial_t v_h, \Pi_h v_h)_Q = (M_h \partial_t v_h, d_T \Pi_h v_h)_Q. \]
Finally, we show the assertion (34). We have for $k = 2, \ldots, N$

$$(M_h v_{k,h}(t_{k-1}), v_{k,h}(t_{k-1}))_\Omega = (M_h \Pi_{k,h} v_{k-1,h}(t_{k-1}), \Pi_{k,h} v_{k-1,h}(t_{k-1}))_\Omega \leq (M_h v_{k-1,h}(t_{k-1}), v_{k-1,h}(t_{k-1}))_\Omega,$$

so that for all $t \in (t_{n-1}, t_n)$ using $v_h(0) = v_{1,h}(t_0) = 0$

$$(M_h v_h(t), v_h(t))_\Omega = (M_h v_{n,h}(t), v_{n,h}(t))_\Omega + \sum_{k=2}^{n} \left( (M_h \Pi_{k,h} v_{k-1,h}(t_{k-1}), \Pi_{k,h} v_{k-1,h}(t_{k-1}))_\Omega - (M_h v_{k,h}(t_{k-1}), v_{k,h}(t_{k-1}))_\Omega \right) - ((M_h v_{1,h}(t_0), v_{1,h}(t_0))_\Omega$$

$$\leq (M_h v_{n,h}(t), v_{n,h}(t))_\Omega + \sum_{k=2}^{n} \left( (M_h v_{k-1,h}(t_{k-1}), v_{k-1,h}(t_{k-1}))_\Omega - (M_h v_{k,h}(t_{k-1}), v_{k,h}(t_{k-1}))_\Omega \right) - ((M_h v_{1,h}(t_0), v_{1,h}(t_0))_\Omega$$

$$= (M_h v_{n,h}(t), v_{n,h}(t))_\Omega - (M_h v_{n,h}(t_{n-1}), v_{n,h}(t_{n-1}))_\Omega + \sum_{k=1}^{n-1} \left( (M_h v_{k,h}(t_k), v_{k,h}(t_k))_\Omega - (M_h v_{k,h}(t_{k-1}), v_{k,h}(t_{k-1}))_\Omega \right)$$

$$= \int_{t_{n-1}}^{t} \partial_s (M_h v_{n,h}(s), v_{n,h}(s))_\Omega ds + \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_{k}} \partial_s (M_h v_{n,h}(s), v_{n,h}(s))_\Omega ds$$

$$= 2 \int_{0}^{t} (M_h \partial_s v_h(s), v_h(s))_\Omega ds$$

and thus using (35), (36), (37), and (38) we obtain (34) by

$$||v_h||_{V_h}^2 = \int_{0}^{T} (M_h v_h(t), v_h(t))_\Omega dt \leq 2 \int_{0}^{T} \int_{0}^{t} (M_h \partial_s v_h(s), v_h(s))_\Omega ds dt$$

$$= 2 \int_{0}^{T} dT(t)(M_h \partial_s v_h(s), v_h(s))_\Omega dt$$

$$= 2(M_h \partial v_h, dT v_h)_Q \leq 2(M_h \Pi_{h} \partial v_h, dT \Pi_{h} v_h)_Q = 2(\Pi_{h} M_h \partial v_h, dT \Pi_{h} v_h)_Q$$

$$\leq 2(\Pi_{h} L_h v_h, dT \Pi_{h} v_h)_Q = 2(M_h^{-1} \Pi_{h} L_h v_h, M_h dT \Pi_{h} v_h)_Q = 2(\Pi_{h} M_h^{-1} L_h v_h, M_h dT \Pi_{h} v_h)_Q$$

$$\leq 2 \|\Pi_{h} M_h^{-1} L_h v_h\|_{W_h} dT \Pi_{h} v_h \|W_h\| \leq 2 T \|\Pi_{h} M_h^{-1} L_h v_h\|_{W_h} \|v_h\|_{W_h}.$$

Now we can prove Thm. 22.

**Proof.** For $v_h \in V_h \setminus \{0\}$ we have

$$b_h(v_h, w_h) = (L_h v_h, w_h)_Q = (M_h^{-1} L_h v_h, w_h)_W = (\Pi_{h} M_h^{-1} L_h v_h, w_h)_W,$$

and we test with $w_h = \Pi_{h} M_h^{-1} L_h v_h$, so that

$$\sup_{w_h \in W_h \setminus \{0\}} \frac{b_h(v_h, w_h)}{||w_h||_{W_h}} \geq \frac{b_h(v_h, \Pi_{h} M_h^{-1} L_h v_h)}{||\Pi_{h} M_h^{-1} L_h v_h||_{W_h}} = ||\Pi_{h} M_h^{-1} L_h v_h||_{W_h} \geq (1 + 4 T^2)^{-1/2} \|v_h\|_{V_h}.$$

\[\square\]
4.4. Convergence for strong solutions

For the error estimate with respect to the norm in $V_h$ we need to extend the norm $\| \cdot \|_{V_h}$. For sufficiently smooth functions the operator $A_h$ can be extended by (30), so that $L_h$ and thus the norm in $V_h$ is well-defined.

**Theorem 26.** Let $u \in V$ be the strong solution of $Lu = f$, and let $u_h \in V_h$ be the approximation solving (33). If the solution is sufficiently smooth, we obtain a priori error estimate

$$
\| u - u_h \|_{V_h} \leq C(\Delta t^p + \Delta x^q) \left( \| \partial_t^{p+1} u \|_Q + \| \partial_t^{q+1} u \|_Q \right)
+ \beta^{-1} \| M_h^{-1/2}(M_h - M)M^{-1/2} \|_{\infty} \| \partial_t u \|_W
+ \beta^{-1} \| M_h^{-1/2}(D_h - D)M^{-1/2} \|_{\infty} \| u \|_W
$$

for $\Delta t$, $\Delta x$ and $p, q \geq 1$ with $\Delta t \geq t_n - t_{n-1}$, $\Delta x \geq \text{diam}(K)$, $p \leq p_R$ and $q \leq q_R$, and with a constant $C > 0$ depending on $\beta$, on the material parameters in $M$, and on the mesh regularity.

**Proof.** For the solution we assume the regularity $u \in H^{p+1}(0, T; L^2(\Omega; \mathbb{R}^m)) \cap L^2(0, T; H^{q+1}(\Omega; \mathbb{R}^m))$. Then, $A_h u$ is well-defined and consistent satisfying (31). We have for all $v_h \in V_h$

$$
b_h(v_h - u, w_h) = b_h(v_h, w_h) - b_h(u_h, w_h) = b_h(v_h, w_h) - (f, w_h)_Q
= (L_h v_h, w_h) - (L_h u, w_h)_Q
= (L_h (v_h - u), w_h)_Q - ((L - L_h)u, w_h)_Q
= (L_h (v_h - u), w_h)_Q - (M_h M_h^{-1} L_h (v_h - u), w_h)_Q
- (M_h M_h^{-1} (M_h - M) \partial_t u, w_h)_Q - (M_h M_h^{-1} (D_h - D) u, w_h)_Q
\leq \left( \| \Pi_h M_h^{-1} L_h (v_h - u) \|_{W_h} + \| M_h^{-1} (M_h - M) \partial_t u \|_{W_h} + \| M_h^{-1} (D_h - D) u \|_{W_h} \right) \| w_h \|_{W_h}
$$

and thus the assertion follows from

$$
\| u - u_h \|_{V_h} \leq \| u - v_h \|_{V_h} + \| v_h - u_h \|_{V_h}
\leq \| u - v_h \|_{V_h} + \beta^{-1} \sup_{w_h \in W_h \setminus \{0\}} \frac{b_h(v_h - u_h, w_h)}{\| w_h \|_{W_h}}
\leq \| u - v_h \|_{V_h} + \beta^{-1} \left( \| \Pi_h M_h^{-1} L_h (v_h - u) \|_{W_h} + \| M_h^{-1} (M_h - M) \partial_t u \|_{W_h} + \| M_h^{-1} (D_h - D) u \|_{W_h} \right)
\leq (1 + \beta^{-1}) \| u - v_h \|_{V_h} + \beta^{-1} \| M_h^{-1/2} (M_h - M) \partial_t u \|_Q + \beta^{-1} \| M_h^{-1/2} (D_h - D) u \|_Q
$$

by inserting for $v_h$ an interpolation of $u$ in $H^{p+1}(0, T; L^2(\Omega; \mathbb{R}^m)) \cap L^2(0, T; H^{q+1}(\Omega; \mathbb{R}^m))$, and using Cor. 17, so that

$$
\| M_h^{-1/2} (M_h - M) \partial_t u \|_Q = \| M_h^{-1/2} (M_h - M) M^{-1/2} M^{1/2} \partial_t u \|_Q \leq \| M_h^{-1/2} (M_h - M) M^{-1/2} \|_{\infty} \| \partial_t u \|_W.
$$

\[ \square \]

**Remark 27.** Since the norm (32) in $V_h + (H^1(Q; \mathbb{R}^m) \cap V)$ is discrete in the derivatives, the topology in the space $V$ with respect to this norm is equivalent to the topology in $L^2$ with mesh dependent bounds for the norm equivalence. Norm equivalent with respect to $\| \cdot \|_V$ is obtained in the limit: Let $(V_h)_{h \in H}$ be a shape regular family of discrete spaces with $0 \in \overline{H}$ such that $(V_h \cap V)_{h \in H}$ is dense in $V$. Then, defining $\| \cdot \|_{V_h} = \sup_{h \in H} \| \cdot \|_{V_h}$, yields a norm equivalent to $\| \cdot \|_V$ for all $v \in H^1(Q; \mathbb{R}^m) \cap V$.

**Remark 28.** The estimate is derived for homogeneous initial and boundary conditions. It transfers to the inhomogeneous case if the initial and boundary data can be extended to $H(L, Q)$, i.e., if $\tilde{u} \in H(L, Q)$ exists such that $\tilde{u}(0, x) = u_0(x)$ for $x \in \Omega$ and $(B_h \tilde{u}(t, x)) = g_j(t, x)$ and for $(t, x) \in (0, T) \times \Gamma_j$. Then, the result is applied to $\tilde{u} \in V$ solving $\tilde{L} \tilde{u} = \tilde{f} - L \tilde{u}$, so that $u = \tilde{u} + \hat{u} \in H(L, Q)$ is a strong solution with inhomogeneous initial and boundary data. The approximation $u_h$ is computed in the affine space $V_h(u_0)$.
4.5. Convergence for weak solutions

Qualitative convergence estimates with respect to the norm in $V \subset H(L, Q)$ require additional regularity, so that these estimates do not apply to weak solutions with discontinuities or singularities. For weak solutions without additional regularity we only can derive asymptotic convergence. Here, this is shown for simplicity only for homogeneous initial and boundary data.

For given $f \in L_2(\Omega; \mathbb{R}^m)$, let $u \in W$ be the weak solution solving

$$(u, L^* z)_Q = (f, z)_Q, \quad z \in V^*,$$

and let $u_h \in V_h$ be the discrete solution of

$$b_h(u_h, w_h) = (f, w_h)_Q, \quad w_h \in W_h.$$

Lemma 29. The discrete solution $u_h \in V_h$ is bounded in $W = L_2(\Omega; \mathbb{R}^m)$ by

$$\|u_h\|_{W_h} \leq 2T \| M_h^{-1} f \|_{W_h}.$$

Proof. Let $B_h \in \mathcal{L}(V_h, W_h)$ be the discrete operator defined by

$$(B_h v_h, w_h)_{W_h} = b_h(v_h, w_h), \quad v_h \in V_h, w_h \in W_h,$$

i.e., $B_h = \Pi_h M_h^{-1} L_h|_{V_h}$. From (34) we observe that the operator $B_h$ is injective, and since $\dim V_h = \dim W_h$, the operator is also surjective. The adjoint operator $B^*_h \in \mathcal{L}(W_h, V_h)$ is defined by duality, i.e.,

$$(B^*_h w_h, v_h)_{W_h} = (w_h, B_h v_h)_{W_h} = b_h(v_h, w_h) = (L_h v_h, w_h)_Q, \quad v_h \in V_h, w_h \in W_h.$$

For the following estimates, we define the dual discrete norm

$$\|w_h\|_{V_h^*} = \|B^*_h w_h\|_{W_h},$$

and we obtain by duality for $v_h \in V_h$

$$\|v_h\|_{W_h} = \sup_{z_h \in V_h \setminus \{0\}} \frac{(v_h, z_h)_{W_h}}{\|z_h\|_{W_h}} = \sup_{w_h \in W_h \setminus \{0\}} \frac{(v_h, B_h^* w_h)_{W_h}}{\|B_h^* w_h\|_{W_h}} = \sup_{w_h \in W_h \setminus \{0\}} \frac{b_h(v_h, w_h)}{\|w_h\|_{V_h^*}}.$$  \hspace{1cm} (40)

The estimate $\|v_h\|_{W_h} \leq 2T \|B_h v_h\|_{W_h}$ in Lem. 25 yields

$$\|w_h\|_{W_h} = \sup_{z_h \in W_h \setminus \{0\}} \frac{(z_h, w_h)_{W_h}}{\|z_h\|_{W_h}} = \sup_{v_h \in V_h \setminus \{0\}} \frac{(B_h v_h, w_h)_{W_h}}{\|B_h v_h\|_{W_h}} = \sup_{v_h \in V_h \setminus \{0\}} \frac{b_h(v_h, w_h)}{\|v_h\|_{V_h^*}} \leq 2T \sup_{v_h \in V_h \setminus \{0\}} \frac{b_h(v_h, w_h)}{\|v_h\|_{W_h}} = 2T \|w_h\|_{V_h^*}.$$  \hspace{1cm} (41)

Together with (40) we obtain the assertion by

$$\|u_h\|_{W_h} = \sup_{w_h \in W_h \setminus \{0\}} \frac{b_h(u_h, w_h)}{\|w_h\|_{V_h^*}} = \sup_{u_h \in V_h \setminus \{0\}} \frac{(f, w_h)_Q}{\|w_h\|_{V_h^*}} \leq \|M_h^{-1} f\|_{W_h} \sup_{w_h \in W_h \setminus \{0\}} \frac{\|w_h\|_{W_h}}{\|w_h\|_{V_h^*}} \leq 2T \|M_h^{-1} f\|_{W_h}.$$  \hspace{1cm} (42)

\[\Box\]

Let $(V_h, W_h), h \in \mathcal{H} \subset (0, h_0)$, be a dense family of nested discretizations with $V_h \subset V_{h'}$ and $W_h \subset W_{h'}$ for $h' < h, h' \in \mathcal{H}$ and $0 \leq h \in \mathcal{H}$. We assume that the assumptions in this section are fulfilled for all discretizations, so that $(V_h, W_h)$ is uniformly inf-sup stable by Thm. 22. Moreover, we only consider the case $P_h(Q_h; \mathbb{R}^m) \subset W_h$, so that $W_h \cap H^1(Q; \mathbb{R}^m)$ includes the continuous linear elements and thus $\bigcup_{h \in \mathcal{H}} (V^* \cap W_h)$ is dense in $V^*$. For simplicity, we assume that the parameters in $M$ and $D$ are piecewise constant on all meshes $\mathcal{K}_h$, so that $M = M_h$ and $D = D_h$ for all $h \in \mathcal{H}$, which implies $\|z_h\|_{W_h} = \|z_h\|_W$ for $z_h \in L_2(0, T; Y_h)$. 

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In the first step, we show that the dual consistency of the DG operator implies dual consistency of the space-time method.

**Lemma 30.** We have

\[
   b_h(v_h, w) = (v_h, L^*w)_Q, \quad v_h \in V_h, \ w \in W_h \cap V^*.
\]  

**Proof.** For \( w_h \in W_h \) we define \( d_h \in L_2(Q; \mathbb{R}^m) \) by \( d_h|_R = L^*w_h|_R \), and we obtain for all \( v \in C^1_0(Q; \mathbb{R}^m) \)

\[
   (w_h, L^*v)_Q - (d_h, v)_Q = \sum_{R \in \mathcal{R}_h} (w_h, L^*v)_R - (L^*w_h, v)_R
\]

\[
   = \sum_{K \in \mathcal{K}_h} (B_n w_h, v)_{I_h \times \partial R} + \sum_{n=1}^{N-1} (w_{n+1,h}(t_n) - w_{n,h}(t_n), v(t_n))_\Omega,
\]

so that \( L^*w_h = d_h \) in \( Q \) and thus \( w_h \in V^* \) if and only if \( B_{nK} [w_h]_{K,F} = 0 \) in \( I_h \times F \) for all \( F \in \mathcal{F}_h \cap \Omega \) and \( w_{n+1,h}(t_n) = w_{n,h}(t_n) \) in \( \Omega \) for all \( n = 1, \ldots, N - 1 \). In particular, this implies

\[
   W_h \cap V^* \subset C^1(I_h; Y_h) \cap C^0(0, T; Y_h) \subset H^1(0, T; Y_h).
\]

Now, we have for \( v_h \in V_h \subset H^1(0, T; Y_h) \) and \( w \in W_h \cap V^* \) using \( M = M_h, \ D = D_h \), and the dual consistency of the DG operator \( A_h \) with upwind flux (see Lem. 21 for acoustics)

\[
   b_h(v_h, w) = (M \partial_t v_h, w)_Q + (Dv_h, w)_Q + (A_h v_h, w)_Q
\]

\[
   = -(v_h, M \partial_t w)_Q + (v_h, Dw)_Q - (v_h, A w)_Q = (v_h, L^*w)_Q.
\]

\[\square\]

**Theorem 31.** The discrete solutions \((u_h)_{h \in \mathcal{H}}\) are converging to the weak solution \( u \in W \) of (39).

**Proof.** Since \((u_h)_{h \in \mathcal{H}}\) is uniformly bounded in \( W \), a subsequence \( \mathcal{H}_0 \subset \mathcal{H} \) and a weak limit \( \tilde{u} \in W \) exists, i.e.,

\[
   \lim_{h \in \mathcal{H}_0} (u_h, w)_W = (\tilde{u}, w)_W, \quad w \in W.
\]

Since we assume that \((W_h \cap V^*)_h \in \mathcal{H} \) is dense in \( V^* \), for all \( z \in V^* \) a sequence \((w_h)_{h \in \mathcal{H}}\) exists with \( w_h \in W_h \cap V^* \) and \( \lim_{h \in \mathcal{H}_0} w_h = z \). This implies, using Lem. 30,

\[
   (\tilde{u}, L^*z)_W = \lim_{h \in \mathcal{H}_0} (u_h, L^*z)_W = \lim_{h \in \mathcal{H}_0} (u_h, L^*w_h)_W = \lim_{h \in \mathcal{H}_0} b_h(u_h, w_h)_W = \lim_{h \in \mathcal{H}_0} (f, w_h)_Q = (f, z)_Q,
\]

so that \( \tilde{u} = u \) is the unique weak solution of (39). In particular this shows that the weak limit of all subsequences in \( \mathcal{H} \) is unique, so that the full sequence is convergent.

\[\square\]
4.6. Goal-oriented adaptivity

In order to find an efficient choice for the polynomial degrees \((p_R, q_R)\), we introduce a dual-weighted residual error indicator with respect to a suitable goal functional. Its construction is based on a dual-primal error representation combined with a priori estimates constructed from an approximation of the dual solution. Note that this corresponds to a problem backward in time, so that the resulting error indicator only refines regions of the space-time domain which are relevant for the evaluation of the chosen goal functional.

**Dual-primal error bound.** Let \(E: W \rightarrow \mathbb{R}\) be a linear error functional. Our goal is to estimate and then to reduce the error with respect to this functional. We define the dual solution by

\[
\mathbf{u}^* \in V^*: \quad \langle \mathbf{w}, L^* \mathbf{u}^* \rangle_Q = \langle E, \mathbf{w} \rangle, \quad \mathbf{w} \in W.
\]

We define the pairing

\[
\langle \mathbf{u}_h, \mathbf{u}^* \rangle_{\partial R} = (L \mathbf{u}_h, \mathbf{u}^*)_R - (\mathbf{u}_h, L^* \mathbf{u}^*)_R
\]

and the norms

\[
\|y\|_y = \sqrt{(M_y, y)_\Omega}, \quad \|y\|_{y_h} = \sqrt{(M_h y, y)_\Omega}, \quad y \in L_2(\Omega; \mathbb{R}^m).
\]

**Lemma 32.** Let \(\mathbf{u} \in V\) be the solution of \(L \mathbf{u} = f\). Then, the error can be represented by

\[
\langle E, \mathbf{u} - \mathbf{u}_h \rangle = \sum_{R \in \mathcal{R}_h} \left( \langle f - L \mathbf{u}_h, \mathbf{u}^* \rangle_R - \langle \mathbf{u}_h, \mathbf{u}^* \rangle_{\partial R} \right),
\]

and if the dual solution is sufficiently regular, the error is bounded by

\[
|\langle E, \mathbf{u} - \mathbf{u}_h \rangle| \leq \sum_{n=1}^N \sum_{R=(t_{n-1}, t_n) \times K} \left( \|f - (M_h \partial_t + A + D_h) \mathbf{u}_h\|_R \|\mathbf{u}^* - \mathbf{w}_h\|_R \right.
\]

\[
+ \sum_{F \in \mathcal{F}_K} \|B_{\mathbf{n}_h} \mathbf{u}_{hR} - B_{\mathbf{n}^\text{num}} \mathbf{u}_{n,h}\|_{(t_{n-1}, t_n) \times F} \|\mathbf{u}^* - \mathbf{w}_h\|_{(t_{n-1}, t_n) \times F}.
\]

Proof. We have

\[
\langle E, \mathbf{u} - \mathbf{u}_h \rangle = (\mathbf{u} - \mathbf{u}_h, L^* \mathbf{u}^*)_Q = (\mathbf{u}_h, L^* \mathbf{u}^*)_Q = (L \mathbf{u}_h, \mathbf{u}^*)_Q - (\mathbf{u}_h, L^* \mathbf{u}^*)_Q
\]

\[
= (f, \mathbf{u}^*)_Q - \sum_{R \in \mathcal{R}_h} (\mathbf{u}_h, L^* \mathbf{u}^*)_R
\]

\[
= (f, \mathbf{u}^*)_Q - \sum_{R \in \mathcal{R}_h} \left( (L \mathbf{u}_h, \mathbf{u}^*)_R + (\mathbf{u}_h, \mathbf{u}^*)_{\partial R} \right)
\]

\[
= \sum_{R \in \mathcal{R}_h} \left( (f - L \mathbf{u}_h, \mathbf{u}^*)_R - (\mathbf{u}_h, \mathbf{u}^*)_{\partial R} \right),
\]
We have, using $u_h(0) = 0$ and $u^*(T) = 0$,

\[
\sum_{n=1}^{N} \sum_{R=(t_{n-1},t_n) \times K} (M \partial_t u_{n,h}, u^*)_R + (M u_{n,h}, \partial_t u^*)_R = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \partial_t (M u_{n,h}, u^*)_{\Omega} dt
\]

\[
= \sum_{n=1}^{N} (M u_{n,h}(t_n), u^*(t_n))_{\Omega} - (M u_{n,h}(t_{n-1}), u^*(t_{n-1}))_{\Omega}
\]

\[
= \sum_{n=1}^{N} (M u_{n,h}(t_n) - u_{n+1,h}(t_n)), u^*(t_n))_{\Omega},
\]

\[
= \sum_{n=1}^{N} (M u_{n,h}(t_n) - \Pi_{n+1,h} u_{n,h}(t_n)), u^*(t_n))_{\Omega},
\]

and in every time slice $(t_{n-1}, t_n)$ we obtain, if the dual solution $u^*$ is sufficiently smooth so that the restriction to the space-time skeleton satisfies $u^*|_{\partial Q_h} \in L_2(\partial Q_h; \mathbb{R}^m)$,

\[
\sum_K (A u_{n,h}^*, u^*)_{(t_{n-1}, t_n) \times K} + (u_{n,h}, A u^*)_{(t_{n-1}, t_n) \times K} = \sum_K (B_{n,k} u_{n,h,K}, u^*)_{(t_{n-1}, t_n) \times \partial K}
\]

\[
= \sum_K \sum_{F \in \mathcal{F}_K} (B_{n,k} u_{n,h,K}, u^*)_{(t_{n-1}, t_n) \times F}
\]

\[
= \sum_K \sum_{F \in \mathcal{F}_K} (B_{n,k} u_{n,h,K} - B_{n,k}^\text{num} u_{n,h,K}, u^*)_{(t_{n-1}, t_n) \times F}
\]

where $u_{n,h,K}$ is the extension of $u_{n,h}|_K$ to $\overline{K}$. This gives

\[
\sum_{R \in \mathcal{R}_h} \langle u_h, u^* \rangle_{\partial R} = \sum_{n=1}^{N} \sum_{R=(t_{n-1}, t_n) \times K} (L u_{n,h}, u^*)_R - (u_{n,h}, L^* u^*)_R
\]

\[
= \sum_{n=1}^{N} \sum_{R=(t_{n-1}, t_n) \times K} (M \partial_t u_{n,h}, u^*)_R + (M u_{n,h}, \partial_t u^*)_R + (A u_{n,h}, u^*)_R + (u_{n,h}, A u^*)_R
\]

\[
= \sum_{n=1}^{N} (M(u_{n,h}(t_n) - \Pi_{n+1,h} u_{n,h}(t_n)), u^*(t_n))_{\Omega},
\]

\[
+ \sum_{R=(t_{n-1}, t_n) \times K} \sum_{F \in \mathcal{F}_K} (B_{n,k} u_{h,R} - B_{n,k}^\text{num} u_{n,h,K}, u^*)_{(t_{n-1}, t_n) \times F},
\]

where $u_{h,R}$ is the extension of $u_h|_R$ to $\overline{R}$.

For the discrete solution $u_h \in V_h$ and any discrete test function $w_h \in W_h$ be have

\[
(f, w_h)_Q = (L_h u_h, w_h)_Q = (M_h \partial_t u_h, w_h)_Q + (A_h u_h, w_h)_Q + (D_h u_h, w_h)_Q
\]

\[
= (M_h \partial_t u_h, w_h)_Q + (D_h u_h, w_h)_Q
\]

\[
+ \sum_{n=1}^{N} \sum_{R=(t_{n-1}, t_n) \times K} \left( (A u_h, w_h)_R + \sum_{F \in \mathcal{F}_K} (B_{n,K}^\text{num} u_{n,h,R} - B_{n,K} u_{h,R}, w_h)_R_{(t_{n-1}, t_n) \times F} \right),
\]

so that

\[
0 = \sum_{n=1}^{N} \sum_{R=(t_{n-1}, t_n) \times K} \left( (A u_h - f, w_h)_R + (M_h \partial_t u_h, w_h)_R + (D_h u_h, w_h)_R
\]

\[
+ \sum_{F \in \mathcal{F}_K} (B_{n,K}^\text{num} u_{n,h,R} - B_{n,K} u_{h,R}, w_h)_R_{(t_{n-1}, t_n) \times F} \right).
\]
Together, this gives
\[
\langle E, \mathbf{u} - \mathbf{u}_h \rangle = \sum_{R \in R_h} \left( (f - Lu_h, \mathbf{u}^*)_R - (\mathbf{u}_h, \mathbf{u}^*)_{\partial R} \right)
\]
\[
= \sum_{R = (t_{n-1}, t_n) \times K} \left( (f - (M_h \partial_t + A + D_h) \mathbf{u}_h, \mathbf{u}^*)_R \\
- ((M - M_h) \partial_t \mathbf{u}_h, \mathbf{u}^*)_R \right. \\
- (D - D_h) \mathbf{u}_h, \mathbf{u}^*)_R \\
+ \sum_{F \in F_K} \left( B_{nK} \mathbf{u}_{h,R} - B_{nK}^{\text{num}} \mathbf{u}_{n,h}, \mathbf{u}^* \right)_{(t_{n-1}, t_n) \times F} \\
+ \sum_{n=1}^{N-1} (M \mathbf{u}_{n,h}(t_n) - \Pi_{n+1,h} \mathbf{u}_{n,h}(t_n), \mathbf{u}^*(t_n) )_{\Omega} \\
\left. + \sum_{n=1}^{N-1} ((M - M_h) \mathbf{u}_{n,h}(t_n) - \Pi_{n+1,h} \mathbf{u}_{n,h}(t_n), \mathbf{u}^*(t_n) )_{\Omega} \\
- ((M - M_h) \partial_t \mathbf{u}_h, \mathbf{u}^*)_Q - ((D - D_h) \mathbf{u}_h, \mathbf{u}^*)_Q \right)
\]
\[
\leq \sum_{R = (t_{n-1}, t_n) \times K} \left( \|f - (M_h \partial_t + A + D_h) \mathbf{u}_h\|_R \|\mathbf{u}^* - \mathbf{w}_h\|_R \\
+ \sum_{F \in F_K} \|B_{nK} \mathbf{u}_{h,R} - B_{nK}^{\text{num}} \mathbf{u}_{n,h}\|_{(t_{n-1}, t_n) \times F} \|\mathbf{u}^* - \mathbf{w}_h\|_{(t_{n-1}, t_n) \times F} \right) \\
+ \sum_{n=1}^{N-1} \|M_h \mathbf{u}_{n,h}(t_n) - \Pi_{n+1,h} \mathbf{u}_{n,h}(t_n)\|_\Omega \|\mathbf{u}^*(t_n) - \mathbf{w}_{n+1,h}(t_n)\|_\Omega \\
+ \sum_{n=1}^{N-1} \|M^{-1}(M - M_h) \mathbf{u}_{n,h}(t_n) - \Pi_{n+1,h} \mathbf{u}_{n,h}(t_n)\|_Y \|\mathbf{u}^*(t_n)\|_Y \\
+ \|(M - M_h) \partial_t \mathbf{u}_h\|_{W'} \|\mathbf{u}^*\|_{W'} - \|(D - D_h) \mathbf{u}_h\|_{W'} \|\mathbf{u}^*\|_{W'}.
\]
**Dual-primal error indicator.** Since $u^* \in V^*$ cannot be computed, it is approximated by

$$u^*_h \in W_h: \quad b_h(v_h, u^*_h) = \langle E, v_h \rangle, \quad v_h \in V_h,$$

and the error indicator $\eta = \sum_R \eta_R$ for $R = (t_{n-1}, t_n) \times K$ is defined by

$$\eta_R = \left( \|(M_h \partial_t + A + D_h)u_h - f\|_R + \|u_{n,h}(t_{n-1}) - \Pi_{n,h}u_{n-1,h}(t_{n-1})\|_\Omega \right) h^{1/2} \|\Pi_0^h u^*_h\|_{(t_{n-1}, t_n) \times \partial K}$$

$$+ \|(B_{n,K} - B_{n,K}^{num}) u_h\|_{(t_{n-1}, t_n) \times \partial K} \|\Pi_0^h u^*_h\|_{(t_{n-1}, t_n) \times \partial K}$$

with the $L_2$ projection $\Pi_0^h: L_2(Q; \mathbb{R}^m) \rightarrow P_0(Q_h; \mathbb{R}^m)$.

This results into the following $p$-adaptive algorithm:

1. choose low order polynomial degrees on the initial mesh
2. **while** $\max_R p_R < p_{\text{max}}$ and $\max_R q_R < q_{\text{max}}$ **do**
3. compute $u_h$
4. compute $u^*_h$ and the projection $\Pi_0^h u^*_h$
5. compute $\eta_R$ on every cell $R$
6. if the error is small enough STOP
7. mark space-time cell $R$ for refinement if $\eta_R > \vartheta_1 \max_R \eta_R$
   and for derefinement if $\eta_R < \vartheta_0 \max_R \eta_R$
8. increase/decrease polynomial degrees on marked cells
9. redistribute cells on processes for better load balancing
4.7. Reliable error estimation for weak solutions

Finally, we derive a posteriori estimates for weak solutions in the general case including inhomogeneous initial and boundary data. For simplicity, we assume that the parameters in $M$ and $D$ are piecewise constant, so that $M = M_h$ and $D = D_h$.

For the data $f \in L^2(Q; \mathbb{R}^m)$, $u_0 \in L^2(\Omega; \mathbb{R}^m)$, $g_k \in L^2((0, T) \times \Gamma_k)$ defining the linear functional $\ell$ by

$$
\langle \ell, z \rangle_Q = \langle f, z \rangle_Q + \langle Mu_0, z(0) \rangle_{\Omega} - \sum_{k=1}^{m} \langle g_k, z_k \rangle_{(0, T) \times \Gamma_k}, \quad z \in V^*
$$

we select approximations $f_h \in P(Q_h; \mathbb{R}^m)$, $u_{0,h} \in P(\Omega_h; \mathbb{R}^m)$, and $g_{k,h} \in P((0, T) \times \Gamma_k)$ defining the discrete linear functional $\ell_h$ by

$$
\langle \ell_h, z_h \rangle_Q = \langle f_h, z_h \rangle_Q + \langle Mu_{0,h}, z_h(0) \rangle_{\Omega} - \sum_{k=1}^{m} \langle g_{k,h}, z_{k,h} \rangle_{(0, T) \times \Gamma_k}, \quad z_h \in V^* + P(Q_h; \mathbb{R}^m).
$$

We assume that $\ell$ is bounded by (21) so that a unique weak solution $u \in W$ exists solving

$$
\langle u, L^* z \rangle_Q = \langle \ell, z \rangle, \quad z \in V^*.
$$

For the approximation $u_h \in V_h(u_{0,h})$ solving

$$
\langle L_h u_h, w_h \rangle_Q = \langle f_h, w_h \rangle_Q - \langle B_{\text{bnd}}^n g_h, w_h \rangle_{(0,T) \times \partial\Omega}, \quad w_h \in W_h,
$$

we now construct a conforming reconstruction in a discrete space $V_{h}^{\text{cf}} \subset H(L, Q) \cap P(Q_h; \mathbb{R}^m)$ as described in the following. Here, we set $g_h = (g_{k,h})_{k=1,\ldots,m} \in L^2((0, T) \times \partial\Omega; \mathbb{R}^m)$ with $g_{k,h} = 0$ on $\partial\Omega \setminus \Gamma_k$.

The reconstruction is defined on local patches. Therefore, let $C_K \subset \bar{K}$ be the corner points of the elements $K \in \mathcal{K}_h$ such that $\bar{K} = \text{conv} C_K$, and define $\bar{C}_K = \bigcup C_K$. For all $c \in \bar{C}_K$ we define $\mathcal{K}_{h,c} = \{ K \in \mathcal{K}_h : c \in C_K \}$ and open subdomains $\omega_c \subset \Omega$ with $\bar{\omega_c} = \bigcup_{K \in \mathcal{K}_{h,c}} K$. This extends to space-time patches $Q_{0,c} = (0, t_1) \times \omega_c$, $Q_{n,c} = (t_{n-1}, t_n+1) \times \omega_c$ for $0 \neq n < N$, and $Q_{N,c} = (t_{N-1}, T) \times \omega_c$.

Let $\psi_{n,c} \in C^0(\bar{Q}) \cap P(Q_h)$ be a corresponding decomposition of $1 \equiv \sum_{n,c} \psi_{n,c}$ with $\text{supp} \psi_{n,c} = \bar{Q}_{n,c}$. On every patch we define discrete conforming local affine spaces

$$
V_{n,c}^{\text{cf}}(\ell_h) = \{ v_h \in V_h^{\text{cf}} : \text{supp}(v_h) \subset \bar{Q}_{n,c} \},
$$

where

$$
v_h(0) = \psi_{n,c} u_{0,h} \text{ in } \Omega \text{ if } n = 0, \quad v_h(t_{n-1}) = 0 \text{ in } \Omega \text{ if } n > 0, \quad v_h(t_{n+1}) = 0 \text{ in } \Omega \text{ if } n < N, \quad (B_{n} v_h)_{k} = \psi_{n,c} g_{k,h} \text{ on } (0, T) \times \Gamma_k, \quad k = 1, \ldots, m, \quad B_{n} v_h = 0 \text{ on } (0, T) \times (\partial \omega_c \setminus \partial \Omega).\n$$

In the following, we assume $V_{n,c}^{\text{cf}}(\ell_h) \neq \emptyset$, which can be achieved by a suitable choice of the data approximation $\ell_h$ depending on the reconstruction space $V_h^{\text{cf}}$.

Now, the local reconstruction of the discrete solution $u_{h} \in V_{h}^{\text{cf}}$ is defined by $u_{h}^{\text{cf}} = \sum_{n,c} u_{n,c}^{\text{cf}}$, where $u_{n,c}^{\text{cf}} \in V_{n,c}^{\text{cf}}(\ell_h)$ is the best approximation of $\psi_{n,c} u_{h}$ in the topology of $W$, i.e.,

$$
\| \psi_{n,c} u_{h} - u_{n,c}^{\text{cf}} \|_W \leq \| \psi_{n,c} u_{h} - v_{n,c} \|_W, \quad v_{n,c} \in V_{n,c}^{\text{cf}}(\ell_h).
$$

Lemma 33. The approximation error of the weak solution can be estimated by

$$
\| u - u_h \|_W \leq \| u - u_h^{\text{cf}} \|_W + 2 \beta \| L u_h^{\text{cf}} - f_h \|_{W^*} + \beta^{-1} \sup_{z \in V^* \setminus \{0\}} \frac{\langle \ell - \ell_h, z \rangle}{\| z \|_{V^*}}.
$$

Proof. For all $z \in V^*$ integration by parts and the boundary conditions in $V^*$ gives

$$
\langle u_h^{\text{cf}}, L^* z \rangle_Q = \langle L u_h^{\text{cf}}, z \rangle_Q + \langle Mu_{0,h}, z(0) \rangle_{\Omega} - \langle B_{n}^{\text{bnd}} g_h, z \rangle_{(0,T) \times \partial\Omega} = \langle L u_h^{\text{cf}} - f_h, z \rangle_Q + \langle \ell_h, z \rangle.
$$
using the consistency of the upwind flux on boundary faces (29).
Since $L_2(Q; \mathbb{R}^m) = M^{-1} L^*(V^*)$ and $V^* \subset V^*$ is dense, we obtain by duality
\[
\| u - u_h \|^W = \sup_{v \in L_2(Q; \mathbb{R}^m) \setminus \{0\}} \frac{(M(u - u_h), v)_Q}{\| v \|^Q} = \sup_{z \in V^*: L^*z \neq 0} \frac{(u - u_h^f, L^*z)_Q}{\| M^{-1} L^*z \|^W}
\]
\[
= \sup_{z \in V^*: L^*z \neq 0} \frac{(Lu_h^f - f_h, z)_Q + \langle \ell - \ell_h, z \rangle}{\| L^*z \|^W}
\]
\[
\leq \sup_{z \in V^*: L^*z \neq 0} \frac{\| Lu_h^f - f_h \|^W \| z \|^W}{\| L^*z \|^W} + \sup_{z \in V^*: L^*z \neq 0} \frac{\langle \ell - \ell_h, z \rangle}{\| L^*z \|^W},
\]
so that the assertion follows from $\| u - u_h \|^W \leq \| u - u_h^f \|^W + \| u_h^f - u_h \|^W, \| z \|^W \leq 2T \| L^*z \|^W, \text{ and } \| z \|^V \leq \beta^{-1} \| L^*z \|^W$, cf. Rem. 12.

The Lemma shows that the corresponding error estimator with local contributions
\[
\eta_{n,K} = \left( \sum_{c \in C_K} \eta_{n-1,c}^2 + \eta_{n,c}^2 \right)^{1/2}, \quad \eta_{n,c} = \| \psi_{n,c} u_h - u_h^f \|^W,
\]
is reliable up to the data approximation error.

Bibliographic comments

This chapter is based on [Dörfler et al., 2016, Dörfler et al., 2018], where also numerical results for the adaptive algorithm are presented. Further applications and several numerical applications are reported in [Findeisen, 2016, Ziegler, 2019, Dörfler et al., 2020]
The exstention to estimates for weak solutions is based on the construction of a right-inverse as it is done in [Ern and Guermond, 2016] for conforming Petrov-Galerkin approximations in reflexive Banach spaces.
The estimate for the Legendre polynomials can also be obtained recursively using [Abramowitz and Stegun, 1964, Lem. 8.5.3], see, e.g., [Dörfler et al., 2016, Lem. 8].
The error estimation based on dual-weighted residuals transfers the approach in [Becker and Rannacher, 2001] to our space-time framework, and for the general concepts on error estimation by conforming reconstructions we refer to [Ern and Vohralík, 2015].
References


