Boundary Element Approximation for Maxwell’s Eigenvalue Problem

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Maxwell’s equations in linear isotropic materials

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with piecewise smooth $\Gamma = \partial \Omega$. We consider Maxwell’s equations in linear isotropic materials with constant permittivity $\epsilon > 0$ and permeability $\mu > 0$: determine $H$ and $E$ such that

$$\mu \partial_t H + \nabla \times E = 0, \quad \epsilon \partial_t E - \nabla \times H = 0,$$
$$\nabla \cdot E = 0, \quad \nabla \cdot H = 0,$$

For monochromatic waves $E(t, x) = \exp(-i\omega t)u(x)$ with frequency $\omega$ we find

$$\nabla \times (\nabla \times u) - k^2 u = 0 \quad \text{and} \quad \nabla \cdot u = 0 \quad (1)$$

with the wave number $k = \omega/c$ and wave speed $c = 1/\sqrt{\epsilon \mu}$.

If $k^2$ is not an eigenvalue of the homogeneous Dirichlet problem with $\mathbf{n} \times u = 0$ on the boundary $\Gamma$, the solution of (1) is uniquely determined by Dirichlet values

$$\mathbf{n} \times u = f \quad \text{on } \Gamma.$$

If $k^2$ is an eigenvalue of the homogeneous Dirichlet problem with $\mathbf{n} \times u = 0$ on the boundary $\Gamma$, a non-trivial solution of (1) exists.
The representation formula for Maxwell’s equations

The fundamental solution for the Maxwell problem is given by

\[
G_k(x, y) = E_k(x, y)I + k^{-2} \nabla_x (\nabla_y E_k(x, y))^T,
\]

\[
E_k(x, y) = \frac{\exp(ik|x - y|)}{4\pi|x - y|}.
\]

This defines the Maxwell single-layer and double-layer potentials

\[
\Psi^k_{\text{SL}}(v)(x) = -\int_{\Gamma} E_k(x, y)v(y)\,ds_y - \frac{1}{k^2} \nabla_x \int_{\Gamma} E_k(x, y)\text{div}_\Gamma(v(y))\,ds_y
\]

\[
\Psi^k_{\text{DL}}(w)(x) = -\nabla_x \times \tilde{V}_k(w)(x) = -\nabla_x \times \int_{\Gamma} E_k(x, y)w(y)\,ds_y, \quad x \in \Omega
\]

for tangential boundary data \(v, w\) where \(v \cdot n = w \cdot n = 0\).

The potentials are solutions of the Maxwell problem, and the representation formula

\[
u(x) = \Psi^k_{\text{SL}}(n \times (\nabla \times u))(x) + \Psi^k_{\text{DL}}(n \times u)(x) \quad x \in \Omega,
\]

determines the Maxwell solution \(u\) in terms of its boundary traces.
Trace operators and function spaces

Let $\Gamma$ be composed of smooth manifolds $\Gamma_j$ with exterior normals $n_j$ such that $\Gamma = \bigcup_j \Gamma_j$ and $\Gamma_j \cap \Gamma_m = \emptyset$, $j \neq m$. For smooth vector fields $v$ in $\Omega$ and $x \in \Gamma_j$ define

$$\gamma_t(v)(x) = \lim_{y \in \Omega \to x \in \Gamma_j} n_j(x) \times v(y), \quad \gamma^k_N(v)(x) = \frac{1}{k} \gamma_t(\nabla \times v)(x).$$

Let $e_{jm} = \partial \Gamma_j \cap \partial \Gamma_m$ be a boundary edge with tangent $t_{jm}$, and let $n_{\partial \Gamma_j} \in \text{affine}(\Gamma_j)$ be the tangential normal along $\partial \Gamma_j$, i.e., $n_{\partial \Gamma_j} = n_j \times t_{jm}$ on the edge $e_{jm}$. From

$$\gamma_t(v)|_{\bar{\Gamma}_j} \cdot n_{\partial \Gamma_j} = (n_j \times v|_{\bar{\Gamma}_j}) \cdot (n_j \times t_{jm}) = t_{jm} \cdot v|_{\bar{\Gamma}_j} \quad \text{on } e_{jm} \quad (2)$$

we obtain

$$\sum_j \int_{\partial \Gamma_j} \gamma_t(v)|_{\bar{\Gamma}_j} \cdot n_{\partial \Gamma_j} w \, dl = \sum_{j \neq m} \int_{e_{jm}} t_{jm} \cdot v|_{\bar{\Gamma}_j} w \, dl = 0. \quad \text{This yields}$$

$$\int_{\Omega} \nabla \times v \cdot \nabla w \, dx = - \int_{\Gamma} \text{div}_\Gamma \gamma_t(v) w \, ds. \quad (3)$$

For $v \in H(\text{curl}, \Omega)$ the weak Dirichlet trace operator $\gamma_t(v) \in H^{-1/2}(\Gamma)$ is defined by

$$\langle \gamma_t(v), w|_{\Gamma} \rangle = \int_{\Omega} \nabla \times v \cdot w \, dx - \int_{\Omega} v \cdot \nabla \times w \, dx, \quad w \in C^\infty(\bar{\Omega}).$$

Extending (3) to $v \in H(\text{curl}, \Omega)$ shows $\text{div}_\Gamma \gamma_t(v) \in H^{-1/2}(\Gamma)$ defined by

$$\langle \text{div}_\Gamma \gamma_t(v), w|_{\Gamma} \rangle = - \int_{\Omega} \nabla \times v \cdot \nabla w \, dx, \quad w \in C^\infty(\bar{\Omega}).$$
Boundary Integral Operators

On the trace space $W^{-1/2}(\Gamma) = \gamma_t(H(\text{curl}, \Omega)) \subset H^{-1/2}(\Gamma)$ we define

$$S_k = \gamma_t \psi^k_{SL} : W^{-1/2}(\Gamma) \rightarrow W^{-1/2}(\Gamma),$$

$$\frac{1}{2}I + C_k = \gamma_t \psi^k_{DL} : W^{-1/2}(\Gamma) \rightarrow W^{-1/2}(\Gamma).$$

Note that we have the identities $\gamma_t \psi^k_{SL} = \gamma^k_N \psi^k_{DL}$ and $\gamma_t \psi^k_{DL} = \gamma^k_N \psi^k_{SL}$.

For smooth tangential vector fields we define the anti-linear pairing

$$\langle v, w \rangle_{\tau, \Gamma} = \int_{\Gamma} (v \times n) \cdot \overline{w} \, ds$$

which extends to $v \in H(\text{curl}^2, \Omega)$ and $v \in H(\text{curl}, \Omega)$ by

$$\langle \gamma^k_N(v), \gamma_t(w) \rangle_{\tau, \Gamma} = \frac{1}{k} \int_{\Omega} (\nabla \times \nabla \times v \cdot \overline{w} - \nabla \times v \cdot \nabla \times \overline{w}) \, dx.$$  

For tangential vector fields $v, w \in L^\infty(\Gamma)$ with $\text{div}_\Gamma v, \text{div}_\Gamma w \in L^\infty(\Gamma)$

$$\langle S_k(v), w \rangle_{\tau, \Gamma} = \int_{\Gamma} \int_{\Gamma} \left( \frac{1}{k} \text{div}_\Gamma v(y) \text{div}_\Gamma \overline{w(x)} - kv(y) \cdot \overline{w(x)} \right) E_k(x, y) \, ds_y ds_x,$$

$$\langle C_k(w), v \rangle_{\tau, \Gamma} = - \int_{\Gamma} \int_{\Gamma} \nabla_x E_k(x, y) \cdot (w(y) \times \overline{v(x)}) \, ds_y ds_x.$$  

... more details by Buffa, Costabel, Hiptmair, Schwab, ...
The Boundary Integral Equation for the Maxwell problem

The Calderon operator $C : W^{-1/2}(\Gamma) \times W^{-1/2}(\Gamma) \rightarrow W^{-1/2}(\Gamma) \times W^{-1/2}(\Gamma)$ is

$$C := \begin{pmatrix} \frac{1}{2} I + C_k & S_k \\ S_k & \frac{1}{2} I + C_k \end{pmatrix} = \begin{pmatrix} \gamma_t \psi^k_{DL} & \gamma_t \psi^k_{SL} \\ \gamma_t \psi^k_{SL} & \gamma_t \psi^k_{DL} \end{pmatrix},$$

and from the representation formula we obtain

$$C \begin{pmatrix} \gamma_t(v) \\ \gamma^k_N(v) \end{pmatrix} = \begin{pmatrix} \gamma_t(v) \\ \gamma^k_N(v) \end{pmatrix}, \quad v \in H(\text{curl}^2, \Omega).$$

For given Dirichlet data $f \in W^{-1/2}(\Gamma)$ find Neumann data $\sigma \in W^{-1/2}(\Gamma)$ with

$$S_k(\sigma) = \left( \frac{1}{2} I - C_k \right)(f).$$

**BIE** In weak form this reads as follows: find $\sigma \in W^{-1/2}(\Gamma)$ such that

$$\langle S_k(\sigma), \chi \rangle_{\tau, \Gamma} = \langle \frac{1}{2} f - C_k(f), \chi \rangle_{\tau, \Gamma}, \quad \chi \in W^{-1/2}(\Gamma).$$

**BEM** For a subspace $W_{h}^{-1/2}(\Gamma) \subset W^{-1/2}(\Gamma)$ find $\sigma_h \in W_{h}^{-1/2}(\Gamma)$ such that

$$\langle S_k(\sigma_h), \chi_h \rangle_{\tau, \Gamma} = \langle \frac{1}{2} f - C_k(f), \chi_h \rangle_{\tau, \Gamma}, \quad \chi_h \in W_{h}^{-1/2}(\Gamma).$$
Lowest Order Raviart-Thomas Elements on Boundaries

We assume that $\Gamma$ is a polygonal boundary, and let $\Gamma_h$ be a triangulation into triangles $\Gamma_j$. Moreover, let $\Omega_h$ be a tetrahedral mesh with boundary $\Gamma_h$, and let $\mathcal{E}_h$ be the edges of the boundary triangulation. The degrees of freedom $\Phi'_e$ on $e$ are $\langle \Phi'_e, v \rangle = \int_e v \cdot n_e \, dl$. On $\text{conv}\{\hat{z}_0 = (0, 0), \hat{z}_1 = (1, 0), \hat{z}_2 = (0, 1)\}$ we have

<table>
<thead>
<tr>
<th>edge $\hat{e}$</th>
<th>$\text{conv}{\hat{z}_0, \hat{z}_1}$</th>
<th>$\text{conv}{\hat{z}_1, \hat{z}_2}$</th>
<th>$\text{conv}{\hat{z}_2, \hat{z}_0}$</th>
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<tr>
<td>basis function $\hat{\Phi}_\hat{e}(\xi, \eta)$</td>
<td>$(\xi, \eta - 1)$</td>
<td>$(\xi, \eta)$</td>
<td>$(\xi - 1, \eta)$</td>
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</table>

Let $\varphi_j: \hat{\Gamma} \rightarrow \Gamma_j$ be the transformation onto a boundary triangle $\Gamma_j$, and let $F_j = D\varphi_j$ and $G_j = F_j^T F_j$. Then, $n_e = \frac{\text{sign}(e)}{|F_j G_j^{-1} \hat{n}_\hat{e}|} F_j G_j^{-1} \hat{n}_\hat{e} \in \text{affine}(\Gamma_j)$ on $e = \partial \Gamma_j \cap \partial \Gamma_m$ depends on the orientation with $\text{sign}(e) = 1$ if $j < m$ and $-1$ else. This defines $\Phi_e \circ \varphi_j = \text{sign}(e) \sqrt{\det G_j^{-1}} F_j \hat{\Phi}_\hat{e}$ and $W_{h^{-1/2}}(\Gamma) = \text{span}\{\Phi_e : e \in \mathcal{E}_h\} = \gamma_t(V_h(\Omega))$, where $V_h(\Omega)$ is the lowest order Nédélec space.

<table>
<thead>
<tr>
<th>$\text{dim } W_{h^{-1/2}}(\Gamma)$</th>
<th>$|\sigma_h - \gamma_N(u)|$</th>
<th>convergence</th>
<th>$|\varphi_h - \gamma_t(u)|$</th>
<th>convergence</th>
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<td></td>
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</table>
The Maxwell eigenvalue problem

Find a non-trivial solution pair \((u, k)\) with \(u \times n = 0\) on \(\Gamma\) and
\[
\nabla \times \nabla \times u + k^2 u(x) = 0 \quad \text{and} \quad \nabla \cdot u = 0 \quad \text{in } \Omega.
\]
A suitable normalization of the eigenfunction \(u\) is required.

**(BIE)** Find \((\sigma, k) \in W^{-1/2}(\Gamma) \times \mathbb{R}^+\) with \(\sigma \neq 0\) satisfying
\[
\langle S_k(\sigma), \chi \rangle_{\tau, \Gamma} = 0, \quad \chi \in W^{-1/2}(\Gamma).
\]

**(BEM)** Find \((\sigma_h, k_h) \in W_h^{-1/2}(\Gamma) \times \mathbb{R}^+\) with \(\sigma_h \neq 0\) such that
\[
\langle S_{k_h}(\sigma_h), \chi_h \rangle_{\tau, \Gamma} = 0, \quad \chi_h \in W_h^{-1/2}(\Gamma).
\]

... introduced and analyzed by Steinbach and Unger for the Helmholtz problem ...

Note that \(S(k) = S_k : \Lambda \subset \mathbb{C} \to \mathcal{L}(W^{-1/2}(\Gamma), W^{-1/2}(\Gamma))\) is holomorphic and satisfies a generalized Garding inequality.
Newton’s Method

The matrix realization yields the problem to find \((\xi, k_h) \in \mathbb{C}^N \times \mathbb{R}\) such that

\[
A(k_h)\xi = 0, \\
\hat{\xi}^H \xi - 1 = 0,
\]

where a suitable fixed vector \(\hat{\xi} \in \mathbb{C}^N\) is a priori chosen.

Newton’s method starts with some initial guess \((\xi^0, k^0) \in \mathbb{C}^N \times \mathbb{R}\).

Then, for \(n = 0, 1, 2, \ldots\), the next iterate \((\xi^{n+1}, k^{n+1}) \in \mathbb{C}^N \times \mathbb{R}\) is computed by

\[
A(k^n)(\xi^{n+1} - \xi^n) + (k^{n+1} - k^n)J_A(k^n)\xi^n = -A(k^n)\xi^n, \\
\hat{\xi}^H (\xi^{n+1} - \xi^n) = -\hat{\xi}^H \xi^n + 1,
\]

where the matrix \(J_A(k) = \lim_{\delta \to 0} \frac{1}{\delta} (A(k + \delta) - A(k))\) is the derivative of \(A(\cdot)\) with respect to \(k\). This can be assembled from the boundary integral representation

\[
\langle \partial_k S_k(v), w \rangle_{\tau, \Gamma} = \int_{\Gamma} \int_{\Gamma} \left( \left( \frac{1}{k} \text{div}_\Gamma v(y) \text{div}_\Gamma w(x) - k v(y) \cdot w(x) \right) \partial_k E_k(x, y) \right. \\
\left. - \left( \frac{1}{k^2} \text{div}_\Gamma v(y) \text{div}_\Gamma w(x) + v(y) \cdot w(x) \right) E_k(x, y) \right) ds_y ds_x.
\]
The contour integral method

Eigenvalues are singular points $\lambda \in \mathbb{C}$ of the holomorphic matrix function $A: \mathbb{C} \rightarrow \mathbb{C}^{N \times N}$. Thus, the matrix function $A(\cdot)^{-1}$ is meromorphic.

Fix $D \subset \mathbb{C}$ such that $A(z)^{-1}$ is regular for $z \in \partial D$, and assume that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues in $D$ with eigenvectors $\xi_j \in \mathbb{C}^N$, i.e., $A(\lambda_j)\xi_j = 0$ and $\xi_j^H \xi_j = 1$.

If all eigenvalues $\lambda_j$ are simple, we have

$$A(z)^{-1} = \sum_{j=1}^n \frac{1}{z - \lambda_j} \xi_j \xi_j^H + B(z),$$

where $B(z)$ is a holomorphic matrix function in $D$.

The residual theorem yields for the contour integral matrices

$$A_0 := \frac{1}{2\pi i} \int_{\partial D} A(z)^{-1} R \, dz = \sum_{j=1}^n \xi_j \xi_j^H R,$$

$$A_1 := \frac{1}{2\pi i} \int_{\partial D} zA(z)^{-1} R \, dz = \sum_{j=1}^n \lambda_j \xi_j \xi_j^H R,$$

where $R \in \mathbb{C}^{N \times n}$ are suitable test matrices such that $A_0$ and $A_1$ have full rank.

By construction we have $A_1 A_0^+ = \sum_{j=1}^n \lambda_j \xi_j \xi_j^H \in \mathbb{C}^{N \times N}$ which allows to recover the eigenvalues from $A_0^+ A_1 \in \mathbb{C}^{n \times n}$.

... more details in a paper by Beyn ...
Eigenvalue Convergence

Let \( \Omega = (0, 1)^3 \), \( \varepsilon = \mu = 1 \), \( \kappa \in \mathbb{N}_0^3 \), \( a \in \mathbb{R}^3 \) with \( a \cdot \kappa = 0 \), and \( k = \pi \sqrt{\kappa_1^2 + \kappa_2^2 + \kappa_3^2} \).

Then, \( u_k = \begin{pmatrix} a_1 \cos(\kappa_1 \pi x_1) \sin(\kappa_2 \pi x_2) \sin(\kappa_3 \pi x_3) \\ a_2 \sin(\kappa_1 \pi x_1) \cos(\kappa_2 \pi x_2) \sin(\kappa_3 \pi x_3) \\ a_3 \sin(\kappa_1 \pi x_1) \sin(\kappa_2 \pi x_2) \cos(\kappa_3 \pi x_3) \end{pmatrix} \) is an eigenfunction.

\( \kappa = (1, 1, 0) \quad \kappa = (1, 1, 1) \)

| \( m \) | \( \dim W_{h_m}^{-1/2}(\Gamma) \) | \( k_m \) | \( \log_2 \frac{|k_{m-1}-k|}{|k_m-k|} \) | \( k_m \) | \( \log_2 \frac{|k_{m-1}-k|}{|k_m-k|} \) |
|---|---|---|---|---|---|
| 1 | 144 | 4.396130 | 3.4003 | 5.319289 | 3.5230 |
| 2 | 576 | 4.438455 | 3.2378 | 5.430776 | 3.2216 |
| 3 | 2304 | 4.442414 | 3.2957 | 5.440259 | 3.2034 |
| 4 | 9216 | 4.442832 | | 5.441274 | |
| \( \infty \) | | 4.442883 | | 5.441398 | |
Comparing Newton and Contour Integral Method

Convergence of the Newton iteration for the first and second eigenvalue on level \( m \).

<table>
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<th>( m )</th>
<th>( \kappa = (1, 1, 0) )</th>
<th>step 1</th>
<th>step 2</th>
<th>step 3</th>
<th>step 4</th>
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<td>( \kappa = (1, 1, 1) )</td>
<td>( k = 4.442883 )</td>
<td>( k = 5.441398 )</td>
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Contour integral method, where \( D \subset \mathbb{C} \) contains the first two eigenvalues, and where \( \partial D \) is approximated by a polygon with \( M \) segments.

<table>
<thead>
<tr>
<th>( \kappa = (1, 1, 0) )</th>
<th>( k = 4.442883 )</th>
<th>( \kappa = (1, 1, 1) )</th>
<th>( k = 5.441398 )</th>
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<tr>
<td>( m )</td>
<td>( M = 10 )</td>
<td>( M = 20 )</td>
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**Application to the Fichera Cube**

Boundary element approximations of the Maxwell eigenvalues on the Fichera cube and comparison with reference values from Zaglmayr and Buffa et. al.

<table>
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<tr>
<th>dof</th>
<th>576</th>
<th>2304</th>
<th>9216</th>
<th>hp FEM</th>
<th>IGA (p = 3)</th>
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# Finite Element Approximation of Fichera Eigenvalues

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Modified LOBPCG Method (including projection)

Let $T_h : X'_h \rightarrow X_h$ be a preconditioner for $A^\delta_h = A_h + \delta M_h : X_h \rightarrow X'_h$.

S0) Choose randomly $u^1_h, \ldots, u^N_h \in X_h$. Compute $v^n_h = P_h u^n_h \in V_h$.

S1) Ritz-step: Set up Hermitian matrices

$$
\hat{A} = \begin{pmatrix} a(v^m_h, v^n_h) \end{pmatrix}_{m,n=1,\ldots,N}, \quad \hat{M} = \begin{pmatrix} m(v^m_h, v^n_h) \end{pmatrix}_{m,n=1,\ldots,N} \in \mathbb{C}^{N \times N}
$$

and solve the matrix eigenvalue problem $\hat{A}\hat{z}^n = \lambda^n\hat{M}\hat{z}^n$.

S2) Compute $y^n_h = \sum_{n=1}^{N} \hat{z}^n_m v^n_h \in V_h$.

S3) Compute $r^n_h = A_h y^n_h - \lambda^n M_h y^n_h \in X'_h$, check for convergence.

S4) Compute $u^n_h := T_h r^n_h \in X_h$ and $w^n_h = P_h u^n_h \in V_h$.

S5) Perform Ritz-step for $\{v^1_h, \ldots, v^N_h, w^1_h, \ldots, w^N_h\} \subset V_h$ of size $2N$.

S6) Go to step S2).

The full algorithm uses orthogonalization, new random vectors, and a Ritz-step of size $3N$.

(the LOBPCG method was introduced by Knyazev 2001)
Multigrid Preconditioner for the Maxwell Operator

Let $X_0 \subset X_1 \subset \cdots \subset X_J$ be finite element spaces of mesh size $h_j = 2^{-j} h_0$.

The multigrid preconditioner $T_j : X'_j \to X_j$ is defined recursively:

For $j = 0$, define $T_0 = \left( A_0 + \delta M_0 \right)^{-1}$.

For $j > 0$, the definition of $T_j$ requires:

1) a prolongation operator $I_j : X_{j-1} \to X_j$

2) the adjoint operator $I'_j : X'_j \to X'_{j-1}$

3) a smoother $R_j : X'_j \to X_j$ for $A_j$

4) a smoother $D_j : Q'_j \to Q_j$ for $C_j$

5) a transfer operator $S_j : Q_j \to X_j$

6) the adjoint transfer operator $S'_j : X'_j \to Q'_j$

Now, define $T_j$ by

$$\text{id} - A_j \delta T_j = \left( \text{id} - A_j \delta I_j T_{j-1} I'_j \right) \left( \text{id} - \delta^{-1} A_j \delta S_j D_{j-1} S'_j \right) \left( \text{id} - A_j \delta R_j \right).$$

(this multigrid variant was introduced and analyzed by Hiptmair 1998)

The multigrid methods works also for $A_j - \mu M_j$ if $h_0$ is sufficiently small.
A Domain Decomposition Method

Let $\Omega = \bigcup_{p=1}^{P} \Omega_p$ with permittivity $\varepsilon_p$ and permeability $\mu_p$ in $\Omega_p$. In $\Omega_p$ solve

$$\nabla \times \nabla \times u^p - (\omega/c_p)^2 u^p = 0 \quad \text{and} \quad \nabla \cdot u^p = 0.$$  

Set $\beta_p = \sqrt{\varepsilon_p/\mu_p}$ and $\Gamma_p = \partial \Omega_p$. On $\Gamma_{pq} = \Gamma_p \cap \Gamma_q$ holds

$$\gamma^p_t(u^p) + \gamma^q_t(u^q) = 0, \quad \beta_p \gamma^k_p(u^p) + \beta_q \gamma^k_q(u^q) = 0.$$  

Dirichlet traces on the interfaces and Neumann traces to subdomain boundaries

Find $\omega, \varphi = (\varphi^{pq})_{p<q} \in \prod_{p<q} W^{-1/2}(\Gamma_{pq})$ and $\sigma = (\sigma^p) \in \prod_p W^{-1/2}(\Gamma_p)$ with

$$\sum_{q} \langle \left(\frac{1}{2} - C^p_{\omega/c_p}\right)(\varphi^{pq}), \tau^p \rangle_{\tau,\Gamma_p} + \langle S^p_{\omega/c_p}(\sigma^p), \tau^p \rangle_{\tau,\Gamma_p} = 0 \quad \text{for all } \tau^p,$$

$$\sum_{p<q} \langle \left(\beta_p S^p_{\omega/c_p} + \beta_q S^q_{\omega/c_q}\right)(\varphi^{pq}), \chi^{pq} \rangle_{\tau,\Gamma_{pq}}$$

$$+ \langle \beta_p \left(\frac{1}{2} + C^p_{\omega/c_p}\right)(\sigma^p) + \beta_q \left(\frac{1}{2} + C^q_{\omega/c_q}\right)(\sigma^q), \chi^{pq} \rangle_{\tau,\Gamma_{pq}} = 0 \quad \text{for all } \chi^{pq}.$$  

... more details by Langer, Of, Steinbach, Zulehner, ...
A Test Example with two subdomains

We consider the previous example with \( \Omega_1 = (1/3, 2/3)^3 \) and \( \Omega_2 = (0, 1)^3 \setminus \overline{\Omega}_1 \): find \( \omega_h > 0 \) and \( (\varphi^1_{12}, \sigma^1_h, \sigma^2_h) \in W_h^{-1/2}(\Gamma_{12}) \times W_h^{-1/2}(\Gamma_1) \times W_h^{-1/2}(\Gamma_2) \) such that

\[
0 = \beta_1 \langle S_{\omega_h/c_1}^1(\varphi^1_{12}), \chi_{12}^1 \rangle_{\tau, \Gamma_{12}} + \beta_2 \langle S_{\omega_h/c_2}^2(\varphi^1_{12}), \chi_{12}^2 \rangle_{\tau, \Gamma_{12}} \\
+ \beta_1 \langle \frac{1}{2} \sigma^1_h + C_{\omega_h/c_1}^1(\sigma^1_h), \chi_{12}^1 \rangle_{\tau, \Gamma_{12}} + \beta_2 \langle \frac{1}{2} \sigma^2_h + C_{\omega_h/c_2}^2(\sigma^2_h), \chi_{12}^2 \rangle_{\tau, \Gamma_{12}} \\
+ \langle \frac{1}{2} \varphi^1_{12} - C_{\omega_h/c_1}(\varphi^1_{12}), \tau^1_h \rangle_{\tau, \Gamma_1} + \langle S_{\omega_h/c_1}^1(\sigma^1_h), \tau^1_h \rangle_{\tau, \Gamma_1} \\
+ \langle - \frac{1}{2} \varphi^1_{12} + C_{\omega_h/c_2}(\varphi^1_{12}), \tau^2_h \rangle_{\tau, \Gamma_2} + \langle S_{\omega_h/c_2}^2(\sigma^2_h), \tau^2_h \rangle_{\tau, \Gamma_2}
\]

for all test functions \( (\chi_{12}^1, \tau^1_h, \tau^2_h) \in W_h^{-1/2}(\Gamma_{12}) \times W_h^{-1/2}(\Gamma_1) \times W_h^{-1/2}(\Gamma_2) \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
m & N_m & \kappa = (1, 1, 0) & \kappa = (1, 1, 1) \\
\hline
\kappa = (1, 1, 0) & \kappa = (1, 1, 1) \\
\hline
0 & 432 & 4.42676 & 3.5593 & 5.37587 & 3.6474 \\
1 & 1728 & 4.44152 & 3.4450 & 5.43617 & 3.3271 \\
2 & 6912 & 4.44276 & & 5.44088 & \\
\infty & 4.44288 & & 5.44140 & \\
\hline
\end{array}
\]
A Test Example with two subdomains

$\sigma_h^2$

$\varphi_h^{12}$

$\sigma_h^1$
Band structure computation for photonic crystals

We consider a periodic medium with \( \varepsilon(x + z) = \varepsilon(x) \) and \( \mu(x + z) = \mu(x) \) for all \( z \in \mathbb{Z}^3 \). As a consequence of the Floquet theory, the spectrum of the Maxwell operator is the union of all eigenvalues in the periodicity cell \( \Omega = (0, 1)^3 \) with quasi-periodic boundary conditions

\[
\gamma_t(u)(x + e_j) + \exp(i\alpha \cdot e_j)\gamma_t(u)(x) = 0, \quad \gamma_N^k(u)(x + e_j) + \exp(i\alpha \cdot e_j)\gamma_N^k(u)(x) = 0
\]
on the faces \( G_1 = [0, 1]^2 \times \{0\}, G_2 = [0, 1] \times \{0\} \times [0, 1] \) and \( G_3 = \{0\} \times [0, 1]^2 \), where \( e_j \) are the unit vectors and \( \alpha \in (-\pi, \pi]^3 \) is a parameter in the Brillouin zone.

For two subdomains (with \( \varepsilon_1 = 1 \) and \( \varepsilon_2 = 13 \)) this leads to the problem to find

\[
(\varphi_h^{12}, \sigma_h^{2} | \Gamma, \sigma^2 | \Gamma_{12}, \varphi_h^{2} | \Gamma, \sigma^1) \in W^{-\frac{1}{2}}_h(\Gamma_{12}) \times W^{-\frac{1}{2}}_h(\Gamma) \times W^{-\frac{1}{2}}_h(\Gamma_{12}) \times W^{-\frac{1}{2}}_h(\Gamma) \times W^{-\frac{1}{2}}_h(\Gamma_1)
\]

and \( \omega_h \) such that

\[
\begin{pmatrix}
A_{11}(\omega_h) & A_{12}(\omega_h)B_{\alpha} & A_{13}(\omega_h) & A_{14}(\omega_h)B_{\alpha} & A_{15}(\omega_h) \\
B_{\alpha}^HA_{21}(\omega_h) & B_{\alpha}^HA_{22}(\omega_h)B_{\alpha} & B_{\alpha}^HA_{23}(\omega_h) & B_{\alpha}^HA_{24}(\omega_h)B_{\alpha} & 0 \\
A_{31}(\omega_h) & A_{32}(\omega_h)B_{\alpha} & A_{33}(\omega_h) & A_{34}(\omega_h)B_{\alpha} & 0 \\
B_{\alpha}^HA_{41}(\omega_h) & B_{\alpha}^HA_{42}(\omega_h)B_{\alpha} & B_{\alpha}^HA_{43}(\omega_h) & B_{\alpha}^HA_{44}(\omega_h)B_{\alpha} & 0 \\
A_{51}(\omega_h) & 0 & 0 & 0 & A_{55}(\omega_h)
\end{pmatrix}
\]
is singular, where the quasi-periodic boundary conditions are realized by the matrix \( B_{\alpha} \) extending the degrees of freedom on \( G_1 \cup G_2 \cup G_3 \) to \( \Gamma = \partial\Omega \).
Band structure computation for photonic crystals

We consider a periodic medium with $\varepsilon(x + z) = \varepsilon(x)$ and $\mu(x + z) = \mu(x)$ for all $z \in \mathbb{Z}^3$. As a consequence of the Floquet theory, the spectrum of the Maxwell operator is the union of all eigenvalues in the periodicity cell $\Omega = (0, 1)^3$ with quasi-periodic boundary conditions

$$\gamma_t(u)(x + e_j) + \exp(i\alpha \cdot e_j)\gamma_t(u)(x) = 0, \quad \gamma_N^k(u)(x + e_j) + \exp(i\alpha \cdot e_j)\gamma_N^k(u)(x) = 0$$

on the faces $G_1 = [0, 1]^2 \times \{0\}$, $G_2 = [0, 1] \times \{0\} \times [0, 1]$ and $G_3 = \{0\} \times [0, 1]^2$, where $e_j$ are the unit vectors and $\alpha \in (-\pi, \pi]^3$ is a parameter in the Brillouin zone.

boundary elements

finite elements

band structure along a path in the Brillouin zone $(-\pi, \pi]^3$ for a photonic crystal