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A short introduction to

Numerical Methods for Maxwell's Equations

Exercises for the Oberwolfach Course 2008

Version of October 17, 2008

This note summarizes some selected chapters of the first part of my lecture on Numerical Methods for Maxwell's Equations from last summer in Karlsruhe. It should provide some background material for the exercises in the Oberwolfach course.

Parts of this note are extremely simple. The examples are designed to provide some insight in the numerical difficulties which are inherent in Maxwell'e equations. Most effects are demonstrated for the scalar wave equation. For more details and proofs of the results we refer to the references below.

The methods presented here cannot successfully applied to photonics; this requires far more analytical and numerical effort. Nevertheless, this note should explain why simple methods do not work.

Parts of the note are not finished. Please check our website

`http://www.mathematik.uni-karlsruhe.de/user/~wieners/MaxwellCourse.pdf`

for an updated and corrected version.

Contents

Introduction to Maxwell's equations **3**

1 Explicit numerical schemes for of the scalar wave equation **10**

2 The FDTD method for Maxwell's equations **20**

3 Time-harmonic solutions and eigenfrequencies **24**

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Introduction to Maxwell's equations

Basic principles

We consider electromagnetic waves in a region $\Omega \subset \mathbb{R}^3$ with no magnetic sources, but Ω may (partly) be filled with materials that absorb electric energy.

Electro-magnetic waves are described by the following quantities:

$$\begin{aligned}
\boldsymbol{\mathcal{E}}\colon \Omega \times [0,\infty) &\to \mathbb{R}^3 && \text{electric field intensity}\\
\boldsymbol{\mathcal{H}}\colon \Omega \times [0,\infty) &\to \mathbb{R}^3 && \text{magnetic field intensity}\\
\boldsymbol{\mathcal{D}}\colon \Omega \times [0,\infty) &\to \mathbb{R}^3 && \text{electric displacement}\\
\boldsymbol{\mathcal{B}}\colon \Omega \times [0,\infty) &\to \mathbb{R}^3 && \text{magnetic field induction}
\end{aligned}$$

We assume that the following quantities are given:

$$\begin{aligned}
\mathbf{J}\colon \Omega \times [0,\infty) &\to \mathbb{R}^3 && \text{electric current density}\\
\rho\colon \Omega \times [0,\infty) &\to \mathbb{R} && \text{electric charge density}
\end{aligned}$$

We assume within this introduction that all fields are sufficiently smooth.

Physical laws

1) Faraday's Law For all (smooth and bounded) oriented two-dimensional manifolds A in Ω we have

$$\partial_t \int_A \boldsymbol{\mathcal{B}} \cdot da + \int_{\partial A} \boldsymbol{\mathcal{E}} \cdot d\ell = 0\,,$$

i.e., an electric field along the line ∂A induces a change of the magnetic introduction through the surface A.

Example Consider the square

$$A = (0,1) \times (0,1) \times \{0\}$$

with boundary

$$\partial A = (0,1) \times \{0\} \times \{0\} \cup \ \{1\} \times \{0\} \times \{0\} \cup \ (0,1) \times \{1\} \times \{0\} \cup \ \{0\} \times (0,1) \times \{0\}$$

For determining the orientation we choose the normal vector on A

$$\boldsymbol{n}(\boldsymbol{x}) \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{x} \in A$$

Then, we have

$$\begin{aligned}
0 \ = \ \partial_t \int_A \boldsymbol{\mathcal{B}} \cdot da + \int_{\partial A} \boldsymbol{\mathcal{E}} \cdot d\ell &= \int_0^1 \int_0^1 \partial_t \boldsymbol{\mathcal{B}}(x_1, x_2, 0, t) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dx_1\, dx_2 \\
&+ \int_0^1 \boldsymbol{\mathcal{E}}(x_1, 0, 0, t) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} dx_1 + \int_0^1 \boldsymbol{\mathcal{E}}(1, x_2, 0, t) \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} dx_2 \\
&+ \int_1^0 \boldsymbol{\mathcal{E}}(x_1, 1, 0, t) \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} dx_1 + \int_1^0 \boldsymbol{\mathcal{E}}(0, x_2, t) \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} dx_2
\end{aligned}$$

2) Ampere's Law For all $A \subset \Omega$ we have

$$\partial_t \int_A \boldsymbol{\mathcal{D}} \cdot da - \int_{\partial A} \boldsymbol{\mathcal{H}} \cdot d\ell + \int_A \boldsymbol{J} \cdot da = 0\,,$$

i.e., an electric current density $\boldsymbol{J}$ through A and a magnetic field $\boldsymbol{\mathcal{H}}$ along ∂A induces a change of the electric displacement $\boldsymbol{\mathcal{D}}$.

3) Gauß's Law for the electric field For all volumes $V \subset \Omega$ with piecewise smooth boundary we have

$$\int_{\partial V} \boldsymbol{\mathcal{D}} \cdot da - \int_V \rho\, dx = 0\,,$$

i.e. the flux of the electric displacement through ∂V coincides with the sources of the charge density in V.

4) Gauß's Law for the magnetic field For all $V \subset \Omega$ we have

$$\int_{\partial V} \boldsymbol{\mathcal{B}} \cdot da = 0\,,$$

i.e., inflow and outflow of the magnetic flux are balanced (no magnetic sources).

Definition 1. *The Maxwell p.d.e. are given by*

$$\begin{array}{rclrcl} \partial_t \boldsymbol{\mathcal{D}} - \nabla \times \boldsymbol{\mathcal{H}} &=& -\boldsymbol{J} & \partial_t \boldsymbol{\mathcal{B}} + \nabla \times \boldsymbol{\mathcal{E}} &=& 0 \\ \nabla \cdot \boldsymbol{\mathcal{D}} &=& \rho & \nabla \cdot \boldsymbol{\mathcal{B}} &=& 0 \end{array}$$

Here, we use for a vector field $\boldsymbol{u}\colon \Omega \to \mathbb{R}^3$ the notation

$$\begin{aligned} \nabla &= \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \\ \nabla \cdot \boldsymbol{u} &= \operatorname{div} \boldsymbol{u} = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 \\ \nabla \times \boldsymbol{u} &= \operatorname{curl} \boldsymbol{u} = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix} \end{aligned}$$

Formally, we can derive the p.d.e. from Maxwell's equations in integral form using the theorems by Gauss and Stokes:

Gauß $\quad \displaystyle\int_V \operatorname{div} \boldsymbol{u}\, dx = \int_{\partial V} \boldsymbol{u} \cdot da$

Stokes $\quad \displaystyle\int_A \operatorname{curl} \boldsymbol{u}\, da = \int_{\partial A} \boldsymbol{u} \cdot d\ell$

Here, we use $\boldsymbol{u} \cdot da = \boldsymbol{u} \cdot \boldsymbol{n}\, da$ and $\boldsymbol{u} \cdot d\ell = \boldsymbol{u} \cdot \boldsymbol{\tau} d\ell$, the normal vector field $\boldsymbol{n}\colon A \longrightarrow \mathbb{R}^3$ and the tangential vector field $\boldsymbol{\tau}\colon \partial A \to \mathbb{R}^3$ (where the orientation of ∂A is given by $\boldsymbol{n}$).

Example (cont.) For the area $A = (0,1) \times (0,1) \times \{0\}$ we have

$$\begin{aligned}\int_A \operatorname{curl} \boldsymbol{u} \cdot da &= \int_0^1 \int_0^1 \big(\partial_1 u_2(x_1,x_2,0) - \partial_2 u_1(x_1,x_2,0)\big) dx_1 dx_2 \\ &= \int_0^1 \big(u_2(1,x_2,0)\big) dx_2 - \int_0^1 \big(u_1(x_1,1,0) - u_1(x_1,0,0)\big) dx_1 = \int_{\partial A} \boldsymbol{u} \cdot d\ell\end{aligned}$$

For the volume $V = (0,1) \times (0,1) \times (0,1)$ we have

$$\begin{aligned}\int_V \operatorname{div} \boldsymbol{u} \cdot dx &= \int_0^1 \int_0^1 \int_0^1 (\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3) dx_1 dx_2 dx_3 \\ &= \int_0^1 \int_0^1 \big(\boldsymbol{u}_1(1,x_2,x_3) - u_1(0,x_2,x_3)\big) dx_2 dx_3 + \cdots = \int_{\partial V} \boldsymbol{u} \cdot \boldsymbol{n}\, da\end{aligned}$$

(general areas A and volumes V can be computed by suitable diffeomorphisms and the transformation theorem).

Thus, the theorems by Gauss and Stokes give Maxwell's equations in the form

$$\begin{aligned}&1) \quad \int_A (\partial_t \boldsymbol{\mathcal{B}} + \operatorname{curl} \boldsymbol{\mathcal{E}}) \cdot da = 0 \\ &2) \quad \int_A (\partial_t \boldsymbol{\mathcal{D}} - \operatorname{curl} \boldsymbol{\mathcal{H}} + \boldsymbol{J}) \cdot da = 0 \\ &3) \quad \int_V (\operatorname{div} \boldsymbol{\mathcal{D}} - \rho) dx = 0 \\ &4) \quad \textstyle\int_V \operatorname{div} \boldsymbol{\mathcal{B}}\, dx = 0\end{aligned}$$

for all A and V. Since A and V are arbitrary and we assume smoothness for all fields, all integrants vanish, i. e., the Maxwell p.d.e. hold for all $\boldsymbol{x} \in \Omega$.

Observation: The p.d.e. provides 8 equations for 12 unknown components of $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{D}}$, $\boldsymbol{\mathcal{H}}$. Thus, the Maxwell system has to be closed by *constitutive relations* (material laws)

$$\begin{aligned}\boldsymbol{\mathcal{D}} &= \boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}) \\ \boldsymbol{\mathcal{B}} &= \boldsymbol{\mathcal{B}}(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}})\end{aligned}$$

Together, we now obtain 14 equations!

Lemma 1. *A solution of Maxwell's equations require the compatibility condition*

$$\partial_t \rho + \operatorname{div} \boldsymbol{J} = 0\,.$$

Proof. We have

$$\partial_t \int_V \rho\, dx = \partial_t \int_{\partial V} \boldsymbol{\mathcal{D}} \cdot da = \int_{\partial V} (\operatorname{curl} \boldsymbol{\mathcal{H}} - \boldsymbol{J}) \cdot da = -\int_V \operatorname{div} \boldsymbol{J}\, dx$$

since $\int_{\partial V} \operatorname{curl} \boldsymbol{\mathcal{H}} \cdot da = \int_V \operatorname{div}(\operatorname{curl} \boldsymbol{\mathcal{H}})\, dx = 0$ for closed surfaces. □

In our lecture, we restrict ourselves to linear constitutive laws (*linear non-dispersive materials*)

$$\begin{aligned}\boldsymbol{\mathcal{D}}(\boldsymbol{x}, t) &= \varepsilon(\boldsymbol{x})\ \boldsymbol{\mathcal{E}}(\boldsymbol{x}, t) \\ \boldsymbol{\mathcal{B}}(\boldsymbol{x}, t) &= \mu(\boldsymbol{x}) \boldsymbol{\mathcal{H}}(\boldsymbol{x}, t)\end{aligned}$$

with

$$\begin{aligned} \varepsilon\colon \Omega &\to \mathbb{R}^{3\times3} \text{ or } \mathbb{R} \qquad \text{electric permittivity} \\ \mu\colon \Omega &\to \mathbb{R}^{3\times3} \text{ or } \mathbb{R} \qquad \text{magnetic permeability} \end{aligned}$$

In general, $\varepsilon(x), \mu(x)$ are symmetric positive definite. For *isotropic materials* we have $\varepsilon(x), \mu(x) \in \mathbb{R}$.
In the special case of *homogeneous materials* we have $\varepsilon(x) \equiv \varepsilon$ and $\mu(x) \equiv \mu$ not depending on $\boldsymbol{x} \in \Omega$.
This gives Maxwell's equations for linear materials:

$$\begin{aligned} \varepsilon\partial_t\boldsymbol{\mathcal{E}} - \nabla\times\boldsymbol{\mathcal{H}} &= -\boldsymbol{J} \qquad & \mu\partial_t\boldsymbol{\mathcal{H}} + \nabla\times\boldsymbol{\mathcal{E}} &= 0 \\ \nabla\cdot(\varepsilon\boldsymbol{\mathcal{E}}) &= \rho & \nabla\cdot(\mu\boldsymbol{\mathcal{H}}) &= 0 \end{aligned}$$

Now we have 6 unknowns and 8 equations!

Lemma 2. *Assume*

$$\begin{aligned} \nabla\cdot\big(\varepsilon\boldsymbol{\mathcal{E}}(\boldsymbol{x},0)\big) &= \rho(\boldsymbol{x},0) \\ \nabla\cdot\big(\mu\boldsymbol{\mathcal{H}}(\boldsymbol{x},0)\big) &= 0 \end{aligned}$$

for the initial values at $t=0$.
Then, we have for every solution of the p.d.e. system

$$\begin{aligned} \varepsilon\partial_t\boldsymbol{\mathcal{E}} - \nabla\times\boldsymbol{\mathcal{H}} &= -\boldsymbol{J} \\ \mu\partial_t\boldsymbol{\mathcal{H}} + \nabla\times\boldsymbol{\mathcal{E}} &= 0 \\ \partial_t\rho + \nabla\cdot\boldsymbol{J} &= 0 \end{aligned}$$

also $\nabla\cdot\big(\varepsilon\boldsymbol{\mathcal{E}}\big) = \rho$ *and* $\nabla\cdot(\mu\boldsymbol{\mathcal{H}}) = 0$ *for all* $t>0$.

Proof. We have for $\nabla = (\partial_1,\partial_2,\partial_3)^T$ and any smooth vector field $\boldsymbol{u} = (u_1,u_2,u_3)$

$$\nabla\cdot(\nabla\times\boldsymbol{u}) = \partial_1(\partial_2 u_3 - \partial_3 u_2) + \partial_2(\partial_3 u - \partial_1 u_3) + \partial_3(\partial_1 u_2 - \partial_2 u_1) = 0\,.$$

This gives

$$\partial_t\big(\nabla\cdot(\mu\boldsymbol{\mathcal{H}})\big) = \nabla\cdot(\mu\partial_t\boldsymbol{\mathcal{H}}) = -\nabla\cdot(\nabla\times\boldsymbol{\mathcal{E}}) = 0$$

and thus $\nabla\cdot(\mu\boldsymbol{\mathcal{H}}) \equiv 0$ for all $t>0$ as a consequence of the initial condition.
Analogously, we obtain from the compatibility equation

$$\partial_t\big(\nabla\cdot(\varepsilon\boldsymbol{\mathcal{E}}) - \rho\big) = \nabla\cdot(\nabla\times\boldsymbol{\mathcal{H}} - \boldsymbol{J}) - \partial_t\rho = \nabla\cdot(\nabla\times\boldsymbol{\mathcal{H}}) = 0$$

which finally proves $\nabla\cdot\big(\varepsilon\boldsymbol{\mathcal{E}}\big) = \rho$. □

A further constitutive law is Ohm's Law $\boldsymbol{J} = \sigma\boldsymbol{\mathcal{E}} + \boldsymbol{J}_a$ for given

$$\begin{aligned} \sigma\colon \Omega &\to \mathbb{R}, \qquad \sigma(x) > 0 \quad \text{conductivity} \\ \boldsymbol{J}_a\colon \Omega &\to \mathbb{R}^3 \qquad\qquad \text{applied current density} \end{aligned}$$

A special case are electro-magnetic waves in the vacuum: there, we have

$$\rho\equiv 0; \qquad \boldsymbol{J}\equiv 0, \qquad \sigma\equiv 0, \qquad \varepsilon\equiv\varepsilon_0, \qquad \mu\equiv\mu_0\,.$$

We choose physical units such that $\varepsilon_0 = \mu_0 = 1$.

A special case for Maxwell's equations: the wave equation

Lemma 3. *Consider a homogeneous and isotropic material (i.e., the parameters $\varepsilon, \mu, \sigma \in \mathbb{R}$ are constant) without charges (i.e., $\rho \equiv 0$) and without applied currents ($\boldsymbol{J}_a \equiv 0$), i.e.,*

$$
\begin{aligned}
\varepsilon \partial_t \boldsymbol{\mathcal{E}} - \nabla \times \boldsymbol{\mathcal{H}} &= -\sigma \boldsymbol{\mathcal{E}} \qquad & \mu \partial_t \boldsymbol{\mathcal{H}} + \nabla \times \boldsymbol{\mathcal{E}} &= 0 \\
\nabla \cdot (\varepsilon \boldsymbol{\mathcal{E}}) &= 0 \qquad & \nabla \cdot (\mu \boldsymbol{\mathcal{H}}) &= 0.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\partial_t^2 \boldsymbol{\mathcal{E}} - c^2 \Delta \boldsymbol{\mathcal{E}} + \frac{\sigma}{\varepsilon} \partial_t \boldsymbol{\mathcal{E}} = 0 \\
&\partial_t^2 \boldsymbol{\mathcal{H}} - c^2 \Delta \boldsymbol{\mathcal{H}} + \frac{\sigma}{\mu} \partial_t \boldsymbol{\mathcal{H}} = 0
\end{aligned}
$$

where $c = \frac{1}{\sqrt{\varepsilon\mu}}$ is the wave velocity and $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplacian.

Observation: We obtain 6 decoupled (damped) scalar wave equations with coupling via the initial conditions

$$
\begin{aligned}
&\boldsymbol{\mathcal{E}}(x,0) = \boldsymbol{\mathcal{E}}_0(x)\,, \qquad \boldsymbol{\mathcal{H}}(x,0) = \boldsymbol{\mathcal{H}}_0(x)\,, \\
&\partial_t \boldsymbol{\mathcal{H}}(x,0) = -\frac{1}{\mu} \nabla \times \boldsymbol{\mathcal{E}}_0(x)\,, \qquad \partial_t \boldsymbol{\mathcal{E}}(x,0) = \frac{1}{\varepsilon} \nabla \times \boldsymbol{\mathcal{H}}_0(x) - \frac{\sigma}{\varepsilon} \boldsymbol{\mathcal{E}}_0(x)
\end{aligned}
$$

Proof. We have for a vector field $\boldsymbol{u}\colon \Omega \to \mathbb{R}^3$

$$
\begin{aligned}
\nabla \times \nabla \times \boldsymbol{u} &= \begin{pmatrix} \partial_2(\partial_1 u_2 - \partial_2 u_1) - \partial_3(\partial_3 u_1 - \partial_1 u_3) \\ \partial_3(\partial_2 u_3 - \partial_3 u_2) - \partial_1(\partial_1 u_2 - \partial_2 u_1) \\ \partial_1(\partial_3 u_1 - \partial_1 u_3) - \partial_2(\partial_2 u_3 - \partial_2 u_3) \end{pmatrix} + \begin{pmatrix} \partial_1^2 u_1 - \partial_1^2 u_1 \\ \partial_2^2 u_2 - \partial_2^2 u_2 \\ \partial_3^2 u_3 - \partial_3^2 u_3 \end{pmatrix} \\
&= \begin{pmatrix} \partial_1 \operatorname{div} \boldsymbol{u} - \Delta u_1 \\ \partial_2 \operatorname{div} \boldsymbol{u} - \Delta u_2 \\ \partial_3 \operatorname{div} \boldsymbol{u} - \Delta u_3 \end{pmatrix} = \nabla(\nabla \cdot \boldsymbol{u}) - \Delta \boldsymbol{u}
\end{aligned}
$$

This gives

$$
\begin{aligned}
\mu\varepsilon \partial_t^2 \boldsymbol{\mathcal{E}} &= \mu \partial_t (\nabla \times \boldsymbol{\mathcal{H}} - \sigma \boldsymbol{\mathcal{E}}) \\
&= \nabla \times (\mu \partial_t \boldsymbol{\mathcal{H}}) - \mu\sigma \partial_t \boldsymbol{\mathcal{E}} \\
&= -\nabla \times \nabla \times \boldsymbol{\mathcal{E}} - \mu\sigma \partial_t \boldsymbol{\mathcal{E}} \\
&= -\nabla\big(\varepsilon^{-1} \underbrace{\nabla \cdot (\varepsilon \boldsymbol{\mathcal{E}})}_{=0}\big) + \Delta \boldsymbol{\mathcal{E}} - \mu\sigma \partial_t \boldsymbol{\mathcal{E}}\,.
\end{aligned}
$$

The rest follows analogously. □

Remark 4. *This result is restricted to smooth solution in homogeneous media, i.e., we obtain only solutions for special cases!*

Harmonic plane waves Consider a scalar function $u\colon \mathbb{R}^3 \to \mathbb{C}$ with

$$Lu := \partial_t^2 u - c^2 \Delta u + \sigma \partial_t u = 0$$

Make the ansatz $u(\boldsymbol{x},t) = \exp\big(\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})\big)$ with *wave number* $\boldsymbol{k} \in \mathbb{R}^3$ and frequency ω. We have

$$\begin{aligned}\partial_t \exp\big(\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})\big) &= \mathrm{i}\omega \exp\big(\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})\big)\\ \partial_j \exp\big(\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})\big) &= -\mathrm{i}k_j \exp\big(\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})\big)\end{aligned}$$

This gives the *dispersion relation*

$$Lu = 0 \iff -\omega^2 + c|\boldsymbol{k}|^2 + \sigma \mathrm{i}\omega = 0$$

Observation: Without damping (i.e., $\sigma = 0$) we have $\omega(\boldsymbol{k}) = c|\boldsymbol{k}|$ and $c = \frac{\omega(\boldsymbol{k})}{|\boldsymbol{k}|}$.
We call a medium *non-dispersive* if the velocity of waves in the medium is not depending on the frequency ω. In fact, only the vacuum is non-dispersive, but in many cases the non-dispersive effect are neglected.

Solutions of the wave equation in 1-d We consider $\Omega \subset \mathbb{R}$ and $u\colon \Omega \to \mathbb{C}$ with $\partial_t^2 u = c^2 \partial_x^2 u$.

A) Travelling waves $u(x,t) = F(x \pm ct)$ on $\Omega = \mathbb{R}$ for a given function $F\colon \mathbb{R} \to \mathbb{R}$

$$\implies \partial_t^2\, F(x \pm ct) = c^2\, F''(x \pm ct) = c^2 \partial_x^2\, F(x \pm ct)$$

B) Harmonic waves $u(x,t) = \exp\big(\mathrm{i}(\omega t - kx)\big)$ on $\Omega = \mathbb{R}$ for $\frac{\omega}{k} = c$. We have

wave length λ	wave number $k = \frac{2\pi}{\lambda}$
frequency f	angular frequency $\omega = 2\pi f$.

C) The Cauchy problem for $\Omega = \mathbb{R}$

$$\partial_t^2 u = c^2 \partial_x u \qquad u(x,0) = u_0(x) \qquad \partial_t u(x,0) = v_0(x)$$

is explicitly solved by

$$u(x,t) = \frac{1}{2}\big(u_0(x-ct) + u_0(x+ct)\big) + \frac{1}{c}\int_{x-ct}^{x+ct} v_0(y)dy$$

Example: $u_0 \equiv 0,\ v_0(x) = \delta_0(x)$ (Delta distribution) gives

$$u(x,t) = \begin{cases} 1 & -ct \le x \le ct \\ 0 & \text{else} \end{cases}$$

D) For the Cauchy problem for $\Omega = (0,\pi)$ with homogeneous boundary conditions $u(0,t) = u(\pi,t) = 0$ for $t > 0$ we make the ansatz

$$u(x,t) = \sum_{n=1}^{\infty} \hat{u}_n(t) \sin(nx)$$

This gives

$$\sum_{n=1}^{\infty} \hat{u}_n''(t)\sin(nx) = -c^2 \sum_{n=1}^{\infty} n^2 \hat{u}_n(t)\sin(nx)\,,$$

i.e., $\hat{u}_n'' + c^2 n^2 \hat{u}_n = 0$ for $t > 0$. The Fourier representation of the initial values

$$u_0(x) = \sum_{n=1}^{\infty} a_n \sin(nx) \quad v_0(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

give $\hat{u}_n(0) = a_n$ and $\hat{u}_n'(0) = b_n$. Then we have

$$\hat{u}_n(t) = a_n \cos(cnt) + \frac{1}{nc} b_n \sin(cnt)$$

Problem 1 Find (analytically) the Fourier representation of the solution of the scalar wave equation for the initial values $u_0 \equiv 1$ and $v_0 \equiv 0$. (Note that these initial values are not compatible with the boundary conditions for $t > 0$.)

Solution of Problem 1 We have

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)dx = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

and $b_n \equiv 0$. This gives

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n \cos(cnt)\sin(nx) \\ &= \tfrac{1}{2}\sum_{n=1}^{\infty} a_n \Big(\sin(nx+nct) + \sin(nx-nct) \Big) \\ &= \tfrac{1}{2}\big(u_0(x+ct) + u_0(x-ct) \big) \end{aligned}$$

with the periodic extension of u_0

$$u_0(x) = \begin{cases} 1 & x \in (0,\pi) + 2\pi\mathbb{Z} \\ 0 & x \in \pi\mathbb{Z} \\ -1 & x \in (-\pi,0) + 2\pi\mathbb{Z} \end{cases}$$

1 Explicit numerical schemes for of the scalar wave equation

Let the wave velocity $c > 0$ be constant. Consider the second order initial value problem

$$\begin{aligned} &\partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t)\,, \qquad (x,t) \in \mathbb{R} \times \mathbb{R} \\ &u(x,0) = u_0(x)\,, \qquad \partial_t u(x,0) = v_0(x)\,, \qquad x \in \mathbb{R} \end{aligned}$$

and an approximation $(u_j^n)_{n,j} \in \mathbb{Z} \times \mathbb{Z}$ on the regular grid $(\Delta x)\mathbb{Z} \times (\Delta t)\mathbb{Z} \subset \mathbb{R} \times \mathbb{R}$, where

$\Delta x > 0$	spatial mesh size,	$x_j = j\Delta x$
$\Delta t > 0$	time step size,	$t_n = n\Delta t$

Problem 2 Let $g \in C^4(\mathbb{R})$ and $h > 0$. Prove

$$\left|g''(x) - \frac{1}{h^2}\big(g(x-h) - 2g(x) + g(x+h)\big)\right| \le \frac{h^2}{12} \max_{\xi \in [x-h,x+h]} \left|g''''(\xi)\right|.$$

Solution of Problem 2 We have the Taylor expansion

$$g(x+h) = g(x) + hg'(x) + \frac{1}{2}h^2 g''(x) + \frac{1}{6}h^3 g'''(x) + \frac{1}{24}h^4 g''''(\xi_1)$$

$$g(x-h) = g(x) - hg'(x) + \frac{1}{2}h^2 g''(x) - \frac{1}{6}h^3 g'''(x) + \frac{1}{24}h^4 g''''(\xi_2)$$

with $\xi_1 \in [x, x+h]$, $\xi_2 \in [x-h, x]$. This gives

$$g(x+h) + g(x-h) - 2g(x) = h^2 g''(x) + \frac{1}{24}h^4\big(g''''(\xi_1) + g''''(\xi_2)\big)\,.$$

This motivates the following *finite difference scheme*: Define the initial values

$$\begin{aligned} &u_j^0 = u_0(x_j) \qquad && j \in \mathbb{Z} \\ &u_j^1 = u_j^0 + \Delta t\, v_0(x_j) \qquad && j \in \mathbb{Z} \end{aligned}$$

and then, for $n = 1, 2, ...$ compute

$$\frac{1}{(\Delta t)^2}\big(u_j^{n+1} - 2u_j^n + u_j^{n-1}\big) = c^2 \frac{1}{(\Delta x)^2}\big(u_{j+1}^n - 2u_j^n + u_{j-1}^n\big) \qquad j \in \mathbb{Z} \qquad n = 2, 3, \dots$$

Problem 3 We consider the special case $\Delta t = \frac{\Delta x}{c}$ (the *magic time step*).
Show that for the magic time step a travelling wave $u(x,t) = F(x \pm ct)$ with consistent initial values

$$u_j^0 = F(j\Delta x),\ u_j^1 = F(j\Delta x \pm c\Delta t) = F\big((j \pm 1)\Delta x\big)$$

is exactly preserved.
Show that the harmonic wave $u(x,t) = \exp\big(\mathrm{i}(\omega t - kx)\big)$ with $\omega/k = \pm c$ has the same property.

Solution of Problem 3 For the magic time step $\Delta t = \frac{\Delta x}{c}$ we have

$$u_j^{n+1} - 2u_j^n + u_j^{n-1} = u_{j+1}^n - 2u_j^n + u_{j-1}^n \quad (1)$$

A) Consider a travelling wave $u(x,t) = F(x \pm ct)$ with consistent initial values

$$u_j^0 = F(j\Delta x),\ u_j^1 = F(j\Delta x \pm c\Delta t) = F\big((j \pm 1)\Delta x\big)$$

Inserting (1) gives for $n > 1$ (by induction)

$$\begin{aligned} u_j^{n+1} &= u_{j+1}^n + u_{j-1}^n - u_j^{n-1} \\ &= F\big((j+1)\Delta x \pm cn\Delta t\big) + F\big((j-1)\Delta x \pm cn\Delta t\big) - F\big(j\Delta x \pm c(n-1)\Delta t\big) \\ &= F\Big(\big(j \pm (n+1)\big)\Delta x\Big)\,. \end{aligned} \quad (2)$$

Thus, the travelling wave is exactly preserved for the magic time step.

B) The harmonic wave $u(x,t) = \exp\big(\mathrm{i}(\omega t - kx)\big) = F(x \pm ct)$ is a travelling wave with $F(x) = \exp(-\mathrm{i}kx)$, since $|\omega/k| = c$.

Program 1 Write a test program for the scalar wave equation in the interval $[0,a]$ with periodic boundary conditions. Choose $J > 1$ and set $\Delta x = a/J$. Choose the magic time step $\Delta t = \Delta x/c$. For given $u_0^0, u_1^0, ..., u_{J-1}^0, u_J^0 = x_0$ and $v_0^0, v_1^0, ..., v_{J-1}^0, v_J^0 = v_0^0$ set

$$u_j^1 = u_j^0 + \Delta t\, v_j^0\,,$$

and then compute u_j^n for $j = 0, ..., J$ and $n = 2, ..., N$ using the finite difference scheme (2).
Use the parameters $f = 1$, $\omega = 2 * \pi * f$, $k = 2 * \pi$, $c = \omega/k = 1/f$, $\lambda = c/f$ and $a = 3\lambda$.
Start with consistent initial values for the harmonic wave $u(x,t) = \sin(\omega t - kx)$.
Show that the method works fine if $N \ll J$.
Why does the method fail for large N?

Solution of Program 1 We provide a short script in *octave* (Matlab-similar syntax). Unfortunately, indexing starts with 1.

```
J = 500;
N = 20;

pi = 4 * atan(1);
f = 1;
omega = 2 * pi * f;
k = 2 * pi;
c = omega / k;
lambda = c / f;

a = 3 * lambda;
dx = a / J;
dt = dx / c;

x = [0:dx:a];
axis("manual",[0,a,-2,2])
```

```
t = 0;
u0 = sin(omega*t-k*x);
v0 = omega * cos(omega*t-k*x);

u = sin(omega*t-k*x);
plot(x,[u;u0]);

t = dt;
u1 = u0 + dt * v0;

u = sin(omega*t-k*x);
plot(x,[u;u1]);

for n=1:N
    t = t + dt;
    u = sin(omega*t-k*x);
    for j=2:J
        u2(j) = u1(j+1) + u(j-1) - u0(j);
    end;
    u2(1) = u1(2) + u(J) - u0(1);
    u2(J+1) = u2(1);
    plot(x,[u;u2]);
    u0 = u1;
    u1 = u2;
end;
```

For $N \geq 70$ oscillations start due to rounding errors, which leads to a complete failure for larger N.

Numerical dispersion

Consider a harmonic wave $u(x,t) = \exp\big(\mathrm{i}(\omega t - kx)\big)$. The equation $\partial_t^2 u = c^2 \partial_x^2 u$ gives the dispersion relation $\omega^2 = c^2 k^2$ (the *dispersion* defines a relation of k and $\omega(k)$), and we define

$$\begin{array}{lll} \text{phase velocity} & v_p = \frac{\omega(k)}{k} & (= \pm c) \\ \text{group velocity} & v_g = \frac{d\omega(k)}{dk} & (= \pm c) \end{array}$$

Note that we have $\frac{d}{dk}\omega^2 = 2\omega\frac{d\omega}{dk} = 2c^2 k$, which gives $v_p v_g = c^2$.
In particular we observe, since the harmonic wave has no dispersion, that $|v_p| = |v_g| = c$.

Now we study phase velocity and group velocity for numerical approximations of the harmonic wave. The ansatz $u_j^n = \exp\big(\mathrm{i}(\omega n\Delta t - \tilde{k} j \Delta x)\big)$ with some $\tilde{k}$ in the finite difference scheme

$$u_j^{n+1} = \big(\frac{c\Delta t}{\Delta x}\big)^2 (u_{j+1}^n - 2u_j^n + u_{j-1}^n) + 2u_j^n - u_j^{n-1} \tag{3}$$

yields

$$\begin{aligned} \exp\big(\mathrm{i}(\omega(n+1)\Delta t - \tilde{k} j\Delta x)\big) &= \big(\tfrac{c\Delta t}{\Delta x}\big)^2 \exp\big(\mathrm{i}(\omega n\Delta t - \tilde{k} j\Delta x)\big)\ \big(\exp(-\mathrm{i}\tilde{k}\Delta x) - 2 + \exp(\mathrm{i}\tilde{k}\Delta x)\big) \\ &\quad + 2\exp\big(\mathrm{i}(\omega n\Delta t - \tilde{k} j\Delta x)\big) - \exp\big(\mathrm{i}(\omega(n-1)\Delta t - \tilde{k} j\Delta x)\big) \end{aligned}$$

which gives the *numerical dispersion relation*

$$\cos(\omega\Delta t) - 1 = \big(\frac{c\Delta t}{\Delta x}\big)^2\big(\cos(\tilde{k}\Delta x) - 1\big)\,.$$

For the magic time step $\Delta t = \frac{\Delta x}{c}$ we obtain $\tilde{k} = k$. In case of $\Delta t \neq \frac{\Delta x}{c}$ we obtain *numerical dispersion*: Consider $\Delta x = ac\Delta t$ and $\Delta t \to 0$

$$\begin{aligned}
&\implies\ 1 - \tfrac{1}{2}(\omega\Delta t)^2 + O(\Delta t^4) - 1 = \tfrac{1}{a^2}\Big(1 - \tfrac{1}{2}\big(\tilde{k}ac\Delta t\big)^2 + O(\Delta x^4) - 1\Big)\\
&\implies\ \tilde{k} = \pm\tfrac{\omega}{c} + O(\Delta t^2),\ \tfrac{\omega(\tilde{k})}{\tilde{k}} = \pm c + O(\Delta t^2)
\end{aligned}$$

This gives asymptotically vanishing numerical dispersion. From

$$\begin{aligned}
&\cos\big(\omega(\tilde{k})\Delta t\big) - 1 = \big(\frac{c\Delta t}{\Delta x}\big)^2\big(\cos(\tilde{k}\Delta x) - 1\big)\\
&\underset{\frac{d}{d\tilde{k}}}{\implies}\ \frac{d\omega(\tilde{k})}{d\tilde{k}}\sin\big(\omega(\tilde{k})\Delta t\big) = c^2\frac{\Delta t}{\Delta x}\sin(\tilde{k}\Delta x)
\end{aligned}$$

we obtain for the group velocity

$$\frac{d\omega(\tilde{k})}{d\tilde{k}} = \begin{cases} \pm c & \text{for} \quad a = 1 \quad \text{(magic time step)}\\ \pm c + O(\Delta t^2) & \text{else}\end{cases}$$

Problem 4 Consider $\Delta x = \frac{\lambda}{10}$ and $\Delta t = \frac{1}{2}\frac{\Delta x}{c}$. Compute an approximation of the numerical phase velocity.

Solution of Problem 4 We have $k = \frac{2\pi}{\lambda}$ and $\omega\Delta t = kc\Delta t = k\frac{\Delta x}{2}$. The approximate solution of the equation $\cos(2\frac{2\pi}{\lambda}\Delta x) - 1 = \frac{1}{4}\,\big(\cos(\tilde{k}\Delta x) - 1\big)$ gives

$$\begin{aligned}
&\tilde{k} \approx \tfrac{0.636}{\Delta x} \qquad \big(\tfrac{\lambda}{\Delta x} = 10\big)\\
\implies\ &\tilde{v}_p = \tfrac{\omega}{\tilde{k}} \approx \tfrac{2\pi f}{0.636/\Delta x} = \tfrac{2\pi\frac{c}{\lambda}\Delta x}{0.636} \approx 0.987c \qquad \text{numerical phase velocity}
\end{aligned}$$

Stability

The harmonic wave is bounded for all times. We observe that this does not hold for the discrete scheme above, if the time step is too large.

Problem 5 Show that for all $q > 0$ the ansatz

$$u_j^n = q^n \exp(\mathrm{i}kj\Delta x)$$

is a solution of the finite difference scheme with some $\Delta t > 0$.
Show that $q \leq 1$ for $\Delta t \leq \frac{\Delta x}{c}$.

Solution of Problem 5 The ansatz gives

$$\begin{aligned}
&\tfrac{1}{(\Delta t)^2}(q^{n+1}-2q^n+q^{n-1})\exp(\mathrm{i}kj\Delta x)\\
&=c^2\tfrac{1}{(\Delta x)^2}q^n\big(\exp(\mathrm{i}k(j+1)\Delta x)-2\exp(\mathrm{i}kj\Delta x)+\exp\big(\mathrm{i}k(j-1)\Delta x\big),\\
\Longrightarrow\quad &q^2-2q+1=\tfrac{2c^2(\Delta t)^2}{(\Delta x)^2}q\big(\cos(k\Delta x)-1\big)\\
\Longrightarrow\quad &q^2-(2+(\Delta t)^2\Lambda_x)q+1=0
\end{aligned}$$

for

$$\Lambda_x=\frac{2c^2}{(\Delta x)^2}\big(\cos(k\Delta x)-1\big)\in\Big[\frac{-4c^2}{(\Delta x)^2},0\Big]\,,$$

so that we have

$$q=p\pm\sqrt{p^2-1}\text{ with }p=\frac{2+\Lambda_x(\Delta t)^2}{2}\,.$$

This yields $|q|\le 1$ for $p^2-1\le 0$, i.e., $-1\le p\le 1$

$$\begin{aligned}
\Longrightarrow\quad &-\tfrac{4}{(\Delta t)^2}\le\Lambda_x\le 0\\
\Longrightarrow\quad &-\tfrac{4}{(\Delta t)^2}\le-\tfrac{4c^2}{(\Delta x)^2}\iff\Delta t\le\tfrac{\Delta x}{c}\,.
\end{aligned}$$

Consequence The magic time step is the stability bound for the numerical scheme!
The interpretation for a point source $u_0(x)=\delta_0(x)$ and $v\equiv 0$ yields:

CFL-condition (Courant-Friedrichs-Lewy 1929) Stability of a finite difference scheme requires that the domain of dependence of the continuous solution contains the numerical domain of dependence.

Program 2 Modify Program 1 for $\Delta t<\Delta x/c$ using the finite difference scheme (3).
Show that the method works for $J\approx N$.
Show that the error at a fixed time $T=N\Delta t$ is reduced by decreasing Δt (and thus increasing N).

Solution of Program 2

```
J = 50;
N = 100;
fac = 0.125;

pi = 4 * atan(1);
f = 1;
omega = 2 * pi * f;
k = 2 * pi;
c = omega / k;
lambda = c / f;
```

```
a = 3 * lambda;
dx = a / J;
dt = fac * dx / c;
q = c*c*dt*dt/(dx*dx);

x = [0:dx:a];
axis("manual",[0,a,-2,2])

t = 0;
u0 = sin(omega*t-k*x);
v0 = omega * cos(omega*t-k*x);

u = sin(omega*t-k*x);
plot(x,[u;u0]);

t = dt;
u1 = u0 + dt * v0;

u = sin(omega*t-k*x);
plot(x,[u;u1]);

for n=1:N
    t = t + dt;
    for j=2:J
        u2(j) = q * (u1(j+1)-2*u1(j)+u(j-1)) + 2*u1(j) - u0(j);
    end;
    u2(1) = q * (u1(2)-2*u1(1)+u(J)) + 2*u1(1) - u0(1);
    u2(J+1) = u2(1);

    u = sin(omega*t-k*x);
    plot(x,[u;u2]);

    u0 = u1;
    u1 = u2;
end;
```

For N = 100 and fac = 0.125 the error gets smaller.

Symplectic schemes

Consider the space of grid functions $V_\Delta = \{(u_j)_{j\in\mathbb{Z}}\}$, and let

$$\begin{aligned} u = (u^n)\colon \mathbb{Z} &\to V_\Delta \\ n &\mapsto (u_j^n) \end{aligned}$$

be the discrete time approximation. We define the spatial finite difference operator

$$(\partial^2_{\Delta x} u^n)_j := \frac{1}{(\Delta x)^2}(u^n_{j+1} - 2u^n_j + u^n_{j-1}) \qquad L_\Delta = c^2 \partial^2_{\Delta x}$$

The scheme above reads $u^{n+1}-2u^n+u^{n-1}=(\Delta t)^2\,L_\Delta u^n$. We give now another interpretation of this scheme, where we split the three term recurrence into two steps. Therefore, we introduce $v^{n+\frac{1}{2}}=\frac{1}{\Delta t}(u^{n+1}-u^n)$.
The Stömer-Verlet scheme (leap-frog method) reads:

$$v^{n+1/2} = v^{n-1/2}+\Delta t\,L_\Delta u^n \tag{4a}$$
$$u^{n+1} = u^n+\Delta t\,v^{n+1/2}\,. \tag{4b}$$

Problem 6 Consider periodic solutions with $u(0)=u(a)$ and $v(0)=v(a)$. Show that the Hamiltonian

$$H(v,u)=\int_0^a \big(\frac{1}{2}|v|^2+\frac{1}{2}c^2|\partial_x u|^2\big)\,dx$$

remains constant along a periodic solution of the wave equation.

Solution of Problem 6 Inserting $v=\partial_t u$ and integration by parts gives

$$\begin{aligned}\partial_t H(v,u) &= \int_0^a l\big(v\partial_t v+c^2\partial_x u\partial_t\partial_x u\big)\,dx\\ &= \int_0^a l\big(\partial_t u\partial_t^2 u-c^2\partial_t u\partial_x^2 u\big)\,dx=0\,.\end{aligned}$$

Program 3 Modify Program 2 for the leap frog scheme (4), starting with

$$v^{1/2} = v^0+\frac{1}{2}\Delta t\,L_\Delta u^0\,.$$

Show that the method works for $J\ll N$, if $\Delta t<\Delta x/c$
Show the effect of numerical dispersion.

Solution of Program 3

```
J = 50;
N = 400;
fac = 0.8;

pi = 4 * atan(1);
f = 1;
omega = 2 * pi * f;
k = 2 * pi;
c = omega / k;
lambda = c / f;

a = 3 * lambda;
dx = a / J;
dt = fac * dx / c;
q = c*c/(dx*dx);

x = [0:dx:a];
```

```
axis("manual",[0,a,-2,2])

t = 0;
u0 = sin(omega*t-k*x);
v0 = omega * cos(omega*t-k*x);
u_n = u0;

u = sin(omega*t-k*x);
plot(x,[u;u_n]);

t = 0.5 * dt;
for j=2:J
    v_n(j) = v0(j) + 0.5 * dt * q * (u_n(j+1)-2*u_n(j)+u_n(j-1));
    end;
v_n(1) = v0(1) + 0.5 * dt * q * (u_n(2)-2*u_n(1)+u_n(J));
v_n(J+1) = v_n(1);

for n=1:N
    t = t + 0.5*dt;
    u = sin(omega*t-k*x);
    u_n = u_n + dt * v_n;
    plot(x,[u;u_n]);
    t = t + 0.5*dt;
    for j=2:J
        v_n(j) = v_n(j) + dt * q * (u_n(j+1)-2*u_n(j)+u_n(j-1));
    end;
    v_n(1) = v_n(1) + dt * q * (u_n(2)-2*u_n(1)+u_n(J));
    v_n(J+1) = v_n(1);
end;
```

The Yee scheme for the wave equation in 1D

Another possibility for the construction of stable schemes are obtained by a reformulation as a first order system. Consider

$$\partial_t \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ w \end{pmatrix} ,$$

i.e., $\partial_t u = c\partial_x w$ and $\partial_t w = c\partial_x u$. This gives

$$\partial_t^2 u = c\partial_x\partial_t w = c^2\partial_x^2 u \, .$$

Now we consider a finite interval $[0, a]$ and homogeneous boundary conditions

$$u(0,t) = u(a,t) = 0, \qquad t > 0$$

and initial conditions

$$u(x,0) = u_0(x) \, , \qquad w(x,0) = w_0(x) \, , \qquad x \in (0,a) \, .$$

This system has a first integral $I(u,w) = \frac{1}{2}\int_0^a (|u|^2 + |w|^2)dx$, since the homogeneous boundary conditions and integration by parts yield

$$\begin{aligned}\partial_t I(u,w) &= \int_0^a (u\partial_t u + w\partial_t w)\,dx \\ &= \int_0^a (c\partial_x w u + c\partial_x u\, w) = 0\,.\end{aligned}$$

A semi-discrete scheme

First we discuss a scheme which is discrete in space but continuous in time:

$$\begin{aligned}\partial_t w_{j+1/2} &= \frac{c}{\Delta x}(u_{j+1} - u_j) \\ \partial_t u_j &= \frac{c}{\Delta x}(w_{j+1/2} - w_{j-1/2})\end{aligned}$$

Here, w is approximated at $x_{j+1/2} = (j+1/2)\Delta x$ and u is approximated at $x_j = j\Delta x$.

Problem 7 Consider in $[0,a]$ with $\Delta x = a/J$ and $u_0(t) \equiv u_J(t) \equiv 0$. Show that the semi-discrete scheme preserves the quantity

$$I_\Delta(u,w) = \sum_{j=1}^{J} \Delta x(|u_j|^2 + |w_{j-1/2}|^2)\,.$$

Solution of Problem 7 Summation by parts and periodic extension gives

$$\frac{d}{dt} I_\Delta(u,w) = \sum_{j=1}^{J} \frac{\Delta x}{\Delta x}\Big(u_j c(w_{j+1/2} - w_{j-1/2}) + w_{j-1/2} c(u_j - u_{j-1}\Big) = 0\,.$$

A staggered scheme in space and time

A fully discrete scheme uses in addition the staggered time discretization (cf. the leap frog scheme)

$$w_{j+1/2}^{n+1/2} = w_{j+1/2}^{n-1/2} + c\frac{\Delta t}{\Delta x}(u_{j+1}^n - u_j^n) \tag{5a}$$

$$u_j^{n+1} = u_j^n + c\frac{\Delta t}{\Delta x}(w_{j+1/2}^{n+1/2} - w_{j-1/2}^{n+1/2})\,. \tag{5b}$$

Program 4 Modify Program 3 for the Yee scheme (5) and for homogeneous boundary conditions, starting with

$$\begin{aligned}u_j^0 &= 1, \qquad j = 1, ..., J-1 \\ w_{j-1/2}^{1/2} &= 0, \qquad j = 1, ..., J-1\end{aligned}$$

and $u_0^0 = u_J^0 = 0$. Compare the numerical solution with the analytical solution in Problem 1.

Solution of Program 4

```
J = 100;
N = 500;
fac = 0.5;

pi = 4 * atan(1);
f = 1;
omega = 2 * pi * f;
k = 2 * pi;
c = omega / k;
lambda = c / f;

a = 3 * lambda;
dx = a / J;
dt = fac * dx / c;
q = c * dt / dx;

x = [0:dx:a];
axis("manual",[0,a,-2,2])

t = 0;
u = x;
w = u;

for j=1:J
    u(j) = 1;
    w(j) = 0;
end;
u(1) = 0;
u(J+1) = 0;

plot(x,u);

for n=1:N
    t = t + dt;
    for j=2:J
        u(j) = u(j) + q * (w(j) - w(j-1));
    end;
    for j=1:J
        w(j) = w(j) + q * (u(j+1) - u(j));
    end;
    plot(x,u);
end;
```

Now, the reader should be prepared to read [5] for a full numerical analysis of this problem.

2 The FDTD method for Maxwell's equations

Before we start to derive a numerical scheme, we summarize some analytical results.
Let $\Omega \subset \mathbb{R}^3$ be a domain, $\varepsilon, \mu \in L_\infty(\Omega, \mathbb{R}^{3,3})$ with

$$\xi^T \varepsilon(x) \xi \geq \varepsilon_0 |\xi|^2, \qquad \xi^T \mu(x) \xi \geq \mu_0 |\xi|^2, \qquad \forall \xi \in \mathbb{R}^3, \ \varepsilon_0, \mu_0 > 0. \tag{6}$$

Consider the Maxwell system

$$\begin{aligned} \varepsilon\, \partial_t \boldsymbol{\mathcal{E}} - \nabla \times \boldsymbol{\mathcal{H}} &= 0 \\ \mu\, \partial_t \boldsymbol{\mathcal{H}} + \nabla \times \boldsymbol{\mathcal{E}} &= 0 \end{aligned}$$

with $\sigma \equiv 0$ (no conductivity) subject to boundary and initial conditions

$$\boldsymbol{\mathcal{E}}(\boldsymbol{x}, t) \times \boldsymbol{n}(\boldsymbol{x}) = 0, \qquad (\boldsymbol{x} \in \partial\Omega), \qquad \boldsymbol{\mathcal{E}}(\boldsymbol{x}, 0) = \boldsymbol{\mathcal{E}}_0(\boldsymbol{x}), \ \boldsymbol{\mathcal{H}}(\boldsymbol{x}, 0) = \boldsymbol{\mathcal{H}}_0(\boldsymbol{x}) \quad (\boldsymbol{x} \in \Omega).$$

Define $U(t) = \begin{pmatrix} \boldsymbol{\mathcal{E}}(t) \\ \boldsymbol{\mathcal{H}}(t) \end{pmatrix} \in X := L_2(\Omega, \mathbb{C}^3) \times L_2(\Omega, \mathbb{C}^3)$, set $U_0 = \begin{pmatrix} \boldsymbol{\mathcal{E}}_0 \\ \boldsymbol{\mathcal{H}}_0 \end{pmatrix}$. We use the inner product $\langle U, V \rangle_X = \int_\Omega U \cdot M\, \bar{V}\, dx$ with $M = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \in L_\infty(\Omega, \mathbb{R}^{6,6})$.

Lemma 5. *The operator* $A = \mathrm{i}\, M^{-1} \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl} & 0 \end{pmatrix}$ *is self-adjoint in* X *with domain*

$$D(A) = H_0(\operatorname{curl}, \Omega) \times H(\operatorname{curl}, \Omega),$$

where

$$\begin{aligned} H(\operatorname{curl}, \Omega) &= \{\boldsymbol{u} \in L_2(\Omega, \mathbb{C}^3) \colon \operatorname{curl} \boldsymbol{u} \in L_2(\Omega, \mathbb{C}^3)\}, \\ H_0(\operatorname{curl}, \Omega) &= \{\boldsymbol{u} \in H(\operatorname{curl}, \Omega) \colon \boldsymbol{u} \times \boldsymbol{n} = 0 \textit{ on } \partial\Omega\}. \end{aligned}$$

Proof. For vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{C}^3$ we have $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = (\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{b}$, for vector fields $\boldsymbol{u}, \boldsymbol{v} \colon \Omega \to \mathbb{C}^3$ we have $\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\nabla \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v})$. Thus, the Gauß theorem gives

$$\int_\Omega \boldsymbol{v} \cdot (\nabla \times \boldsymbol{u}) dx - \int_\Omega \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v}) dx = \int_\Omega \nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) dx = \int_{\partial\Omega} (\boldsymbol{u} \times \boldsymbol{v}) \cdot da = \int_{\partial\Omega} \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{n}) da$$

which implies for $\boldsymbol{u} \in H(\operatorname{curl}, \Omega)$ and $\boldsymbol{v} \in H_0(\operatorname{curl}, \Omega)$

$$\int_\Omega \boldsymbol{v} \cdot \operatorname{curl} \boldsymbol{u}\, dx = \int_\Omega \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{v}\, dx.$$

Thus, we have

$$\begin{aligned} \langle A\, U, V \rangle_X &= \mathrm{i} \int_\Omega \big((-\operatorname{curl} U_2) \cdot \bar{V}_1 + \operatorname{curl} U_1 \cdot \bar{V}_2 \big) dx \\ &= -\mathrm{i} \int_\Omega (U_2 \cdot \operatorname{curl} \bar{V}_1 - U_1 \cdot \operatorname{curl} \bar{V}_2) dx = \langle U, A\, V \rangle_X, \end{aligned}$$

i.e., A is symmetric. Thus, we have $A \subset A^*$, where $A^* \colon D(A^*) \to X$ is defined on $D(A^*) = \{U \in X \colon$ there exists $V \in X$ with $\langle V, W \rangle_X = \langle U, AW \rangle \quad \forall\, W \in D(A)\}$; then set

$A^*U = V$ (this defines V uniquely, since $D(A)$ is dense in X).
We finally have to prove $D(A^*) = D(A)$. For $U \in D(A^*)$ we consider

$$W = \begin{pmatrix} \varepsilon \boldsymbol{\mathcal{E}} \\ 0 \end{pmatrix} \in D(A) \quad \Longrightarrow \quad \int_\Omega V_1 \cdot \varepsilon \bar{\boldsymbol{\mathcal{E}}} dx = -\mathrm{i} \int_\Omega U_2 \cdot \operatorname{curl} \bar{\boldsymbol{\mathcal{E}}}\, dx\,.$$

Thus, the weak derivative $\operatorname{curl} U_2$ is in L_2 and $\varepsilon V_1 = -\mathrm{i} \operatorname{curl} U_2$. Analogously, we get $\mu V_2 = \mathrm{i} \operatorname{curl} U_1$ and $U \times \boldsymbol{n} = 0$ on $\partial\Omega$. □

The Maxwell Cauchy problem is of the form

$$\dot{U} + i\, A\, U = 0 \qquad U(0) = U_0\,.$$

It has a unique solution, and since A is self-adjoint, it has the representation

$$U(t) = \exp(-\mathrm{i}A\, t) U_0 = \int_{-\infty}^{\infty} \exp(-\mathrm{i}\lambda\, t) dP(\lambda) U_0\,.$$

The energy $E(t) := \|U(t)\|_X^2 \equiv E(0)$ is conserved, since we have

$$\partial_t E(t) = \langle \partial_t U, U\rangle + \langle U, \partial_t U\rangle = -\mathrm{i}\langle AU, U\rangle + \mathrm{i}\langle U, AU\rangle = 0\,.$$

Problem 8 Prove that the kernel $N(A) = \{U \in M \colon AU = 0\}$ is infinite-dimensional.

Solution of Problem 8 Choose open sets $\omega_n \subset \Omega$ with $|\omega_n| < \frac{1}{n}$ $(n \in \mathbb{N})$ and such that $\omega_n \cap \omega_m = \emptyset$ for $n \neq m$. Choose $\phi_n \in C_0^\infty(\omega_n)$ such that $U_n = \begin{pmatrix} \nabla\, \phi_n \\ 0 \end{pmatrix}$ is normalized with $\|U_n\| = 1$. This gives $\operatorname{curl} \nabla \phi_n = 0$ and thus $AU_n = 0$. Since $(U_n)_{n \in \mathbb{N}}$ is orthonormal in X, we have infinitely many linear independent kernel vectors.

The semi-discrete scheme

Now we consider the Finite-Difference-Time-Domain method (introduced 1966 by K.S. Yee) for the system

$$\begin{aligned} \varepsilon\, \partial_t \boldsymbol{\mathcal{E}} - \nabla \times \boldsymbol{\mathcal{H}} &= \boldsymbol{J} \\ \mu\, \partial_t \boldsymbol{\mathcal{H}} + \nabla \times \boldsymbol{\mathcal{E}} &= 0\,. \end{aligned}$$

For $\Delta x > 0$ define the grid $\mathcal{Z}_\Delta = \Delta x \mathbb{Z}^3$. Set $x_j = j\, \Delta x$.
We consider Maxwell's equations on the domain $\Omega = (0, J_1\Delta x) \times (0, J_2\Delta x) \times (0, J_3\Delta x)$ with absorbing boundary conditions $\boldsymbol{\mathcal{E}} \times \boldsymbol{n} = 0$ on $\partial\Omega$. The grid points $\mathcal{Z}_\Delta \cap \bar{\Omega}$ are the corners of a uniform hexahedral mesh with $J_1 J_2 J_3$ cells.
We assume that the solution $\big(\boldsymbol{\mathcal{E}}(t), \boldsymbol{\mathcal{H}}(t)\big) = \big(\mathcal{E}_1(t), \mathcal{E}_2(t), \mathcal{E}_3(t), \mathcal{H}_1(t), \mathcal{H}_2(t), \mathcal{H}_3(t)\big)$ of the continuous problem is sufficiently smooth. We compute approximations $\boldsymbol{E} = (E_1, E_2, E_3)$ on edge midpoints and $\boldsymbol{H} = (H_1, H_2, H_3)$ on face midpoints of $\boldsymbol{\mathcal{H}}$, i.e., approximations

$E_{i+1/2,\, j,\, k}(t)$	of	$\mathcal{E}_1(x_{i+1/2},\ x_j,\ x_k,\ t)$
$E_{i,\, j+1/2,\, k}(t)$	of	$\mathcal{E}_2(x_i,\ x_{j+1/2},\ x_k,\ t)$
$E_{i,\, j,\, k+1/2}(t)$	of	$\mathcal{E}_3(x_i,\ x_j,\ x_{k+1/2},\ t)$
$H_{i,\, j+1/2,\, k+1/2}(t)$	of	$\mathcal{H}_1(x_i,\ x_{j+1/2},\ x_{k+1/2},\ t)$
$H_{i+1/2,\, j,\, k+1/2}(t)$	of	$\mathcal{H}_2(x_{i+1/2},\ x_j,\ x_{k+1/2},\ t)$
$H_{i+1/2,\, j+1/2,\, k}(t)$	of	$\mathcal{H}_3(x_{i+1/2},\ x_{j+1/2},\ x_k,\ t)\,,$

by evaluating Maxwell's equations in integral form with midpoint quadrature formulae.

The application of Faraday's law $\partial_t \int_A \mu \boldsymbol{\mathcal{H}} \cdot da = -\int_{\partial A} \boldsymbol{\mathcal{E}} \cdot dl$ with the face

$$A = A_{i,j+1/2,k+1/2} := \{x_i\} \times [x_j, x_{j+1}] \times [x_k, x_{k+1}]$$

together with midpoint quadrature gives

$$(\Delta x)^2 \mu_{i,j+1/2,k+1/2} \partial_t H_{i,j+1/2,k+1/2} = \Delta x(-E_{i,j+1,k+1/2} + E_{i,j,k+1/2} - E_{i,j+1/2,k} + E_{i,j+1/2,k+1}) .$$

This motivates the definition of the finite difference approximation of the curl component in direction of the normal $\boldsymbol{n}_{i,j+1/2,k+1/2}$ of $A_{i,j+1/2,k+1/2}$ at the face midpoint $(x_i, x_{j+1/2}, x_{k+1/2})$ by

$$(\Delta x)^2 \operatorname{curl}_{i,j+1/2,k+1/2} E = \Delta x(E_{i,j+1,k+1/2} - E_{i,j,k+1/2} + E_{i,j+1/2,k} - E_{i,j+1/2,k+1}) .$$

Thus, we can rewrite the finite difference equation as

$$\mu_{i,j+1/2,k+1/2} \partial_t H_{i,j+1/2,k+1/2} = -\operatorname{curl}_{i,j+1/2,k+1/2} E .$$

In the same way, the application to the face $A_{i+1/2,j,k+1/2} = [x_i, x_{i+1}] \times \{x_j\} \times [x_k, x_{k+1}]$ gives

$$(\Delta x)^2 \mu_{i+1/2,j,k+1/2} \partial_t H_{i+1/2,j,k+1/2} = \Delta x(-E_{i,j,k+1/2} + E_{i+1,j,k+1/2} - E_{i+1/2,j,k+1} + E_{i+1/2,j,k}) ,$$

which rewrites (by introducing a discrete curl component) as

$$\mu_{i+1/2,j,k+1/2} \partial_t H_{i+1/2,j,k+1/2} = -\operatorname{curl}_{i+1/2,j,k+1/2} E .$$

Analogously, we obtain

$$(\Delta t)^2 \mu_{i+1/2,j+1/2,k} \partial_t H_{i+1/2,j+1/2,k} = \Delta t(-E_{i+1,j+1/2,k} + E_{i,j+1/2,k} - E_{i+1/2,j,k} + E_{i+1/2,j+1,k}) ,$$

i. e.,

$$\mu_{i+1/2,j+1/2,k} \partial_t H_{i+1/2,j+1/2,k} = -\operatorname{curl}_{i+1/2,j+1/2,k} E .$$

Now, the application to Ampère's law $\int_A (\varepsilon \partial_t \boldsymbol{\mathcal{E}} - J) \cdot da = \int_{\partial A} \boldsymbol{\mathcal{H}} \cdot dl$ to the surface

$$A_{i+1/2,j,k} = \{x_{i+1/2}\} \times [x_{j+1/2}, x_{j+1/2}] \times [x_{k-1/2}, x_{k+1/2}]$$

(orthogonal to the edge midpoint $(x_{i+1/2}, x_j, x_k)$) gives

$$(\Delta x)^2 (\varepsilon_{i+1/2,j,k} \partial_t E_{i+1/2,j,k} - J_{i+1/2,j,k}) = \Delta x(H_{i+1/2,j+1/2,k} - H_{i+1/2,j-1/2,k} + H_{i+1/2,j,k-1/2} - H_{i+1/2,j,k+1/2}) .$$

As above, we write

$$\varepsilon_{i+1/2,j,k} \partial_t E_{i+1/2,j,k} - J_{i+1/2,j,k} = \operatorname{curl}_{i+1/2,j,k} H ,$$

and integration along $A_{i,j+1/2,k}$ and $A_{i+1/2,j,k+1/2}$ gives

$$\begin{aligned} \varepsilon_{i,j+1/2,k} \partial_t E_{i,j+1/2,k} - J_{i,j+1/2,k} &= \operatorname{curl}_{i,j+1/2,k} H , \\ \varepsilon_{i,j+1/2,k} \partial_t E_{i,j+1/2,k} - J_{i,j+1/2,k} &= \operatorname{curl}_{i,j+1/2,k} H . \end{aligned}$$

We set $E_{\alpha,\beta,\gamma} = 0$ for all boundary points $(x_\alpha, x_\beta, x_\gamma) \in \partial\Omega$.

Discrete compatibility

We have $\int_V \operatorname{div} \boldsymbol{J}\, dx = \int_{\partial V} \boldsymbol{J} \cdot da$. Evaluating this for the volume

$$V = (x_{i-1/2}, x_{i+1/2}) \times (x_{j-1/2}, x_{j+1/2}) \times (x_{k-1/2}, x_{k+1/2})$$

motivates the following definition for the discrete divergence at (x_i, x_j, x_k) by

$$(\Delta x)^3 \operatorname{div}_{i,j,k} \boldsymbol{J} = (\Delta x)^2 (J_{i+1/2,j,k} - J_{i-1/2,j,k} + J_{i,j+1/2,k} - J_{i,j-1/2,k} + J_{i,j,k+1/2} - J_{i,j,k-1/2}) \,.$$

It can be shown that for compatible data $\operatorname{div}_{i,j,k} \boldsymbol{J} = 0$ and compatible initial values $\operatorname{div}_{i,j,k} \varepsilon E(0) = 0$ and $\operatorname{div}_{i+1/2,j+1/2,k+1/2} \mu H(0) = 0$ we have

$$\operatorname{div}_{i,j,k} \varepsilon E(t) = 0, \qquad \operatorname{div}_{i+1/2,j+1/2,k+1/2} \mu H(t) = 0$$

for all $t > 0$.

Convergence

It can be shown (for suitable norms) that we have

$$\|\boldsymbol{\mathcal{E}}(t) - \boldsymbol{E}(t)\|_E + \|\boldsymbol{\mathcal{H}}(t) - \boldsymbol{H}(t)\|_H \le (C_0 + t\, C_1)\Delta x$$

for $t \in [0, T]$ and C_0, C_1 depending on T (see [8]).

Problem 9 Construct the corresponding fully discrete scheme.

Solution of Problem 9 Again, together with leap frog we get the scheme

$$\begin{aligned}
\mu_{i,j+1/2,k+1/2} H^{n+1/2}_{i,j+1/2,k+1/2} &= \mu_{i,j+1/2,k+1/2} H^{n-1/2}_{i,j+1/2,k+1/2} - \Delta t \operatorname{curl}_{i,j+1/2,k+1/2} E^n \,, \\
\mu_{i+1/2,j,k+1/2} H^{n+1/2}_{i+1/2,j,k+1/2} &= \mu_{i+1/2,j,k+1/2} H^{n-1/2}_{i+1/2,j,k+1/2} - \Delta t \operatorname{curl}_{i+1/2,j,k+1/2} E^n \,, \\
\mu_{i+1/2,j+1/2,k} H^{n+1/2}_{i+1/2,j+1/2,k} &= \mu_{i+1/2,j+1/2,k} H^{n-1/2}_{i+1/2,j+1/2,k} - \Delta t \operatorname{curl}_{i+1/2,j+1/2,k} E^n \,, \\
\varepsilon_{i+1/2,j,k} E^{n+1}_{i+1/2,j,k} &= \varepsilon_{i+1/2,j,k} E^{n}_{i+1/2,j,k} + J^{n+1/2}_{i+1/2,j,k} + \Delta t \operatorname{curl}_{i+1/2,j,k} H^{n+1/2} \,, \\
\varepsilon_{i,j+1/2,k} E^{n+1}_{i,j+1/2,k} &= \varepsilon_{i,j+1/2,k} E^{n}_{i,j+1/2,k} + J^{n+1/2}_{i,j+1/2,k} + \Delta t \operatorname{curl}_{i,j+1/2,k} H^{n+1/2} \,, \\
\varepsilon_{i,j+1/2,k} E^{n+1}_{i,j+1/2,k} &= \varepsilon_{i,j+1/2,k} E^{n}_{i,j+1/2,k} + J^{n+1/2}_{i,j+1/2,k} + \Delta t \operatorname{curl}_{i,j+1/2,k} H^{n+1/2} \,.
\end{aligned}$$

Remark

The error is (slowly) growing in time, so that the application to photonics (where we have small wave lengths) is very restrictive. On the other hand, it is extremely difficult to construct schemes with better properties, so that the spectral analysis of the Maxwell operator is the only safe method for a qualitative numerical analysis of photonic waves.

3 Time-harmonic solutions and eigenfrequencies

We insert into the Maxwell system

$$\begin{aligned} \varepsilon\partial_t\boldsymbol{\mathcal{E}} - \operatorname{curl}\boldsymbol{\mathcal{H}} &= 0 \qquad & \mu\partial_t\boldsymbol{\mathcal{H}} + \operatorname{curl}\boldsymbol{\mathcal{E}} &= 0 \\ \operatorname{div}(\varepsilon\boldsymbol{\mathcal{E}}) &= 0 & \operatorname{div}(\mu\boldsymbol{\mathcal{H}}) &= 0 \end{aligned}$$

a time-harmonic ansatz for a monochromatic wave with frequency ω:

$$\boldsymbol{\mathcal{E}}(\boldsymbol{x},t) = \boldsymbol{E}(\boldsymbol{x})\exp(\mathrm{i}\omega t)\,, \qquad \boldsymbol{\mathcal{H}}(\boldsymbol{x},t) = \boldsymbol{H}(\boldsymbol{x})\exp(\mathrm{i}\omega t)\,.$$

This gives

$$\partial_t\boldsymbol{\mathcal{E}} = \mathrm{i}\omega\boldsymbol{\mathcal{E}}\,, \qquad \partial_t\boldsymbol{\mathcal{H}} = \mathrm{i}\omega\boldsymbol{\mathcal{H}}\,, \qquad \operatorname{curl}\boldsymbol{\mathcal{E}} = \exp(\mathrm{i}\omega t)\operatorname{curl}\boldsymbol{E}\,,$$

and from $\mathrm{i}\omega\,\varepsilon\boldsymbol{\mathcal{E}} - \operatorname{curl}\boldsymbol{\mathcal{H}} = 0$, $\mathrm{i}\omega\,\mu\boldsymbol{\mathcal{H}} + \operatorname{curl}\boldsymbol{\mathcal{E}} = 0$ we obtain

$$\begin{aligned} \mathrm{i}\omega\,\varepsilon\boldsymbol{E} - \operatorname{curl}\boldsymbol{H} &= 0 \quad & \mathrm{i}\omega\,\mu\boldsymbol{H} + \operatorname{curl}\boldsymbol{E} &= 0 \\ \operatorname{div}(\varepsilon\boldsymbol{H}) &= 0 & \operatorname{div}(\mu\boldsymbol{H}) &= 0\,. \end{aligned} \tag{7}$$

Now, $\operatorname{curl}\varepsilon^{-1}\operatorname{curl}\boldsymbol{H} = \mathrm{i}\omega\operatorname{curl}\boldsymbol{E} = \mathrm{i}\omega(-i\omega)\mu\boldsymbol{H} = \omega^2\mu\boldsymbol{H}$ yields a decoupled system

$$\begin{aligned} \operatorname{curl}\varepsilon^{-1}\operatorname{curl}\boldsymbol{H} &= \omega^2\mu\boldsymbol{H} \qquad & \operatorname{div}(\mu\boldsymbol{H}) &= 0 \\ \operatorname{curl}\mu^{-1}\operatorname{curl}\boldsymbol{E} &= \omega^2\varepsilon\boldsymbol{E} & \operatorname{div}(\varepsilon\boldsymbol{E}) &= 0 \end{aligned}$$

These are eigenvalue problems with eigenvalue $\lambda = \omega^2$. Thus, it suffices to find Maxwell eigenfunctions and corresponding eigenvalues in order to construct time-harmonic solutions of the full Maxwell system.

The Maxwell eigenvalue problem

Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain, and consider the spaces

$$\begin{aligned} &H_0(\operatorname{curl},\Omega) = \{\boldsymbol{u} \in L^2(\Omega,\mathbb{C}^3)\colon\ \operatorname{curl}\boldsymbol{u} \in L^2(\Omega,\mathbb{C}^3),\ \boldsymbol{u}\times\boldsymbol{n} = 0 \text{ on } \partial\Omega\} \\ &V = \{u \in H_0(\operatorname{curl},\Omega)\colon\ \textstyle\int \mu u\nabla p dx = 0\ \forall\, p \in C_0^\infty(\Omega)\} \\ &H_0^1 = \{p \in L_2(\Omega)\colon \nabla p \in L_2(\Omega,\mathbb{C}^3) \quad p = 0 \text{ on } \partial\Omega\} \end{aligned}$$

and assume (6) for the parameters ε, μ.
Then, the eigenvalue problem in weak form

$$\boldsymbol{H} \in V\colon\ \int_\Omega \operatorname{curl}\boldsymbol{H}\cdot\varepsilon^{-1}\cdot\operatorname{curl}\boldsymbol{\varphi}dx = \lambda\int_\Omega \mu\boldsymbol{H}\boldsymbol{\varphi}\,dx$$

has the null-space $\nabla H_0^1(\Omega)$ and a discrete spectrum $\sigma = \{0,\lambda_1,\lambda_2\dots\}$ with eigenvalues $0 < \lambda_1 \le \lambda_2 \le \dots$.

Problem 9 Consider a 3-d layered structure in x_1-direction, i.e., all quantities are independent of the variables x_2 and x_3, i.e., $\varepsilon = \varepsilon(x_1) \in \mathbb{R}$, $\mu = \mu(x_1) \in \mathbb{R}$. Show that we have in this case for the components of the magnetic field $H_1 \equiv 0$ and $-\partial_1\varepsilon^{-1}\partial_1 H_j = \omega^2\mu H_j$ $(j = 2,3)$.

Solution of Problem 9 We have in this case $\boldsymbol{H} = H(x_1)$, which gives $\operatorname{curl} \boldsymbol{H} = \begin{pmatrix} 0 \\ -\partial_1 H_3 \\ \partial_1 H_2 \end{pmatrix}$.

Then, the result follows from

$$\operatorname{curl}\left(\varepsilon^{-1} \operatorname{curl} \boldsymbol{H}\right) = -\begin{pmatrix} 0 \\ \partial_1 \varepsilon^{-1} \partial_1 H_2 \\ \partial_1 \varepsilon^{-1} \partial_1 H_3 \end{pmatrix} = \omega^2 \, \mu \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} .$$

A special case: the 2d reduction

Now we consider a structure with infinite columns in x_3-direction, i.e., all quantities are independent of x_3, so that we have $\varepsilon = \varepsilon(x_1, x_2) \in \mathbb{R}$, $\mu = \mu(x_1, x_2) \in \mathbb{R}$, $\boldsymbol{H} = \boldsymbol{H}(x_1, x_2)$ and $\boldsymbol{E} = \boldsymbol{E}(x_1, x_2)$. This gives $\operatorname{curl} \boldsymbol{E} = \begin{pmatrix} \partial_2 E_3 \\ -\partial_1 E_3 \\ \partial_1 E_2 - \partial_2 E_1 \end{pmatrix}$ and thus

$$\operatorname{curl}(\mu^{-1} \operatorname{curl} \boldsymbol{E}) = \begin{pmatrix} \partial_2 \mu^{-1}(\partial_1 E_2 - \partial_2 E_1) \\ -\partial_1 \mu^{-1}(\partial_1 E_2 - \partial_2 E_1) \\ -\partial_2 \mu^{-1} \partial_2 E_3 - \partial_1 \mu^{-1} \partial_1 E_3 \end{pmatrix} = \omega^2 \, \varepsilon \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} .$$

so that we have $-\operatorname{div} \mu^{-1} \nabla E_3 = \omega^2 \varepsilon E_3$. In the same way we obtain $-\operatorname{div} \varepsilon^{-1} \nabla H_3 = \omega^2 \mu H_3$. The remaining components are determined by (7):

$$\varepsilon E_1 = \frac{1}{i\omega} \partial_2 H_3 \,, \quad \varepsilon E_2 = -\frac{1}{i\omega} \partial_1 H_3 \,, \quad \mu H_1 = -\frac{1}{i\omega} \partial_2 E_3 \,, \quad \mu H_2 = \frac{1}{i\omega} \partial_1 E_3 \,.$$

A finite difference approximation of the 2-D eigenvalue problem

Let $\Omega = (0, L_x) \times (0, L_y)$ be a rectangle, and let $\Delta x = \frac{L_x}{J_x}$, $\Delta y = \frac{L_y}{J_y}$ be step sizes.
We consider a finite difference approximation of the 2-d reduction of the Maxwell eigenvalue problem with homogeneous Dirichlet boundary conditions

$$-\nabla \cdot \mu^{-1} \, \nabla u = \lambda \, \varepsilon \, u \text{ in } \Omega \,, \qquad u = 0 \text{ on } \partial\Omega \,.$$

on the grid $\Delta x \mathbb{Z} \times \Delta y \mathbb{Z} \cap \bar{\Omega}$ by

$$\begin{aligned} &\frac{1}{(\Delta x)^2}\Big((\mu^{-1}_{i+1/2,j} + \mu^{-1}_{i-1/2,j}) u_{i,j} - \mu^{-1}_{i+1/2,j} u_{i+1,j} - \mu^{-1}_{i-1/2,j} u_{i-1,j}\Big) \\ &+ \frac{1}{(\Delta y)^2}\Big((\mu^{-1}_{i,j+1/2} + \mu^{-1}_{i,j-1/2}) u_{i,j} - \mu^{-1}_{i,j+1/2} u_{i,j+1} - \mu^{-1}_{i,j-1/2} u_{i,j-1}\Big) = \lambda \varepsilon_{i,j} u_{i,j} \end{aligned}$$

for $i = 1, ..., J_x - 1$, $j = 1, ..., J_y - 1$ and

$$u_{i,0} = u_{i,J_y} = u_{0,j} = u_{J_x,j} = 0 \,.$$

This yields a matrix eigenvalue problem with a sparse symmetric positive matrix.

Program 5 Compute the eigenvalues of the 2-d reduced problem in the unit square $\Omega = (0,1)^2$ with $J_x = J_y = J$ for $\mu \equiv 1$ and

$$\varepsilon(x,y) = \begin{cases} 7 & 0.25 \leq x, y \leq 0.75 \\ 1 & \text{else.} \end{cases}$$

Study the convergence properties of the smallest eigenvalue in dependence of J.

Solution of Program 5

```
J = 40;
dx = 1 / J;
N = (J-1)*(J-1);

A = zeros(N,N);
M = ones(N,1);

for i=1:N
    A(i,i) = 4;
end;
for i=1:J-1
    for j=1:J-2
        k = (i-1)*(J-1) + j;
        A(k,k+1) = -1;
        A(k+1,k) = -1;
        k = (j-1)*(J-1) + i;
        A(k,k+J-1) = -1;
        A(k+J-1,k) = -1;
    end;
end;

for i=1:J-1
    x = i * dx;
    if (x >= 0.25)
        if (x <= 0.75)
            for j=1:J-1
                y = j * dx;
                if (y >= 0.25)
                    if (y <= 0.75)
                        k = (i-1)*(J-1) + j;
                        M(k) = 1 / sqrt(7);
                    end;
                end;
            end;
        end;
    end;
end;

for i=1:N
    for j=1:N
        A(i,j) = A(i,j) * M(i) * M(j);
```

```
    end;
end;

ev = eig(A);

for i=1:N
    ev(i) = ev(i) / (dx*dx);
end;
```

This yields

	$J=5$	$J=10$	$J=20$	$J=40$	$J=80$	J=100	$J=120$
λ_1	4.0593	3.6638	3.4684	3.6069	3.6796	3.6945	3.7044
$\lambda_2=\lambda_3$	11.627	11.175	10.408	11.177	11.573	11.653	11.706
λ_4	19.445	20.321	18.937	20.753	21.684	21.871	21.996

For more precise results, more efficient methods are required.