Karlsruhe Institute of Technology

Waves in materials with heterogeneous microstructures

Christian Wieners

Institut für Angewandte und Numerische Mathematik

Linear waves in 1D and in layered media

In simple cases for wave propagation in layered media the effective wave speed can be derived explicitly: for the *linear wave equation in 1D*

$$\frac{1}{\kappa_\delta(x)}\partial_t^2 p^\delta(t,x) - \partial_x \frac{1}{\varrho_\delta(x)}\partial_x p^\delta(t,x) = 0$$

the wave speed is given locally by $c_\delta(x) = \sqrt{\kappa_\delta(x)/\varrho_\delta(x)}$.
In case of $\kappa_\delta(x) = \kappa(\frac{x}{\delta})$, $\varrho_\delta(x) = \varrho(\frac{x}{\delta})$ with $\mathbb{Z}$-periodic coefficients $\kappa(\cdot)$ and $\varrho(\cdot)$, we get in the limit $\delta \longrightarrow 0$

$$c_{\mathrm{eff}} = \sqrt{\kappa_{\mathrm{eff}}/\varrho_{\mathrm{eff}}}\,, \qquad \kappa_{\mathrm{eff}} = \left(\int_{-1/2}^{1/2} \kappa(y)^{-1}\,\mathrm{d}y\right)^{-1}, \qquad \varrho_{\mathrm{eff}} = \int_{-1/2}^{1/2} \varrho(y)\,\mathrm{d}y\,.$$

This also applies to *compressional elastic waves* in layered media

$$\varrho_\delta(x_1)\partial_t \mathbf{v}^\delta(t,\mathbf{x}) - \nabla p^\delta(t,\mathbf{x}) = \mathbf{b}(t,\mathbf{x})\,,$$
$$\partial_t p^\delta(t,\mathbf{x}) - \kappa_\delta(x_1)\nabla\cdot\mathbf{v}^\delta(t,\mathbf{x}) = 0$$

with $p = \frac{1}{3}\,\mathrm{trace}\,\boldsymbol{\sigma}$ and $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon} = 2\mu\,\mathrm{dev}\,\boldsymbol{\varepsilon} + \frac{\kappa}{3}\,\mathrm{trace}(\boldsymbol{\varepsilon})\mathbf{I}_3$, where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

Fouque / Garnier / Papanicolaou / Solna:
Wave propagation and time reversal in randomly layered media. Springer 2007.

Periodic homogenization for diffusion

We assume that $\boldsymbol{\kappa} \in \mathrm{L}_\infty(\mathbb{R}^d; \mathbb{R}^{d\times d}_{\text{sym}})$ is $\mathbb{Z}^d$-periodic and symmetric uniformly positive definite with fundamental cell $Y = (-0.5, 0.5)^d$.
For $\delta > 0$ we set $\boldsymbol{\kappa}_\delta \in \mathrm{L}_\infty(\Omega; \mathbb{R}^{d\times d}_{\text{sym}})$ is in a bounded domain $\Omega \subset \mathbb{R}^d$

$$\boldsymbol{\kappa}_\delta(\mathbf{x}) = \boldsymbol{\kappa}\left(\frac{\mathbf{x}}{\delta}\right) \qquad \text{a.e. in } \Omega\,.$$

We consider the elliptic equation with periodic coefficients

$$-\nabla\cdot\boldsymbol{\kappa}_\delta(\mathbf{x})\nabla p^\delta(\mathbf{x}) = b(\mathbf{x})\,, \qquad \mathbf{x}\in\Omega$$

for the pressure p^δ with homogeneous Dirichlet boundary condition.

A weak limit $\lim\limits_{\delta\longrightarrow 0} p^\delta$ exists and solves

$$-\nabla\cdot\boldsymbol{\kappa}_{\text{eff}}\nabla p^{\text{eff}}(\mathbf{x}) = b(\mathbf{x})\,, \qquad \mathbf{x}\in\Omega$$

with

$$\boldsymbol{\kappa}_{\text{eff}} = \left(\int_Y \boldsymbol{\kappa}(\mathbf{y})\big(\mathbf{e}_j + \nabla w_j(\mathbf{y})\big)\cdot\mathbf{e}_k\,\mathrm{d}\mathbf{y}\right)_{j,k} \in \mathbb{R}^{d\times d}\,, \qquad \mathbf{e}_j \in \mathbb{R}^d \text{ unit vectors,}$$

and $w_j \in \mathrm{H}^1_{\text{per},0}(Y) = \big\{\phi\in\mathrm{H}^1(Y)\colon \phi \text{ periodic and } \int_Y \phi(\mathbf{y})\,\mathrm{d}\mathbf{y} = 0\big\}$, and

$$\int_Y \boldsymbol{\kappa}(\mathbf{y})\big(\mathbf{e}_j + \nabla w_j(\mathbf{y})\big)\cdot\nabla\phi(\mathbf{y})\,\mathrm{d}\mathbf{y} = 0\,, \qquad \phi\in\mathrm{H}^1_{\text{per},0}(Y)$$

Periodic homogenization for diffusion

For the formulation as symmetric Friedrichs system we define $\mathbf{q}^\delta = -\boldsymbol{\kappa}_\delta \nabla p^\delta$,

$$\mathbf{u}^\delta(\mathbf{x}) = \begin{pmatrix} p^\delta(\mathbf{x}) \\ \mathbf{q}^\delta(\mathbf{x}) \end{pmatrix}, \qquad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} b(\mathbf{x}) \\ \mathbf{0} \end{pmatrix}, \qquad \boldsymbol{\Phi}(\mathbf{x}) = \begin{pmatrix} \phi(\mathbf{x}) \\ \boldsymbol{\psi}(\mathbf{x}) \end{pmatrix},$$

$$M_\delta(\mathbf{x})\mathbf{u}^\delta(\mathbf{x}) = \begin{pmatrix} 0 \\ \boldsymbol{\kappa}_\delta(\mathbf{x})^{-1}\mathbf{q}^\delta(\mathbf{x}) \end{pmatrix}, \qquad A\mathbf{u}^\delta(\mathbf{x}) = \begin{pmatrix} \nabla \cdot \mathbf{q}^\delta(\mathbf{x}) \\ \nabla p^\delta(\mathbf{x}) \end{pmatrix},$$

$$\ell(\boldsymbol{\Phi}) = \int_\Omega b(\mathbf{x})\phi(\mathbf{x})\,\mathrm{d}\mathbf{x} = \big(\mathbf{f}, \boldsymbol{\Phi}\big)_\Omega,$$

and the operators $L_\delta \mathbf{u}^\delta = M_\delta \mathbf{u}^\delta + A\mathbf{u}^\delta$ and $L_\delta^* \boldsymbol{\Phi} = M_\delta \boldsymbol{\Phi} - A\boldsymbol{\Phi}$.

The strong solution $\mathbf{u}^\delta \in \mathrm{H}_0^1(\Omega) \times \mathrm{H}(\mathrm{div};\Omega)$ is given by $L_\delta \mathbf{u}^\delta = \mathbf{f}$, i.e.

$$\operatorname{div} \mathbf{q}^\delta(\mathbf{x}) = b(\mathbf{x}), \qquad \boldsymbol{\kappa}_\delta(\mathbf{x})^{-1}\mathbf{q}^\delta(\mathbf{x}) + \nabla p^\delta(\mathbf{x}) = \mathbf{0}.$$

Defining the ansatz space and test space (in case of homogeneous Dirichlet boundary conditions for the primal problem and thus homogeneous Neumann boundary conditions for the dual problem)

$$W = \mathrm{L}_2(\Omega; \mathbb{R}^{1+d}), \qquad W^* = \mathrm{H}^1(\Omega) \times \mathrm{H}_0(\mathrm{div};\Omega),$$

the weak solution $\mathbf{u}^\delta \in W$ is defined by the variational equation

$$(\mathbf{u}^\delta, L_\delta^* \boldsymbol{\Phi})_\Omega = \ell(\boldsymbol{\Phi}), \qquad \boldsymbol{\Phi} \in W^*.$$

Periodic homogenization for diffusion

Karlsruhe Institute of Technology

The weak solution exists, it is unique and bounded (independently of δ) by

$$\|\mathbf{u}^\delta\|_\Omega \le C_{\mathrm{data}}$$

with $C_{\mathrm{data}} > 0$ depending on $\boldsymbol{\kappa}$, the domain, and the data $\mathbf{f}$.
In order to compute $\boldsymbol{\kappa}_{\text{eff}}^{-1}$, we define $\mathbf{V}^\perp = \big(V^\perp\big)^d$ with

$$V^\perp = \Big\{\mathbf{w} \in \mathrm{L}_2(Y;\mathbb{R}^d)\colon \big(\mathbf{w}, \nabla\phi\big)_\Omega = 0 \text{ for all } \phi \in \mathrm{H}^1_{\text{per},0}(Y) \text{ and } \int_Y \mathbf{w}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \mathbf{0}\Big\}\,.$$

Then, we define $\boldsymbol{\xi}_j \in V^\perp$ by

$$\int_Y \boldsymbol{\kappa}(\mathbf{y})^{-1}\big(\mathbf{e}_j + \boldsymbol{\xi}_j(\mathbf{y})\big)\cdot\boldsymbol{\psi}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \mathbf{0}\,, \qquad \boldsymbol{\psi} \in V^\perp\,, \tag{1}$$

we set $\boldsymbol{\Xi} = (\boldsymbol{\xi}_1|\cdots|\boldsymbol{\xi}_d) \in \mathbf{V}^\perp$, and this defines

$$\boldsymbol{\kappa}_{\text{eff}}^{-1} = \int_Y \boldsymbol{\kappa}(\mathbf{y})^{-1}\big(\mathbf{I}_d + \boldsymbol{\Xi}_j(\mathbf{y})\big)\,\mathrm{d}\mathbf{y}\,.$$

Selecting $\boldsymbol{\psi} = \boldsymbol{\xi}_k$ in (1) we observe

$$\boldsymbol{\kappa}_{\text{eff}}^{-1} = \left(\int_Y \boldsymbol{\kappa}(\mathbf{y})^{-1}\big(\mathbf{e}_j + \boldsymbol{\xi}_j(\mathbf{y})\big)\cdot\big(\mathbf{e}_k + \boldsymbol{\xi}_k(\mathbf{y})\big)\,\mathrm{d}\mathbf{y}\right)_{j,k=1,\dots,d}$$

$$= \int_Y \big(\mathbf{I}_d + \boldsymbol{\Xi}(\mathbf{y})\big)^\top \boldsymbol{\kappa}(\mathbf{y})^{-1}\big(\mathbf{I}_d + \boldsymbol{\Xi}(\mathbf{y})\big)\,\mathrm{d}\mathbf{y}\,.$$

The Maxwell-Debey model

We consider the linear Maxwell system

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = \mathbf{0}\,, \qquad \partial_t \mathbf{D} - \nabla \times \mathbf{H} = -\mathbf{J}\,, \qquad \nabla \cdot \mathbf{B} = 0\,, \qquad \nabla \cdot \mathbf{D} = \rho$$

in the special case of a nonmagnetic medium and

$$\mathbf{D} = \varepsilon_0(\mathbf{E} + \mathbf{P})\,, \qquad \mathbf{P} = \chi \mathbf{E}$$

with linear *polarization* $\mathbf{P}$ depending on the *susceptibility* χ. Depending on the *conductivity* σ and the external current $\mathbf{J}_0$ we have for the electric current density

$$\mathbf{J} = \sigma \mathbf{E} + \mathbf{J}_0\,.$$

In case of $\chi_\delta(\mathbf{x}) = \chi(\frac{\mathbf{x}}{\delta})$, $\sigma_\delta(\mathbf{x}) = \sigma(\frac{\mathbf{x}}{\delta})$ with $\mathbb{Z}$-periodic coefficients $\chi(\cdot)$ and $\sigma(\cdot)$ in

$$\mu_0 \partial_t \mathbf{H} + \nabla \times \mathbf{E} = \mathbf{0}\,, \qquad \varepsilon_0(1 + \chi_\delta)\partial_t \mathbf{E} + \sigma_\delta \mathbf{E} - \nabla \times \mathbf{H} = -\mathbf{J}_0\,,$$

we get for the limit $\delta \longrightarrow 0$

$$\mu_0 \partial_t \mathbf{H} + \nabla \times \mathbf{E} = \mathbf{0}\,, \qquad \partial_t(\varepsilon_{\text{eff}} \mathbf{E} + \varepsilon_0 \chi_{\text{eff}} * \mathbf{P}) + \sigma_{\text{eff}} \mathbf{E} - \nabla \times \mathbf{H} = -\mathbf{J}_0\,.$$

Wellander: Homogenization of the Maxwell equations. Appl. of Mathematics 2001

Symmetric Friedrichs systems

For the Maxwell system with initial values $\mathbf{E}^\delta(0) = \mathbf{E}_0$ and $\mathbf{H}^\delta(0) = \mathbf{H}_0$ we have

$$\mathbf{u} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \mathbf{f} = \begin{pmatrix} -\mathbf{J}_0 \\ \mathbf{0} \end{pmatrix}, \mathbf{u}_0 = \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}, M_\delta \mathbf{u} = \begin{pmatrix} \varepsilon_\delta \mathbf{E} \\ \mu_0 \mathbf{H} \end{pmatrix}, D_\delta \mathbf{u} = \begin{pmatrix} \sigma_\delta \mathbf{E} \\ \mathbf{0} \end{pmatrix}, A\mathbf{u} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ \nabla \times \mathbf{E} \end{pmatrix},$$

$L_\delta \mathbf{u} = M_\delta \partial_t \mathbf{u} + D_\delta \mathbf{u} + A\mathbf{u}$ and its adjoint $L_\delta^* \boldsymbol{\phi} = -M_\delta \partial_t \boldsymbol{\phi} + D_\delta \boldsymbol{\phi} - A\boldsymbol{\phi}$.
This defines the *weak solution* $\mathbf{u}^\delta \in \mathrm{L}_2((0,T) \times \mathbb{R}^3; \mathbb{R}^6)$ by

$$\int_{(0,T)\times\mathbb{R}^3} \mathbf{u}^\delta \cdot L_\delta^* \boldsymbol{\phi} \, \mathrm{d}(t,\mathbf{x}) = \int_{(0,T)\times\mathbb{R}^3} \mathbf{f} \cdot \boldsymbol{\phi} \, \mathrm{d}(t,\mathbf{x}) + \int_{\mathbb{R}^3} \mathbf{u}_0 \cdot \boldsymbol{\phi}(0) \, \mathrm{d}\mathbf{x}$$

for all test functions $\boldsymbol{\phi} \in \mathcal{V}^* = \mathrm{H}^1([0,T] \times \mathbb{R}^3; \mathbb{R}^6)$ with $\boldsymbol{\phi}(T) = \mathbf{0}$.
In case of periodic coefficients we obtain for $\delta \longrightarrow 0$

$$(L_{\mathrm{eff}} \mathbf{u}^{\mathrm{eff}})(t,\mathbf{x}) = M_{\mathrm{eff}} \partial_t \mathbf{u}^{\mathrm{eff}}(t,\mathbf{x}) + \partial_t \int_0^t C_{\mathrm{eff}}(t-s) \mathbf{u}^{\mathrm{eff}}(s,\mathbf{x}) \, \mathrm{d}s + D_{\mathrm{eff}} \mathbf{u}^{\mathrm{eff}}(t,\mathbf{x}) + A\mathbf{u}^{\mathrm{eff}}(t,\mathbf{x}) \,,$$

$$(L_{\mathrm{eff}}^* \boldsymbol{\phi})(t,\mathbf{x}) = -M_{\mathrm{eff}} \partial_t \boldsymbol{\phi}(t,\mathbf{x}) - \int_t^T C_{\mathrm{eff}}(s-t) \partial_t \boldsymbol{\phi}(s,\mathbf{x}) \, \mathrm{d}s + D_{\mathrm{eff}}(\mathbf{x}) \boldsymbol{\phi}(t,\mathbf{x}) - A\boldsymbol{\phi}(t,\mathbf{x}) \,.$$

Bokil, Banks, Cioranescu, Griso: A multiscale method for computing effective parameters of composite electromagnetic materials. Quart. of Appl. Math. 2018

The effective parameters

$V^\perp = \big\{\mathbf{v} \in \mathrm{L}_2(Y,\mathbb{R}^m)\colon (\mathbf{v}, A\boldsymbol{\phi})_Y = 0$ for $\boldsymbol{\phi} \in \mathrm{H}^1_{\mathrm{per},0}(Y;\mathbb{R}^m)$ and $\int_Y \mathbf{v}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \mathbf{0}\big\}$.

We define the tensors $\boldsymbol{\eta}_M, \boldsymbol{\eta}_D \in \mathbf{V}^\perp = (V^\perp)^m$ by

$$\int_Y \big(\mathbf{I}_m + \boldsymbol{\eta}_M(\mathbf{y})\big) M(\mathbf{y})\boldsymbol{\phi}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \mathbf{0}\,,\quad \int_Y \big(\mathbf{I}_m + \boldsymbol{\eta}_D(\mathbf{y})\big) D(\mathbf{y})\boldsymbol{\phi}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \mathbf{0}\,,\quad \boldsymbol{\phi} \in V^\perp,$$

and the tensor function $\boldsymbol{\eta}_C \in \mathrm{H}^1(0,T;\mathbf{V}^\perp)$ solving the evolution equation

$$\int_Y \big(\partial_t \boldsymbol{\eta}_C(t,\mathbf{y}) M(\mathbf{y}) + \boldsymbol{\eta}_C(t,\mathbf{y}) D(\mathbf{y})\big)\boldsymbol{\phi}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \mathbf{0}\,, \qquad \boldsymbol{\phi} \in V^\perp$$

with initial value

$$\int_Y \boldsymbol{\eta}_C(0,\mathbf{y}) M(\mathbf{y})\boldsymbol{\phi}(\mathbf{y})\,\mathrm{d}\mathbf{y} = -\int_Y \big(\mathbf{I}_m + \boldsymbol{\eta}_M(\mathbf{y})\big) D(\mathbf{y})\boldsymbol{\phi}(\mathbf{y})\,\mathrm{d}\mathbf{y}\,, \qquad \boldsymbol{\phi} \in V^\perp\,.$$

This defines the tensors $M_{\mathrm{eff}}, D_{\mathrm{eff}} \in \mathbb{R}^{m\times m}$ and $C_{\mathrm{eff}} \in \mathrm{H}^1(0,T;\mathbb{R}^{m\times m})$ by

$$M_{\mathrm{eff}} = \int_Y \big(\mathbf{I}_m + \boldsymbol{\eta}_M(\mathbf{y})\big) M(\mathbf{y})\,\mathrm{d}\mathbf{y}\,,$$

$$D_{\mathrm{eff}} = \int_Y \big(\mathbf{I}_m + \boldsymbol{\eta}_D(\mathbf{y})\big) D(\mathbf{y})\,\mathrm{d}\mathbf{y}\,,$$

$$C_{\mathrm{eff}}(t-s) = \int_Y \big(\mathbf{I}_m + \boldsymbol{\eta}_D(\mathbf{y})\big) M(x)\boldsymbol{\eta}_C(t-s,\mathbf{y})\,\mathrm{d}\mathbf{y}\,.$$

The effective parameters: Examples

Maxwell $\mu_0 \partial_t \mathbf{H}^\delta + \nabla \times \mathbf{E}^\delta = \mathbf{0}$ and $\varepsilon_\delta \partial_t \mathbf{E}^\delta - \nabla \times \mathbf{H}^\delta = -\mathbf{J}_0$

We have $V^\perp = \big\{\nabla\phi \in \mathrm{L}_2(Y;\mathbb{R}^3)\colon \phi \in \mathrm{H}^1_{\text{per},0}(Y)\big\}$, and $\boldsymbol{\eta}^{\text{perm}} \in \mathbf{V}^\perp = (V^\perp)^3$ solves

$$\int_Y \varepsilon(\mathbf{y})\Big(\mathbf{I}_3 + \boldsymbol{\eta}^{\text{perm}}(\mathbf{y})\Big) \cdot \boldsymbol{\phi}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \mathbf{0}\,, \qquad \boldsymbol{\phi} \in V^\perp\,.$$

This constructs $\varepsilon^{\text{eff}} = \displaystyle\int_Y \varepsilon(\mathbf{y})\big(\mathbf{I}_3 + \boldsymbol{\eta}^{\text{perm}}(\mathbf{y})\big)\,\mathrm{d}\mathbf{y}$.

Elasticity $\varrho_\delta \partial_t \mathbf{v}^\delta - \operatorname{div}\boldsymbol{\sigma}^\delta = \mathbf{f}$ and $\partial_t \boldsymbol{\sigma}^\delta = \mathbf{C}_\delta \varepsilon(\mathbf{v}^\delta)$, i.e., $\mathbf{C}_\delta^{-1}\partial_t\boldsymbol{\sigma}^\delta - \varepsilon(\mathbf{v}^\delta) = \mathbf{0}$

$\{\mathbf{v} \in \mathrm{L}_2(Y;\mathbb{R}^3)\colon \int_Y \mathbf{v}\,\mathrm{d}\mathbf{y} = \mathbf{0}$ and $\int_Y \mathbf{v} : \operatorname{div}\boldsymbol{\sigma}\,\mathrm{d}\mathbf{y} = \mathbf{0}$ for $\boldsymbol{\sigma} \in \mathrm{H}^1_{\text{per},0}(Y;\mathbb{R}^{3\times3}_{\text{sym}})\} = \{\mathbf{0}\}$
implies $\varrho_{\text{eff}} = \displaystyle\int_Y \varrho(\mathbf{y})\,\mathrm{d}\mathbf{y}$.
In $V^\perp = \big\{\boldsymbol{\sigma} \in \mathrm{L}_2(Y;\mathbb{R}^{3\times3}_{\text{sym}})\colon \int_Y \boldsymbol{\sigma}\,\mathrm{d}\mathbf{y} = \mathbf{0}, \int_Y \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{v})\,\mathrm{d}\mathbf{y} = 0$ for $\mathbf{v} \in \mathrm{H}^1_{\text{per},0}(Y;\mathbb{R}^3)\big\}$
for basis tensors $\boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_6 \in \mathbb{R}^{3\times3}_{\text{sym}}$ we define $\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_6 \in V^\perp$ solving

$$\int_Y \mathbf{C}(\mathbf{y})^{-1}\big(\boldsymbol{\eta}_j + \boldsymbol{\sigma}_j(\mathbf{y})\big) : \boldsymbol{\Psi}(\mathbf{y})\,\mathrm{d}\mathbf{y} = 0\,, \quad \boldsymbol{\Psi} \in V^\perp\,, \quad j = 1, \dots, 6\,,$$

and we get $\mathbf{C}_{\text{eff}}^{-1} = \displaystyle\sum_{j,k=1}^{6} \bigg(\int_Y \mathbf{C}(\mathbf{y})^{-1}\big(\boldsymbol{\eta}_j + \boldsymbol{\sigma}_j(\mathbf{y})\big) : \boldsymbol{\eta}_k\,\mathrm{d}\mathbf{y}\bigg)\,\boldsymbol{\eta}_j \otimes \boldsymbol{\eta}_k$.

Visco-elasticity

We consider $\varrho_\delta \partial_t \mathbf{v}^\delta - \operatorname{div} \boldsymbol{\sigma}^\delta = \mathbf{f}$ and $\mathbf{C}_\delta^{-1} \partial_t \boldsymbol{\sigma}^\delta + \mathbf{D}_\delta \boldsymbol{\sigma}^\delta - \varepsilon(\mathbf{v}^\delta) = \mathbf{0}$.

For basis tensors $\boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_6 \in \mathbb{R}^{3\times 3}_{\text{sym}}$ we define $\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_6, \boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_6 \in V^\perp$ solving

$$\int_Y \mathbf{C}(\mathbf{y})^{-1}\big(\boldsymbol{\eta}_j + \boldsymbol{\sigma}_j(\mathbf{y})\big) : \boldsymbol{\Psi}(\mathbf{y})\, \mathrm{d}\mathbf{y} = 0\,,$$

$$\int_Y \mathbf{D}(\mathbf{y})\big(\boldsymbol{\eta}_j + \boldsymbol{\tau}_j(\mathbf{y})\big) : \boldsymbol{\Psi}(\mathbf{y})\, \mathrm{d}\mathbf{y} = 0\,, \quad \boldsymbol{\Psi} \in V^\perp\,, \quad j = 1, \dots, 6\,.$$

This defines

$$\mathbf{C}_{\text{eff}}^{-1} = \sum_{j,k=1}^{6} \left(\int_Y \mathbf{C}(\mathbf{y})^{-1}\big(\boldsymbol{\eta}_j + \boldsymbol{\sigma}_j(\mathbf{y})\big) : \boldsymbol{\eta}_k\, \mathrm{d}\mathbf{y} \right) \boldsymbol{\eta}_j \otimes \boldsymbol{\eta}_k\,,$$

$$\mathbf{D}_{\text{eff}} = \sum_{j,k=1}^{6} \left(\int_Y \mathbf{D}(\mathbf{y})\big(\boldsymbol{\eta}_j + \boldsymbol{\tau}_j(\mathbf{y})\big) : \boldsymbol{\eta}_k\, \mathrm{d}\mathbf{y} \right) \boldsymbol{\eta}_j \otimes \boldsymbol{\eta}_k\,.$$

Visco-elasticity

We define $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_6 \in \mathrm{H}^1(0,T;V^\perp)$ solving for $t > 0$ the evolution equation

$$\int_Y \big(\mathbf{C}^{-1}(\mathbf{y})\partial_t\boldsymbol{\xi}_j(t,\mathbf{y}) + \mathbf{D}(\mathbf{y})\boldsymbol{\xi}_j(t,\mathbf{y})\big) : \boldsymbol{\Psi}(\mathbf{y})\,\mathrm{d}\mathbf{y} = 0\,, \qquad \boldsymbol{\Psi} \in V^\perp$$

with initial value for $t = 0$

$$\int_Y \mathbf{C}^{-1}(\mathbf{y})\boldsymbol{\xi}_j(0,\mathbf{y}) : \boldsymbol{\Psi}(\mathbf{y})\,\mathrm{d}\mathbf{y} = -\int_Y \mathbf{D}(\mathbf{y})\big(\boldsymbol{\eta}_j + \boldsymbol{\xi}_j(\mathbf{y})\big) : \boldsymbol{\Psi}(\mathbf{y})\,\mathrm{d}\mathbf{y}\,, \qquad \boldsymbol{\Psi} \in V^\perp\,.$$

This defines $\mathbf{E}_{\mathrm{eff}} \in \mathrm{H}^1(0,T;\mathbb{R}^{d\times d}_{\mathsf{sym}})$ by

$$\mathbf{E}_{\mathrm{eff}}(t) = \sum_{j,k=1}^{6} \bigg(\int_Y \mathbf{C}(\mathbf{y})^{-1}\big(\boldsymbol{\eta}_j + \boldsymbol{\tau}_j(\mathbf{y})\big) : \boldsymbol{\xi}_k(t,\mathbf{y})\,\mathrm{d}\mathbf{y}\bigg)\,\boldsymbol{\eta}_j \otimes \boldsymbol{\eta}_k\,.$$

This results into the effective system

$$\varrho_{\mathrm{eff}}\partial_t\mathbf{v}^{\mathrm{eff}}(t) - \operatorname{div}\boldsymbol{\sigma}^{\mathrm{eff}}(t) = \mathbf{f}(t)\,,$$

$$\partial_t\bigg(\mathbf{C}_{\mathrm{eff}}^{-1}\boldsymbol{\sigma}^{\mathrm{eff}}(t) + \int_0^t \mathbf{E}_{\mathrm{eff}}(t-s)\boldsymbol{\sigma}^{\mathrm{eff}}(s)\,\mathrm{d}s\bigg) + \mathbf{D}_{\mathrm{eff}}\boldsymbol{\sigma}^{\mathrm{eff}} - \varepsilon(\mathbf{v}^\delta) = \mathbf{0}\,.$$

Visco-elasticity

We define $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_6 \in \mathrm{H}^1(0,T;V^\perp)$ solving for $t > 0$ the evolution equation

$$\int_Y \big(\mathbf{C}^{-1}(\mathbf{y})\partial_t \boldsymbol{\xi}_j(t,\mathbf{y}) + \mathbf{D}(\mathbf{y})\boldsymbol{\xi}_j(t,\mathbf{y})\big) : \boldsymbol{\Psi}(\mathbf{y})\,\mathrm{d}\mathbf{y} = 0\,, \qquad \boldsymbol{\Psi} \in V^\perp$$

with initial value for $t = 0$

$$\int_Y \mathbf{C}^{-1}(\mathbf{y})\boldsymbol{\xi}_j(0,\mathbf{y}) : \boldsymbol{\Psi}(\mathbf{y})\,\mathrm{d}\mathbf{y} = -\int_Y \mathbf{D}(\mathbf{y})\big(\boldsymbol{\eta}_j + \boldsymbol{\xi}_j(\mathbf{y})\big) : \boldsymbol{\Psi}(\mathbf{y})\,\mathrm{d}\mathbf{y}\,, \qquad \boldsymbol{\Psi} \in V^\perp\,.$$

This defines $\mathbf{E}_{\text{eff}} \in \mathrm{H}^1(0,T;\mathbb{R}^{d\times d}_{\text{sym}})$ by

$$\mathbf{E}_{\text{eff}}(t) = \sum_{j,k=1}^{6} \left(\int_Y \mathbf{C}(\mathbf{y})^{-1}\big(\boldsymbol{\eta}_j + \boldsymbol{\tau}_j(\mathbf{y})\big) : \boldsymbol{\xi}_k(t,\mathbf{y})\,\mathrm{d}\mathbf{y}\right) \boldsymbol{\eta}_j \otimes \boldsymbol{\eta}_k\,.$$

Expanding $\boldsymbol{\xi}_j$ by eigenfunctions $\mathbf{w}_n$ with eigenvalues λ_n solving

$$\int_Y \mathbf{C}^{-1}(\mathbf{y})\mathbf{w}_n(\mathbf{y}) : \boldsymbol{\Psi}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \lambda_n \int_Y \mathbf{D}(\mathbf{y})\mathbf{w}_n(\mathbf{y}) : \boldsymbol{\Psi}(\mathbf{y})\,\mathrm{d}\mathbf{y}\,, \qquad \boldsymbol{\Psi} \in V^\perp$$

results in a representation of the form

$$\mathbf{E}_{\text{eff}}(t) = \sum_n \exp(-\lambda_n t)\mathbf{E}_n\,.$$

Generalized Standard Linear Solids

Defining the *relaxation tensor*

$$\dot{\mathbf{C}}(s) = -\sum_{n=1}^{r} \frac{1}{\tau_n} \exp\Big(-\frac{s}{\tau_n}\Big)\mathbf{C}_n\,, \qquad \mathbf{C}(0) = \mathbf{C}_0 + \mathbf{C}_1 + \cdots + \mathbf{C}_r\,.$$

and introducing the corresponding stress decomposition $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + \cdots + \boldsymbol{\sigma}_r$ with

$$\boldsymbol{\sigma}_n(t) = \int_0^t \exp\Big(\frac{s-t}{\tau_n}\Big)\,\mathbf{C}_n\boldsymbol{\varepsilon}(\mathbf{v}(s))\,\mathrm{d}s\,, \qquad n = 1,\dots,r\,,$$

results in the first-order system for visco-elastic waves

$$\begin{aligned} \varrho\,\partial_t\mathbf{v} - \nabla\cdot\big(\boldsymbol{\sigma}_0 + \cdots + \boldsymbol{\sigma}_r\big) &= \mathbf{f}\,, \\ \partial_t\boldsymbol{\sigma}_0 - \mathbf{C}_0\boldsymbol{\varepsilon}(\mathbf{v}) &= \mathbf{0}\,, \\ \partial_t\boldsymbol{\sigma}_n - \mathbf{C}_j\boldsymbol{\varepsilon}(\mathbf{v}) + \tau_n^{-1}\boldsymbol{\sigma}_j &= \mathbf{0}\,, \qquad n = 1,\dots,r \end{aligned}$$

in form of a symmetric Friedrichs system.
Equivalently, this can be formulated with a convolution in time and $\partial_t\mathbf{C}_{\text{eff}}(s) = \dot{\mathbf{C}}(s)$

$$\varrho\,\partial_t\mathbf{v} - \nabla\cdot\boldsymbol{\sigma} = \mathbf{f}\,, \qquad \partial_t\boldsymbol{\sigma}(t) = \partial_t\int_0^t \mathbf{C}_{\text{eff}}(t-s)\boldsymbol{\varepsilon}(\mathbf{v}(s))\,\mathrm{d}s\,.$$

Fichtner: Full Seismic Waveform Modelling and Inversion. Springer 2011