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An adaptive Petrov-Galerkin space-time approximation for linear hyperbolic systems

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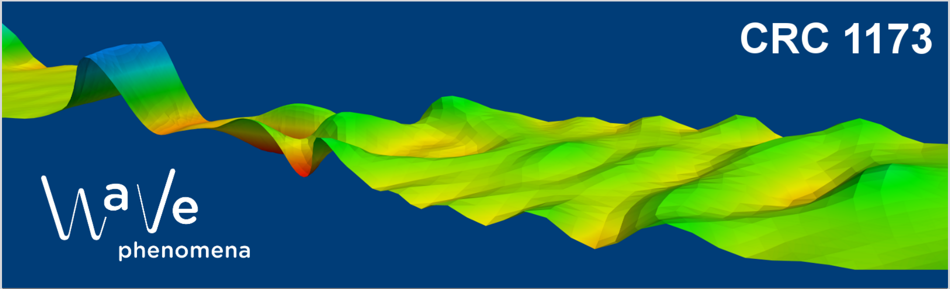

Decomposition of the space-time cylinder

For $0 = t_0 < t_1 < \cdots < t_N = T$ define $I_h = (t_0, t_1) \cup \cdots \cup (t_{N-1}, t_N) \subset I = (0, T)$.

Let $\Omega_h = \bigcup_{K \in \mathcal{K}} K$ be a decomposition in open cells $K \subset \Omega \subset \mathbb{R}^d$ with $\partial\Omega_h = \overline{\Omega} \setminus \Omega_h$.

Let $Q_h = \bigcup_{R \in \mathcal{R}} R = I_h \times \Omega_h$ be a decomposition of $Q = I \times \Omega$.

For $R = (t_{n-1}, t_n) \times K$ select $p_R = p_{n,K} \geq 1$ in time and $q_R = q_{n,K} \geq 0$ in space.

Define the discontinuous space $W_h = \prod_{R \in \mathcal{R}} \mathbb{P}_{p_R - 1} \otimes \mathbb{P}_{q_R}(K; \mathbb{R}^m) \subset \mathbb{P}(I_h \times \Omega_h; \mathbb{R}^m)$
and $Y_{n,h} = \prod_{K \in \mathcal{K}} \mathbb{P}_{q_{n,K}}(K; \mathbb{R}^m)$ in (t_{n-1}, t_n).

Let $M_h \in \mathrm{L}_\infty(\Omega; \mathbb{R}^{m \times m}_{\mathrm{sym}})$ be pos. def. and $\Pi_{n,h} \colon \mathrm{L}_2(\Omega; \mathbb{R}^m) \longrightarrow Y_{n,h}$ the projection

$$\big(M_h \Pi_{n,h} \mathbf{y}, \mathbf{z}_h\big)_\Omega = \big(M_h \mathbf{y}, \mathbf{z}_h\big)_\Omega\,, \qquad \mathbf{y} \in \mathrm{L}_2(\Omega; \mathbb{R}^m)\,,\ \mathbf{z}_h \in Y_{n,h}\,.$$

For $\mathbf{v}_h \in \mathbb{P}(I_h \times \Omega_h; \mathbb{R}^m)$ let $\mathbf{v}_{n,h}$ be the extension of $\mathbf{v}_h|_{(t_{n-1}, t_n) \times \Omega_h}$ to $[t_{n-1}, t_n]$, define

$$V_h = \Big\{ \mathbf{v}_h \in \prod_{R \in \mathcal{R}} \mathbb{P}_{p_R} \otimes \mathbb{P}_{q_R}(K; \mathbb{R}^m) \subset \mathbb{P}(I_h \times \Omega_h; \mathbb{R}^m) \colon$$
$$\mathbf{v}_h(0) = \mathbf{0} \text{ for } t = 0\,,\ \mathbf{v}_{n,h}(t_{n-1}) = \Pi_{n,h} \mathbf{v}_{n-1,h}(t_{n-1}) \text{ for } n = 2, \ldots, N \Big\}\,.$$

By construction, we have $\partial_t V_h = W_h$ in I_h and $\dim V_h = \dim W_h$.

The Petrov-Galerkin setting

Let $L_h = M_h\partial_t + D_h + A_h$ be an approximation of $L = M\partial_t + D + A$ with:

a) $M_h \in \mathrm{L}_\infty(\Omega; \mathbb{R}^{m\times m}_{\mathrm{sym}})$ is uniformly positive definite, i.e., $c_M > 0$ exists with

$$(M_h\mathbf{y}_h, \mathbf{y}_h)_\Omega \ge c_M \|\mathbf{y}_h\|^2_W\,, \qquad \mathbf{y}_h \in Y_h = Y_{1,h} + \cdots + Y_{N,h}\,;$$

b) $D_h \in \mathrm{L}_\infty(\Omega; \mathbb{R}^{m\times m}_{\mathrm{sym}})$ is monotone, i.e.,

$$(D_h\mathbf{y}_h, \mathbf{y}_h)_Q \ge 0\,, \qquad \mathbf{y}_h \in Y_h\,;$$

c) $A_h \in \mathcal{L}(Y_h, Y_h)$ is monotone and consistent, i.e.,

$$(A_h\mathbf{y}_h, \mathbf{y}_h)_Q \ge 0\,, \qquad \mathbf{y}_h \in Y_h\,,$$
$$(A_h\mathbf{z}_h, \mathbf{y}_h)_Q = (A\mathbf{z}_h, \mathbf{y}_h)_Q\,, \qquad \mathbf{z}_h \in Y_h \cap \mathcal{D}(A)\,;$$

Let $\Pi_h\colon \mathrm{L}_2(Q;\mathbb{R}^m) \longrightarrow W_h$ be the projection with $\big(M_h\Pi_h\mathbf{v}, \mathbf{w}_h\big)_Q = \big(M_h\mathbf{v}, \mathbf{w}_h\big)_Q$.

Define $\|\mathbf{v}\|^2_{W_h} = \big(M_h\mathbf{v}, \mathbf{v}\big)_Q$ and $\|\mathbf{v}\|^2_{V_h} = \|\mathbf{v}\|^2_{W_h} + \|\Pi_h M_h^{-1} L_h\mathbf{v}\|^2_{W_h}$ for $\mathbf{v} \in V_h + V$.

Theorem

The bilinear form $b_h\colon V_h \times W_h \longrightarrow \mathbb{R}$, $b_h(\mathbf{v}_h, \mathbf{w}_h) = (L_h\mathbf{v}_h, \mathbf{w}_h)_Q$ is inf-sup stable:

$$\sup_{\mathbf{w}_h \in W_h \setminus \{0\}} \frac{b_h(\mathbf{v}_h, \mathbf{w}_h)}{\|\mathbf{w}_h\|_{W_h}} \ge \beta\, \|\mathbf{v}_h\|_{V_h}\,, \quad \mathbf{v}_h \in V_h \quad \textit{with } \beta \ge \frac{1}{4T^2 + 1}\,.$$

Inf-sup stability

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The proof of the inf-sup stability is based on the following estimates.

Lemma

Let $\lambda_{n,k} \in \mathbb{P}_k$ be the orthonormal Legendre polynomials in $\mathrm{L}_2(t_{n-1}, t_n)$. Then, we have $\big(t\partial_t\lambda_{n,k}, \lambda_{n,k}\big)_{(t_{n-1},t_n)} = k$ for $k = 0, 1, 2, \dots$.

Define $d_T(t) = T - t$. Then, $\int_0^T \int_0^t \phi(s)\,\mathrm{d}s\mathrm{d}t = \int_0^T d_T(t)\phi(t)\mathrm{d}t$ for $\phi \in \mathrm{L}_1(0, T)$ and

$$\|\mathbf{v}_h\|_{W_h}^2 \le \int_0^T \int_0^t \partial_t\big(M_h\mathbf{v}_h(s), \mathbf{v}_h(s)\big)_\Omega \mathrm{d}s\mathrm{d}t = 2\int_0^T d_T(t)\big(M_h\partial_t\mathbf{v}_h(t), \mathbf{v}_h(t)\big)_\Omega \mathrm{d}t\,, \quad \mathbf{v}_h \in V_h\,.$$

Lemma

We have

$$\begin{aligned}
\big(M_h\partial_t\mathbf{v}_h, d_T\mathbf{v}_h\big)_Q &\le \big(M_h\partial_t\mathbf{v}_h, d_T\Pi_h\mathbf{v}_h\big)_Q\,,\\
0 &\le \big(\Pi_h A_h\mathbf{v}_h, d_T\Pi_h\mathbf{v}_h\big)_Q\,,\\
0 &\le \big(\Pi_h D_h\mathbf{v}_h, d_T\Pi_h\mathbf{v}_h\big)_Q\,,\\
\|\mathbf{v}_h\|_{W_h} &\le 2T\,\big\|\Pi_h M_h^{-1}L_h\mathbf{v}\big\|_{W_h}\,, \qquad \mathbf{v}_h \in V_h\,.
\end{aligned}$$

Error analysis

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Theorem

For given $\mathbf{f} \in \mathrm{L}_2(Q;\mathbb{R}^m)$ *there exists a unique solution* $\mathbf{u}_h \in V_h$ *of*

$$(L_h\mathbf{u}_h, \mathbf{w}_h)_Q = (\mathbf{f}, \mathbf{w}_h)_Q\,, \qquad \mathbf{w}_h \in W_h$$

satisfying the a priori bound $\|\mathbf{u}_h\|_{V_h} \le \beta^{-1}\|\Pi_h M_h^{-1}\mathbf{f}\|_{W_h}$.

Theorem

The error is bounded by

$$\|\mathbf{u}-\mathbf{u}_h\|_{V_h} \le \inf_{\mathbf{v}_h\in V_h}\left(\big(1+\beta^{-1}\big)\|\mathbf{u}-\mathbf{v}_h\|_{V_h} + \beta^{-1}\sup_{\mathbf{w}_h\in W_h\setminus\{\mathbf{0}\}}\frac{b_h(\mathbf{v}_h,\mathbf{w}_h)-b(\mathbf{u},\mathbf{w}_h)}{\|\mathbf{w}_h\|_{W_h}}\right).$$

If in addition the solution is sufficiently smooth, we obtain the a priori error estimate

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_{V_h} \le C\big(\triangle t^p + \triangle x^q\big)&\Big(\|\partial_t^{p+1}\mathbf{u}\|_Q + \|\mathrm{D}^{q+1}\mathbf{u}\|_Q\Big)\\ &+ \beta^{-1}\Big\|M_h^{-1/2}(M_h-M)M^{-1/2}\Big\|_\infty\|\partial_t\mathbf{u}\|_W\\ &+ \beta^{-1}\Big\|M_h^{-1/2}(D_h-D)M^{-1/2}\Big\|_\infty\|\mathbf{u}\|_W\end{aligned}$$

for $\triangle t$, $\triangle x$ *and* $p,q\ge 1$ *with* $\triangle t \ge t_n - t_{n-1}$, $\triangle x \ge \mathrm{diam}(K)$, $p \le p_R$ *and* $q \le q_R$.

Dual-primal error bound

Let $E \in W'$ be a linear error functional. We define the dual solution by

$$\mathbf{u}^* \in V^* : \qquad (\mathbf{w}, L^*\mathbf{u}^*)_Q = \langle E, \mathbf{w} \rangle\,, \qquad \mathbf{w} \in W\,.$$

Define $\langle \mathbf{u}_h, \mathbf{u}^* \rangle_{\partial R} = (L\mathbf{u}_h, \mathbf{u}^*)_R - (\mathbf{u}_h, L^*\mathbf{u}^*)_R$ and $\|\mathbf{y}\|_Y^2 = (M\mathbf{y}, \mathbf{y})_\Omega$, $\mathbf{y} \in \mathrm{L}_2(\Omega; \mathbb{R}^m)$.

Lemma

The error is represented by $\langle E, \mathbf{u} - \mathbf{u}_h \rangle = \sum_{R \in \mathcal{R}} \Big(\big(\mathbf{f} - L\mathbf{u}_h, \mathbf{u}^* \big)_R - \langle \mathbf{u}_h, \mathbf{u}^* \rangle_{\partial R} \Big)$, *and if the dual solution is sufficiently regular, the error is bounded by*

$$\begin{aligned}
\big|\langle E, \mathbf{u} - \mathbf{u}_h \rangle\big| \le \sum_{n=1}^{N} \sum_{R=(t_{n-1},t_n)\times K} \Bigg(& \big\|\mathbf{f} - (M_h\partial_t + A + D_h)\mathbf{u}_h\big\|_R \big\|\mathbf{u}^* - \mathbf{w}_h\big\|_R \\
& + \sum_{F \in \mathcal{F}_K} \big\|B_{\mathbf{n}_K}\mathbf{u}_{h,R} - B_{\mathbf{n}_K}^{\mathrm{num}}\mathbf{u}_{n,h}\big\|_{(t_{n-1},t_n)\times F} \big\|\mathbf{u}^* - \mathbf{w}_h\big\|_{(t_{n-1},t_n)\times F} \Bigg) \\
+ \sum_{n=1}^{N-1} \big\| M_h \big(\mathbf{u}_{n,h}(t_n) - \Pi_{n+1,h}\mathbf{u}_{n,h}(t_n)\big) \big\|_\Omega & \big\|\mathbf{u}^*(t_n) - \mathbf{w}_{n+1,h}(t_n)\big\|_\Omega \\
+ \big\|M_h^{-1/2}(M - M_h)M^{-1/2}\big\|_\infty \sum_{n=1}^{N-1} & \big\|\mathbf{u}_{n,h}(t_n) - \Pi_{n+1,h}\mathbf{u}_{n,h}(t_n)\big\|_Y \big\|\mathbf{u}^*(t_n)\big\|_Y \\
+ \Big(\big\|M_h^{-1/2}(M - M_h)M^{-1/2}\big\|_\infty \big\|\partial_t\mathbf{u}_h\big\|_W & + \big\|M_h^{-1/2}(D - D_h)M^{-1/2}\big\|_\infty \big\|\mathbf{u}_h\big\|_W \Big) \big\|\mathbf{u}^*\big\|_W\,.
\end{aligned}$$

Dual-primal error indicator

Since $\mathbf{u}^* \in V^*$ cannot be computed, it is approximated by

$$\mathbf{u}_h^* \in W_h\colon \qquad b_h(\mathbf{v}_h, \mathbf{u}_h^*) = \langle E, \mathbf{v}_h \rangle\,, \qquad \mathbf{v}_h \in V_h\,,$$

and the error indicator $\eta = \sum_R \eta_R$ for $R = (t_{n-1}, t_n) \times K$ is defined by

$$\eta_R = \Big(\big\| L_h \mathbf{u}_h - \mathbf{f} \big\|_R + \big\| \mathbf{u}_{n,h}(t_{n-1}) - \Pi_{n,h} \mathbf{u}_{n-1,h}(t_{n-1}) \|_K \Big) h_K^{1/2} \big\| [\Pi_h^0 \mathbf{u}_h^*] \big\|_{(t_{n-1},t_n)\times \partial K}$$
$$+ \big\| \big(B_{\mathbf{n}_K} - B_{\mathbf{n}_K}^{\mathrm{num}} \big) \mathbf{u}_h \|_{(t_{n-1},t_n)\times \partial K} \big\| [\Pi_h^0 \mathbf{u}_h^*] \big\|_{(t_{n-1},t_n)\times \partial K}$$

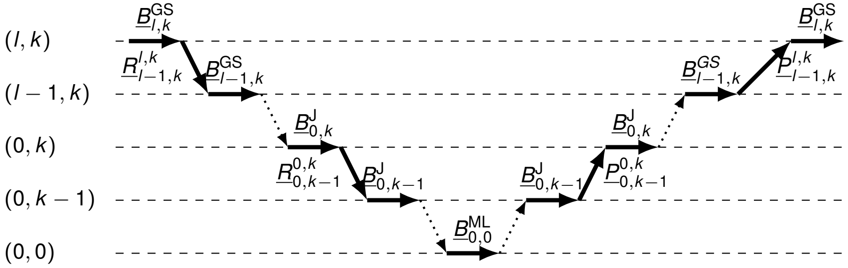
where $\Pi_h^0\colon \mathrm{L}_2(Q;\mathbb{R}^m) \longrightarrow \mathbb{P}_0(Q_h;\mathbb{R}^m)$ is the L_2 projection.

This results into the following *p*-adaptive algorithm:

1: choose low order polynomial degrees on the initial mesh
2: **while** $\max_R(p_R) < p_{\max}$ and $\max_R(q_R) < q_{\max}$ **do**
3: compute $\mathbf{u}_h$
4: compute $\mathbf{u}_h^*$ and the projection $\Pi_h^0 \mathbf{u}_h^*$
5: compute η_R on every cell R
6: if the error is small enough STOP
7: mark space-time cell R for refinement if $\eta_R > \vartheta_1 \max_{R'} \eta_{R'}$
and for derefinement if $\eta_R < \vartheta_0 \max_{R'} \eta_{R'}$
8: increase/decrease polynomial degrees on marked cells
9: redistribute cells on processes for better load balancing

Space-time multilevel preconditioner

Let $\mathcal{R}_{0,0}$ be the coarse space-time mesh, and let $\mathcal{R}_{l,k}$ be the mesh obtained by $l = 1, \dots, l_{\max}$ refinements in space and $k = 1, \dots, k_{\max}$ refinements in time.



- $\underline{L}_{l,k}$ approximates L on $\mathcal{R}_{l,k}$
- the prolongation $\underline{P}^{l,k}_{l-1,k}$ from $\mathcal{R}_{l-1,k}$ to $\mathcal{R}_{l,k}$ represents the natural injection
- the restriction $\underline{R}^{l,k}_{l-1,k}$ from $\mathcal{R}_{l,k}$ to $\mathcal{R}_{l-1,k}$ represents the L_2 projection
- $\underline{B}^{\mathrm{J}}_{l,k} = \theta^{\mathrm{J}}_{l,k}\,\text{block\_diag}(\underline{L}_{l,k})^{-1}$ is the block-Jacobi smoother with damping $\theta_{l,k}$
- $\underline{B}^{\mathrm{GS}}_{l,k} = \theta^{\mathrm{GS}}_{l,k}\left(\text{block\_lower}(\underline{L}_{l,k}) + \text{block\_diag}(\underline{L}_{l,k})\right)^{-1}$ Gauss-Seidel smoother