Space-time solutions for linear hyperbolic systems

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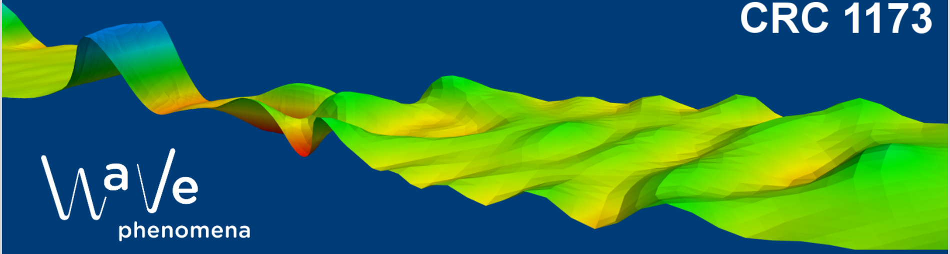

Wave phenomena

Linear hyperbolic first-order systems $L = M\partial_t + A$

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Configuration

domain in space $\Omega \subset \mathbb{R}^d$, time interval $I = (0, T)$, $Q = I \times \Omega$, $\Gamma_j \subset \partial\Omega$

Symmetric Friedrichs systems

$L = M\partial_t + A$ with $A\mathbf{y} = \sum_{j=1}^{d} B_j \partial_j \mathbf{y}$ with $M \in \mathrm{L}_\infty(\Omega; \mathbb{R}^{m\times m}_{\text{sym}})$ pos. def., $B_j \in \mathbb{R}^{m\times m}_{\text{sym}}$

Example: Acoustic waves

$$
\begin{aligned}
\rho\,\partial_t \mathbf{v} - \nabla p &= \mathbf{f} && \text{in } (0,T)\times\Omega\\
\partial_t p - \kappa \nabla\cdot\mathbf{v} &= 0 && \text{in } (0,T)\times\Omega\\
p(0) &= p_0 && \text{in } \Omega \text{ at } t = 0\\
\mathbf{v}(0) &= \mathbf{v}_0 && \text{in } \Omega \text{ at } t = 0\\
p(t) &= p_{\text{S}} && \text{on } \Gamma_{\text{S}} \subset \partial\Omega\,,\ t \in (0,T)\\
\mathbf{n}\cdot\mathbf{v}(t) &= g_{\text{D}} && \text{on } \Gamma_{\text{D}} = \partial\Omega \setminus \Gamma_{\text{S}}\,,\ t \in (0,T)
\end{aligned}
$$

First-order system

$$\mathbf{y} = \begin{pmatrix} \mathbf{v} \\ p \end{pmatrix},\ M\mathbf{y} = \begin{pmatrix} \rho\mathbf{v} \\ \kappa^{-1} p \end{pmatrix},\ A\mathbf{y} = \begin{pmatrix} -\nabla p \\ -\nabla\cdot\mathbf{v} \end{pmatrix},\ \mathbf{n}\cdot B\mathbf{y} = \begin{pmatrix} -p\mathbf{n} \\ -\mathbf{n}\cdot\mathbf{v} \end{pmatrix}$$

in 2d:

$$M = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \kappa^{-1} \end{pmatrix} \quad B_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \begin{matrix} \Gamma_1 = \Gamma_{\text{S}} \\ \Gamma_2 = \Gamma_{\text{D}} \\ \Gamma_3 = \Gamma_{\text{D}} \end{matrix}$$

Linear conservation laws $M\partial_t \mathbf{y} + \mathrm{div}(B\mathbf{y}) = \mathbf{f}$

With $B = (B_1|\cdots|B_d) \in \mathbb{R}^{d\times m\times m}$, i.e., $A\mathbf{y} = \sum_{j=1}^d B_j\partial_j\mathbf{y} = \sum_{j=1}^d \partial_j B_j\mathbf{y} = \mathrm{div}\, B\mathbf{y}$, we get

$$(A\mathbf{y},\mathbf{z})_\Omega = \sum_j \int_\Omega B_j\partial_j\mathbf{y}\cdot\mathbf{z}\,\mathrm{d}x$$

$$= -\sum_j \int_\Omega \mathbf{y}\cdot B_j\partial_j\mathbf{z}\,\mathrm{d}x = -(\mathbf{y},A\mathbf{z})_\Omega\,, \qquad \mathbf{y},\mathbf{z}\in \mathrm{C}^1_\mathrm{c}(\Omega;\mathbb{R}^m)\,,$$

so that $A^* = -A$ on $\mathrm{C}^1_\mathrm{c}(\Omega;\mathbb{R}^m)$.

Defining $B_\mathbf{n} = \mathbf{n}\cdot B = \sum_{j=1}^d n_jB_j \in \mathbb{R}^{m\times m}_{\mathrm{sym}}$ for $\mathbf{n}\in\mathbb{R}^d$, integration by parts yields

$$(A\mathbf{y},\mathbf{z})_\Omega + (\mathbf{y},A\mathbf{z})_\Omega = \sum_{j=1}^d\sum_{k,l=1}^m \int_\Omega \partial_j B_{jkl}y_kz_l\,\mathrm{d}x$$

$$= \int_{\partial\Omega} B_\mathbf{n}\mathbf{y}\cdot\mathbf{z}\,\mathrm{d}a = (B_\mathbf{n}\mathbf{y},\mathbf{z})_{\partial\Omega}\,, \qquad \mathbf{y},\mathbf{z}\in\mathrm{C}^1(\Omega;\mathbb{R}^m)\cap\mathrm{C}^0(\overline{\Omega};\mathbb{R}^m)\,,$$

where $\mathbf{n}$ is the outer unit normal at $\partial\Omega$.

Together, we obtain in space and time for $L = M\partial_t + A$ and its adjoint $L^* = -L$

$$(L\mathbf{w},\mathbf{z})_Q - (\mathbf{w},L^*\mathbf{z})_Q = \big(M\mathbf{w}(T),\mathbf{z}(T)\big)_\Omega - \big(M\mathbf{w}(0),\mathbf{z}(0)\big)_\Omega + (B_\mathbf{n}\mathbf{w},\mathbf{z})_{(0,T)\times\partial\Omega}\,,$$

$$\mathbf{w},\mathbf{z}\in\mathrm{C}^1(Q;\mathbb{R}^m)\cap\mathrm{C}^0(\overline{Q};\mathbb{R}^m)\,.$$

Solution spaces

We define the Hilbert spaces

$$\mathrm{H}(A,\Omega) = \big\{\mathbf{y} \in \mathrm{L}_2(\Omega;\mathbb{R}^m) \colon A\mathbf{y} \in \mathrm{L}_2(\Omega;\mathbb{R}^m)\big\}\,, \qquad \|\mathbf{y}\|_{\mathrm{H}(A,\Omega)} = \sqrt{\|\mathbf{y}\|_\Omega^2 + \|A\mathbf{y}\|_\Omega^2}\,,$$

$$\mathrm{H}(L,Q) = \big\{\mathbf{w} \in \mathrm{L}_2(Q;\mathbb{R}^m) \colon L\mathbf{w} \in \mathrm{L}_2(Q;\mathbb{R}^m)\big\}\,, \qquad \|\mathbf{w}\|_{\mathrm{H}(L,\Omega)} = \sqrt{\|\mathbf{w}\|_Q^2 + \|L\mathbf{w}\|_Q^2}\,.$$

Depending on homogeneous boundary conditions, we define

$$\begin{aligned}\mathcal{V} = \big\{\mathbf{w} \in \mathrm{C}^1(Q;\mathbb{R}^m) \cap \mathrm{C}^0(\overline{Q};\mathbb{R}^m) \colon\ & \mathbf{w}(0) = \mathbf{0}\,,\\ & (B_\mathbf{n}\mathbf{w})_j = 0 \text{ on } I \times \Gamma_j\,, j = 1,\dots,m\big\}\,,\\ \mathcal{V}^* = \big\{\mathbf{z} \in \mathrm{C}^1(Q;\mathbb{R}^m) \cap \mathrm{C}^0(\overline{Q};\mathbb{R}^m) \colon\ & \mathbf{z}(T) = \mathbf{0}\,,\\ & (B_\mathbf{n}\mathbf{z})_j = 0 \text{ on } I \times \Gamma_j^*\,, j = 1,\dots,m\big\}\,,\end{aligned}$$

where $\Gamma_j^* \subset \partial\Omega$ is minimal such that $(B_\mathbf{n}\mathbf{w},\mathbf{z})_{(0,T)\times\partial\Omega} = 0$ for $\mathbf{w} \in \mathcal{V}$ and $\mathbf{z} \in \mathcal{V}^*$.

Let $V \subset \mathrm{H}(L,Q)$ be the closure of $\mathcal{V}$, and $V^* \subset \mathrm{H}(L^*,Q)$ be the closure of $\mathcal{V}^*$.
In $Y = \mathrm{L}_2(\Omega;\mathbb{R}^m)$ and $W = \mathrm{L}_2(Q;\mathbb{R}^m)$ we use the energy norm and its adjoint

$$\|\mathbf{y}\|_Y = \sqrt{(M\mathbf{y},\mathbf{y})_\Omega}\,, \quad \|\mathbf{w}\|_W = \sqrt{(M\mathbf{w},\mathbf{w})_Q}\,, \quad \|\mathbf{w}\|_{W^*} = \sqrt{(M^{-1}\mathbf{w},\mathbf{w})_Q}\,.$$

In V and V^* we use the weighted norms

$$\|\mathbf{w}\|_V = \sqrt{\|\mathbf{w}\|_W^2 + \|L\mathbf{w}\|_{W^*}^2}\,, \qquad \|\mathbf{z}\|_{V^*} = \sqrt{\|\mathbf{z}\|_W^2 + \|L^*\mathbf{z}\|_{W^*}^2}\,.$$

Solution concepts

Classical solutions $\mathbf{u} \in \mathrm{C}^1(Q;\mathbb{R}^m) \cap \mathrm{C}^0(\overline{Q};\mathbb{R}^m)$

$$\begin{aligned} L\mathbf{u} &= \mathbf{f} && \text{in } Q = (0,T)\times\Omega\,,\\ \mathbf{u}(0) &= \mathbf{u}_0 && \text{in } \Omega \text{ at } t = 0\,,\\ (B_\mathbf{n}\mathbf{u})_j &= g_j && \text{on } (0,T)\times\Gamma_j\,, j = 1,\dots,m \end{aligned}$$

for $\mathbf{f} \in \mathrm{C}^0(Q;\mathbb{R}^m)$, $\mathbf{u}_0 \in \mathrm{C}^0(\Omega;\mathbb{R}^m)$, $g_j \in \mathrm{C}^0((0,T)\times\Gamma_j)$

Strong solutions $\mathbf{u} \in \mathrm{H}(L,Q)$

$$\begin{aligned} L\mathbf{u} &= \mathbf{f} && \text{in } Q = (0,T)\times\Omega\,,\\ \mathbf{u}(0) &= \mathbf{u}_0 && \text{in } \Omega \text{ at } t = 0\,,\\ (B_\mathbf{n}\mathbf{u})_j &= g_j && \text{on } (0,T)\times\Gamma_j\,, j = 1,\dots,m \end{aligned}$$

for $\mathbf{f} \in \mathrm{L}_2(Q;\mathbb{R}^m)$, $\mathbf{u}_0 \in \mathrm{L}_2(\Omega;\mathbb{R}^m)$, $g_j \in \mathrm{L}_2((0,T)\times\Gamma_j)$

Weak solutions $\mathbf{u} \in \mathrm{L}_2(Q;\mathbb{R}^m)$

$$(\mathbf{u}, L^*\mathbf{z})_Q = (\mathbf{f},\mathbf{z})_Q + \big(\mathbf{u}_0,\mathbf{z}(0)\big)_\Omega - (\mathbf{g},\mathbf{z})_{(0,T)\times\partial\Omega}\,, \qquad \mathbf{z} \in \mathcal{V}^*$$

for $\mathbf{f} \in \mathrm{L}_2(Q;\mathbb{R}^m)$, $\mathbf{u}_0 \in \mathrm{L}_2(\Omega;\mathbb{R}^m)$, $\mathbf{g} \in \mathrm{L}_2((0,T)\times\partial\Omega;\mathbb{R}^m)$

A weak solution in 1d

We consider weak solutions of $\partial_t^2 u - c^2 \partial_x^2 u = 0$ in $(0,T) \times (0,L)$ with piecewise constant initial values for $(v,\sigma) = (\partial_t u, c\partial_x u)$ at $t = 0$ and $u(t,0) = u(t,L) = 0$.

We consider $L = cT$, $N \in \mathbb{N}$, $\triangle x = c\triangle t$, $\triangle t = T/N$.

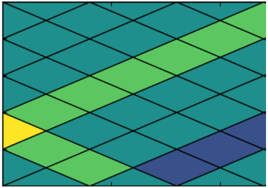
Starting with $v(0,x) = v^0_{j-\frac12}$ and $\sigma(0,x) = \sigma^0_{j-\frac12}$ for $(j-1)\triangle x < x < j\triangle x$

we compute recursively for $n = 1, 2, \dots, N$

$$
\begin{aligned}
v_j^{n-\frac12} &= \frac12\Big(v^{n-1}_{j+\frac12} + v^{n-1}_{j-\frac12} + \sigma^{n-1}_{j+\frac12} - \sigma^{n-1}_{j-\frac12}\Big)\,, & & v^n_{-\frac12} = -v^n_{\frac12}\,,\\
\sigma_j^{n-\frac12} &= \frac12\Big(v^{n-\frac12}_{j+\frac12} - v^{n-\frac12}_{j-\frac12} + \sigma^{n-\frac12}_{j+\frac12} + \sigma^{n-\frac12}_{j-\frac12}\Big)\,,\ j = 0,\dots,N\,, & & v^n_{N+\frac12} = -v^n_{N-\frac12}\,,\\
v^n_{j-\frac12} &= \frac12\Big(v_j^{n-\frac12} + v_{j-1}^{n+\frac12} + \sigma_j^{n-\frac12} - \sigma_{j-1}^{n-\frac12}\Big)\,, & & \sigma^n_{-\frac12} = \sigma^n_{\frac12}\,,\\
\sigma^n_{j-\frac12} &= \frac12\Big(v_j^{n-\frac12} - v_{j-1}^{n-\frac12} + \sigma_j^{n-\frac12} + \sigma_{j-1}^{n-\frac12}\Big)\,,\ j = 1,\dots,N\,, & & \sigma^n_{N+\frac12} = \sigma^n_{N-\frac12}\,.
\end{aligned}
$$

This defines a piecewise constant weak solution $(v,\sigma) \in \mathrm{L}_2(Q,\mathbb{R}^2)$ which is discontinuous along the characteristics

$$\big(t,\chi(t)\big) = (t, j\triangle x \pm ct) \in (0,T) \times (0,L)\,.$$



Existence and uniqueness of space-time solutions

Define $\langle \ell, \mathbf{z}\rangle = (\mathbf{f}, \mathbf{z})_Q + \big(\mathbf{u}_0, \mathbf{z}(0)\big)_\Omega - (\mathbf{g}, \mathbf{z})_{(0,T)\times\partial\Omega}$ and

$$J(\mathbf{w}) = \frac{1}{2}\|L\mathbf{w} - \mathbf{f}\|_{W^*}^2\,, \quad \mathbf{w} \in \mathrm{H}(L, Q)\,, \qquad J^*(\mathbf{z}) = \frac{1}{2}\|L^*\mathbf{z}\|_{W^*}^2 - \langle \ell, \mathbf{z}\rangle\,, \quad \mathbf{z} \in \mathcal{V}^*\,.$$

Theorem

a) *Assume that $C_L > 0$ exists with*

$$\|\mathbf{w}\|_W \le C_L \|L\mathbf{w}\|_{W^*}\,, \qquad \mathbf{w} \in V\,.$$

Then, a unique minimizer $\mathbf{u} \in V$ of $J(\cdot)$ exists, and if $L(V) = W$, the minimizer $\mathbf{u} \in V$ is the unique strong solution of

$$(L\mathbf{u}, \mathbf{w})_Q = (\mathbf{f}, \mathbf{w})_Q\,, \qquad \mathbf{w} \in W\,.$$

b) *Assume that $C_{L^*} > 0$ and $C_\ell > 0$ exists with*

$$\|\mathbf{z}\|_W \le C_{L^*}\|L^*\mathbf{z}\|_{W^*}\,, \qquad |\langle \ell, \mathbf{z}\rangle| \le C_\ell \|\mathbf{z}\|_{V^*}\,, \qquad \mathbf{z} \in \mathcal{V}^*\,.$$

Then, $J^*(\cdot)$ extends to V^*, a unique minimizer $\mathbf{z}^* \in V^*$ of $J^*(\cdot)$ exists, and if $L^*(\mathcal{V}^*) \subset W$ is dense, $\mathbf{u} = L^*\mathbf{z}^* \in \mathrm{L}_2(Q;\mathbb{R}^m)$ is the unique weak solution of

$$(\mathbf{u}, L^*\mathbf{z})_Q = \langle \ell, \mathbf{z}\rangle\,, \qquad \mathbf{z} \in V^*\,.$$

Mapping properties of the space-time operator

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Lemma

$\|\mathbf{w}\|_W \le C_L \|L\mathbf{w}\|_{W^*}$ for $\mathbf{w} \in V$ holds with $C_L = 2T$.

Thus, $L\colon\ V \longrightarrow \mathrm{L}_2(Q;\mathbb{R}^m)$ is injective, continuous, and $L(V) \subset \mathrm{L}_2(Q;\mathbb{R}^m)$ is closed.

Let $\mathcal{D}(A) = Z \subset \mathrm{H}(A,\Omega)$ be the closure of

$$\mathcal{Z} = \left\{\mathbf{z} \in \mathrm{C}^1(\Omega;\mathbb{R}^m) \cap \mathrm{C}^0(\overline{\Omega};\mathbb{R}^m)\colon\ (B_\mathbf{n}\mathbf{w})_j = 0 \text{ on } \Gamma_j\,, j = 1,\dots,m\right\},$$

and select $\Gamma_j \subset \partial\Omega$ such that $(A\mathbf{z},\mathbf{z})_\Omega = \frac{1}{2}(B_\mathbf{n}\mathbf{z},\mathbf{z})_{\partial\Omega} = 0$ for $\mathbf{z} \in Z$.
Then, $\big((M+A)\mathbf{z},\mathbf{z}\big)_\Omega = \big(M\mathbf{z},\mathbf{z}\big)_\Omega > 0$ for $\mathbf{z} \neq 0$, i.e., $M+A$ is injective.

Lemma

Assume that $M + A\colon\ Z \longrightarrow \mathrm{L}_2(\Omega;\mathbb{R}^m)$ is surjective.
Then, $L(V) \subset \mathrm{L}_2(Q;\mathbb{R}^m)$ is dense.

Together, $L(V) = \mathrm{L}_2(Q;\mathbb{R}^m)$, i.e., $L\colon\ V \longrightarrow \mathrm{L}_2(Q;\mathbb{R}^m)$ is surjective.

Construction   For $\mathbf{f} \in \mathrm{L}_2(Q;\mathbb{R}^m)$, $N \in \mathbb{N}$ and $t_{N,n} = n\frac{T}{N}$ let $\mathbf{f}_N \in \mathrm{L}_2(Q;\mathbb{R}^m)$ be piecewise constant in time with $\mathbf{f}_{N,n} = \mathbf{f}_N|_{(t_{N,n-1},t_{N,n})}$ so that $\lim\limits_{N\longrightarrow\infty} \mathbf{f}_N = \mathbf{f}$.

Let $\mathbf{u}_N \in \mathrm{H}^1(0,T;Z)$ be piecewise linear in $(t_{N,n-1},t_{N,n})$ with $\mathbf{u}_N(0) = \mathbf{0}$ and $\big(M + \frac{T}{N}A\big)\mathbf{u}_N(t_{N,n}) = M\mathbf{u}_N(t_{N,n-1}) + \frac{T}{N}\mathbf{f}_{N,n}$. Then, $\lim\limits_{N\longrightarrow\infty} L\mathbf{u}_N = \mathbf{f}$.

Inf-sup stability

Corollary

With these assumptions, we have $C_L = C_{L^*}$ *and*

$$V = \left\{ \mathbf{v} \in \mathrm{H}(L,Q) \colon (L\mathbf{v}, \mathbf{z})_Q = (\mathbf{v}, L^*\mathbf{z})_Q \ \text{ for } \ \mathbf{z} \in \mathcal{V}^* \right\},$$

$$V^* = \left\{ \mathbf{z} \in \mathrm{H}(L^*,Q) \colon (L^*\mathbf{z}, \mathbf{v})_Q = (\mathbf{z}, L\mathbf{v})_Q \ \text{ for } \ \mathbf{v} \in \mathcal{V} \right\}.$$

Theorem

The bilinear form $b \colon V \times W \longrightarrow \mathbb{R}$, $b(\mathbf{v},\mathbf{w}) = (L\mathbf{v},\mathbf{w})_Q$ *is inf-sup stable:*

$$\inf_{\mathbf{v}\in V\setminus\{0\}} \sup_{\mathbf{w}\in W\setminus\{0\}} \frac{b(\mathbf{v},\mathbf{w})}{\|\mathbf{v}\|_V \|\mathbf{w}\|_W} = \inf_{\mathbf{w}\in W\setminus\{0\}} \sup_{\mathbf{v}\in V\setminus\{0\}} \frac{b(\mathbf{v},\mathbf{w})}{\|\mathbf{v}\|_V \|\mathbf{w}\|_W} = \beta\,, \qquad \beta \geq \frac{1}{C_L^2 + 1}\,.$$

Thus, for all $\mathbf{f} \in \mathrm{L}_2(Q, \mathbb{R}^m)$ a unique Petrov-Galerkin solution $\mathbf{u} \in V$ of

$$b(\mathbf{u},\mathbf{w}) = \left(\mathbf{f},\mathbf{w}\right)_Q, \qquad \mathbf{w} \in W$$

exists, and the solution is bounded by $\|\mathbf{u}\|_V \leq \beta^{-1} \|\mathbf{f}\|_{W^*}$.

Corollary

$\mathbf{f} \in \mathrm{H}^1(0,T;\mathrm{L}_2(\Omega;\mathbb{R}^m)) \implies \mathbf{u} \in \mathrm{H}^1(0,T;\mathrm{L}_2(\Omega;\mathbb{R}^m))$ *and* $\|\partial_t \mathbf{u}\|_W \leq C_L \|\partial_t \mathbf{f}\|_{W^*}$

Applications

Acoustics $\quad Z = \big\{(\mathbf{v}, p) \in \mathrm{H}(\mathrm{div}, \Omega) \times \mathrm{H}^1(\Omega) \colon \mathbf{n}\cdot\mathbf{v} = 0 \text{ on } \Gamma_{\mathrm{D}}\,,\ p = 0 \text{ on } \Gamma_{\mathrm{S}}\big\}$

For all $(\mathbf{f}, g) \in \mathrm{L}_2(\Omega; \mathbb{R}^{d+1})$ we define $(\mathbf{v}, p) = \big(\rho^{-1}(\nabla p - \mathbf{f}), p\big) \in Z$ by solving

$$(\rho^{-1}\nabla p, \nabla\phi)_\Omega + (\kappa^{-1} p, \phi)_\Omega = (g, \phi)_\Omega - (\rho^{-1}\mathbf{f}, \nabla\phi)_\Omega\,, \quad \phi \in \mathrm{H}^1(\Omega) \text{ with } \phi = 0 \text{ on } \Gamma_{\mathrm{S}}\,.$$

Visco-elastic waves

The space-time setting extends to $L = M\partial_t + A + D$
with $D \in \mathrm{L}_\infty(\Omega; \mathbb{R}^{m\times m}_{\mathrm{sym}})$ positive semi-definite. For the system

$$\begin{aligned} \rho\,\partial_t\mathbf{v} - \nabla\cdot(\boldsymbol{\sigma}_0 + \cdots + \boldsymbol{\sigma}_r) &= \mathbf{f}\,,\\ \partial_t\boldsymbol{\sigma}_0 - \mathbf{C}_0\boldsymbol{\varepsilon}(\mathbf{v}) &= \mathbf{0}\,,\\ \partial_t\boldsymbol{\sigma}_j - \mathbf{C}_j\boldsymbol{\varepsilon}(\mathbf{v}) + \tau_j^{-1}\boldsymbol{\sigma}_j &= \mathbf{0}\,, \qquad j = 1, \dots, r\,, \end{aligned}$$

we obtain $L^* = -M\partial_t - A + D$ with $\mathbf{y} = \big(\mathbf{v}, \boldsymbol{\sigma}_0, \dots, \boldsymbol{\sigma}_r\big)^\top$ and

$$M = \begin{pmatrix} \rho & 0 & \cdots & 0\\ 0 & \mathbf{C}_0^{-1} & & \\ \vdots & & \ddots & \\ 0 & & & \mathbf{C}_r^{-1} \end{pmatrix} \quad A = -\begin{pmatrix} 0 & \mathrm{div} & \cdots & \mathrm{div}\\ \boldsymbol{\varepsilon} & 0 & & \\ \vdots & & \ddots & \\ \boldsymbol{\varepsilon} & 0 & & 0 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 0 & \cdots & 0\\ 0 & \tau_0^{-1}\mathbf{C}_0^{-1} & & \\ \vdots & & \ddots & \\ 0 & & & \tau_r^{-1}\mathbf{C}_r^{-1} \end{pmatrix}$$