A SPACE-TIME DISCONTINUOUS GALERKIN DISCRETIZATION FOR THE LINEAR TRANSPORT EQUATION

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Abstract. We consider a full-upwind DG approximation in space and time for the linear transport equation. Based on our results for linear symmetric Friedrichs systems we establish inf-sup stability and convergence in a mesh-dependent DG norm, and we construct an error indicator with respect to this norm. Numerical results of test problems with known solution demonstrate the efficiency of the a priori and a posteriori results as well for smooth and for non-smooth solutions. Then, we show that by introducing suitable degrees of freedom on the space-time element boundaries the corresponding hybrid formulation yields a reduction to a considerably smaller linear system.

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1. Introduction

Full upwind discretizations in space for convection-diffusion systems and more general for hyperbolic conservation laws are well established, and for upwind discretizations in time equivalence to implicit Runge-Kutta methods can be shown. Combining these to space-time upwind discretizations has the advantage that this allows for a numerical analysis of adaptive schemes where the discretization in space can be modified in every time step.

For linear symmetric Friedrichs systems and H^s regularity of the solutions, $\mathcal{O}(h^{s-1/2})$ convergence can be established for discontinuous Galerkin approximations in space with respect to a suitable mesh-dependent DG norm [Ern and Guermond, 2021, Chap. 57], [Di Pietro and Ern, 2011, Chap. 7]. This applies also to a space-time approximation for acoustics, where estimates for all discrete time steps in a DG semi-norm are presented in [Bansal et al., 2021, Prop. 6.5].

This is extended to DG full upwind space-time discretizations for linear wave equations in [Corallo et al., 2023], where inf-sup stability and convergence in the mesh-dependent DG norm is proved and a corresponding error indicator is constructed. Convergence order $\mathcal{O}(h^{s-1/2})$ in the mesh-dependent DG norm requires only H^s regularity in the space-time cylinder and thus only $\mathrm{H}^{s-1/2}$ regularity for the solution in space at fixed time. Since the L_2 norm at the time steps can be bounded by the mesh-dependent DG norm, this rate is optimal.

Here, these results are adapted to the linear transport equation, where the main difference in the analysis is the dependence of the flux on the spatial coordinate which requires additional estimates and suitable assumptions on the flux vector. Moreover, we show that the unified hybridized discontinuous Galerkin framework [Bui-Thanh, 2015] transfers to our space-time method, i.e., by solving local problems in every space-time cell a reduced global system for the face degrees of freedom is derived.

The numerical analysis in this work includes the lowest-order case with piece-wise constant approximations in every space-time cell corresponding to finite volumes in space and the implicit Euler method in time. The stability of the method is obtained by the upwind flux. This differs from other methods, where stability requires an appropriate choice of the basis functions and the polynomial degree in the ansatz and/or test spaces.

A class of discontinuous Petrov–Galerkin methods (DPG) is introduced and applied to the transport equation in [Demkowicz and Gopalakrishnan, 2010], where for given discontinuous ansatz spaces an optimal test space is constructed, and the continuity requirements of the solution are approximated weakly by introducing element interface degrees of freedom. Solving local element problems, the system can be reduced to a symmetric positive definite system for the interface variables. The DPG analysis proves convergence in the graph norm and is based on inf-sup stability with respect to a sufficiently large test space. Qualitative convergence estimates require additional regularity of the solution. A corresponding adaptive method is analyzed in [Dahmen et al., 2012], inf-sup stability is established in [Broersen et al., 2018], and reliability and efficiency of an error estimator up to oscillations is shown in [Dahmen and Stevenson, 2019].

Meanwhile, the DPG method is applied to a large class of equations including space-time discretizations for acoustic waves [Demkowicz et al., 2017, Gopalakrishnan and Sepúlveda, 2019, Ernesti and Wieners, 2019a, Ernesti and Wieners, 2019b]. Recent applications to convection-dominated problems and the extension to the L_p-DPG method are considered in [Li and Demkowicz, 2022, Muñoz-Matute et al., 2022, Demkowicz et al., 2023].

Several approaches for convection-diffusion problems in space also apply to the transport equation. Here, hybrid high-order methods (HHO) methods are well established for this problem class. An overview to this method and comparison with other approaches are discussed in [Di Pietro et al., 2016].

The paper is organized as follows. In Sect. 2 we introduce the notation for weak and strong solutions of the linear transport equation. In Sect. 3 we introduce the DG discretization in time and in space. In particular, two representations for the full upwind method are derived, where the differential operator and the jump terms on the space-time element interfaces are applied to the ansatz functions (primal representation) or to the test functions (dual representation). This is essential for the analysis of well-posedness and stability in Sect. 4 as well as for the existence proof of weak solutions and the qualitative convergence estimates in Sect. 5. Adapted to the a priori analysis in the mesh-depending DG norm, we introduce in Sect. 6 an a posteriori error indicator, and numerical results in Sect. 7 illustrate the convergence of the DG scheme for smooth solutions as well for discontinuous Riemann problems. The hybridization is addressed in Sect. 8. In Sect. 9 we conclude with a short discussion of possible applications if the numerical analysis in mesh-dependent DG norms also applies to other discretizations.

2. The linear transport equation

We consider the linear transport equation in a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$ and in the time interval $I = (0,T)$. The space-time cylinder is denoted by $Q = (0,T) \times \Omega$. For $S \subset Q$ the L_2 norm and inner product are denoted by $\|\cdot\|_S$ and $(\cdot,\cdot)_S$. Let $\mathbf{n} \in \mathrm{L}_\infty(\partial\Omega;\mathbb{R}^d)$ be the outer normal vector field a.e. defined on $\partial\Omega$, and for convex subsets $K \subset \Omega$ outer normal vector field is denoted by $\mathbf{n}_K \in \mathrm{L}_\infty(\partial K;\mathbb{R}^d)$.

We aim to compute the transport of a quantity $u\colon [0,T] \times \overline{\Omega} \longrightarrow \mathbb{R}$ along a given vector field $\mathbf{q}\colon \overline{\Omega} \longrightarrow \mathbb{R}^d$. The corresponding flux function $\mathbf{f}$ is given by $\mathbf{f}(u)(t,\mathbf{x}) = u(t,\mathbf{x})\,\mathbf{q}(\mathbf{x})$ for $(t,\mathbf{x}) \in Q$, and the evolution is characterized by the conservation property for all convex subsets $K \subset \Omega$ and time intervals $(t_1,t_2) \subset (0,T)$

$$\int_K \big(u(t_2,\mathbf{x}) - u(t_1,\mathbf{x})\big)\,\mathrm{d}\mathbf{x} + \int_{t_1}^{t_2}\int_{\partial K} \mathbf{f}(u)(t,\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\,\mathrm{d}\mathbf{a}\,\mathrm{d}t = 0\,. \tag{2.1}$$

In order to obtain a unique solution, the conservation property is complemented by the initial condition

$$u(0,\mathbf{x}) = u^0(\mathbf{x})\,, \qquad \mathbf{x} \in \Omega \tag{2.2}$$

and the boundary condition on the inflow boundary $\Gamma_{\mathrm{in}} = \{\mathbf{x} \in \partial\Omega\colon \mathbf{q}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\}$

$$\mathbf{f}(u)(t,\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = g_{\mathrm{in}}(t,\mathbf{x})\,, \qquad (t,\mathbf{x}) \in (0,T) \times \Gamma_{\mathrm{in}}\,. \tag{2.3}$$

If the solution u and the vector field $\mathbf{q}$ are sufficiently smooth, the conservation property (2.1) is equivalent to

$$\partial_t u + \operatorname{div} \mathbf{f}(u) = 0 \;\text{ in } Q\,. \tag{2.4}$$

Multiplication with a smooth test function $v\colon [0,T] \times \overline{\Omega} \longrightarrow \mathbb{R}$ and integration by parts in the space-time cylinder yields

$$\begin{aligned} 0 &= \int_Q \Big(\partial_t u(t,\mathbf{x}) + \operatorname{div}\big(\mathbf{f}(u)(t,\mathbf{x})\big) v(t,\mathbf{x})\,\mathrm{d}(t,\mathbf{x}) \\ &= \int_Q u(t,\mathbf{x})\Big(-\partial_t v(t,\mathbf{x}) - \mathbf{q}(\mathbf{x}) \cdot \nabla v(t,\mathbf{x})\Big)\,\mathrm{d}(t,\mathbf{x}) \\ &\qquad + \int_\Omega \big(u(T,\mathbf{x})v(T,\mathbf{x}) - u(0,\mathbf{x})v(0,\mathbf{x})\big)\,\mathrm{d}\mathbf{x} + \int_0^T\int_{\partial\Omega} \mathbf{f}(u)(t,\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) v(t,\mathbf{x})\,\mathrm{d}\mathbf{a}\,\mathrm{d}t\,. \end{aligned} \tag{2.5}$$

This motivates to define the test space with complementary homogeneous space-time boundary conditions

$$\mathcal{V}^* = \big\{v \in \mathrm{C}^1(\overline{Q})\colon v(T,\mathbf{x}) = 0 \text{ for } \mathbf{x} \in \Omega\,, v(t,\mathbf{x}) = 0 \text{ on } (t,\mathbf{x}) \in (0,T) \times \Gamma_{\mathrm{out}}\big\}\,, \qquad \Gamma_{\mathrm{out}} = \partial\Omega \setminus \Gamma_{\mathrm{in}}\,.$$

Definition 2.1. *For given vector field $\mathbf{q} \in \mathrm{L}_2(\Omega;\mathbb{R}^d)$, initial value $u^0 \in \mathrm{L}_2(\Omega)$, and inflow data $g_{\mathrm{in}} \in \mathrm{L}_2((0,T) \times \Gamma_{\mathrm{in}})$, $u \in \mathrm{L}_2(Q)$ is a* weak solution *of the linear transport equation, if*

$$\int_Q u(t,\mathbf{x})\Big(-\partial_t v(t,\mathbf{x}) - \mathbf{q}(\mathbf{x}) \cdot \nabla v(t,\mathbf{x})\Big)\,\mathrm{d}(t,\mathbf{x}) = \int_\Omega u^0(\mathbf{x})v(0,\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_0^T\int_{\Gamma_{\mathrm{in}}} g_{\mathrm{in}}(t,\mathbf{x})v(t,\mathbf{x})\,\mathrm{d}\mathbf{a}\,\mathrm{d}t\,, \qquad v \in \mathcal{V}^*\,. \tag{2.6}$$

Defining $Lu = \partial_t u + \operatorname{div}(\mathbf{f}(u))$ and the adjoint $L^*v = -\partial_t v - \mathbf{q} \cdot \nabla v$, we directly obtain the following result from (2.5).

Lemma 2.2. *Let $u \in \mathrm{L}_2(Q)$ be a weak solution. Then we obtain by (2.6) that the weak derivative $Lu \in \mathrm{L}_2(Q)$ exists satisfying $(Lu,v)_Q = (u,L^*v)_Q = 0$ for $v \in \mathrm{C}^1_c(Q)$, so that $Lu = 0$ in Q. If in addition the solution is sufficiently regular with initial value $u(0) \in \mathrm{L}_2(\Omega)$ and trace $\mathbf{f}(u) \cdot \mathbf{n}|_{I\times\Gamma_{\mathrm{in}}} \in \mathrm{L}_2(I \times \Gamma_{\mathrm{in}})$, the weak solution is also a* strong solution *satisfying the equation (2.4), the initial condition (2.2), and the boundary condition (2.3).*

Remark 2.3. *For weak solutions of symmetric Friedrichs systems the definition of the test space with adjoint boundary conditions is discussed in detail in [Corallo et al., 2023]. This directly corresponds to the boundary conditions of the adjoint problem $L^*v = r$ backward in time within dual-primal error estimation; see [Dörfler et al., 2023, Chap. 4] for the application to space-time discretizations of linear hyperbolic systems.*

Remark 2.4. *In porous media applications the pressure distribution $p \in \mathrm{H}^1(\Omega)$ and the flux vector $\mathbf{q} \in \mathrm{H}(\operatorname{div},\Omega)$ for given uniformly positive definite permeability tensor $\boldsymbol{\kappa} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ are determined by the elliptic problem*

$$\mathbf{q} = -\boldsymbol{\kappa}\nabla p \;\text{ and }\; \operatorname{div}\mathbf{q} = 0 \;\text{ in }\; \Omega\,, \qquad p = p_{\mathrm{D}} \;\text{ on }\; \Gamma_{\mathrm{D}} \subset \partial\Omega\,, \qquad \mathbf{q} \cdot \mathbf{n} = g_{\mathrm{N}} \;\text{ on }\; \Gamma_{\mathrm{N}} = \partial\Omega \setminus \Gamma_{\mathrm{D}}\,. \tag{2.7}$$

3. The space-time full-upwind discontinuous Galerkin discretization for linear transport

3.1. Discontinuous finite element spaces in the space-time cylinder

We use the notation introduced in [Corallo et al., 2023] for tensor product space-time meshes combining the mesh in space Ω_h with a decomposition in time I_h.

Let the elements $K \subset \Omega$, $K \in \mathcal{K}_h$ be open polygons, and we assume that also the domain Ω is a polygon so that $\Omega_h = \bigcup_{K\in\mathcal{K}_h} K$ and $\overline{\Omega} = \Omega_h \cup \partial\Omega_h$. For $N \in \mathbb{N}$ let $0 = t_0 < t_1 < \cdots < t_N = T$ be a time series.
We define the open time intervals $I_{n,h} = (t_{n-1}, t_n)$, $n = 1, \dots, N$, the decomposition

$$[0,T] = I_h \cup \partial I_h\,, \qquad I_h = (t_0,t_1) \cup \cdots \cup (t_{N-1},t_N) \subset I = (0,T)\,, \qquad \partial I_h = \{t_0, t_1, \dots, t_{N-1}, t_N\}$$

with time-step sizes $\triangle t_n = t_n - t_{n-1}$. Let $\triangle t = \max \triangle t_n$ be the maximal time step. We assume quasi-uniformity in I_h, i.e., $\triangle t_n \in [c_{\rm qu}\triangle t, \triangle t]$ with $c_{\rm qu} \in (0,1]$ independent of N.

In the space-time cylinder Q we define a tensor-product decomposition into space-time cells

$$\mathcal{R}_h = \big\{R = I_{n,h} \times K \colon n = 1,\dots,N,\ K \in \mathcal{K}_h\big\}\,,$$
$$Q_h = I_h \times \Omega_h = \bigcup_{n=1}^N Q_{n,h} = \bigcup_{R\in\mathcal{R}_h} R \ \subset\ Q = I\times\Omega \subset \mathbb{R}^{1+d}\,, \qquad \text{with} \qquad Q_{n,h} = \bigcup_{K\in\mathcal{K}_h} I_{n,h}\times K \ \subset\ I_{n,h}\times\Omega\,.$$

Let $F \in \mathcal{F}_K$ be the faces of the element K, and we set $\mathcal{F}_h = \bigcup_{K\in\mathcal{K}_h} \mathcal{F}_K$, so that $\partial\Omega_h = \bigcup_{F\in\mathcal{F}_h} \overline{F}$ is the skeleton in space and $\partial Q_h = \partial I_h \times \Omega_h \cup I_h \times \partial\Omega_h$ is the corresponding space-time skeleton.

For inner faces $F \in \mathcal{F}_h \cap \Omega$ and $K \in \mathcal{K}_h$, let K_F be the neighboring cell such that $\overline{F} = \partial K \cap \partial K_F$. We assume that the inflow boundary is compatible with the mesh, i.e., $\overline{\Gamma}_{\rm in} = \bigcup_{F\in\mathcal{F}_h\cap\Gamma_{\rm in}} \overline{F}$.

The DG discretization is defined for a finite dimensional subspace $V_h \subset \mathcal{V}_h \subset {\rm C}^1(I_h;\mathcal{S}_h)$, where

$$\mathcal{S}_h = \big\{v_h \in {\rm C}^1(\Omega_h)\colon v_{h,K} = v_h|_K \text{ extends continuously to } v_{h,K} \in {\rm C}^0(\overline{K})\big\}$$
$$\mathcal{V}_h = \big\{v_h \in {\rm C}^1(Q_h)\colon v_{n,h,K} = v_h|_{I_{n,h}\times K} \text{ extends continuously to } v_{n,h,K} \in {\rm C}^0(\overline{I_{n,h}\times K})\big\}\,.$$

We set $h_K = \operatorname{diam} K$, $h_F = \operatorname{diam} F$, and $h = \max h_K$. We assume quasi-uniform meshes and shape-regularity, i.e., $h_F \geq c_{\rm sr} h_K$ for $F \in \mathcal{F}_K$ and $h_K \geq c_{\rm mr} h$ with $c_{\rm sr}, c_{\rm mr} > 0$ independent of $K \in \mathcal{K}_h$. In the following, we use the mesh-dependent weighted ${\rm L}_2$ norm in the space-time cylinder and the ${\rm L}_2$ norm on the space-time skeleton ∂Q_h

$$\big\|h^{-1/2} v_h\big\|_{Q_h} = \Big(\sum_{n=1}^N \sum_{K\in\mathcal{K}_h} h_K^{-1} \|v_h\|^2_{I_{n,h}\times K}\Big)^{1/2}, \qquad \big\|v_h\big\|_{\partial Q_h} = \Big(\sum_{n=1}^N \sum_{K\in\mathcal{K}_h} \|v_{n,h,K}\|^2_{\partial(I_{n,h}\times K)}\Big)^{1/2}, \qquad v_h \in \mathcal{V}_h\,.$$

For the vector field $\mathbf{q} \in {\rm L}_2(\Omega;\mathbb{R}^d)$ let $\mathbf{q}_h \in {\rm L}_\infty(\Omega_h;\mathbb{R}^d) \cap {\rm H}(\operatorname{div},\Omega)$ be a ${\rm H}(\operatorname{div},\Omega)$ conforming finite element approximation, so that $\mathbf{q}_{h,K}\cdot\mathbf{n}_K = \mathbf{q}_{h,K_F}\cdot\mathbf{n}_K$ on all inner faces $\overline{F} = \partial K \cap \partial K_F$. We assume for simplicity that $\mathbf{q}_{h,K}\cdot\mathbf{n}_K|_F$ is constant for all $F \in \mathcal{F}_h$, since we use lowest-order Raviart-Thomas approximations for the flux vector in our implementation.

For every space-time cell $R = I_{n,h} \times K$ we select polynomial degrees $p_R = p_{n,K} \geq 0$ in time and $s_R = s_{n,K} \geq 0$ in space. With this we define $V_{h,R} = \mathbb{P}_{p_{n,K}} \otimes \mathbb{P}_{s_{n,K}}(K)$ and the discontinuous finite element spaces in Ω and in Q

$$S_{n,h} = \prod_{K\in\mathcal{K}_h} \mathbb{P}_{s_{n,K}}(K)\,, \qquad S_h = S_{1,h} + \cdots + S_{N,h} \subset \mathcal{S}_h\,,$$
$$V_{n,h} = \prod_{K\in\mathcal{K}_h} \mathbb{P}_{p_{n,K}} \otimes \mathbb{P}_{s_{n,K}}(K)\,, \qquad V_h = V_{1,h} + \cdots + V_{N,h} \subset \mathcal{V}_h\,,$$

where $\mathbb{P}_p$ denotes the space of polynomials up to order p. For the following, we fix $p = \max p_R$ and $s = \max s_R$, so that

$$S_{n,h} \subset S_h \subset \mathbb{P}_s(\Omega_h) \subset \mathcal{S}_h\,, \qquad V_h \subset \mathbb{P}_p(I_h) \otimes S_h \subset \mathbb{P}_p(I_h) \otimes \mathbb{P}_s(\Omega_h) \subset \mathcal{V}_h\,.$$

3.2. The discontinuous Galerkin method with full upwind for linear transport

We construct a discretization for the linear problem to find $u \in \mathrm{L}_2(Q)$ solving

$$b(u,w) = \ell(w)\,, \qquad w \in \mathcal{V}^*$$

with $b(v,w) = m(v,w) + \int_0^T a(v(t),w(t))\,\mathrm{d}t$ and, using the notation $v(t) = v(t,\cdot) \in \mathrm{L}_2(\Omega)$,

$$m(v,w) = -\int_Q v\partial_t w\,\mathrm{d}(t,\mathbf{x})\,,\quad a(v(t),w(t)) = -\int_\Omega \mathbf{f}(v(t))\cdot\nabla w(t)\,\mathrm{d}\mathbf{x}\,,\quad \ell(w) = \int_\Omega u^0 w(0)\,\mathrm{d}\mathbf{x} - \int_0^T\int_{\Gamma_{\mathrm{in}}} g_{\mathrm{in}}(t)w(t)\,\mathrm{d}\mathbf{a}\,\mathrm{d}t\,.$$

The Riemann problem. In the first step, we consider the special case $\mathbf{q}\in\mathbb{R}^d$ and constant initial values $u^-,u^+\in\mathbb{R}$ for $\mathbf{x}\cdot\mathbf{n}<0$ and $\mathbf{x}\cdot\mathbf{n}>0$ with $\mathbf{n}\in\mathbb{R}^d$, $\mathbf{n}\cdot\mathbf{n}=1$. Then we define the piecewise constant function $u\in\mathrm{L}_2(Q)$ by

$$u(t,\mathbf{x}) = \begin{cases} u^-\,, & (\mathbf{x}-t\mathbf{q})\cdot\mathbf{n}<0\,,\\ u^+\,, & (\mathbf{x}-t\mathbf{q})\cdot\mathbf{n}>0\,,\end{cases}$$

and we observe for $v\in\mathrm{C}^1_{\mathrm{c}}(Q)$

$$\big(u,-\partial_t v-\mathbf{q}\cdot\nabla v\big)_Q = \int_Q u(t,\mathbf{x})\begin{pmatrix}1\\ \mathbf{q}\end{pmatrix}\cdot\begin{pmatrix}-\partial_t v(t,\mathbf{x})\\ -\nabla v(t,\mathbf{x})\end{pmatrix}\mathrm{d}(t,\mathbf{x}) = \int_{\{(t,\mathbf{x})\in Q\colon\,(\mathbf{x}-t\mathbf{q})\cdot\mathbf{n}=0\}}(u^+-u^-)v(t,\mathbf{x})\begin{pmatrix}1\\ \mathbf{q}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{q}\cdot\mathbf{n}\\ -\mathbf{n}\end{pmatrix}\mathrm{d}\mathbf{a}(t,\mathbf{x}) = 0\,,$$

i.e., u is a weak solution of the transport equation. In particular, for $t>0$ and $\mathbf{x}\cdot\mathbf{n}=0$ we obtain $u(t,\mathbf{x})=u^-$ if $\mathbf{q}\cdot\mathbf{n}>0$ and $u(t,\mathbf{x})=u^+$ if $\mathbf{q}\cdot\mathbf{n}<0$. This now defines the upwind flux.

Full upwind in space. For $v_h,w_h\in\mathcal{S}_h$ we observe for the discrete flux $\mathbf{f}_h(v_h)=v_h\,\mathbf{q}_h$, integrating by parts for $K\in\mathcal{K}_h$,

$$\big(\operatorname{div}\mathbf{f}_h(v_h),w_h\big)_{\Omega_h} = \sum_{K\in\mathcal{K}_h}\bigg(-\big(\mathbf{f}_h(v_{h,K}),\nabla w_{h,K}\big)_K + \sum_{F\in\mathcal{F}_K}\big(\mathbf{f}_h(v_{h,K})\cdot\mathbf{n}_K,w_{h,K}\big)_F\bigg)\,.$$

For conforming functions $v\in\mathrm{H}^1(\Omega)$, the flux $\mathbf{f}_h(v)$ is well defined on inner faces $F\in\mathcal{F}_h\cap\Omega$.
For discontinuous functions $v_h\in V_h$, this is approximated by the upwind flux

$$\mathbf{f}^{\mathrm{up}}_{K,F}(v_h) = \begin{cases}\mathbf{f}_h(v_{h,K})\,, & F\in\mathcal{F}_K^{\mathrm{out}}\,,\\ \mathbf{f}_h(v_{h,K_F})\,, & F\in\mathcal{F}_K^{\mathrm{in}}\setminus\Gamma_{\mathrm{in}}\,,\\ \mathbf{0}\,, & F\in\mathcal{F}_K^{\mathrm{in}}\cap\Gamma_{\mathrm{in}}\,,\end{cases}\qquad\text{with}\qquad \begin{cases}\mathcal{F}_K^{\mathrm{out}} = \big\{F\in\mathcal{F}_K\colon\,\mathbf{q}_h\cdot\mathbf{n}_K\ge 0\text{ on }F\big\}\,,\\ \mathcal{F}_K^{\mathrm{in}} \;= \big\{F\in\mathcal{F}_K\colon\,\mathbf{q}_h\cdot\mathbf{n}_K<0\text{ on }F\big\}\,,\end{cases} \tag{3.1}$$

and we set

$$a_h(v_h,w_h) = \sum_{K\in\mathcal{K}_h}\bigg(-\big(\mathbf{f}_h(v_{h,K}),\nabla w_{h,K}\big)_K + \sum_{F\in\mathcal{F}_K}\big(\mathbf{f}^{\mathrm{up}}_{K,F}(v_h)\cdot\mathbf{n}_K,w_{h,K}\big)_F\bigg)\,. \tag{3.2}$$

We set $\mathbf{f}^{\mathrm{up}}_{K,F}(v_h)=\mathbf{0}$ on $F\in\mathcal{F}_K^{\mathrm{in}}\cap\Gamma_{\mathrm{in}}$ since we insert g_{in} on the inflow boundary which is included in the right-hand side.

Integrating (3.2) by parts yields

$$a_h(v_h,w_h) = \sum_{K\in\mathcal{K}_h}\bigg(\big(\operatorname{div}\mathbf{f}_h(v_{h,K}),w_{h,K}\big)_K + \sum_{F\in\mathcal{F}_K}\big((\mathbf{f}^{\mathrm{up}}_{K,F}(v_h)-\mathbf{f}_h(v_{h,K}))\cdot\mathbf{n}_K,w_{h,K}\big)_F\bigg)\,. \tag{3.3}$$

Defining $[v_h]_{K,F}=v_{h,K_F}-v_{h,K}$ on inner faces $F\in\mathcal{F}_h\cap\Omega$, we obtain

$$\big(\mathbf{f}^{\mathrm{up}}_{K,F}(v_h)-\mathbf{f}_h(v_{h,K})\big)\cdot\mathbf{n}_K = [v_h]_{K,F}\frac{1}{2}\big(\mathbf{q}_h\cdot\mathbf{n}_K-|\mathbf{q}_h\cdot\mathbf{n}_K|\big)\,.$$

This yields the following equality.

Lemma 3.1. *We have*

$$a_h(v_h,v_h) = \frac12\int_{\Omega_h}v_h^2\operatorname{div}\mathbf{q}_h\,\mathrm{d}\mathbf{x} + \frac14\sum_{K\in\mathcal{K}_h}\sum_{F\in\mathcal{F}_K\cap\Omega}\int_F\big([v_h]_{K,F}\big)^2\,|\mathbf{q}_h\cdot\mathbf{n}_K|\,\mathrm{d}\mathbf{a} + \frac12\int_{\partial\Omega}v_h^2\,|\mathbf{q}_h\cdot\mathbf{n}|\,\mathrm{d}\mathbf{a}\,,\qquad v_h\in\mathcal{S}_h\,.$$

Proof. We obtain from (3.2) and (3.3) and $\operatorname{div}\mathbf{f}_h(v_h) = v_h \operatorname{div}\mathbf{q}_h + \mathbf{q}_h\cdot\nabla v_h$ in Ω_h

$$\begin{aligned}
a_h(v_h,v_h) &= \frac{1}{2}\sum_{K\in\mathcal{K}_h}\bigg(-\big(\mathbf{f}_h(v_{h,K}),\nabla v_{h,K}\big)_K + \sum_{F\in\mathcal{F}_K}\big(\mathbf{f}^{\mathrm{up}}_{K,F}(v_h)\cdot\mathbf{n}_K, v_{h,K}\big)_F \\
&\qquad\qquad + \big(\operatorname{div}\mathbf{f}_h(v_{h,K}),v_{h,K}\big)_K + \sum_{F\in\mathcal{F}_K}\big((\mathbf{f}^{\mathrm{up}}_{K,F}(v_h)-\mathbf{f}_h(v_{h,K}))\cdot\mathbf{n}_K, v_{h,K}\big)_F\bigg) \\
&= \frac{1}{2}\int_{\Omega_h} v_h^2 \operatorname{div}\mathbf{q}_h\,\mathrm{d}\mathbf{x} + \frac{1}{2}\sum_{K\in\mathcal{K}_h}\sum_{F\in\mathcal{F}_K}\big((2\mathbf{f}^{\mathrm{up}}_{K,F}(v_h)-\mathbf{f}_h(v_{h.K}))\cdot\mathbf{n}_K, v_{h,K}\big)_F\,,
\end{aligned}$$

so that we obtain the assertion by proving the identity

$$\sum_{K\in\mathcal{K}_h}\sum_{F\in\mathcal{F}_K}\big((2\mathbf{f}^{\mathrm{up}}_{K,F}(v_h)-\mathbf{f}_h(v_{h,K}))\cdot\mathbf{n}_K, v_{h,K}\big)_F = \frac{1}{2}\sum_{K\in\mathcal{K}_h}\sum_{F\in\mathcal{F}_K\cap\Omega}\big([v_h]_{K,F}|\mathbf{q}_h\cdot\mathbf{n}_K|,[v_h]_{K,F}\big)_F + \int_{\partial\Omega} v_h^2|\mathbf{q}_h\cdot\mathbf{n}|\,\mathrm{d}\mathbf{a}\,.$$

On inner faces $F\in\mathcal{F}_h\cap\Omega$, this follows from

$$\begin{aligned}
&\big((2\mathbf{f}^{\mathrm{up}}_{K,F}(v_h)-\mathbf{f}_h(v_{h,K}))\cdot\mathbf{n}_K, v_{h,K}\big)_F + \big((2\mathbf{f}^{\mathrm{up}}_{K_F,F}(v_h)-\mathbf{f}_h(v_{h,K_F}))\cdot\mathbf{n}_{K_F}, v_{h,K_F}\big)_F \\
&\quad= \big([v_h]_{K,F}(\mathbf{q}_h\cdot\mathbf{n}_K-|\mathbf{q}_h\cdot\mathbf{n}_K|),v_{h,K}\big)_F + \big([v_h]_{K_F,F}(\mathbf{q}_h\cdot\mathbf{n}_{K_F}-|\mathbf{q}_h\cdot\mathbf{n}_{K_F}|),v_{h,K_F}\big)_F \\
&\qquad + \big(\mathbf{f}_h(v_{h,K})\cdot\mathbf{n}_K,v_{h,K}\big)_F + \big(\mathbf{f}_h(v_{h,K_F})\cdot\mathbf{n}_{K_F},v_{h,K_F}\big)_F \\
&\quad= \big([v_h]_{K,F}\mathbf{q}_h\cdot\mathbf{n}_K,v_{h,K}\big)_F - \big([v_h]_{K,F}|\mathbf{q}_h\cdot\mathbf{n}_K|,v_{h,K}\big)_F + \big([v_h]_{K,F}\mathbf{q}_h\cdot\mathbf{n}_K,v_{h,K_F}\big)_F + \big([v_h]_{K,F}|\mathbf{q}_h\cdot\mathbf{n}_K|,v_{h,K_F}\big)_F \\
&\qquad + \big(v_{h,K}\mathbf{q}_h\cdot\mathbf{n}_K,v_{h,K}\big)_F - \big(v_{h,K_F}\mathbf{q}_h\cdot\mathbf{n}_K,v_{h,K_F}\big)_F \\
&\quad= \big([v_h]_{K,F}|\mathbf{q}_h\cdot\mathbf{n}_K|,[v_h]_{K,F}\big)_F
\end{aligned}$$

since $\big([v_h]_{K,F}\mathbf{q}_h\cdot\mathbf{n}_K,v_{h,K}\big)_F + \big([v_h]_{K,F}\mathbf{q}_h\cdot\mathbf{n}_K,v_{h,K_F}\big)_F + \big(v_{h,K}\mathbf{q}_h\cdot\mathbf{n}_K,v_{h,K}\big)_F - \big(v_{h,K_F}\mathbf{q}_h\cdot\mathbf{n}_K,v_{h,K_F}\big)_F = 0\,.$
On the boundary, we have

$$\begin{aligned}
\big((2\mathbf{f}^{\mathrm{up}}_{K,F}(v_h)-\mathbf{f}_h(v_h))\cdot\mathbf{n}_K,v_h\big)_F &= -\big(v_h\mathbf{q}_h\cdot\mathbf{n}_K,v_h\big)_F = \big(v_h|\mathbf{q}_h\cdot\mathbf{n}_K|,v_h\big)_F\,, &&F\in\mathcal{F}_h\cap\Gamma_{\mathrm{in}}\,,\\
\big((2\mathbf{f}^{\mathrm{up}}_{K,F}(v_h)-\mathbf{f}_h(v_h))\cdot\mathbf{n}_K,v_h\big)_F &= \big(v_h\mathbf{q}_h\cdot\mathbf{n}_K,v_h\big)_F = \big(v_h|\mathbf{q}_h\cdot\mathbf{n}_K|,v_h\big)_F\,, &&F\in\mathcal{F}_h\cap\Gamma_{\mathrm{out}}\,.
\end{aligned}$$

□

Full upwind in time. For $v_h,w_h\in\mathcal{V}_h$ we obtain after integration by parts in all intervals $I_{n,h}\subset I_h$

$$\big(\partial_t v_h,w_h\big)_{Q_h} = \sum_{n=1}^{N}\Big(-\big(v_{n,h},\partial_t w_{n,h}\big)_{Q_{n,h}} + \big(v_{n,h}(t_n),w_{n,h}(t_n)\big)_\Omega - \big(v_{n,h}(t_{n-1}),w_{n,h}(t_{n-1})\big)_\Omega\Big)\,.$$

Introducing the jump terms $[w_h]_n = w_{n+1,h}(t_n)-w_{n,h}(t_n)$ for $n=1,\dots,N-1$ and $[w_h]_N = -w_{N,h}(t_N)$, we define

$$m_h(v_h,w_h) = \sum_{n=1}^{N}\Big(-\big(v_{n,h},\partial_t w_{n,h}\big)_{Q_{n,h}} - \big(v_{n,h}(t_n),[w_h]_n\big)_\Omega\Big)\,. \tag{3.4}$$

Again integrating by parts and defining $[v_h]_0 = v_{1,h}(0)$ yields

$$m_h(v_h,w_h) = \big(\partial_t v_h,w_h\big)_{Q_h} + \sum_{n=1}^{N}\big([v_h]_{n-1},w_{n,h}(t_{n-1})\big)_\Omega\,. \tag{3.5}$$

Together, we obtain for $v_h\in\mathcal{V}_h$ from (3.4) and (3.5)

$$\begin{aligned}
2\,m_h(v_h,v_h) &= m_h(v_h,v_h)+m_h(v_h,v_h) = \sum_{n=1}^{N}\Big(\big([v_h]_{n-1},v_{n,h}(t_{n-1})\big)_\Omega - \big(v_{n,h}(t_n),[v_h]_n\big)_\Omega\Big) \qquad (3.6)\\
&= \big(v_h(0),v_h(0)\big)_\Omega + \big(v_h(T),v_h(T)\big)_\Omega + \sum_{n=1}^{N-1}\Big(\big([v_h]_n,v_{n+1,h}(t_n)\big)_\Omega - \big(v_{n,h}(t_n),[v_h]_n\big)_\Omega\Big) = \sum_{n=0}^{N}\big\|[v_h]_n\big\|_\Omega^2 \ge 0\,.
\end{aligned}$$

The full upwind method in space and time. The discrete bilinear form is defined by

$$b_h(v_h,w_h) = m_h(v_h,w_h) + \int_0^T a_h(v_h(t),w_h(t))\,\mathrm{d}t\,, \qquad v_h,w_h \in \mathcal{V}_h\,. \tag{3.7}$$

From (3.2) and (3.4) we obtain consistency up to the data error by the approximation of the vector field $\mathbf{q}$

$$b_h(v_h,w) = b(v_h,w) + \int_Q v_h(\mathbf{q}-\mathbf{q}_h)\cdot\nabla w\,\mathrm{d}(t,\mathbf{x})\,, \qquad v_h\in\mathcal{V}_h,\ w\in\mathcal{V}^* \tag{3.8}$$

for smooth test functions, and from Lem. 3.1 and (3.6) we get for discontinuous functions $v_h\in\mathcal{V}_h$

$$b_h(v_h,v_h) = \frac12\sum_{n=0}^N \big\|[v_h]_n\big\|_\Omega^2 + \frac12\big(v_h\operatorname{div}\mathbf{q}_h,v_h\big)_Q + \frac12\sum_{F\in\mathcal{F}_h\cap\Omega}\big\||\mathbf{q}_h\cdot\mathbf{n}_K|^{1/2}[v_h]_{K,F}\big\|_{I\times F}^2 + \frac12\big\||\mathbf{q}_h\cdot\mathbf{n}|^{1/2}v_h\big\|_{I\times\partial\Omega}^2. \tag{3.9}$$

The following analysis relies on a stability estimate weighted in time by $d_T(t)=T-t$.

Lemma 3.2. *If* $\operatorname{div}\mathbf{q}_h\ge 0$, *we have*

$$\|v_h\|_{Q_h}^2 + T\,\|v_h(0)\|_{\Omega_h}^2 \le 2\,b_h(v_h,d_Tv_h)\,, \qquad v_h\in\mathcal{V}_h\,.$$

Proof. The identity for $v_h\in\mathcal{V}_h$

$$\begin{aligned}
\|v_h\|_{Q_h}^2 &= -\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\big(v_h(t),v_h(t)\big)_{\Omega_h}\partial_t d_T(t)\,\mathrm{d}t\\
&= \sum_{n=1}^N\bigg(2\int_{t_{n-1}}^{t_n}\big(\partial_t v_h(t),v_h(t)\big)_{\Omega_h}d_T(t)\,\mathrm{d}t - d_T(t_n)\big(v_{n,h}(t_n),v_{n,h}(t_n)\big)_{\Omega_h} + d_T(t_{n-1})\big(v_{n,h}(t_{n-1}),v_{n,h}(t_{n-1})\big)_{\Omega_h}\bigg)\\
&= 2\big(\partial_t v_h,d_Tv_h\big)_{Q_h} + \sum_{n=1}^{N-1}d_T(t_n)\Big(\big(v_{n+1,h}(t_n),v_{n+1,h}(t_n)\big)_{\Omega_h} - \big(v_{n,h}(t_n),v_{n,h}(t_n)\big)_{\Omega_h}\Big) - T\,\|v_{1,h}(t_0)\|_{\Omega_h}^2
\end{aligned}$$

is bounded by

$$\begin{aligned}
&\sum_{n=1}^{N-1}d_T(t_n)\Big(\big(v_{n+1,h}(t_n),v_{n+1,h}(t_n)\big)_{\Omega_h} - \big(v_{n,h}(t_n),v_{n,h}(t_n)\big)_{\Omega_h}\Big)\\
&\qquad= \sum_{n=1}^{N-1}d_T(t_n)\big(v_{n+1,h}(t_n)-v_{n,h}(t_n),v_{n+1,h}(t_n)+v_{n,h}(t_n)\big)_{\Omega_h}\\
&\qquad= \sum_{n=1}^{N-1}d_T(t_n)\big([v_h]_n,v_{n+1,h}(t_n)+v_{n,h}(t_n)\big)_{\Omega_h}\\
&\qquad= \sum_{n=1}^{N-1}d_T(t_n)\Big(2\big([v_h]_n,v_{n+1,h}(t_n)\big)_{\Omega_h} - \big([v_h]_n,[v_h]_n\big)_{\Omega_h}\Big) \le 2\sum_{n=2}^N d_T(t_{n-1})\big([v_h]_{n-1},v_{n,h}(t_{n-1})\big)_\Omega\,,
\end{aligned}$$

so that the assertion follows from $\operatorname{div}\mathbf{q}_h\ge0$ and Lem. 3.1 by

$$\begin{aligned}
\big(v_h,v_h\big)_{Q_h} + T\,\|v_{1,h}(t_0)\|_{\Omega_h}^2 &\le 2\big(\partial_t v_h,d_Tv_h\big)_{Q_h} + 2\sum_{n=1}^N d_T(t_{n-1})\big([v_h]_{n-1},v_{n,h}(t_{n-1})\big)_{\Omega_h} = 2\,m_h(v_h,d_Tv_h)\\
&\le 2\,m_h(v_h,d_Tv_h) + 2\int_0^T d_T(t)a(v_h(t),v_h(t))\,\mathrm{d}t = 2\,b_h(v_h,d_Tv_h)\,.
\end{aligned}$$

□

Remark 3.3. *The* DG *method in time with fixed polynomial degree is equivalent to the Radau Ia collocation method. This is used in [Corallo et al., 2023, Sect. 4.2] to construct an interpolation operator* $\mathcal{I}_h\colon\mathcal{V}_h\longrightarrow V_h$ *which can be applied to extend the stability result in Lem. 3.2 as follows: if* $p_{n,K}=p_n$ *for all* $K\in\mathcal{K}_h$ *and* $n=1,\dots,N$, *we have*

$$\big\|v_h\big\|_Q^2 \le 2\,b_h\big(v_h,\mathcal{I}_h(d_Tv_h)\big)\,, \qquad v_h\in V_h\,.$$

4. Well-posedness and stability for the DG method

In this section we show that a unique discrete solution exists and that the solution is stable with respect to a mesh-dependent DG norm. Therefore, we assume for the H(div)-conforming approximation of the flux vector

A1) $\operatorname{div} \mathbf{q}_h \geq 0$,
A2) for the discontinuous function $\operatorname{div}(v_h \mathbf{q}_h)$ in Q_h we assume $\operatorname{div}(v_h \mathbf{q}_h) \in V_h$,
A3) inflow and outflow boundary characterized from the continuous and the discrete flux coincide, i.e.,

$$\overline{\Gamma}_{\text{in}} = \bigcup_{F \in \mathcal{F}_h^{\text{in}}} \overline{F}\,, \qquad \overline{\Gamma}_{\text{out}} = \bigcup_{F \in \mathcal{F}_h^{\text{out}}} \overline{F} \qquad \text{with} \qquad \mathcal{F}_h^{\text{in}} = \bigcup_{K \in \mathcal{K}_h} \mathcal{F}_K^{\text{in}}\,, \qquad \mathcal{F}_h^{\text{out}} = \bigcup_{K \in \mathcal{K}_h} \mathcal{F}_K^{\text{out}}\,.$$

4.1. Well-posedness of the full-upwind space-time DG discretization for linear transport

Lemma 4.1. *A unique Galerkin approximation $u_h \in V_h$ exists solving*

$$b_h(u_h, v_h) = \ell(v_h)\,, \qquad v_h \in V_h\,. \tag{4.1}$$

Proof. $\dim V_h < \infty$, so it is sufficient to show that $u_h = 0$ is the unique solution of the homogeneous problem $\ell = 0$. Then, $b_h(u_h, u_h) = 0$, so that by (3.6), Lem. 3.1 and assumption A1) all jump terms and boundary traces are vanishing, and we obtain from (3.2), (3.3), (3.4), and (3.5)

$$0 = b_h(u_h, v_h) = \big(u_h, -\partial_t v_h - \mathbf{q}_h \cdot \nabla v_h\big)_{Q_h} = \big(\partial_t u_h + \operatorname{div}(u_h \mathbf{q}_h), v_h\big)_{Q_h}\,, \qquad v_h \in V_h\,. \tag{4.2}$$

Now, defining $v_h = \partial_t u_h + \operatorname{div}(u_h \mathbf{q}_h)$ in Q_h we observe by assumption A2) that $v_h \in V_h$.
This yields $0 = b_h(u_h, v_h) = \|\partial_t u_h + \operatorname{div}(u_h \mathbf{q}_h)\|_{Q_h}^2$, i.e., $\partial_t u_h + \operatorname{div}(u_h \mathbf{q}_h) = 0$. Thus, (4.2) extends to $\mathcal{V}_h$, i.e., we have

$$b_h(u_h, v_h) = \big(\partial_t u_h + \operatorname{div}(u_h \mathbf{q}_h), v_h\big)_{Q_h} = 0\,, \qquad v_h \in \mathcal{V}_h\,.$$

Testing with $v_h = d_T u_h \in \mathcal{V}_h$ we obtain from Lem. 3.2 that $\|u_h\|_{Q_h}^2 \leq 2 b_h(u_h, d_T u_h) = 0$, which finally proves $u_h = 0$. $\square$

4.2. Inf-sup stability of the full-upwind method in the DG norm

For all $v_h \in \mathcal{V}_h$ we define the DG semi-norms and norms

$$\begin{aligned}
|v_h|_{h,\mathrm{DG}} &= \left(\frac{1}{2}\sum_{n=0}^{N} \big\|[v_h]_n\big\|_\Omega^2 + \frac{1}{4}\sum_{K \in \mathcal{K}_h}\sum_{F \in \mathcal{F}_K \cap \Omega} \big\||\mathbf{q}_h \cdot \mathbf{n}_K|^{1/2}[v_h]_{K,F}\big\|_{I_h \times F}^2 + \frac{1}{2}\big\||\mathbf{q}_h \cdot \mathbf{n}|^{1/2} v_h\big\|_{I_h \times \partial\Omega}^2\right)^{1/2},\\
|v_h|_{h,\mathrm{DG}^+} &= \left(\frac{1}{2}\sum_{n=1}^{N}\Big(\big\|v_{n,h}(t_{n-1})\big\|_\Omega^2 + \big\|v_{n,h}(t_n)\big\|_\Omega^2\Big) + \frac{1}{2}\sum_{K \in \mathcal{K}_h}\big\||\mathbf{q}_h \cdot \mathbf{n}_K|^{1/2} v_h\big\|_{I_h \times \partial K}^2\right)^{1/2},\\
\|v_h\|_{h,\mathrm{DG}} &= \sqrt{|v_h|_{h,\mathrm{DG}}^2 + \big\|h^{1/2}\left(\partial_t v_h + \operatorname{div}(v_h \mathbf{q}_h)\right)\big\|_{Q_h}^2}\,, \qquad \|v_h\|_{h,\mathrm{DG}^+} = \sqrt{|v_h|_{h,\mathrm{DG}^+}^2 + \big\|h^{-1/2} v_h\big\|_{Q_h}^2}\,.
\end{aligned} \tag{4.3}$$

By construction, we have $|v_h|_{h,\mathrm{DG}} \leq |v_h|_{h,\mathrm{DG}^+}$, $b_h(v_h, w_h) \leq \|v_h\|_{h,\mathrm{DG}}\|w_h\|_{h,\mathrm{DG}^+}$, and using (3.3) and (3.5), we obtain

$$\big|b_h(v_h, w_h) - \big(\partial_t v_h + \operatorname{div}(v_h \mathbf{q}_h), w_h\big)_{Q_h}\big| \leq |v_h|_{h,\mathrm{DG}}|w_h|_{h,\mathrm{DG}^+}\,, \qquad v_h, w_h \in \mathcal{V}_h\,. \tag{4.4}$$

Analogously to the proof of Lem. 4.1 we observe that $\|v_h\|_{h,\mathrm{DG}} = 0$ implies $v_h = 0$, so that $\|\cdot\|_{h,\mathrm{DG}}$ indeed is a norm.

In order to calibrate the accuracy in space and time, we assume for the maximal mesh size in time $\triangle t$ and in space h

$$c_{\text{ref}} \triangle t \leq h\,, \tag{4.5}$$

where $c_{\text{ref}} > 0$ is a reference velocity depending on the flux vector. Since in the following all estimates depend on c_{ref}, we use in our applications appropriate physical units in space and time so that c_{ref} is a moderate number.

Depending on the space-time mesh regularity (and thus also on c_{ref}) and on $\mathbf{q}_h$, constants $C_{\text{inv}}, C_{\text{tr}} > 0$ exists such that

$$\big\|h^{1/2}\left(\partial_t v_h + \operatorname{div}(v_h \mathbf{q}_h)\right)\big\|_{Q_h} \leq C_{\text{inv}}\big\|h^{-1/2} v_h\big\|_{Q_h}\,, \qquad \|v_h\|_{\partial Q_h} \leq C_{\text{tr}}\big\|h^{-1/2} v_h\big\|_{Q_h}\,, \qquad v_h \in V_h\,. \tag{4.6}$$

Inf-sup stability in the DG norm is introduced for the advection equation in [Di Pietro and Ern, 2011, Lem. 2.35]. This is transferred to symmetric Friedrichs systems in [Corallo et al., 2023] and also applies to linear transport.

Theorem 4.2. *We have*

$$\sup_{w_h \in V_h \setminus \{0\}} \frac{b_h(v_h, w_h)}{\|w_h\|_{h,\mathrm{DG}}} \geq c_{\text{inf-sup}} \|v_h\|_{h,\mathrm{DG}}\,, \qquad v_h \in V_h$$

with $c_{\text{inf-sup}} > 0$ *independent of the mesh size* h *and only depending on the mesh regularity and the flux vector.*

Proof. For given $v_h \in V_h \setminus \{0\}$ we define $z_h = h\big(\partial_t v_h + \mathrm{div}(v_h \mathbf{q}_h)\big)$ in Q_h, and by assumption A2) we have $z_h \in V_h$. The discrete trace inequality (4.6) yields

$$|z_h|_{h,\mathrm{DG}^+} \leq C_{\mathrm{tr}} \|h^{-1/2} z_h\|_{Q_h} = C_{\mathrm{tr}} \|h^{1/2}\big(\partial_t v_h + \mathrm{div}(v_h \mathbf{q}_h)\big)\|_{Q_h} \leq C_{\mathrm{tr}} \|v_h\|_{h,\mathrm{DG}}\,.$$

Together with the inverse inequality (4.6) we obtain

$$\begin{aligned} \|z_h\|^2_{h,\mathrm{DG}} &= |z_h|^2_{h,\mathrm{DG}} + \|h^{1/2}\big(\partial_t z_h + \mathrm{div}(z_h \mathbf{q}_h)\big)\|^2_{Q_h} \\ &\leq |z_h|^2_{h,\mathrm{DG}^+} + C^2_{\mathrm{inv}} \|h^{-1/2} z_h\|^2_{Q_h} \leq \big(C^2_{\mathrm{tr}} + C^2_{\mathrm{inv}}\big) \|v_h\|^2_{h,\mathrm{DG}}\,. \end{aligned} \tag{4.7}$$

Using (4.4) we get

$$\begin{aligned} \big(\partial_t v_h + \mathrm{div}(v_h \mathbf{q}_h), z_h\big)_{Q_h} - b_h(v_h, z_h) &\leq \big|\big(\partial_t v_h + \mathrm{div}(v_h \mathbf{q}_h), z_h\big)_{Q_h} - b_h(v_h, z_h)\big| \\ &\leq |v_h|_{h,\mathrm{DG}} |z_h|_{h,\mathrm{DG}^+} \leq \frac{C^2_{\mathrm{tr}}}{2} |v_h|^2_{h,\mathrm{DG}} + \frac{1}{2C^2_{\mathrm{tr}}} |z_h|^2_{h,\mathrm{DG}^+} \leq \frac{C^2_{\mathrm{tr}}}{2} |v_h|^2_{h,\mathrm{DG}} + \frac{1}{2} \|v_h\|^2_{h,\mathrm{DG}}\,. \end{aligned}$$

Inserting $\|h^{1/2}(\partial_t v_h + \mathrm{div}(v_h \mathbf{q}_h))\|^2_{Q_h} = \big(\partial_t v_h + \mathrm{div}(v_h \mathbf{q}_h), z_h\big)_{Q_h}$ this yields

$$\|v_h\|^2_{h,\mathrm{DG}} = |v_h|^2_{h,\mathrm{DG}} + \big(\partial_t v_h + \mathrm{div}(v_h \mathbf{q}_h), z_h\big)_{Q_h} \leq |v_h|^2_{h,\mathrm{DG}} + \frac{C^2_{\mathrm{tr}}}{2} |v_h|^2_{h,\mathrm{DG}} + \frac{1}{2} \|v_h\|^2_{h,\mathrm{DG}} + b_h(v_h, z_h)\,. \tag{4.8}$$

Using A1) implies $|v_h|^2_{h,\mathrm{DG}} \leq b_h(v_h, z_h)$ by (3.9), and together with (4.8) we obtain, defining $C = 2 + C^2_{\mathrm{tr}}$,

$$\|v_h\|^2_{h,\mathrm{DG}} \leq C |v_h|^2_{h,\mathrm{DG}} + 2\, b_h(v_h, z_h) \leq b_h(v_h, C v_h + 2 z_h)\,. \tag{4.9}$$

From (4.7) we obtain $\|C v_h + 2 z_h\|_{h,\mathrm{DG}} \leq \big(C + 2\sqrt{C^2_{\mathrm{tr}} + C^2_{\mathrm{inv}}}\big) \|v_h\|_{h,\mathrm{DG}}$, and we observe $C v_h + 2 z_h \neq 0$ for $v_h \neq 0$. Together, this yields the assertion by

$$\|v_h\|^2_{h,\mathrm{DG}} \leq \|C v_h + 2 z_h\|_{h,\mathrm{DG}} \frac{b_h(v_h, C v_h + 2 z_h)}{\|C v_h + 2 z_h\|_{h,\mathrm{DG}}} \leq \Big(C + 2\sqrt{C^2_{\mathrm{tr}} + C^2_{\mathrm{inv}}}\Big) \|v_h\|_{h,\mathrm{DG}} \sup_{w_h \in V_h \setminus \{0\}} \frac{b_h(v_h, w_h)}{\|w_h\|_{h,\mathrm{DG}}}\,,$$

i.e., $c_{\text{inf-sup}} = \big(C + 2\sqrt{C^2_{\mathrm{tr}} + C^2_{\mathrm{inv}}}\big)^{-1}$. □

Remark 4.3. *In the numerical experiments we use lowest-order Raviart-Thomas approximations of the flux vector, so that the assumption A2) is satisfied. Nevertheless, this restricts the order of the consistency error* $\mathbf{q}_h - \mathbf{q}$*, so that for higher-order convergence a better approximation of the flux vector is required. But then it is additionally required that for all* $v_h \in V_h$ *with* $\mathrm{div}(v_h \mathbf{q}) \neq 0$ *the discontinuous function* $\mathrm{div}(v_h \mathbf{q}_h)|_{Q_h}$ *is not* L_2*-orthogonal to* V_h*. In analogy to the DPG or HHO method, this may require to modify the discrete ansatz and/or test space.*

5. Convergence of the DG space-time approximation for linear transport

We show for a sequence of meshes that uniform stability in L_2 implies asymptotic convergence, and then, in case that the solution is sufficiently regular, we prove qualitative convergence in the DG norm.

5.1. Asymptotic convergence

For a sequence of mesh sizes $\mathcal{H} = \{h_0, h_1, h_2, \dots\} \subset (0,\infty)$ and $0 \in \overline{\mathcal{H}}$, let $\big(Q_h\big)_{h\in\mathcal{H}}$ be a shape-regular family of space-time meshes and $\big(V_h\big)_{h\in\mathcal{H}}$ the corresponding DG finite element spaces, so that

$$\lim_{h\in\mathcal{H}} \inf_{v_h\in V_h} \big\|v - v_h\big\|_Q = 0\,, \qquad v \in \mathrm{L}_2(Q)\,. \tag{5.1}$$

For $h \in \mathcal{H}$, let $u_h \in V_h$ be the solution of the discrete problem (4.1), i.e.,

$$b_h(u_h, v_h) = \big(u^0, v_h(0)\big)_\Omega - \big(g_{\mathrm{in}}, v_h\big)_{I\times\Gamma_{\mathrm{in}}}\,, \qquad v_h \in V_h\,.$$

Since $\frac12\big\|v_h(0)\big\|_\Omega^2 + \frac12\big\||\mathbf{q}_h\cdot\mathbf{n}|^{1/2}v_h\big\|^2_{I_h\times\Gamma_{\mathrm{in}}} \le \big|v_h\big|^2_{h,\mathrm{DG}}$, inf-sup stability implies for the discrete solution $u_h \in V_h$

$$\begin{aligned} c_{\text{inf-sup}}\big\|u_h\big\|_{h,\mathrm{DG}} &\le \sup_{v_h\in V_h\setminus\{0\}} \frac{b_h(u_h,v_h)}{\big\|v_h\big\|_{h,\mathrm{DG}}} = \sup_{v_h\in V_h\setminus\{0\}} \frac{\big(u^0, v_h(0)\big)_\Omega - \big(g_{\mathrm{in}}, v_h\big)_{I_h\times\Gamma_{\mathrm{in}}}}{\big\|v_h\big\|_{h,\mathrm{DG}}} \\ &\le \sqrt{2}\,\big\|u^0\big\|_\Omega + \sqrt{2}\,\big\||\mathbf{q}_h\cdot\mathbf{n}|^{-1/2}g_{\mathrm{in}}\big\|_{I_h\times\Gamma_{\mathrm{in}}}\,. \end{aligned} \tag{5.2}$$

Lemma 5.1. *Assume for the approximation of the flux vector $\mathbf{q}_h$ and the solution discrete $u_h \in V_h$*

1) *$C_{\mathrm{in}} > 0$ exists such that $\big\|\,|\mathbf{q}_h\cdot\mathbf{n}|^{-1/2}g_{\mathrm{in}}\big\|_{I\times\Gamma_{\mathrm{in}}} \le C_{\mathrm{in}}$ is uniformly bounded for $h \in \mathcal{H}$;*
2) *$C_{\mathbf{q}} > 0$ exists such that $\big\|\,|\mathbf{q}_h\cdot\mathbf{n}|^{1/2}u_h\big\|_{I_h\times\partial\Omega_h} \le C_{\mathbf{q}}\big\|u_h\big\|_{I_h\times\partial\Omega_h}$ for all $h \in \mathcal{H}$;*
3) *strong convergence in L_2, i.e., $\lim_{h\in\mathcal{H}}\big\|\mathbf{q}_h - \mathbf{q}\big\|_\Omega = 0$.*

Then,

a) *$(u_h)_{h\in\mathcal{H}}$ is weakly converging in $\mathrm{L}_2(Q)$;*
b) *the weak limit $u \in \mathrm{L}_2(Q)$ is a weak solution;*
c) *the weak solution $u \in \mathrm{L}_2(Q)$ is unique;*
d) *the weak solution is also a strong solution satisfying $Lu = 0$ in Q, $\mathbf{f}(u)\cdot\mathbf{n} = g_{\mathrm{in}}$ on $I\times\Gamma_{\mathrm{in}}$, and $u(0) = u^0$ in Ω.*

Proof. From 1), 2) and $2\big|u_h\big|^2_{h,\mathrm{DG}^+} = \big\|u_h\big\|^2_{\partial I_h\times\Omega_h} + \big\|\,|\mathbf{q}_h\cdot\mathbf{n}|^{1/2}u_h\big\|^2_{I_h\times\partial\Omega_h} \le \big\|u_h\big\|^2_{\partial I_h\times\Omega_h} + C_{\mathbf{q}}^2\big\|u_h\big\|^2_{I_h\times\partial\Omega_h}$ we obtain by (5.2) and (4.6) for the discrete solution

$$\big\|u_h\big\|_{h,\mathrm{DG}} \le \frac{\sqrt{2}}{c_{\text{inf-sup}}}\Big(\big\|u^0\big\|_\Omega + C_{\mathrm{in}}\Big)\,, \qquad \big\|u_h\big\|_{h,\mathrm{DG}^+} \le \sqrt{1 + \frac12\max\{1, C_{\mathbf{q}}^2\}C_{\mathrm{tr}}^2}\,\big\|h^{-1/2}u_h\big\|_{Q_h}\,, \qquad h \in \mathcal{H}\,, \tag{5.3}$$

and depending on the mesh regularity we get $c_{\mathrm{mr}}\big\|h^{-1/2}u_h\big\|^2_{Q_h} \le h^{-1}\,\big\|u_h\big\|^2_Q$. We set $C_+ = \sqrt{1 + \frac12\max\{1, C_{\mathbf{q}}^2\}C_{\mathrm{tr}}^2}$.

For $h \in \mathcal{H}$ let $t_{n,h}$, $n = 0,\dots,N_h$ be the time steps in $[0,T]$, and set $\triangle t_h = \max t_{n,h}$. For $d_{T,h}(t) = t_{n-1,h}$ in $(t_{n-1,h}, t_{n,h})$ we obtain from (4.5) the bound $0 \le d_{T,h}(t) - d_T(t) \le \triangle t_h \le c_{\mathrm{ref}}^{-1}h$, and together with (5.3) this yields $\big\|(d_T - d_{T,h})u_h\big\|_{h,\mathrm{DG}^+} \le c_{\mathrm{ref}}^{-1}h\,\big\|u_h\big\|_{h,\mathrm{DG}^+} \le c_{\mathrm{ref}}^{-1}h\,C_+\big\|h^{-1/2}u_h\big\|_{Q_h} \le c_{\mathrm{ref}}^{-1}C_+c_{\mathrm{mr}}^{-1/2}h^{1/2}\big\|u_h\big\|_Q$. By Lem. 3.2 we get

$$\begin{aligned} \big\|u_h\big\|^2_Q + T\,\big\|u_h(0)\big\|^2_\Omega &\le 2\,b_h(u_h, d_Tu_h) = 2\,b_h(u_h, d_{T,h}u_h) + 2\,b_h(u_h, (d_T - d_{T,h})u_h) \\ &\le 2\big(u^0, d_{T,h}(0)u_h(0)\big)_\Omega - 2\big(g_{\mathrm{in}}, d_{T,h}u_h\big)_{I\times\Gamma_{\mathrm{in}}} + 2\,\big\|u_h\big\|_{h,\mathrm{DG}}\big\|(d_T - d_{T,h})u_h\big\|_{h,\mathrm{DG}^+} \\ &\le 2T\,\big\|u^0\big\|_\Omega\big\|u_h(0)\big\|_\Omega + 2T\,C_{\mathrm{in}}\big|u_h\big|_{h,\mathrm{DG}} + 2\,c_{\mathrm{ref}}^{-1}C_+c_{\mathrm{mr}}^{-1/2}h^{1/2}\big\|u_h\big\|_{h,\mathrm{DG}}\big\|u_h\big\|_Q \\ &\le 2T\,\big\|u^0\big\|^2_\Omega + \frac{T}{2}\,\big\|u_h(0)\big\|^2_\Omega + 2T\,C_{\mathrm{in}}\big\|u_h\big\|_{h,\mathrm{DG}} + \frac{2C_+^2h}{c_{\mathrm{ref}}^2c_{\mathrm{mr}}}\,\big\|u_h\big\|^2_{h,\mathrm{DG}} + \frac12\,\big\|u_h\big\|^2_Q\,, \end{aligned}$$

so that, using (5.3), the discrete solution is bounded by

$$\begin{aligned} \frac12\,\big\|u_h\big\|^2_Q + \frac{T}{2}\,\big\|u_h(0)\big\|^2_\Omega &\le 2T\,\big\|u^0\big\|^2_\Omega + 2T\,C_{\mathrm{in}}\big\|u_h\big\|_{h,\mathrm{DG}} + \frac{2C_+^2h}{c_{\mathrm{ref}}^2c_{\mathrm{mr}}}\,\big\|u_h\big\|^2_{h,\mathrm{DG}} \\ &\le 2T\,\big\|u^0\big\|^2_\Omega + \frac{2\sqrt{2}T\,C_{\mathrm{in}}}{c_{\text{inf-sup}}}\Big(\big\|u^0\big\|_\Omega + C_{\mathrm{in}}\Big) + \frac{4C_+^2h}{c_{\text{inf-sup}}^2c_{\mathrm{ref}}^2c_{\mathrm{mr}}}\Big(\big\|u^0\big\|_\Omega + C_{\mathrm{in}}\Big)^2. \end{aligned} \tag{5.4}$$

Then a subsequence $\mathcal{H}_0 \subset \mathcal{H}$ with $0 \in \overline{\mathcal{H}}_0$ exists such that $(u_h)_{h\in\mathcal{H}_0}$ is weakly converging to $u \in \mathrm{L}_2(Q)$ and as well $\big(u_h(0)\big)_{h\in\mathcal{H}_0}$ is weakly converging to $u(0) \in \mathrm{L}_2(\Omega)$.

For all smooth test functions $v \in \mathcal{V}^*$ and corresponding discrete approximations $v_h \in V_h$, $h \in \mathcal{H}$ with

$$\lim_{h\in\mathcal{H}} \Big(\|\mathbf{q}_h \cdot \nabla(v - v_h)\|_{Q_h} + \|\partial_t(v - v_h)\|_{Q_h} + \|v - v_h\|_Q + \|v - v_h\|_{\partial Q_h} \Big) = 0$$

we obtain, using (3.8) and strong convergence 3) of $\mathbf{q}_h$, the identity

$$\begin{aligned} b(u,v) &= \lim_{h\in\mathcal{H}_0} b(u_h,v) = \lim_{h\in\mathcal{H}_0} \Big(b_h(u_h,v) + \big(u_h, (\mathbf{q}_h - \mathbf{q})\cdot\nabla v\big)_\Omega \Big) = \lim_{h\in\mathcal{H}_0} b_h(u_h,v_h) \\ &= \lim_{h\in\mathcal{H}_0} \Big(\big(u^0, v_h(0)\big)_\Omega - \big(g_{\mathrm{in}}, v_h\big)_{I_h\times\Gamma_{\mathrm{in}}} \Big) = \big(u^0, v(0)\big)_\Omega - \big(g_{\mathrm{in}}, v\big)_{I\times\Gamma_{\mathrm{in}}}, \end{aligned} \tag{5.5}$$

i.e., u is a weak solution. Moreover, $b(u,v) = \big(u, -\partial_t v - \mathbf{q}\cdot\nabla v\big)_Q = 0$ for $v \in \mathrm{C}^1_{\mathrm{c}}(Q)$, so that the weak derivative $\partial_t u + \operatorname{div}\mathbf{f}(u) \in \mathrm{L}_2(Q)$ exists and $\partial_t u + \operatorname{div}\mathbf{f}(u) = 0$, and we obtain for smooth test functions $v \in \mathcal{V}^*$

$$\begin{aligned} \big(u^0, v(0)\big)_\Omega - \big(g_{\mathrm{in}}, v\big)_{I\times\Gamma_{\mathrm{in}}} &= b(u,v) = \big(u, -\partial_t v - \mathbf{q}\cdot\nabla v\big)_Q \\ &= \big(u, -\partial_t v - \mathbf{q}\cdot\nabla v\big)_Q - \big(\partial_t u + \operatorname{div}\mathbf{f}(u), v\big)_Q = \big(u(0), v(0)\big)_\Omega - \big(\mathbf{f}(u)\cdot\mathbf{n}, v\big)_{I\times\Gamma_{\mathrm{in}}}, \end{aligned}$$

so that $u(0) = u^0$ in Ω and $\mathbf{f}(u)\cdot\mathbf{n} = g_{\mathrm{in}}$ on $I\times\Gamma_{\mathrm{in}}$, i.e., u is also a strong solution.

Next we show uniqueness. Therefore, assume that $\widetilde{u} \in \mathrm{L}_2(Q)$ is also a weak solution, i.e., $b(u - \widetilde{u}, v) = 0$ for $v \in \mathcal{V}^*$. With the same arguments as above, we can show that a solution $v^* \in \mathrm{L}_2(\Omega)$ of the dual problem backward in time

$$-\partial_t v^* - \mathbf{q}\cdot\nabla v^* = u - \widetilde{u} \text{ in } Q\,, \qquad v^*(T) = 0 \text{ in } \Omega\,, \qquad v^* = 0 \text{ on } (0,T)\times\Gamma_{\mathrm{out}}$$

exists. Then, we have for all for $v \in \mathcal{V}^*$

$$0 = b(u - \widetilde{u}, v) = b(u - \widetilde{u}, v^*) + b(u - \widetilde{u}, v - v^*) = \big\|u - \widetilde{u}\big\|_Q^2 + b(u - \widetilde{u}, v - v^*)\,.$$

Since $\inf_{v\in\mathcal{V}^*} \big|b(u - \widetilde{u}, v - v^*)\big| = 0$, this implies $u = \widetilde{u}$. Thus, the weak solution u is unique, and therefore also for the discrete solutions $(u_h)_{h\in\mathcal{H}}$ only one limit exists. □

Remark 5.2. *The assumptions 1) and 2) in Lem. 5.1 are weighted with $|\mathbf{q}_h\cdot\mathbf{n}|^{1/2}$ and, by duality with $|\mathbf{q}_h\cdot\mathbf{n}|^{-1/2}$. This is a consequence of the upwind flux and the corresponding choice of the DG semi-norm, so that the numerical analysis uses weighted L_2 norms [Dörfler et al., 2023, Thm. 2.8] for the traces in space, as it is also used for the boundary semi-norm in [Ern and Guermond, 2021, Chap. 57.3.2].*

Here, the objective is to show that the standard approach used for hyperbolic conservation laws applies to our setting: the discrete solutions are uniformly bounded, and by consistency of the discretization the weak limit is a weak solution.

Alternatively, the existence of a weak solutions can by shown by the LL^* approach, as is is proved for our application class in [Dörfler et al., 2023, Thm. 2.8]. Therefore, one shows that $L^*(\mathcal{V}^*) \subset \mathrm{L}_2(Q)$ is surjective, see [Dörfler et al., 2023, Lem. 2.12] for the wave equation.

Let V^* be the closure of $\mathcal{V}^*$ in $\mathrm{H}(L^*;Q) = \{v \in \mathrm{L}_2(Q) \colon L^*v \in \mathrm{L}_2(Q)\}$ with respect to the graph norm. Then it can be shown that $L^*\colon V^* \longrightarrow \mathrm{L}_2(Q)$ is an isomorphism. This is the basis to apply the theory in [Broersen et al., 2018], where the linear transport equation in space is analyzed; defining $\hat{\mathbf{q}} = (1,\mathbf{q})^\top$ directly transfers the analysis in space for the DPG approximation into our space-time setting and provides an alternative for the numerical analysis in a different norm.

Remark 5.3. *Explicit time stepping schemes for hyperbolic equations require a CFL condition; nevertheless, for qualitative estimates it is required that the transport velocity c_{ref} and the relation of mesh size and time steps $h/\triangle t_h$ is well balanced. This is related with assumption 2) in Lem. 5.1: the constant $C_{\mathbf{q}}$ directly corresponds to the transport velocity, and if assumption A2) is not valid, the transport velocity cannot be bounded.*

Remark 5.4. *In applications where the flux is determined by Eqn. (2.7), the flux vector is discretized and approximated by $\mathbf{q}_h$, and the results in Lem. 5.1 depend on uniform bounds for $\mathbf{q}_h$. Alternatively, one can assume sufficient regularity for $\mathbf{q}$, e.g., $\mathbf{q} \in \mathrm{W}^{1,\infty}(\mathrm{div};\Omega)$ and $|\mathbf{q}|^{-1} \in \mathrm{L}_\infty(\Omega)$, see [Broersen et al., 2018].*

Remark 5.5. *The proof of Lem. 5.1 follows the classical Lax equivalence theorem for linear equations: stability and consistency of numerical approximations imply convergence. For a class of nonlinear conservation laws this is generalized to $\mathcal{K}$-convergence in [Feireisl et al., 2020].*

5.2. Qualitative convergence estimates in the DG norm

Theorem 5.6. *Assume that the solution of (2.6) is sufficiently smooth satisfying $u \in \mathrm{H}^r(Q)$ with*

$$1 \le r \le \min_{n,K}\{p_{n,K}, s_{n,K}\} + 1 .$$

Then, the error for the discrete solution $u_h \in V_h$ of (4.1) is bounded by

$$\|u - u_h\|_{h,\mathrm{DG}} \le C_1 h^{r-1/2} \|\mathrm{D}^r u\|_Q + C_2 T h^{-1/2} \| \operatorname{div} \mathbf{f}(u) - \operatorname{div} \mathbf{f}_h(u)\|_Q$$

with $C_1, C_2 > 0$ depending on the mesh regularity, the polynomial degrees in V_h, and the flux vector.

Proof. By the assumption $u \in \mathrm{H}^1(Q)$, for the solution all jump terms are vanishing, so that by (3.3) and (3.5)

$$\begin{aligned} b_h(u, w_h) &= \big(\partial_t u + \operatorname{div} \mathbf{f}_h(u), w_h\big)_Q + \big(u(0), w_h(0)\big)_\Omega - \big(\mathbf{f}(u)\cdot\mathbf{n}, w_h\big)_{I\times\Gamma_{\mathrm{in}}} \\ &= \big(\partial_t u + \operatorname{div} \mathbf{f}_h(u), w_h\big)_Q - \big(\partial_t u + \operatorname{div} \mathbf{f}(u), w_h\big)_Q + \big(u^0, w_h(0)\big)_\Omega - \big(g_{\mathrm{in}}, w_h\big)_{I\times\Gamma_{\mathrm{in}}} \\ &= \big(\operatorname{div} \mathbf{f}_h(u) - \operatorname{div} \mathbf{f}(u), w_h\big)_Q + b_h(u_h, w_h)\,, \qquad w_h \in V_h\,, \end{aligned} \tag{5.6}$$

i.e., we have Galerkin orthogonality up to the data error.

By (5.3) we obtain $\|w_h\|^2_{h,\mathrm{DG}^+} \le C_+ \|h^{-1/2} w_h\|^2_Q \le C_+ c_{\mathrm{mr}}^{-1} h^{-1} \|w_h\|^2_Q$, so that by Lem. 3.2

$$\begin{aligned} \|w_h\|_Q^2 &\le 2\, b_h(w_h, d_T w_h) \le 2\, \|w_h\|_{h,\mathrm{DG}} \|d_T w_h\|_{h,\mathrm{DG}^+} \le 2T\, \|w_h\|_{h,\mathrm{DG}} \|w_h\|_{h,\mathrm{DG}^+} \\ &\le 2T\, \|w_h\|_{h,\mathrm{DG}} \sqrt{C_+ c_{\mathrm{mr}}^{-1} h^{-1}}\, \|w_h\|_Q \le 2T^2 C_+ c_{\mathrm{mr}}^{-1} h^{-1}\, \|w_h\|^2_{h,\mathrm{DG}} + \frac{1}{2}\|w_h\|_Q^2\,, \end{aligned}$$

i.e., $\|w_h\|_Q \le C_3 T h^{-1/2} \|w_h\|_{h,\mathrm{DG}}$ with $C_3 = 2\sqrt{C_+ c_{\mathrm{mr}}^{-1}}$. Thus, the consistency term can be bounded by

$$\big(\operatorname{div} \mathbf{f}_h(u) - \operatorname{div} \mathbf{f}(u), w_h\big)_Q \le \|\operatorname{div} \mathbf{f}_h(u) - \operatorname{div} \mathbf{f}(u)\|_Q \|w_h\|_Q \le C_3 T h^{-1/2} \|\operatorname{div} \mathbf{f}_h(u) - \operatorname{div} \mathbf{f}(u)\|_Q \|w_h\|_{h,\mathrm{DG}}\,.$$

Using Thm. 4.2, (5.6) and continuity of the bilinear form $b_h(\cdot,\cdot)$ in the DG norms, we obtain for all $v_h \in V_h$

$$\begin{aligned} c_{\text{inf-sup}} \|u_h - v_h\|_{h,\mathrm{DG}} &\le \sup_{w_h \in V_h\setminus\{0\}} \frac{b_h(u_h - v_h, w_h)}{\|w_h\|_{h,\mathrm{DG}}} = \sup_{w_h \in V_h\setminus\{0\}} \frac{b_h(u - v_h, w_h) + \big(\operatorname{div} \mathbf{f}_h(u) - \operatorname{div} \mathbf{f}(u), w_h\big)_Q}{\|w_h\|_{h,\mathrm{DG}}} \\ &\le \|u - v_h\|_{h,\mathrm{DG}^+} + C_3 T h^{-1/2} \|\operatorname{div} \mathbf{f}_h(u) - \operatorname{div} \mathbf{f}(u)\|_Q\,. \end{aligned} \tag{5.7}$$

Let $v_h = \Pi_h^{\mathrm{Cl}} u$ be a stable quasi-interpolation of Clement-type [Bartels, 2016, Sect. 4.4.2] with

$$h^{-1}\|u - \Pi_h^{\mathrm{Cl}} u\|_Q + \|\partial_t\big(u - \Pi_h^{\mathrm{Cl}} u\big)\|_{Q_h} + \|\operatorname{div} \mathbf{f}(u) - \operatorname{div} \mathbf{f}(\Pi_h^{\mathrm{Cl}} u)\|_{Q_h} \le C_{\mathrm{Cl}}\, \|\mathrm{D} u\|_Q\,, \qquad \mathrm{D}u = \begin{pmatrix} \partial_t u \\ \nabla u \end{pmatrix}$$

with a constant $C_{\mathrm{Cl}} > 0$ depending on the mesh regularity, the polynomial degrees in V_h, and $\mathbf{q}$.

For $r \le \min\{p, s\} + 1$ the interpolation estimates [Di Pietro and Ern, 2011, Lem. 1.59] yield

$$\|u - \Pi_h^{\mathrm{Cl}} u\|_Q + h^{1/2} \|u - \Pi_h^{\mathrm{Cl}} u\|_{\partial Q_h} + h \|\partial_t(u - \Pi_h^{\mathrm{Cl}} u) + \operatorname{div} \mathbf{f}_h(u - \Pi_h^{\mathrm{Cl}} u)\|_{Q_h} \le C_{\mathrm{int}} h^r \|\mathrm{D}^r u\|_Q$$

with $C_{\mathrm{int}} > 0$ depending on the mesh regularity. Then, the result follows from (5.7) by

$$\begin{aligned} \|u - u_h\|_{h,\mathrm{DG}} &\le \|u - \Pi_h^{\mathrm{Cl}} u\|_{h,\mathrm{DG}} + \|u_h - \Pi_h^{\mathrm{Cl}} u\|_{h,\mathrm{DG}} \\ &\le \|u - \Pi_h^{\mathrm{Cl}} u\|_{h,\mathrm{DG}} + c_{\text{inf-sup}}^{-1} \Big(\|u - \Pi_h^{\mathrm{Cl}} u\|_{h,\mathrm{DG}^+} + C_3 T h^{-1/2} \|\operatorname{div} \mathbf{f}_h(u) - \operatorname{div} \mathbf{f}(u)\|_Q \Big) \\ &\le C_1 h^{r-1/2} \|\mathrm{D}^r u\|_Q + C_2 T h^{-1/2} \|\operatorname{div} \mathbf{f}_h(u) - \operatorname{div} \mathbf{f}(u)\|_Q\,. \end{aligned}$$

□

Remark 5.7. *The error analysis in the graph norm $\sqrt{\|v\|_Q^2 + \|\partial_t v + \operatorname{div} \mathbf{f}(v)\|_Q^2}$ is more restrictive with respect to the regularity assumptions, e.g., $\mathcal{O}(h)$ convergence requires for the solution the regularity $u \in \mathrm{H}^2(Q)$, cf. [Dörfler et al., 2016].*

Remark 5.8. *If the flux vector* $\mathbf{q}$ *is sufficiently smooth, the consistency term can be estimated by*

$$\big\| \operatorname{div} \mathbf{f}_h(u) - \operatorname{div} \mathbf{f}(u)\big\|_Q \le \big\| \operatorname{div} \mathbf{q} - \operatorname{div} \mathbf{q}_h\big\|_\infty \big\|u\big\|_Q + \big\|\mathbf{q} - \mathbf{q}_h\big\|_\infty \big\|\nabla u\big\|_Q\,.$$

In general applications, only $\mathbf{q} \in \mathrm{H}(\operatorname{div},\Omega)$ *can be assumed. Then, the consistency error can be estimated in case of higher regularity of the solution: if* $u \in \mathrm{L}_2(0,T;\mathrm{W}^1_r(\Omega))$ *with* $r > 2$*, we obtain*

$$\big\| \operatorname{div} \mathbf{f}_h(u) - \operatorname{div} \mathbf{f}(u)\big\|_Q \le \big\| \operatorname{div}(\mathbf{q} - \mathbf{q}_h)\big\|_\Omega \big\|u\big\|_Q + \big\|\mathbf{q} - \mathbf{q}_h\big\|_{\mathrm{L}_{r/(r-1)}(\Omega;\mathbb{R}^d)} \big\|\nabla u\big\|_{\mathrm{L}_2(0,T;\mathrm{L}_r(\Omega;\mathbb{R}^d))}\,.$$

Since we use lowest-order Raviart-Thomas approximations of the flux vector, this is limited by $\mathcal{O}(h)$*, cf. Rem. 4.3.*

6. An error indicator of residual type

In case that the solution u is sufficiently smooth and the traces on ∂Q_h are well-defined, the error $u - u_h$ in the DG semi-norm takes the form

$$\begin{aligned}\big|u - u_h\big|_{h,\mathrm{DG}} = \Bigg(&\frac12\big\|u^0 - u_h(0)\big\|_\Omega^2 + \frac12\sum_{n=1}^{N-1}\big\|[u_h]_n\big\|_\Omega^2 + \frac12\big\|u(T) - u_h(T)\big\|_\Omega^2 \\ &+ \frac14\sum_{K\in\mathcal{K}_h}\sum_{F\in\mathcal{F}_K\cap\Omega}\big\||\mathbf{q}_h\cdot\mathbf{n}_K|^{1/2}[u_h]_{K,F}\big\|^2_{I_h\times F} + \frac12\big\||\mathbf{q}_h\cdot\mathbf{n}|^{1/2}(u-u_h)\big\|^2_{I_h\times\partial\Omega}\Bigg)^{1/2}.\end{aligned} \tag{6.1}$$

On the inflow boundary we can estimate $\big\||\mathbf{q}_h\cdot\mathbf{n}|^{1/2}(u-u_h)\big\|_{I_h\times\Gamma_{\mathrm{in}}} \le \big\||\mathbf{q}_h\cdot\mathbf{n}|^{-1/2}\mathbf{f}_h(u-u_h)\big\|_{I_h\times\Gamma_{\mathrm{in}}}$ and thus

$$\frac12\big\||\mathbf{q}_h\cdot\mathbf{n}|^{1/2}(u-u_h)\big\|^2_{I_h\times\Gamma_{\mathrm{in}}} \le \big\||\mathbf{q}_h\cdot\mathbf{n}|^{-1/2}(g_{\mathrm{in}} - \mathbf{f}_h(u_h)\cdot\mathbf{n})\big\|^2_{I_h\times\Gamma_{\mathrm{in}}} + \big\||\mathbf{q}_h\cdot\mathbf{n}|^{-1/2}(\mathbf{f}(u) - \mathbf{f}_h(u))\cdot\mathbf{n}\big\|^2_{I_h\times\Gamma_{\mathrm{in}}}\,,$$

and for the DG norm we get

$$\begin{aligned}\big\|u - u_h\big\|^2_{h,\mathrm{DG}} &= \big|u - u_h\big|^2_{h,\mathrm{DG}} + \big\|h^{1/2}\left(\partial_t(u-u_h) + \operatorname{div}\mathbf{f}_h(u-u_h)\right)\big\|^2_{Q_h} \\ &\le \big|u - u_h\big|^2_{h,\mathrm{DG}} + 2\,\big\|h^{1/2}\left(\partial_t u_h + \operatorname{div}\mathbf{f}_h(u_h)\right)\big\|^2_{Q_h} + 2\,\big\|h^{1/2}\,\operatorname{div}(\mathbf{f}(u) - \mathbf{f}_h(u))\big\|^2_{Q_h}\,.\end{aligned} \tag{6.2}$$

Up to the error $u_h - u$ at final time T and on the outflow boundary in (6.1) and without estimating the approximation error of the discrete flux in (6.1) and (6.2), this is explicitly evaluated by the residual error indicator

$$\eta_{\mathrm{res},h} = \Big(\sum_{R\in\mathcal{R}_h}\eta^2_{\mathrm{res},R}\Big)^{1/2}$$

given by the local contributions

$$\begin{aligned}\eta^2_{\mathrm{res},R} = \eta^2_{\mathrm{res},n,K} &+ 2h_K\,\big\|\partial_t u_h + \operatorname{div}\mathbf{f}_h(u_h)\big\|^2_R \\ &+ \frac14\sum_{F\in\mathcal{F}_K\cap\Omega}\big\||\mathbf{q}_h\cdot\mathbf{n}_K|^{1/2}[u_h]_{K,F}\big\|^2_{I_h\times F} + \big\||\mathbf{q}_h\cdot\mathbf{n}|^{-1/2}(g_{\mathrm{in}} - \mathbf{f}_h(u_h)\cdot\mathbf{n})\big\|^2_{I_h\times(\partial K\cap\Gamma_{\mathrm{in}})}\end{aligned}$$

for $R = (t_{n-1},t_n)\times K$, $n = 1,\dots,N$ and residuals at t_{n-1} and t_n

$$\begin{aligned}\eta^2_{\mathrm{res},1,K} &= \frac12\big\|u^0 - u_h(0)\big\|^2_K + \frac14\big\|[u_h]_1\big\|^2_K\,, && R = (0,t_1)\times K\,,\\ \eta^2_{\mathrm{res},n,K} &= \frac14\big\|[u_h]_{n-1}\big\|^2_K + \frac14\big\|[u_h]_n\big\|^2_K\,, && R = (t_{n-1},t_n)\times K\,,\ 1<n<N\,,\\ \eta^2_{\mathrm{res},N,K} &= \frac14\big\|[u_h]_{N-1}\big\|^2_K\,, && R = (t_{N-1},T)\times K\,.\end{aligned}$$

Lemma 6.1. *Let* $u \in \mathrm{L}_2(Q)$ *be the weak solution of (2.6) and* $u_h \in V_h$ *the discrete solution of (4.1). Then, if* u *is sufficiently smooth, the error in the* DG *norm is bounded by*

$$\begin{aligned}\big\|u - u_h\big\|_{h,\mathrm{DG}} \le \Big(&\eta^2_{\mathrm{res},h} + \frac12\big\|(u_h(T) - u(T))\big\|^2_\Omega + \frac12\big\||\mathbf{q}_h\cdot\mathbf{n}|^{1/2}(u-u_h)\big\|^2_{I_h\times\Gamma_{\mathrm{out}}} \\ &+ 2\,\big\|h^{1/2}\,\operatorname{div}(\mathbf{f}(u) - \mathbf{f}_h(u))\big\|^2_{Q_h} + \big\||\mathbf{q}_h\cdot\mathbf{n}|^{-1/2}(\mathbf{f}(u) - \mathbf{f}_h(u))\cdot\mathbf{n}\big\|^2_{I_h\times\Gamma_{\mathrm{in}}}\Big)^{1/2}.\end{aligned}$$

7. Numerical experiments

The convergence results are illustrated by two numerical experiments in the domain $\Omega = (-0.5, 0.5)^2$ and $T = 1$.

In the first test we consider a linear flux vector and the smooth solution

$$u(t,\mathbf{x}) = \begin{cases} \exp\left(-\dfrac{|\mathbf{x}-\mathbf{y}(t)|}{1/8 - |\mathbf{x}-\mathbf{y}(t)|}\right), & |\mathbf{x}-\mathbf{y}(t)| < 1/8\,, \\ 0\,, & \text{else,} \end{cases} \qquad \mathbf{y}(t) = (1/4)\begin{pmatrix}\cos(2\pi t)\\ \sin(2\pi t)\end{pmatrix}, \qquad \mathbf{q}(\mathbf{x}) = \begin{pmatrix}-2\pi x_2\\ 2\pi x_1\end{pmatrix},$$

cf. Fig. 1. Starting with linear elements $p = s = 1$, we clearly observe linear convergence in the DG norm which is very closely approximated by the error indicator. This is compared with results of a p-adaptive strategy, see [Dörfler et al., 2023][Chap. 4.6] for the algorithmic details. Then, two refinement/derefinement steps improve the order of convergence, using quadratic and cubic approximations of the smooth solution and only finite volumes where the solution vanishes.

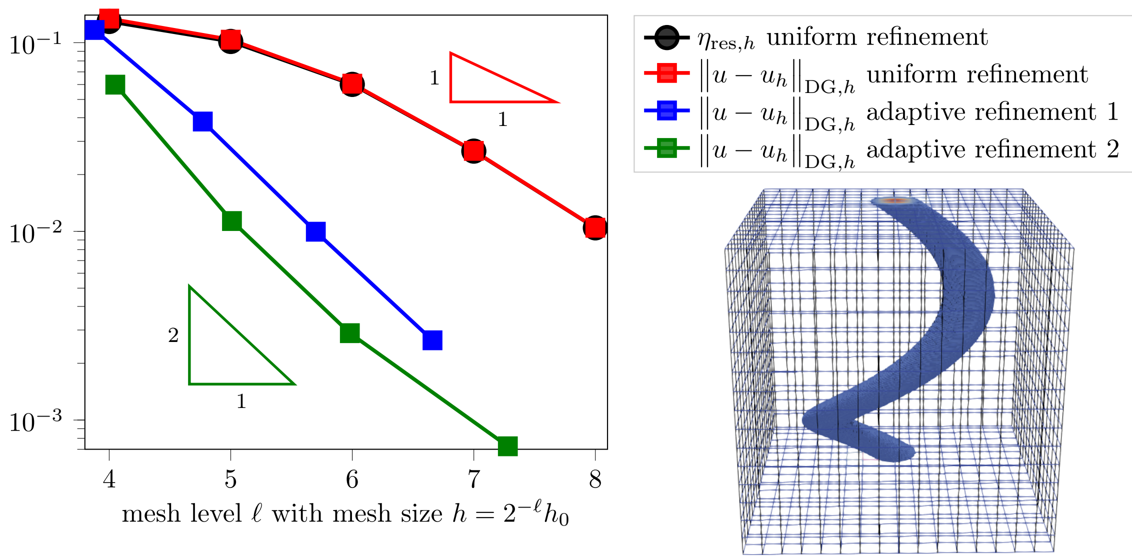

FIGURE 1. Convergence test for smooth test example.

In the second test we approximate the discontinuous piecewise constant solution of the Riemann problem

$$u^0(\mathbf{x}) = \begin{cases} 1\,, & x_1 + 2x_2 + 5/9 < 0\,, \\ 0\,, & \text{else,} \end{cases} \qquad \mathbf{q} = \begin{pmatrix} 2\\ 1/2\end{pmatrix}$$

where the interface cannot be resolved by the mesh, cf. Fig. 2. Since the solution is not in $\mathrm{H}^1(Q)$, Thm. 5.6 cannot be applied. Numerically we observe convergence with a rate smaller than $1/2$, and adaptivity can reduce only the problem size for a fixed accuracy but not the order of convergence.

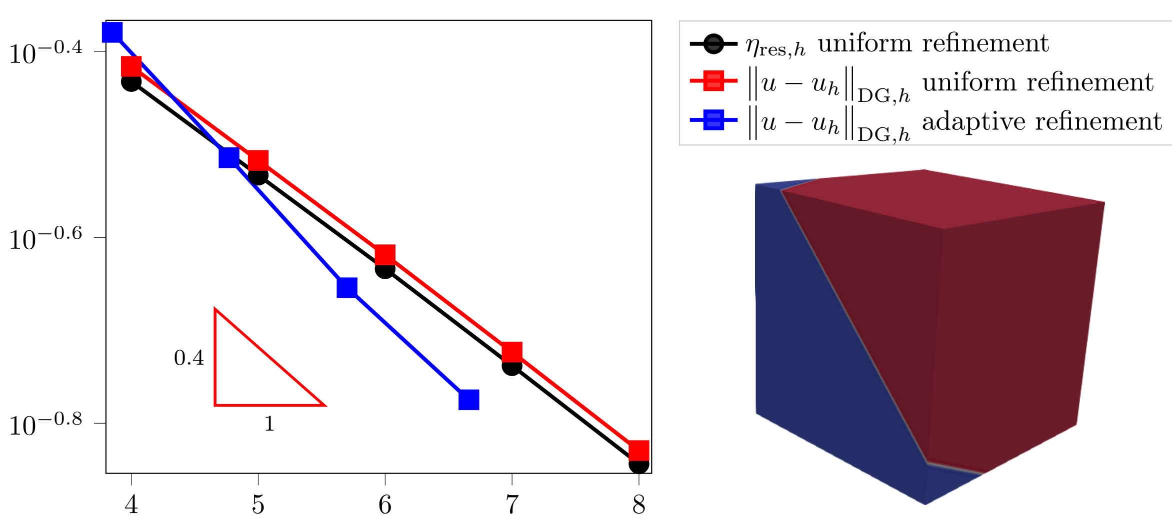

FIGURE 2. Convergence test for the Riemann problem.

8. Hybridization to the space-time skeleton

By (3.2) and (3.4) we obtain for the discrete bilinear form (3.7) the dual representation

$$b_h(v_h, w_h) = -\big(v_h, \partial_t w_h + \mathbf{q}_h \cdot \nabla w_h\big)_{Q_h} - \sum_{n=1}^{N} \big(v_{n,h}(t_n), [w_h]_n\big)_\Omega + \sum_{K\in\mathcal{K}_h} \sum_{F\in\mathcal{F}_K} \big(\mathbf{f}^{\mathrm{up}}_{K,F}(v_h) \cdot \mathbf{n}_K, w_{h,K}\big)_{I_h\times F}$$

For the hybridization we introduce appropriate trace variables in space and time. Therefore, we define the linear mapping $\Pi_h^{\mathrm{up}} \colon V_h \longrightarrow \mathrm{L}_2(\partial Q_h \cap Q)$ onto the space-time skeleton in the interior of the space-time domain by

$$\Pi_h^{\mathrm{up}} v_h = \begin{cases} v_{n,h}(t_n) & \text{in } \Omega_h \text{ for } n = 1, \dots, N-1\,, \\ v_{h,K} & \text{on } I_h \times F,\ F \in \mathcal{F}_K^{\mathrm{out}} \cap \Omega\,, \\ v_{h,K_F} & \text{on } I_h \times F,\ F \in \mathcal{F}_K^{\mathrm{in}} \cap \Omega\,, \end{cases}$$

and this defines the discrete space $\widehat{V}_h = \Pi_h^{\mathrm{up}}(V_h)$.

For the discontinuous discrete solution $u_h = (u_{h,R})_{R\in\mathcal{R}_h} \in V_h$ we define $\widehat{u}_h = \Pi_h^{\mathrm{up}} u_h$. If $\widehat{u}_h = \Pi_h^{\mathrm{up}} u_h$ is known, in every space-time cell $R = (t_{n-1}, t_n) \times K$, $n = 1, \dots, N$ the local solutions $u_{h,R} \in V_{h,R}$ can be recovered from the local equations

$$b_{h,R}(u_{h,R}, w_{h,R}) = -\widehat{b}_{h,R}(\widehat{u}_h, w_{h,R}) + \ell_{h,R}(w_{h,R})\,, \qquad w_{h,R} \in V_{h,R} \tag{8.1}$$

with

$$b_{h,R}(v_{h,R}, w_{h,R}) = -\big(v_{h,R}, \partial_t w_{h,R} + \mathbf{q}_h \cdot \nabla w_{h,R}\big)_R + \big(v_{h,R}(t_n), w_{h,R}(t_n)\big)_K + \big(\mathbf{f}_h(v_{h,R}) \cdot \mathbf{n}_K, w_{h,R}\big)_{I_{n,h}\times(F\cap\Gamma_{\mathrm{out}})}\,,$$

for $n > 1$

$$\widehat{b}_{h,R}(\widehat{u}_h, w_{h,R}) = -\big(\widehat{u}_h(t_{n-1}), w_{h,R}(t_{n-1})\big)_K + \big(\mathbf{f}_h(\widehat{u}_h) \cdot \mathbf{n}_K, w_{h,R}\big)_{I_{n,h}\times(\partial K\cap\Omega)}\,, \quad \ell_{h,R}(\widehat{u}_h, w_{h,R}) = -\big(g_{\mathrm{in}}, w_{h,R}\big)_{I_{n,h}\times(F\cap\Gamma_{\mathrm{in}})}$$

and for $n = 1$

$$\widehat{b}_{h,R}(\widehat{u}_h, w_{h,R}) = \big(\mathbf{f}_h(\widehat{u}_h) \cdot \mathbf{n}_K, w_{h,R}\big)_{I_{n,h}\times(\partial K\cap\Omega)}\,, \qquad \ell_{h,R}(\widehat{u}_h, w_{h,R}) = \big(u^0, w_{h,R}(0)\big)_\Omega - \big(g_{\mathrm{in}}, w_{h,R}\big)_{I_{n,h}\times(F\cap\Gamma_{\mathrm{in}})}\,.$$

Lem. 4.1 transfers to $b_{h,R}(\cdot,\cdot)$ and $V_{h,R}$, so that the local problems (8.1) have a unique solution.

Defining $B_h \in \mathcal{L}(V_h, V_h')$, $\widehat{B}_h \in \mathcal{L}(\widehat{V}_h, V_h')$ and $\ell_h \in V_h'$ by

$$\langle B_h v_h, w_h\rangle = \sum_{R\in\mathcal{R}_h} b_{h,R}(v_{h,R}, w_{h,R})\,, \qquad \langle \widehat{B}_h \widehat{v}_h, w_h\rangle = \sum_{R\in\mathcal{R}_h} \widehat{b}_{h,R}(\widehat{v}_{h,R}, w_{h,R})\,, \qquad \langle \ell_h, w_h\rangle = \sum_R \ell_{h,R}(w_{h,R})$$

yields $(B_h + \widehat{B}_h \Pi_h^{\mathrm{up}}) u_h = \ell_h$ for the discrete equation (4.1), so that

$$B_h u_h + \widehat{B}_h \widehat{u}_h = \ell_h\,, \qquad \Pi_h^{\mathrm{up}} u_h = \widehat{u}_h \qquad \text{and} \qquad u_h = B_h^{-1}\big(\ell_h - \widehat{B}_h \widehat{u}_h\big)\,.$$

Lemma 8.1. *$\widehat{u}_h \in \widehat{V}_h$ is the unique solution of the sparse linear system in $\widehat{V}_h$*

$$\big(\mathrm{id}_{\widehat{V}_h} + \Pi_h^{\mathrm{up}} B_h^{-1} \widehat{B}_h\big)\widehat{u}_h = \Pi_h^{\mathrm{up}} B_h^{-1} \ell_h\,. \tag{8.2}$$

Proof. By construction we have $\widehat{u}_h = \Pi_h^{\mathrm{up}} B_h^{-1}\big(\ell_h - \widehat{B}_h \widehat{u}_h\big)$, so that $\widehat{u}_h$ solves (8.2). It remains to show uniqueness, i.e., that $\mathrm{id}_{\widehat{V}_h} + \Pi_h^{\mathrm{up}} B_h^{-1} \widehat{B}_h \in \mathcal{L}(\widehat{V}_h, \widehat{V}_h)$ is injective. Therefore, consider a solution $\widehat{v}_h \in \widehat{V}_h$ of the homogeneous problem $\big(\mathrm{id}_{\widehat{V}_h} + \Pi_h^{\mathrm{up}} B_h^{-1} \widehat{B}_h\big)\widehat{v}_h = 0$. For $v_h = -B_h^{-1} \widehat{B}_h \widehat{v}_h$ we get $\widehat{v}_h = \Pi_h^{\mathrm{up}} v_h$. This yields $B_h v_h + \widehat{B}_h \Pi_h^{\mathrm{up}} v_h = \widehat{B}_h \widehat{v}_h - \widehat{B}_h \widehat{v}_h = 0$ and thus $b_h(v_h, w_h) = \sum_{R\in\mathcal{R}_h} \Big(b_{h,R}(v_{h,R}, w_{h,R}) + \widehat{b}_{h,R}(\Pi_h^{\mathrm{up}} v_h, w_{h,R})\Big) = \langle B_h v_h + \widehat{B}_h \Pi_h^{\mathrm{up}} v_h, w_h\rangle = 0$ for all $w_h \in V_h$, so that $v_h = 0$ by Lem. 4.1. This proves injectivity and, since $\dim \widehat{V}_h < \infty$ also unique solvability of (8.2). $\square$

Remark 8.2. *Here we only introduce a reduction to skeleton degrees of freedom $\widehat{V}_h = \Pi_h^{\mathrm{up}}(V_h)$ for the upwind* DG *method, see also [Bui-Thanh, 2015] for more applications. An extension corresponding to the* ideal DPG method *yields a formulation in $\mathcal{V}_h \times \widehat{V}_h$ and defines the reduced system in $\widehat{V}_h$ by exact solutions of the transport equation in the space-time cells R. Corresponding to the* practical DPG method *a fine space $V_h \subset \mathcal{V}_h$ and a smaller subspace $\widehat{V}_h \subset \Pi_h^{\mathrm{up}}(V_h)$ can be selected [Gopalakrishnan and Qiu, 2014]. Then, the local transport problems are solved very accurately, but the overall conservation property depends on $\widehat{V}_h$. Note that the* DPG *method yields a symmetric positive definite system for the skeleton reduction also for hyperbolic problems while the hybridization of the upwind* DG *method is not symmetric.*

9. Conclusion

We established for the transport equation well-posedness and stability of the space-time DG method with full upwind, and we showed for solutions in $\mathrm{H}^r(Q)$ convergence in the DG norm. This implies for every time step $u(t_{n,h})$ regularity in $\mathrm{H}^{r-1/2}(\Omega)$ and convergence of order $\mathcal{O}(h^{r-1/2})$ in $\mathrm{L}_2(\Omega)$.

First numerical results show that the error control in the DG norm with the indicator η_h is very efficient in the case of smooth solutions and is also quite accurate for solutions with low regularity. In the next steps, the efficiency of different adaptive strategies and a comparison with the hybrid formulation will be considered. In case of smooth solutions, it is required to extend the convergence analysis to higher-order approximations of the flux vector. In addition we will investigate how to estimate the remaining term $u_h(T) - u(T)$ in the error control, e.g., by solving the (discrete) adjoint problem backward in time.

The hybrid formulation of the full upwind DG method for linear transport is closely related to the DPG method since it ensures weak continuity on the element faces in space and in time. On the other hand, the hybrid reduction is not symmetric, and the analysis in the mesh-dependent DG norm is different from the analysis with respect to the graph norm and yields estimates with lower regularity requirements. This relies on the construction of the DG norm which also provides a bound for the jump terms on the element faces. If a corresponding error analysis including inf-sup stability with respect to a suitable mesh-dependent norm can be transferred to the general DPG setting remains an open question which will be investigated in future work.

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Data availability statement

The code used for the numerical experiments are available in `https://git.scc.kit.edu/mpp/mpp/-/tags/3.1.4`.

Conflict of interest

The author declare that there is no conflict of interest.

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