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Adaptive parallel space-time discontinuous Galerkin Methods for the linear transport equation

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Application: Transport in porous media

Let $\Omega \subset (0,1)^2$ be a simplified configuration intersecting the top earth layers of sand with different permeability $\kappa \colon \overline{\Omega} \longrightarrow (\kappa_{\text{min}}, \kappa_{\text{max}}) \subset (0,\infty)$.

In this configuration $(0,1)^2 \setminus \Omega$ are impermeable stones and rocks.

Let $\Gamma_{\text{top}} = (0,1) \times \{1\} \subset \partial\Omega$ be the surface where it is raining, and let $\Gamma_{\text{bottom}} = [0,1] \times \{0\} \subset \partial\Omega$ be the groundwater level.

First step

Compute the flux vector

$$\mathbf{q} \colon \overline{\Omega} \longrightarrow \mathbb{R}^2.$$

Therefore, solve $-\nabla \cdot \kappa \nabla p = 0$ in Ω
with $p = p_{\text{D}}$ on $\Gamma_{\text{D}} = \Gamma_{\text{bottom}}$
and $-\kappa \nabla p = g_{\text{N}}$ on $\Gamma_{\text{N}} = \partial\Omega \setminus \Gamma_{\text{D}}$.
This defines $\mathbf{q} = -\kappa \nabla p$.

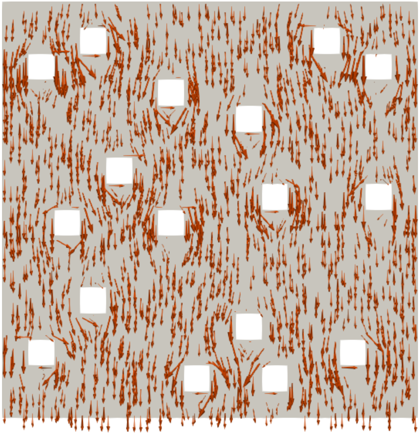
Second step

Starting with a pollution density

$$u_0 \colon \Omega \longrightarrow \mathbb{R}$$

compute the transport along $\mathbf{q}$

$$u \colon (0,T) \times \Omega \longrightarrow \mathbb{R}.$$



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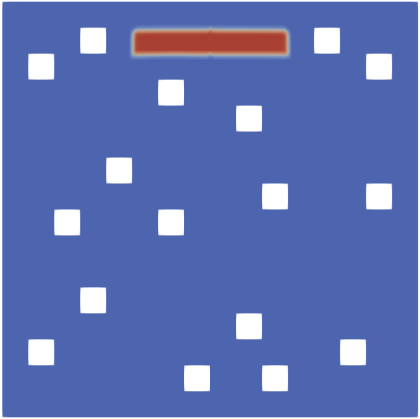

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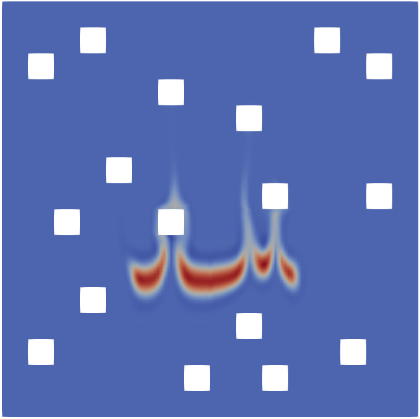

Second step

Starting with a pollution density

$$u_0\colon \Omega \longrightarrow \mathbb{R}$$

compute the transport along $\mathbf{q}$

$$u\colon (0,T) \times \Omega \longrightarrow \mathbb{R}.$$

Application: Transport in porous media

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain in space with Lipschitz boundary.
Let $\mathbf{n}$ be the outer normal vector.

First step
For given Dirichlet data p_{D} on $\Gamma_{\mathsf{D}} \subset \partial\Omega$, Neumann data g_{N} on $\Gamma_{\mathsf{N}} = \partial\Omega \setminus \Gamma_{\mathsf{D}}$, and permeability $\kappa \in \mathrm{L}_\infty(\Omega)$ determine the pressure $p \in \mathrm{H}^1(\Omega)$ and the flux vector $\mathbf{q} = -\kappa\nabla p \in \mathrm{H}(\mathrm{div}, \Omega)$ with $\mathrm{div}\, \mathbf{q} = 0$, $p = p_{\mathsf{D}}$ on Γ_{D}, and and $\mathbf{n} \cdot \mathbf{q} = g_{\mathsf{N}}$ on Γ_{N}.

Second step
Let $I = (0, T)$ be a time interval, and $Q = (0, T) \times \Omega$ the space-time cylinder.
For given flux function $\mathbf{f}(u) = u\mathbf{q}$ and initial pollution density $u_0 \in \mathrm{L}_2(\Omega)$ determine $u \in \mathrm{H}^1(0, T; \mathrm{L}_2(\Omega))$ solving the linear hyperbolic conservation law

$$\partial_t u + \mathrm{div}\, \mathbf{f}(u) = 0 \quad \text{in } Q\,,$$

$u(0) = u_0$ in Ω, and $u = u_{\mathsf{in}}$ on $(0, T) \times \Gamma_{\mathsf{in}} = \{\mathbf{x} \in \partial\Omega \colon \mathbf{n}(\mathbf{x}) \cdot \mathbf{q}(\mathbf{x}) > 0\}$.

Application
In our application the $p_{\mathsf{D}} = 0$ on Γ_{bottom} and $g_{\mathsf{N}} = 1$ on Γ_{top} we obtain $p \geq 0$ in Ω by monotonicity principle and thus $\mathbf{n} \cdot \mathbf{f}(u) \geq 0$ on $(0, T) \times \Gamma_{\mathsf{bottom}}$.

For the goal functional $G_{\mathsf{out}}(t) = \displaystyle\int_{\Gamma_{\mathsf{bottom}}} \mathbf{n} \cdot \mathbf{f}(u(t))\, \mathrm{d}\mathbf{a}$ we have $\displaystyle\int_\Omega u_0\, \mathrm{d}\mathbf{x} = \int_0^\infty G_{\mathsf{out}}(t)\, \mathrm{d}t$.

Application: Transport in porous media

For $h \in \mathcal{H} \subset (0, h_0)$ let $\Omega_h = \bigcup_{K \in \mathcal{K}_h} K$ be meshes where the elements $K \subset \Omega$, $K \in \mathcal{K}_h$ are open triangles/tetrahedra.

Let $F \in \mathcal{F}_K$ be the faces of the element K, and we set $\mathcal{F}_h = \bigcup_{K \in \mathcal{K}_h} \mathcal{F}_K$, so that $\partial\Omega_h = \overline{\bigcup_{F \in \mathcal{F}_h} F}$ is the skeleton in space, $\overline{\Omega} = \Omega_h \cup \partial\Omega_h$, and $\Gamma_{\mathsf{D}} = \overline{\bigcup_{F \in \mathcal{F}_h \cap \Gamma_{\mathsf{in}}} F}$.

First step

We use Raviart-Thomas finite elements. Let

$$W_h = \Big\{ (p_h, \mathbf{q}_h) \in \mathrm{L}_2(\Omega) \times \mathrm{H}^1(\mathrm{div}, \Omega) \colon p_h|_K \in \mathrm{P}_0(K) \text{ for all } K \in \mathcal{K}_h \text{ and } \mathbf{q}_h|_K \in \mathrm{P}_1(K)^d \text{ such that } \mathbf{n}_F \cdot \mathbf{q}_h|_F \in \mathrm{P}_0(F) \text{ for all } F \in \mathcal{F}_K \Big\},$$

and compute $(p_h, \mathbf{q}_h) \in W_h$ with $\mathbf{n} \cdot \mathbf{q}_h = g_{\mathsf{N}}$ on Γ_{N} solving

$$\int_\Omega \Big(\boldsymbol{\kappa}^{-1} \mathbf{q}_h \cdot \boldsymbol{\phi}_h - p_h \,\mathrm{div}\, \boldsymbol{\phi}_h - \mathrm{div}\, \mathbf{q}_h \psi_h \Big) \,\mathrm{d}\mathbf{x} = -\int_{\Gamma_{\mathsf{D}}} p_{\mathsf{D}} \mathbf{n} \cdot \boldsymbol{\phi} \,\mathrm{d}\mathbf{a}$$

for all $(\psi_h, \boldsymbol{\phi}_h) \in W_h$ with $\mathbf{n} \cdot \boldsymbol{\phi}_h = 0$ on Γ_{N}.

Second step

Full-upwind finite volumes in space and implicit Euler method in time or discontinuous Galerkin in space with implicit midpoint rule in time or adaptive parallel space-time discontinuous Galerkin.

Discontinuous Galerkin for the transport equation

For $v_h, w_h \in \mathcal{S}_h = \big\{ v_h \in \mathrm{L}_2(\Omega) \colon v_{h,K} = v_h|_K \in \mathrm{P}(K) \text{ for all } K \in \mathcal{K}_h \big\}$
and the discrete flux $\mathbf{f}_h(v_h) = v_h \, \mathbf{q}_h$ we observe

$$\big(\operatorname{div} \mathbf{f}_h(v_h), w_h\big)_{\Omega_h} = \sum_{K \in \mathcal{K}_h} \Big(-\big(\mathbf{f}_h(v_{h,K}), \nabla w_{h,K}\big)_K + \sum_{F \in \mathcal{F}_K} \big(\mathbf{f}_h(v_{h,K}) \cdot \mathbf{n}_K, w_{h,K}\big)_F \Big).$$

For discontinuous functions $v_h \in \mathcal{S}_h$, this is approximated by the upwind flux

$$a_h(v_h, w_h) = \sum_{K \in \mathcal{K}_h} \Big(-\big(\mathbf{f}_h(v_{h,K}), \nabla w_{h,K}\big)_K + \sum_{F \in \mathcal{F}_K} \big(\mathbf{f}^{\text{up}}_{K,F}(v_h) \cdot \mathbf{n}_K, w_{h,K}\big)_F \Big),$$

$$\mathbf{f}^{\text{up}}_{K,F}(v_h) = \begin{cases} \mathbf{f}_h(v_{h,K}), & F \in \mathcal{F}_K^{\text{out}} \\ \mathbf{f}_h(v_{h,K_F}), & F \in \mathcal{F}_K^{\text{in}} \setminus \Gamma_{\text{in}} \\ \mathbf{0}, & F \in \mathcal{F}_K^{\text{in}} \cap \Gamma_{\text{in}} \end{cases} \quad \text{with} \quad \begin{cases} \mathcal{F}_K^{\text{out}} = \big\{F \in \mathcal{F}_K \colon \mathbf{q}_h \cdot \mathbf{n}_K \geq 0 \text{ on } F\big\} \\ \mathcal{F}_K^{\text{in}} = \big\{F \in \mathcal{F}_K \colon \mathbf{q}_h \cdot \mathbf{n}_K < 0 \text{ on } F\big\} \end{cases}$$

with $\overline{F} = \partial K \cap K_F$ and assuming that $\mathbf{q}_h \cdot \mathbf{n}_K$ is constant on F.

By construction, this is *consistent*, i.e., for $v_h, w_h \in \mathcal{S}_h \cap \mathrm{H}^1(\Omega)$ we have

$$\begin{aligned} \big(\operatorname{div} \mathbf{f}_h(v_h), w_h\big)_\Omega &= a_h(v_h, w_h) + \big(\mathbf{n} \cdot \mathbf{f}_h(v_h), w_h\big)_{\Gamma_{\text{out}}} \\ &= -\big(\mathbf{f}_h(v_h), \nabla w_h\big)_\Omega + \big(\mathbf{n} \cdot \mathbf{f}_h(v_h), w_h\big)_{\partial\Omega}. \end{aligned}$$

Discontinuous Galerkin for the transport equation

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with $\overline{F} = \partial K \cap K_F$ and assuming that $\mathbf{q}_h \cdot \mathbf{n}_K$ is constant on F.

Lemma

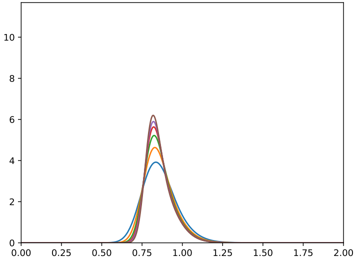
$$a_h(v_h, v_h) = \frac{1}{2} \int_{\Omega_h} v_h^2 \operatorname{div} \mathbf{q}_h \,\mathrm{d}\mathbf{x} + \frac{1}{4} \sum_{K \in \mathcal{K}_h} \sum_{F \in \mathcal{F}_K \cap \Omega} \int_F \big([v_h]_{K,F}\big)^2 \,|\mathbf{q}_h \cdot \mathbf{n}_K| \,\mathrm{d}\mathbf{a}$$
$$+ \frac{1}{2} \int_{\partial\Omega} v_h^2 \,|\mathbf{q}_h \cdot \mathbf{n}| \,\mathrm{d}\mathbf{a}$$

for $v_h \in \mathcal{S}_h$ *with* $[v_h]_{K,F} = v_{h,K_F} - v_{h,K}$.

Classical lowest-order approximation

For $N \in \mathbb{N}_0$ we use fixed time steps $\triangle t = T/N$ and $t_n = n\triangle t$.
Starting with $u_h^0 \in S_h^{p=0} = \big\{v_h \in \mathrm{L}_2(\Omega) \colon v_{h,K} = v_h|_K \in \mathrm{P}_0(K) \text{ for all } K \in \mathcal{K}_h\big\}$
we determine for $n = 1, \dots, N$ by the implicit Euler method

$$u_h^n \in S_h^{p=0} \colon \qquad (u_h^n, v_h)_\Omega + \triangle t\, a_h(u_h^n, v_h) = (u_h^{n-1}, v_h)_\Omega\,, \qquad v_h \in S_h^{p=0}\,.$$



goal functional $G_{\text{out}}(u_h)$ for $t \in [0,2]$
and $k = 1, \dots, 7$

number of triangles $\#\mathcal{K}_h = 122\,368$, number of time steps $N_k = 160 \cdot 2^k$
finite volume approximation with implicit Euler for $\triangle t_k = 2^{-k} \triangle t_0$, $\triangle t_0 = 1/160$

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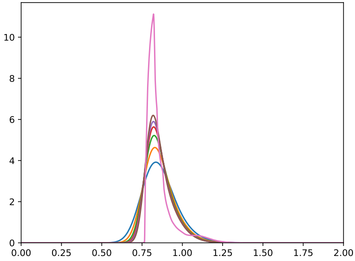

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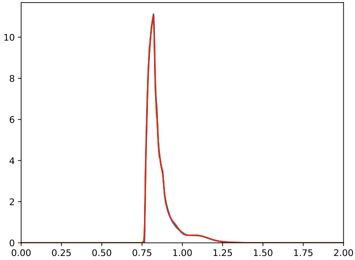
comparison with reference solution on 489 472 triangles, $p = 2$, $k = 7$

DG convergence

For $N \in \mathbb{N}_0$ we use fixed time steps $\triangle t = T/N$ and $t_n = n\triangle t$.
Starting with $u_h^0 \in S_h^{p=2} = \big\{v_h \in \mathrm{L}_2(\Omega) \colon v_{h,K} = v_h|_K \in \mathrm{P}_2(K) \text{ for all } K \in \mathcal{K}_h\big\}$
we determine for $n = 1, \dots, N$ by the implicit midpoint rule

$$u_h^n \in S_h^{p=2}\colon \qquad (u_h^n, v_h)_\Omega + \triangle t a_h(0.5(u_h^n + u_h^{n-1}), v_h) = (u_h^{n-1}, v_h)_\Omega\,, \quad v_h \in S_h^{p=2}.$$



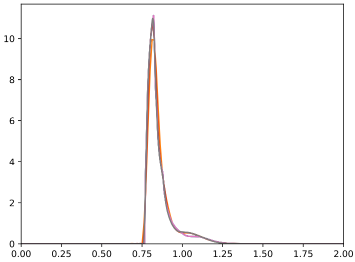
goal functional $G_{\text{out}}(u_h)$ for $t \in [0, 2]$
and $k = 4, \dots, 7$

number of triangles $\#\mathcal{K}_h = 122\,368$, number of time steps $N_k = 160 \cdot 2^k$
DG approximation with implicit midpoint rule for $\triangle t_k = 2^{-k}\triangle t_0$, $\triangle t_0 = 1/160$
comparison with reference solution on 489 472 triangles, $p = 2$, $k = 7$

Adaptive space-time convergence

For $N \in \mathbb{N}_0$ we use fixed time steps $\triangle t = T/N$ and $t_n = n\triangle t$.
Starting in $V_h^{p=0} = \big\{v_h \in \mathrm{L}_2(Q)\colon v_{n,h,K} = w_h|_{(t_{n-1},t_n)\times K} \in \mathrm{P}_0((t_{n-1},t_n)\times K)\big\}$
we determine adaptively $u_h \in V_h \subset \mathrm{L}_2(Q)$ with $u_{n,h,K} \in \mathrm{P}((t_{n-1},t_n)\times K)$ with

$$b_h(u_h, v_h) = (u_0, v_h)_\Omega\,, \qquad v_h \in V_h\,.$$



goal functional $G_{\text{out}}(u_h)$ for $t \in [0,2]$
and $k = 0,\dots,3$

number of triangles $\#\mathcal{K}_h = 122\,368$, number of time steps $N_k = 160\cdot 2^k$
DoFs 37$\cdot 10^6$, 94$\cdot 10^6$, 127$\cdot 10^6$, 247$\cdot 10^6$, L_1 Error 0.1262, 0.0570, 0.0236, 0.0146
comparison with reference solution on 489 472 triangles, $p = 2$, $k = 7$

The DG finite element space in the space-time cylinder

For $0 = t_0 < t_1 < \cdots < t_N = T$, we define time intervals $I_{n,h} = (t_{n-1}, t_n)$ and

$$I_h = (t_0, t_1) \cup \cdots \cup (t_{N-1}, t_N) \subset I = (0, T)\,, \qquad \partial I_h = \{t_0, t_1, \dots, t_{N-1}, t_N\}\,.$$

Let $\mathcal{K}_h$ be a mesh so that $\Omega_h = \bigcup_{K \in \mathcal{K}_h} K$ is a decomposition in space into open cells $K \subset \Omega \subset \mathbb{R}^d$. We obtain a decomposition into $R = I_{n,h} \times K$ and

$$Q_h = I_h \times \Omega_h = \bigcup_{n=1}^{N} Q_{n,h} = \bigcup_{R \in \mathcal{R}_h} R\,, \qquad Q_{n,h} = \bigcup_{K \in \mathcal{K}_h} I_{n,h} \times K \ \subset \ I_{n,h} \times \Omega\,.$$

In order to calibrate the accuracy in space and time, we assume

$$c_{\text{ref}} \triangle t \le h\,, \qquad \triangle t = \max(t_n - t_{n-1})\,, \qquad h = \max \operatorname{diam}(K)\,,$$

where $c_{\text{ref}} > 0$ is a reference velocity depending on the flux vector $\mathbf{q}$.

For the DG discretization in space and time

$$b_h(v_h, w_h) = m_h(v_h, w_h) + \int_0^T a_h(v_h(t), w_h(t))\,\mathrm{d}t\,,$$
$$v_h, w_h \in \mathcal{V}_h \subset \Big\{ v_h \in \mathrm{L}_2(Q) \colon v_{n,h,K} = v_h|_{(t_{n-1},t_n)\times K} \in \mathrm{P}((t_{n-1}, t_n) \times K) \Big\}$$

we establish *consistency* and *stability* in a mesh-dependent DG norm.

Full upwind in time

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For $v_h, w_h \in \mathrm{H}^1(Q_h)$ we obtain after integration by parts in all intervals $I_{n,h}$

$$\big(\partial_t v_h, w_h\big)_{Q_h} = \sum_{n=1}^{N} \Big(- \big(v_{n,h}, \partial_t w_{n,h}\big)_{Q_{n,h}} + \big(v_{n,h}(t_n), w_{n,h}(t_n)\big)_\Omega - \big(v_{n,h}(t_{n-1}), w_{n,h}(t_{n-1})\big)_\Omega \Big) \,.$$

With $[w_h]_n = w_{n+1,h}(t_n) - w_{n,h}(t_n)$, $n = 1, \dots, N-1$ and $[w_h]_N = -w_{N,h}(t_N)$ set

$$m_h(v_h, w_h) = \sum_{n=1}^{N} \Big(- \big(v_{n,h}, \partial_t w_{n,h}\big)_{Q_{n,h}} - \big(v_{n,h}(t_n), [w_h]_n\big)_\Omega \Big) \,.$$

Integrating by parts and defining $[v_h]_0 = v_{1,h}(0)$ yields

$$m_h(v_h, w_h) = \big(\partial_t v_h, w_h\big)_{Q_h} + \sum_{n=1}^{N} \big([v_h]_{n-1}, w_{n,h}(t_{n-1})\big)_\Omega \,.$$

Together, we obtain

$$m_h(v_h, v_h) = \frac{1}{2} \sum_{n=0}^{N} \big\| [v_h]_n \big\|_\Omega^2 \geq 0 \,, \qquad v_h \in \mathrm{H}^1(Q_h) \,.$$

For functions $w \in \mathrm{H}^1\big(0,T; \mathrm{L}_2(\Omega)\big)$ we get *consistency*, i.e.,

$$m_h(v_h, w) = \big(\partial_t v_h, w\big)_{Q_h} + \big(v_h(0), w(0)\big)_\Omega = -\big(v_h, \partial_t w\big)_Q - \big(v_h(T), w(T)\big)_\Omega \,.$$

The full upwind method in space and time

For the discrete bilinear form

$$b_h(v_h, w_h) = m_h(v_h, w_h) + \int_0^T a_h(v_h(t), w_h(t))\,\mathrm{d}t\,, \qquad v_h, w_h \in \mathcal{V}_h\,.$$

we have consistency up to the data error

$$b_h(v_h, w) = b(v_h, w) + \int_Q v_h(\mathbf{q} - \mathbf{q}_h) \cdot \nabla w\,\mathrm{d}(t, \mathbf{x})\,, \qquad v_h \in \mathcal{V}_h,\ w \in \mathcal{V}^*$$

for smooth test fuctions in $\mathcal{V}^* = \big\{w \in \mathrm{C}^1(\overline{Q})\colon w = 0$ on $\{T\} \times \Omega \cup (0, T) \times \Gamma_{\text{out}}\big\}$, and combining the results for $m_h(v_h, v_h)$ and $a_h(v_h, v_h)$ we obtain

$$\begin{aligned} b_h(v_h, v_h) &= \frac{1}{2}\sum_{n=0}^{N} \big\|[v_h]_n\big\|_\Omega^2 + \frac{1}{2}\big(v_h \operatorname{div} \mathbf{q}_h, v_h\big)_Q \\ &\quad + \frac{1}{2}\sum_{F \in \mathcal{F}_h \cap \Omega} \big\||\mathbf{q}_h \cdot \mathbf{n}_K|^{1/2}[v_h]_{K,F}\big\|_{I \times F}^2 + \frac{1}{2}\big\||\mathbf{q}_h \cdot \mathbf{n}|^{1/2} v_h\big\|_{I \times \partial\Omega}^2\,. \end{aligned}$$

Lemma

Define $d_T(t) = T - t$. *If* $\operatorname{div} \mathbf{q}_h \geq 0$*, we have*

$$\|v_h\|_{Q_h}^2 + T\,\|v_h(0)\|_{\Omega_h}^2 \leq 2\,b_h(v_h, d_T v_h)\,, \qquad v_h \in \mathcal{V}_h\,.$$

Well-posedness of the DG method

We assume for the approximation of the flux vector

A1) $\operatorname{div} \mathbf{q}_h \geq 0$,

A2) $\operatorname{div}(v_h \mathbf{q}_h) \in V_h$ for all $v_h \in V_h$,

A3) inflow and outflow boundary characterized from $\mathbf{q}$ and $\mathbf{q}_h$ coincide:

$$\overline{\Gamma}_{\text{in}} = \bigcup_{F \in \mathcal{F}_h^{\text{in}}} \overline{F}\,, \quad \overline{\Gamma}_{\text{out}} = \bigcup_{F \in \mathcal{F}_h^{\text{out}}} \overline{F} \quad \text{with} \quad \mathcal{F}_h^{\text{in}} = \bigcup_{K \in \mathcal{K}_h} \mathcal{F}_K^{\text{in}}\,, \quad \mathcal{F}_h^{\text{out}} = \bigcup_{K \in \mathcal{K}_h} \mathcal{F}_K^{\text{out}}\,.$$

Lemma

A unique Galerkin approximation $u_h \in V_h$ exists solving

$$b_h(u_h, v_h) = \ell(v_h)\,, \qquad v_h \in V_h\,.$$

For a sequence of mesh sizes $\mathcal{H} = \{h_0, h_1, h_2, \cdots\} \subset (0, \infty)$ and $0 \in \overline{\mathcal{H}}$,
let $\big(Q_h\big)_{h \in \mathcal{H}}$ be a shape-regular family of space-time meshes and
let $\big(V_h\big)_{h \in \mathcal{H}}$ the corresponding DG finite element spaces, so that

$$\lim_{h \in \mathcal{H}} \inf_{v_h \in V_h} \big\|v - v_h\big\|_Q = 0\,, \qquad v \in \mathrm{L}_2(Q)\,.$$

Inf-sup stability of the full-upwind method

For all $v_h \in \mathcal{V}_h$ we define the DG semi-norm and mesh-dependent DG norm

$$
\begin{aligned}
\left|v_h\right|_{h,\mathrm{DG}}^2 &= \frac{1}{2}\sum_{n=0}^{N}\left\|[v_h]_n\right\|_\Omega^2 \\
&\quad + \frac{1}{4}\sum_{K\in\mathcal{K}_h}\sum_{F\in\mathcal{F}_K\cap\Omega}\left\|\left|\mathbf{q}_h\cdot\mathbf{n}_K\right|^{1/2}[v_h]_{K,F}\right\|_{I_h\times\partial K}^2 + \frac{1}{2}\left\|\left|\mathbf{q}_h\cdot\mathbf{n}\right|^{1/2}v_h\right\|_{I_h\times\partial\Omega}^2\,,
\end{aligned}
$$

$$
\left\|v_h\right\|_{h,\mathrm{DG}} = \sqrt{\left|v_h\right|_{h,\mathrm{DG}}^2 + \left\|h^{1/2}\left(\partial_t v_h + \operatorname{div}(v_h\mathbf{q}_h)\right)\right\|_{Q_h}^2}\,.
$$

Theorem

A constant $c_{\text{inf-sup}} > 0$ independent of the mesh size h exists such that

$$
\sup_{w_h\in V_h\setminus\{0\}}\frac{b_h(v_h,w_h)}{\left\|w_h\right\|_{h,\mathrm{DG}}} \geq c_{\text{inf-sup}}\left\|v_h\right\|_{h,\mathrm{DG}}\,, \qquad v_h\in V_h\,.
$$

Convergence of the DG space-time approximation

The space-time trace estimate

$$\frac{1}{2}\left\|v_h(0)\right\|_\Omega^2 + \frac{1}{2}\left\||\mathbf{q}_h \cdot \mathbf{n}|^{1/2} v_h\right\|_{I_h \times \Gamma_{\text{in}}}^2 \leq \left|v_h\right|_{h,\mathrm{DG}}^2 , \qquad v_h \in V_h ,$$

and inf-sup stability implies for the discrete solution $u_h \in V_h$

$$c_{\text{inf-sup}} \left\|u_h\right\|_{h,\mathrm{DG}} \leq 2\left\|u^0\right\|_\Omega + 2\left\||\mathbf{q}_h \cdot \mathbf{n}|^{-1/2} g_{\text{in}}\right\|_{I_h \times \Gamma_{\text{in}}} .$$

Lemma

Assume for the approximation of the flux vector

1) $C_{\mathrm{in}} > 0$ *exists s.t.* $\left\| |\mathbf{q}_h \cdot \mathbf{n}|^{-1/2} g_{\mathrm{in}}\right\|_{I \times \Gamma_{in}} \leq C_{\mathrm{in}}$ *is uniformly bounded for* $h \in \mathcal{H}$*;*
2) $C_{\mathbf{q}} > 0$ *exists such that* $\left\| |\mathbf{q}_h \cdot \mathbf{n}|^{1/2} u_h\right\|_{I_h \times \partial\Omega_h} \leq C_{\mathbf{q}} \left\|u_h\right\|_{I_h \times \partial\Omega_h}$ *for all* $h \in \mathcal{H}$*;*
3) *strong convergence in* L_2*, i.e.,* $\lim_{h \in \mathcal{H}} \left\|\mathbf{q}_h - \mathbf{q}\right\|_\Omega = 0$.

Then,

a) $(u_h)_{h \in \mathcal{H}}$ *is weakly converging in* $\mathrm{L}_2(Q)$*;*
b) *the weak limit* $u \in \mathrm{L}_2(Q)$ *is a weak solution;*
c) *the weak solution* $u \in \mathrm{L}_2(Q)$ *is unique;*
d) *the weak solution is also a strong solution and* $\partial_t u + \operatorname{div} \mathbf{f}(u) \in \mathrm{L}_2(Q)$.

Qualitative convergence estimates in the DG norm

Theorem

Assume that the solution u is sufficiently smooth satisfying $u \in \mathrm{H}^r(Q)$ with

$$1 \le r \le \min_{n,K}\{p_{n,K}, s_{n,K}\} + 1\,.$$

Then, the error for the discrete solution $u_h \in V_h$ is bounded by

$$\big\|u - u_h\big\|_{h,\mathrm{DG}} \le C_1 h^{r-1/2}\,\big\|\mathrm{D}^r u\big\|_Q + C_2 T h^{-1/2}\big\|\operatorname{div}\mathbf{f}(u) - \operatorname{div}\mathbf{f}_h(u)\big\|_Q$$

with $C_1, C_2 > 0$ depending on mesh regularity, the polynomial degrees in V_h, and $\mathbf{q}$.

$\big\|u(t_n) - u_h(t_n)\big\|_\Omega \le t_n\big\|u - u_h\big\|_{h,\mathrm{DG}} = \mathcal{O}(h^{r-1/2})$ is optimal for $u(t_n) \in \mathrm{H}^{r-1/2}(Q)$.

If the flux vector $\mathbf{q}$ is sufficiently smooth, the consistency term can be estimated by

$$\big\|\operatorname{div}\mathbf{f}_h(u) - \operatorname{div}\mathbf{f}(u)\big\|_Q \le \big\|\operatorname{div}\mathbf{q} - \operatorname{div}\mathbf{q}_h\big\|_\infty\big\|u\big\|_Q + \big\|\mathbf{q} - \mathbf{q}_h\big\|_\infty\big\|\nabla u\big\|_Q\,.$$

In general only $\mathbf{q} \in \mathrm{H}(\mathrm{div},\Omega)$ can be assumed. If $u \in \mathrm{L}_2(0,T;\mathrm{W}^1_r(\Omega))$ with $r > 2$

$$\big\|\operatorname{div}\mathbf{f}_h(u) - \operatorname{div}\mathbf{f}(u)\big\|_Q \le \big\|\operatorname{div}(\mathbf{q} - \mathbf{q}_h)\big\|_\Omega\big\|u\big\|_Q + \big\|\mathbf{q} - \mathbf{q}_h\big\|_{\mathrm{L}_{r/(r-1)}(\Omega;\mathbb{R}^d)}\big\|\nabla u\big\|_{\mathrm{L}_2(0,T;\mathrm{L}_r(\Omega;\mathbb{R}^d))}\,.$$

Error control

The error $u - u_h$ in the DG semi-norm takes the form

$$\big|u - u_h\big|_{h,\mathrm{DG}} = \Bigg(\frac{1}{2}\big\|u^0 - u_h(0)\big\|_\Omega^2 + \frac{1}{2}\sum_{n=1}^{N-1}\big\|[u_h]_n\big\|_\Omega^2 + \frac{1}{2}\big\|u(T) - u_h(T)\big\|_\Omega^2$$

$$+ \frac{1}{4}\sum_{K\in\mathcal{K}_h}\sum_{F\in\mathcal{F}_K\cap\Omega}\big\||\mathbf{q}_h\cdot\mathbf{n}_K|^{1/2}[u_h]_{K,F}\big\|_{I_h\times\partial K}^2 + \frac{1}{2}\big\||\mathbf{q}_h\cdot\mathbf{n}|^{1/2}(u - u_h)\big\|_{I_h\times(\Gamma_{\text{in}}\cup\Gamma_{\text{out}})}^2$$

and for the DG norm we get

$$\big\|u - u_h\big\|_{h,\mathrm{DG}}^2 \le \big|u - u_h\big|_{h,\mathrm{DG}}^2 + 2\,\big\|h^{1/2}\,(\partial_t u_h + \operatorname{div}\mathbf{f}_h(u_h))\big\|_{Q_h}^2$$
$$+ 2\,\big\|h^{1/2}\,\operatorname{div}(\mathbf{f}(u) - \mathbf{f}_h(u))\big\|_{Q_h}^2 .$$

Error control

The error $u - u_h$ in the DG semi-norm takes the form

$$\big|u-u_h\big|_{h,\mathrm{DG}} = \Bigg(\frac{1}{2}\big\|u^0 - u_h(0)\big\|_\Omega^2 + \frac{1}{2}\sum_{n=1}^{N-1}\big\|[u_h]_n\big\|_\Omega^2 + \frac{1}{2}\big\|u(T)-u_h(T)\big\|_\Omega^2$$

$$+\frac{1}{4}\sum_{K\in\mathcal{K}_h}\sum_{F\in\mathcal{F}_K\cap\Omega}\big\||\mathbf{q}_h\cdot\mathbf{n}_K|^{1/2}[u_h]_{K,F}\big\|_{I_h\times\partial K}^2 + \frac{1}{2}\big\||\mathbf{q}_h\cdot\mathbf{n}|^{1/2}(u-u_h)\big\|_{I_h\times(\Gamma_{\text{in}}\cup\Gamma_{\text{out}})}^2$$

and for the DG norm we get

$$\big\|u-u_h\big\|_{h,\mathrm{DG}}^2 \le \big|u-u_h\big|_{h,\mathrm{DG}}^2 + 2\,\big\|h^{1/2}\left(\partial_t u_h + \operatorname{div}\mathbf{f}_h(u_h)\right)\big\|_{Q_h}^2$$
$$+\,2\,\big\|h^{1/2}\ \operatorname{div}(\mathbf{f}(u)-\mathbf{f}_h(u))\big\|_{Q_h}^2\,.$$

Up to the error $u_h - u$ at final time T and on the outflow boundary
and without estimating the consistency error
this is explicitly evaluated by the residual error indicator

$$\eta_{\mathrm{res},h} = \Bigg(\sum_{R\in\mathcal{R}_h}\eta_{\mathrm{res},R}^2\Bigg)^{1/2}$$

Error control

$\eta_{\mathrm{res},h}$ is given by the local contributions for $R=(t_{n-1},t_n)\times K$, $n=1,\dots,N$

$$\eta^2_{\mathrm{res},R} = \eta^2_{\mathrm{res},n,K} + 2h_K\,\big\|\partial_t u_h + \operatorname{div}\mathbf{f}_h(u_h)\big\|^2_R$$
$$+\frac14\big\||\mathbf{q}_h\cdot\mathbf{n}_K|^{1/2}[u_h]_{K,F}\big\|^2_{I_h\times\partial K\cap\Omega} + \big\||\mathbf{q}_h\cdot\mathbf{n}|^{-1/2}(g_{\text{in}}-\mathbf{f}_h(u_h)\cdot\mathbf{n})\big\|^2_{I_h\times\partial K\cap\Gamma_{\text{in}}}$$

$$\eta^2_{\mathrm{res},1,K} = \frac12\big\|u^0-u_h(0)\big\|^2_K + \frac14\big\|[u_h]_1\big\|^2_K\,,\qquad R=(0,t_1)\times K\,,$$
$$\eta^2_{\mathrm{res},n,K} = \frac14\big\|[u_h]_{n-1}\big\|^2_K + \frac14\big\|[u_h]_n\big\|^2_K\,,\qquad R=(t_{n-1},t_n)\times K\,,\ 1<n<N\,,$$
$$\eta^2_{\mathrm{res},N,K} = \frac14\big\|[u_h]_{N-1}\big\|^2_K\,,\qquad R=(t_{N-1},T)\times K\,.$$

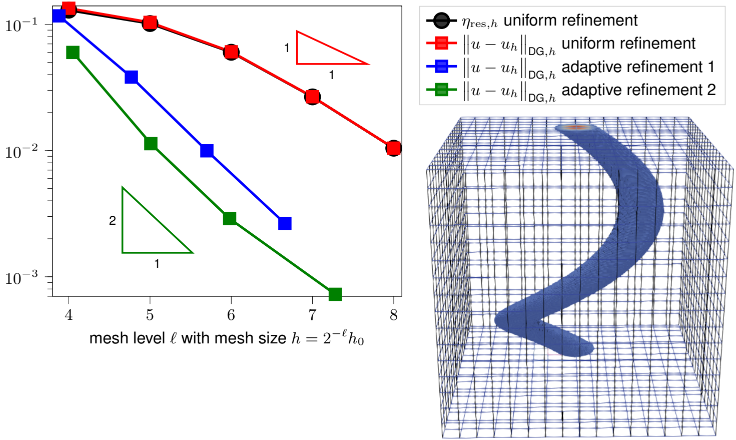
Lemma

Let $u\in \mathrm{L}_2(Q)$ be the weak solution and $u_h\in V_h$ the discrete solution. Then, if u is sufficiently smooth, the error in the DG norm is bounded by

$$\big\|u-u_h\big\|_{h,\mathrm{DG}} \le \Big(\eta^2_{\mathrm{res},h} + \frac12\big\|(u_h(T)-u(T))\big\|^2_\Omega + \frac12\big\||\mathbf{q}_h\cdot\mathbf{n}|^{1/2}(u-u_h)\big\|^2_{I_h\times\Gamma_{\mathrm{out}}}$$
$$+\,2\,\big\|h^{1/2}\,\operatorname{div}(\mathbf{f}(u)-\mathbf{f}_h(u))\big\|^2_{Q_h} + \big\||\mathbf{q}_h\cdot\mathbf{n}|^{-1/2}(\mathbf{f}(u)-\mathbf{f}_h(u))\cdot\mathbf{n}\big\|^2_{I_h\times\Gamma_{\mathrm{in}}}\Big)^{1/2}.$$

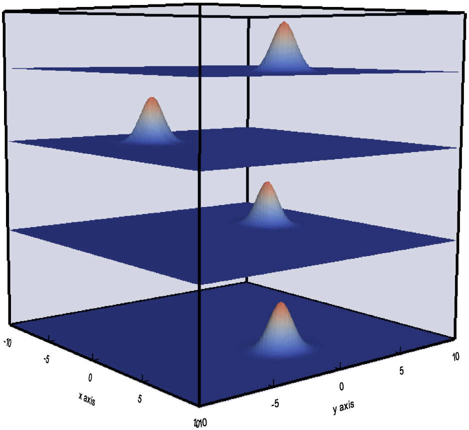
Example: Rotation cone

$$u(t,\mathbf{x}) = \begin{cases} \exp\left(-\dfrac{|\mathbf{x}-\mathbf{y}(t)|}{\frac{1}{8}-|\mathbf{x}-\mathbf{y}(t)|}\right), & |\mathbf{x}-\mathbf{y}(t)| < \frac{1}{8}, \quad \mathbf{y}(t) = \frac{1}{4}\begin{pmatrix}\cos(2\pi t)\\ \sin(2\pi t)\end{pmatrix}, \\ 0, & \text{else}, \qquad \mathbf{q}(\mathbf{x}) = \begin{pmatrix}-2\pi x_2\\ 2\pi x_1\end{pmatrix}. \end{cases}$$



Example: Rotation cone

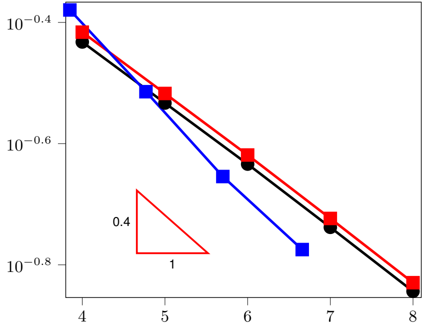
The adaptive solution with 3 303 810 degrees of freedom computes the same solution as the uniform computation on $524\,288 = 4\,096 \times 128$ space-time cells and 31 703 040 degrees of freedom.

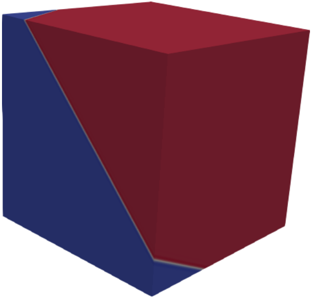


solution sliced at times $t = 0, 0.3, 0.6, 1$

Example: The Riemann problem

Since the Riemann solution is not in $\mathrm{H}^1(Q)$, the theorem does not apply.





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