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Space-time discontinuous Galerkin methods for weak solutions of hyperbolic linear symmetric Friedrichs systems

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The linear transport equation

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Let $\Omega \subset \mathbb{R}^d$ be a bounded domain in space with Lipschitz boundary,
$I = (0, T)$ a time interval, and $Q = (0, T) \times \Omega$ the space-time cylinder.

Let $\mathbf{a} \in \mathrm{L}_\infty(\Omega, \mathbb{R}^d)$ be such that $\big(\mathbf{a} \cdot \nabla v, w\big)_\Omega = -\big(v, \mathbf{a} \cdot \nabla w\big)_\Omega$ for $v, w \in \mathrm{C}^1_c(\Omega)$.

Let $\Gamma = \overline{\big\{\mathbf{x} \in \partial\Omega \colon \mathbf{a}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\big\}}$ be the inflow boundary.

For the linear transport equation

$$Lu = \partial_t u + \mathbf{a} \cdot \nabla u = f \text{ in } Q\,, \qquad u(0) = u_0 \text{ in } \Omega \qquad u = g \text{ on } (0, T) \times \Gamma$$

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$u \in \mathrm{L}_2(Q)$ is a *weak solution*, if

$$\big(u, L^* v\big)_Q = \big(f, v\big)_Q + \big(u_0, v(0)\big)_Q - \big(g, v\big)_{(0,T)\times\partial\Omega}\,, \qquad v \in \mathcal{V}^*$$

with

$$\mathcal{V}^* = \big\{v \in \mathrm{C}^1(\overline{Q}) \colon v(T) = 0 \text{ in } \Omega\,,\ v = 0 \text{ on } (0, T) \times \Gamma^*\big\}\,.$$

$u \in \mathrm{L}_2(Q)$ is a *strong solution*, if in addition $u(0) \in \mathrm{L}_2(\Omega)$ and $u|_\Gamma \in \mathrm{L}_2((0, T) \times \Gamma)$.

Hyperbolic linear symmetric Friedrichs systems

Let $L = M\partial_t + A$ be given by $(M\mathbf{v})(t,\mathbf{x}) = \underline{M}(\mathbf{x})\mathbf{v}(t,\mathbf{x})$ with positive definite matrix $\underline{M} \in \mathrm{L}_\infty(\Omega;\mathbb{R}^{m\times m}_{\mathrm{sym}})$ and $A\mathbf{v} = \sum_{j=1}^{d} \underline{A}_j \partial_j \mathbf{v}$ with matrices $\underline{A}_j \in \mathbb{R}^{m\times m}_{\mathrm{sym}}$.

Boundary conditions are imposed on $\Gamma_k \subset \partial\Omega$, $k = 1,\dots,m$.

$L^* = -L$ is the adjoint differential operator.

For the unit normal vector $\mathbf{n} \in \mathrm{L}_\infty(\partial\Omega;\mathbb{R}^d)$ we define $\underline{A}_\mathbf{n} = \sum_{j=1}^{d} n_j \underline{A}_j \in \mathbb{R}^{m\times m}_{\mathrm{sym}}$.

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$$\mathcal{S}^* = \big\{\mathbf{w}\in\mathrm{C}^1(\overline{\Omega};\mathbb{R}^m)\colon (\underline{A}_{\mathbf{n}}\mathbf{w})_k = 0 \text{ on } \Gamma_k^*\,,\ k = 1,\dots,m\big\}\,.$$

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If in addition $\mathbf{u}(0) \in \mathrm{L}_2(\Omega;\mathbb{R}^m)$ and $\underline{A}_{\mathbf{n}}\mathbf{u}|_{(0,T)\times\Gamma_k} \in \mathrm{L}_2((0,T)\times\Gamma_k)$ for $k = 1,\dots,m$, the weak solution is also a *strong solution* characterized by

$$L\mathbf{u} = \mathbf{f} \text{ in } \mathrm{L}_2(Q;\mathbb{R}^m)\,,\quad \mathbf{u}(0) = \mathbf{u}_0 \text{ in } \mathrm{L}_2(\Omega;\mathbb{R}^m)\,,\quad (\underline{A}_{\mathbf{n}}\mathbf{u})_k = g_k \text{ on } \mathrm{L}_2((0,T)\times\Gamma_k)\,.$$

Examples

Acoustic waves The second-order wave equation

$$\varrho \partial_t^2 \phi - \nabla \cdot (\kappa \nabla \phi) = b$$

is considered as first-order system with $p = \partial_t \phi$ and $\mathbf{q} = -\kappa \nabla \phi$, i.e.,

$$\mathbf{u} = \begin{pmatrix} p \\ \mathbf{q} \end{pmatrix}, \ M\mathbf{u} = \begin{pmatrix} \varrho p \\ \kappa^{-1}\mathbf{q} \end{pmatrix}, \ A\mathbf{u} = \begin{pmatrix} \nabla \cdot \mathbf{q} \\ \nabla p \end{pmatrix}, \ \underline{A}_{\mathbf{n}}\mathbf{u} = \begin{pmatrix} \mathbf{n} \cdot \mathbf{q} \\ p\mathbf{n} \end{pmatrix}, \ \mathbf{f} = \begin{pmatrix} b \\ \mathbf{0} \end{pmatrix}, \ \mathbf{g} = \begin{pmatrix} g_{\mathrm{N}} \\ p_{\mathrm{D}}\mathbf{n} \end{pmatrix}$$

with mass density ϱ and permeability κ.

Elastic waves Linear elastic waves are described by velocity $\mathbf{v}$ and stress $\boldsymbol{\sigma}$ and

$$\mathbf{u} = \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \end{pmatrix}, \quad M\mathbf{u} = \begin{pmatrix} \varrho \mathbf{v} \\ \mathbf{C}^{-1}\boldsymbol{\sigma} \end{pmatrix}, \quad A\mathbf{u} = \begin{pmatrix} -\nabla \cdot \boldsymbol{\sigma} \\ -\varepsilon(\mathbf{v}) \end{pmatrix}, \quad \underline{A}_{\mathbf{n}}\mathbf{u} = \begin{pmatrix} -\boldsymbol{\sigma}\mathbf{n} \\ -\mathbf{n}\mathbf{v}^\top - \mathbf{v}\mathbf{n}^\top \end{pmatrix}$$

with mass density ϱ and Hookian tensor $\mathbf{C}\boldsymbol{\varepsilon} = 2\mu\boldsymbol{\varepsilon} + \lambda \,\mathrm{trace}(\boldsymbol{\varepsilon})\mathbf{I}_3$.

Electro-magnetic waves Electric field $\mathbf{E}$ and the magnetic field intensity $\mathbf{H}$ define

$$\mathbf{u} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \quad M\mathbf{u} = \begin{pmatrix} \varepsilon \mathbf{E} \\ \mu \mathbf{H} \end{pmatrix}, \quad A\mathbf{u} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ \nabla \times \mathbf{E} \end{pmatrix}, \quad \underline{A}_{\mathbf{n}}\mathbf{u} = \begin{pmatrix} -\mathbf{n} \times \mathbf{H} \\ \mathbf{n} \times \mathbf{E} \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} -\mathbf{J} \\ \mathbf{0} \end{pmatrix}$$

with permittivity ε and permeability μ.

A discontinuous Galerkin method in time

For $0 = t_0 < t_1 < \cdots < t_N = T$, we define time intervals $I_{n,h} = (t_{n-1}, t_n)$ and

$$I_h = (t_0, t_1) \cup \cdots \cup (t_{N-1}, t_N) \subset I = (0, T)\,, \qquad \partial I_h = \{t_0, t_1, \dots, t_{N-1}, t_N\}\,.$$

For $\mathbf{v}_h, \mathbf{w}_h \in \mathrm{H}^1\big(I_h; \mathrm{L}_2(\Omega)\big)$ we obtain after integration by parts

$$\begin{aligned}\big(M_h \partial_t \mathbf{v}_h, \mathbf{w}_h\big)_{I_h \times \Omega} &= -\big(M_h \mathbf{v}_{n,h}, \partial_t \mathbf{w}_{n,h}\big)_{I_h \times \Omega} \\ &\quad + \sum_{n=1}^{N} \Big(\big(M_h \mathbf{v}_{n,h}(t_n), \mathbf{w}_{n,h}(t_n)\big)_\Omega - \big(M_h \mathbf{v}_{n,h}(t_{n-1}), \mathbf{w}_{n,h}(t_{n-1})\big)_\Omega \Big)\,.\end{aligned}$$

With $[\mathbf{w}_h]_n = \mathbf{w}_{n+1,h}(t_n) - \mathbf{w}_{n,h}(t_n)$ for $n = 1, \dots, N-1$ and $[\mathbf{w}_h]_N = -\mathbf{w}_{N,h}(t_N)$, set

$$m_h(\mathbf{v}_h, \mathbf{w}_h) = -\big(M_h \mathbf{v}_h, \partial_t \mathbf{w}_h\big)_{Q_h} - \sum_{n=1}^{N} \big(M_h \mathbf{v}_{n,h}(t_n), [\mathbf{w}_h]_n\big)_\Omega\,.$$

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Properties

$$m_h(\mathbf{v}_h, \mathbf{v}_h) = \frac{1}{2} \sum_{n=0}^{N} \big(M_h [\mathbf{v}_h]_n, [\mathbf{v}_h]_n\big)_\Omega \geq 0\,,$$

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Let $\Omega_h = \bigcup_{K \in \mathcal{K}_h} K$ be a decomposition into open cells $K \subset \Omega$ with $\partial\Omega_h = \bigcup_{F \in \mathcal{F}_h} F$.

Let $S_h \subset \mathbb{P}_q(\Omega_h; \mathbb{R}^m) = \prod_{K \in \mathcal{K}_h} \mathbb{P}_q(K; \mathbb{R}^m)$ be a discontinuous finite element space.

For the transport equation ($m = 1$) we observe for $v_h, w_h \in S_h$ in space

$$\big(\mathbf{a} \cdot \nabla v_h, w_h\big)_{\Omega_h} = \sum_{K \in \mathcal{K}_h} \left(-\big(v_{h,K}, \mathbf{a} \cdot \nabla w_{h,K}\big)_K + \sum_{F \in \mathcal{F}_K} \big(\mathbf{a} \cdot \mathbf{n}_K \, v_{h,K}, w_{h,K}\big)_F \right) .$$

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Now we define the upwind flux $\mathbf{a}^{\mathrm{up}}_{\mathbf{n}_K}$ and

$$a_h(v_h, w_h) = -\big(v_h, \mathbf{a}\cdot\nabla w_h\big)_{\Omega_h} + \sum_{K\in\mathcal{K}_h}\sum_{F\in\mathcal{F}_K}\big(v_{h,K}, \mathbf{a}^{\mathrm{up}}_{\mathbf{n}_K}[w_h]_{K,F}\big)_F$$

with jump term $[w_h]_{K,F} = w_{h,K_F} - w_{h,K}$ on on inner faces $F \subset \Omega$, $\overline{F} = \partial K \cap \partial K_F$; on boundary faces $F \subset \partial\Omega$ we set $[v_h]_F = -2v_h$ on $F \subset \Gamma$ and $[v_h]_F = 0$ else.

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A discontinuous Galerkin method in space

For $\mathbf{v}_h, \mathbf{w}_h \in S_h$ we observe, integrating by parts for all elements $K \in \mathcal{K}_h$,

$$\big(A\mathbf{v}_h, \mathbf{w}_h\big)_{\Omega_h} = \sum_{K \in \mathcal{K}_h} \left(-\big(\mathbf{v}_{h,K}, A\mathbf{w}_{h,K}\big)_K + \sum_{F \in \mathcal{F}_K} \big(\underline{A}_{\mathbf{n}_K} \mathbf{v}_{h,K}, \mathbf{w}_{h,K}\big)_F \right) .$$

We use the discontinuous method with full upwind discretization in space

$$a_h(\mathbf{v}_h, \mathbf{w}_h) = -\big(\mathbf{v}_h, A\mathbf{w}_h\big)_{\Omega_h} + \sum_{K \in \mathcal{K}_h} \sum_{F \in \mathcal{F}_K} \big(\mathbf{v}_{h,K}, \underline{A}^{\mathrm{up}}_{\mathbf{n}_K} [\mathbf{w}_h]_{K,F}\big)_F ,$$

where the upwind flux $\underline{A}^{\mathrm{up}}_{\mathbf{n}_K} \in \mathbb{R}^{m \times m}$ is obtained by solving local Riemann problems.

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Properties

$$a_h(\mathbf{v}_h, \mathbf{v}_h) \geq c_1 \|\underline{A}_{\mathbf{n}}[\mathbf{v}_h]\|^2_{\partial\Omega_h}$$

$$a_h(\mathbf{v}_h, \mathbf{v}_h) = 0 \quad \Longrightarrow \quad a_h(\mathbf{v}_h, \mathbf{w}_h) = -(\mathbf{v}_h, A\mathbf{w}_h)_{\Omega_h} = (A\mathbf{v}_h, \mathbf{w}_h)_{\Omega_h}$$

$$\left| a_h(\mathbf{v}_h, \mathbf{w}_h) + (\mathbf{v}_h, A\mathbf{w}_h)_{\Omega_h} \right| \leq C_1 \|M_h^{1/2} \mathbf{v}_h\|_{\partial\Omega_h} \|\underline{A}_{\mathbf{n}}[\mathbf{w}_h]\|_{\partial\Omega_h}$$

$$\left| a_h(\mathbf{v}_h, \mathbf{w}_h) + (A\mathbf{v}_h, \mathbf{w}_h)_{\Omega_h} \right| \leq C_1 \|\underline{A}_{\mathbf{n}}[\mathbf{v}_h]\|_{\partial\Omega_h} \|M_h^{1/2} \mathbf{w}_h\|_{\partial\Omega_h}$$

Examples

Acoustics   impedance $Z_K = \sqrt{\kappa_{h,K}\varrho_{h,K}}$

$$a_h\big((p_h,\mathbf{q}_h),(\varphi_h,\boldsymbol{\psi}_h)\big) = \sum_{K\in\mathcal{K}_h}\Big(-\big(\mathbf{q}_{h,K},\nabla\varphi_{h,K}\big)_K - \big(p_{h,K},\nabla\cdot\boldsymbol{\psi}_{h,K}\big)_K$$
$$-\sum_{F\in\mathcal{F}_K}(Z_K+Z_{K_F})^{-1}\big(p_{K,h}+Z_{K_F}\mathbf{n}_K\cdot\mathbf{q}_{K,h},[\varphi_h]_{K,F}+Z_K\mathbf{n}_K\cdot[\boldsymbol{\psi}_h]_{K,F}\big)_F\Big)$$

Elasticity   impedances $Z_K^{\mathrm{p}} = \sqrt{(2\mu_{h,K}+\lambda_{h,K})\varrho_{h,K}}$ and $Z_K^{\mathrm{s}} = \sqrt{\mu_{h,K}\varrho_{h,K}}$

$$a_h\big((\mathbf{v}_h,\boldsymbol{\sigma}_h),(\mathbf{w}_h,\boldsymbol{\eta}_h)\big) = \sum_{K\in\mathcal{K}_h}\Big(\big(\boldsymbol{\sigma}_{h.K},\varepsilon(\mathbf{w}_{h,K})\big)_K + \big(\mathbf{v}_{h,K},\nabla\cdot\boldsymbol{\eta}_{h,K}\big)_K$$
$$-\sum_{F\in\mathcal{F}_K}(Z_K^{\mathrm{p}}+Z_{K_F}^{\mathrm{p}})^{-1}\Big(\mathbf{n}_K\cdot\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K - Z_{K_F}^{\mathrm{p}}\mathbf{v}_{h,K}\big),\mathbf{n}_K\cdot\big([\boldsymbol{\eta}_h]_{K,F}\mathbf{n}_K - Z_K^{\mathrm{p}}[\mathbf{w}_h]_{K,F}\big)\Big)_F$$
$$-\sum_{F\in\mathcal{F}_K}(Z_K^{\mathrm{s}}+Z_{K_F}^{\mathrm{s}})^{-1}\Big(\mathbf{n}_K\times\big(\boldsymbol{\sigma}_{h,K}\mathbf{n}_K - Z_{K_F}^{\mathrm{s}}\mathbf{v}_{h,K}\big),\mathbf{n}_K\times\big([\boldsymbol{\eta}_h]_{K,F}\mathbf{n}_K - Z_K^{\mathrm{s}}[\mathbf{w}_h]_{K,F}\big)\Big)_F\Big)$$

Maxwell system   impedance $Z_K = \sqrt{\varepsilon_K/\mu_K}$

$$a_h\big((\mathbf{E}_h,\mathbf{H}_h),(\boldsymbol{\varphi}_h,\boldsymbol{\psi}_h)\big) = \sum_{K\in\mathcal{K}_h}\Big(\big(\mathbf{E}_{h,K},\nabla\times\boldsymbol{\psi}_{h,K}\big)_K - \big(\mathbf{H}_{h,K},\nabla\times\boldsymbol{\varphi}_{h,K}\big)_K$$
$$+\sum_{F\in\mathcal{F}_K}(Z_K+Z_{K_F})^{-1}\Big(\big(Z_K\mathbf{E}_{h,K}-\mathbf{n}_K\times\mathbf{H}_{h,K},\mathbf{n}_K\times[\boldsymbol{\psi}_h]_{K,F}\big)_F$$
$$-\big(Z_K\mathbf{n}_K\times\mathbf{E}_{h,K}+\mathbf{H}_{h,K},Z_{K_F}\mathbf{n}_K\times[\boldsymbol{\varphi}_h]_{K,F}\big)_F\Big)\Big)$$

Examples

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Acoustics impedance $Z_K = \sqrt{\kappa_{h,K}\varrho_{h,K}}$

$$a_h((p_h,\mathbf{q}_h),(\varphi_h,\psi_h)) = \sum_{K\in\mathcal{K}_h}\Big(\ (\nabla\cdot\mathbf{q}_{h,K},\varphi_{h,K})_K + (\nabla p_{h,K},\psi_{h,K})_K$$
$$-\sum_{F\in\mathcal{F}_K}(Z_K+Z_{K_F})^{-1}\big([p_{K,h}]_{K,F} - Z_{K_F}\mathbf{n}_K\cdot[\mathbf{q}_{K,h}]_{K,F},\varphi_h - Z_K\mathbf{n}_K\cdot\psi_h\big)_F\Big)$$

Elasticity impedances $Z_K^{\mathrm{p}} = \sqrt{(2\mu_{h,K}+\lambda_{h,K})\varrho_{h,K}}$ and $Z_K^{\mathrm{s}} = \sqrt{\mu_{h,K}\varrho_{h,K}}$

$$a_h((\mathbf{v}_h,\boldsymbol{\sigma}_h),(\mathbf{w}_h,\boldsymbol{\eta}_h)) = \sum_{K\in\mathcal{K}_h}\Big(- (\nabla\cdot\boldsymbol{\sigma}_{h.K},\mathbf{w}_{h,K})_K - (\varepsilon(\mathbf{v}_{h,K}),\boldsymbol{\eta}_{h,K})_K$$
$$-\sum_{F\in\mathcal{F}_K}(Z_K^{\mathrm{p}}+Z_{K_F}^{\mathrm{p}})^{-1}\Big(\mathbf{n}_K\cdot\big([\boldsymbol{\sigma}_{h,K}]_{K,F}\mathbf{n}_K + Z_{K_F}^{\mathrm{p}}[\mathbf{v}_{h,K}]_{K,F}\big),\mathbf{n}_K\cdot\big(\boldsymbol{\eta}_h\mathbf{n}_K + Z_K^{\mathrm{p}}\mathbf{w}_h\big)\Big)_F$$
$$-\sum_{F\in\mathcal{F}_K}(Z_K^{\mathrm{s}}+Z_{K_F}^{\mathrm{s}})^{-1}\Big(\mathbf{n}_K\times\big([\boldsymbol{\sigma}_{h,K}]_{K,F}\mathbf{n}_K + Z_{K_F}^{\mathrm{s}}[\mathbf{v}_{h,K}]_{K,F}\big),\mathbf{n}_K\times\big(\boldsymbol{\eta}_h\mathbf{n}_K + Z_K^{\mathrm{s}}\mathbf{w}_h\big)\Big)_F\Big)$$

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$$a_h((\mathbf{E}_h,\mathbf{H}_h),(\varphi_h,\psi_h)) = \sum_{K\in\mathcal{K}_h}\Big((\nabla\times\mathbf{E}_{h,K},\psi_{h,K})_K - (\nabla\times\mathbf{H}_{h,K},\varphi_{h,K})_K$$
$$+\sum_{F\in\mathcal{F}_K}(Z_K+Z_{K_F})^{-1}\Big(\big(Z_{K_F}\mathbf{n}_K\times[\mathbf{E}_h]_{K,F},\psi_{h,K} - Z_K\mathbf{n}_K\times\varphi_{h,K}\big)_F$$
$$-\big(\mathbf{n}_K\times[\mathbf{H}_h]_{K,F},Z_K\mathbf{n}_K\times\varphi_{h,K} + \mathbf{n}_K\times\psi_{h,K}\big)_F\Big)\Big)$$

A discontinuous Galerkin method in time and space

Let $Q_h = I_h \times \Omega_h$ be a tensor product time-space mesh with $c_{\text{ref}}(t_n - t_{n-1}) \le h$.

Let $V_h \subset \mathbb{P}_q(Q_h;\mathbb{R}^m) = \prod_{n=1}^N \prod_{K\in\mathcal{K}_h} \mathbb{P}_q((t_{n-1},t_n)\times K;\mathbb{R}^m)$ an define

$$b_h(\mathbf{v}_h,\mathbf{w}_h) = m_h(\mathbf{v}_h,\mathbf{w}_h) + \int_0^T a_h\big(\mathbf{v}_h(t),\mathbf{w}_h(t)\big)\,\mathrm{d}t\,, \qquad \mathbf{v}_h\,,\ \mathbf{w}_h \in V_h\,.$$

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Let M_h be an approximation of M and $L_h = M_h\partial_t + A$.

Properties

$$b_h\big(\mathbf{v}_h,\mathbf{v}_h\big) \ge \frac{1}{2}\sum_{n=0}^N \big\|M_h^{1/2}[\mathbf{v}_h]_n\big\|_\Omega^2 + c_1\big\|\underline{A}_{\mathbf{n}}[\mathbf{v}_h]\big\|_{\partial\Omega_h}^2$$

$$b_h\big(\mathbf{v}_h,\mathbf{v}_h\big) = 0 \quad\Longrightarrow\quad b_h\big(\mathbf{v}_h,\mathbf{w}_h\big) = \big(\mathbf{v}_h, L_h^*\mathbf{w}_h\big)_{Q_h} = \big(L_h\mathbf{v}_h,\mathbf{w}_h\big)_{Q_h}$$

$$\big|b_h\big(\mathbf{v}_h,\mathbf{w}_h\big) - \big(\mathbf{v}_h, L_h^*\mathbf{w}_h\big)_{\Omega_h}\big| \le \big\|M_h^{1/2}\mathbf{v}_h\big\|_{\partial Q_h}\sqrt{\big\|M_h^{1/2}[\mathbf{w}_h]\big\|_{\partial I_h\times\Omega}^2 + C_1\big\|\underline{A}_{\mathbf{n}}[\mathbf{w}_h]\big\|_{\partial\Omega_h}^2}$$

$$\big|b_h\big(\mathbf{v}_h,\mathbf{w}_h\big) - \big(L_h\mathbf{v}_h,\mathbf{w}_h\big)_{\Omega_h}\big| \le \sqrt{\big\|M_h^{1/2}[\mathbf{v}_h]\big\|_{\partial I_h\times\Omega}^2 + C_1\big\|\underline{A}_{\mathbf{n}}[\mathbf{v}_h]\big\|_{\partial\Omega_h}^2}\,\big\|M_h^{1/2}\mathbf{w}_h\big\|_{\partial Q_h}$$

Well-posedness and stability

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Lemma

Define $d_T(t) = T - t$. We have

$$\left\| M_h^{1/2} \mathbf{v}_h \right\|_Q^2 + T \left\| M_h^{1/2} \mathbf{v}_h(0) \right\|_\Omega^2 \le 2\, m_h(\mathbf{v}_h, d_T \mathbf{v}_h)\,, \qquad \mathbf{v}_h \in V_h\,.$$

Well-posedness and stability

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Lemma

A unique discrete approximation $\mathbf{u}_h \in V_h$ exists solving

$$b_h(\mathbf{u}_h, \mathbf{v}_h) = \left\langle \ell_h, \mathbf{v}_h \right\rangle , \qquad \mathbf{v}_h \in V_h \,.$$

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The mesh-dependent DG semi-norm and norm is defined for $\mathbf{v}_h \in \mathcal{V}_h$ by

$$\left|\mathbf{v}_h\right|_{h,\mathrm{DG}} = \sqrt{b_h(\mathbf{v}_h, \mathbf{v}_h)}\,, \qquad \left\|\mathbf{v}_h\right\|_{h,\mathrm{DG}} = \sqrt{\left|\mathbf{v}_h\right|_{h,\mathrm{DG}}^2 + h\left\|M_h^{-1/2}L_h\mathbf{v}_h\right\|_{Q_h}^2}\,.$$

Theorem

$c_{\text{inf-sup}} > 0$ exists such that $\displaystyle\sup_{\mathbf{w}_h \in V_h\setminus\{\mathbf{0}\}} \frac{b_h(\mathbf{v}_h, \mathbf{w}_h)}{\left\|\mathbf{w}_h\right\|_{h,\mathrm{DG}}} \ge c_{\text{inf-sup}}\left\|\mathbf{v}_h\right\|_{h,\mathrm{DG}}$ *for* $\mathbf{v}_h \in V_h$.

Convergence

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Theorem

Assume that the solution is sufficiently smooth satisfying $\mathbf{u} \in \mathrm{H}^s(Q;\mathbb{R}^m)$ *with* $s \geq 1$*. Then, the error for the discrete solution* $\mathbf{u}_h \in V_h$ *is bounded by*

$$\|\mathbf{u} - \mathbf{u}_h\|_{h,\mathrm{DG}} \leq Ch^{s-1/2}\,\|\mathrm{D}^s\mathbf{u}\|_Q + CT\,h^{-1/2}\|M_h^{-1/2}(M_h - M)\partial_t\mathbf{u}\|_Q\,.$$

$C > 0$ *depends on mesh regularity / polynomial degree / material parameters.*

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$C > 0$ *depends on mesh regularity / polynomial degree / material parameters.*

Corollary

$$\left\|\mathbf{u}(t_n) - \mathbf{u}_h(t_n)\right\|_\Omega \leq Ch^{s-1/2} \left\|\mathrm{D}^s\mathbf{u}\right\|_{(0,t_n)\times\Omega} + Ct_n\, h^{-1/2}\left\|M_h^{-1/2}(M_h - M)\partial_t\mathbf{u}\right\|_{(0,t_n)\times\Omega}$$

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Corollary

$$\|\mathbf{u}(t_n) - \mathbf{u}_h(t_n)\|_\Omega \leq Ch^{s-1/2} \|\mathrm{D}^s\mathbf{u}\|_{(0,t_n)\times\Omega} + Ct_n\, h^{-1/2} \|M_h^{-1/2}(M_h - M)\partial_t\mathbf{u}\|_{(0,t_n)\times\Omega}$$

If M is discontinuous and the parameter are not resolved by the mesh, the consistency error can be estimated in case of higher regularity:

$$\|(M_h^{-1/2}(M_h - M)\partial_t\mathbf{u}\|_Q \leq \|M_h^{-1/2}(M - M_h)M^{-1/2}\|_{\mathrm{L}_{2q/(2-q)}(\Omega)} \|M^{1/2}\partial_t\mathbf{u}\|_{\mathrm{L}_2(0,T;\mathrm{L}_q(\Omega))}$$

in case of $\partial_t\mathbf{u} \in \mathrm{L}_2(0,T;\mathrm{L}_q(\Omega;\mathbb{R}^m))$ with $q > 2$.

Convergence in the limit

Let $\big(Q_h\big)_{h\in\mathcal{H}}$ be a shape-regular family of space-time meshes with

$$\mathcal{H} = \{h_0, h_1, h_2, \cdots\} \subset (0,\infty)\,, \qquad 0 \in \overline{\mathcal{H}}\,.$$

Let $\big(V_h\big)_{h\in\mathcal{H}}$ be corresponding DG finite element spaces, so that

$$\lim_{h\in\mathcal{H}} \inf_{\mathbf{v}_h\in V_h} \big\|\mathbf{v}-\mathbf{v}_h\big\|_Q = 0\,, \qquad \mathbf{v}\in\mathcal{V}^*\,.$$

For $h \in \mathcal{H}$, let $\mathbf{u}_h \in V_h$ be the solution of the discrete problem.

Theorem

Assume that $p_{n,K} = p_n \geq 1$ *and* $q_{n,K} \geq 1$.
Assume homogeneous boundary data $\mathbf{g} = \mathbf{0}$.
Assume for the material parameters $M_h \longrightarrow M$, $M_h^{-1} \longrightarrow M^{-1}$ *in* $\mathrm{L}_\infty(Q;\mathbb{R}^m)$.
Then, the discrete solutions $\big(\mathbf{u}_h\big)_{h\in\mathcal{H}}$ *are uniformly bounded in* $\mathrm{L}_2(Q;\mathbb{R}^m)$ *and converging to the a weak solution* $\mathbf{u} \in \mathrm{L}_2(Q;\mathbb{R}^m)$.
Moreover, $\mathbf{u}$ *is a strong solution, and the strong solution is unique.*

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Moreover, $\mathbf{u}$ is a strong solution, and the strong solution is unique.

This extends to $\mathbf{g}\neq\mathbf{0}$, if an extension $\mathbf{u}_\mathbf{g}\in\mathrm{L}_2(Q;\mathbb{R}^m)$ exists with $L\mathbf{u}_\mathbf{g}\in\mathrm{L}_2(Q;\mathbb{R}^m)$ and $(A_\mathbf{n}\mathbf{u}_\mathbf{g})_k\in\mathrm{L}_2(I\times\Gamma_k)$ satisfying $(A_\mathbf{n}\mathbf{u}_\mathbf{g})_k = g_k$, $k=1,\dots,m$.

Error control

The residual error indicator

$$\eta_{\mathrm{res},h} = \Big(\sum_{R\in\mathcal{R}_h} \eta_{\mathrm{res},R}^2 \Big)^{1/2}$$

is given by the local contributions

$$\eta_{\mathrm{res},R}^2 = \eta_{\mathrm{res},n,K}^2 + C_1 \big\| A_{\mathbf{n}}[\mathbf{u}_h] \big\|^2_{(t_{n-1},t_n)\times\Omega\cap\partial K} + \sum_{k=1}^{m} \big\| (A_{\mathbf{n}}\mathbf{u}_h)_k - g_k \big\|^2_{(t_{n-1},t_n)\times\Gamma_k\cap\partial K}$$
$$+ 2h \big\| M_h^{-1/2}(L_h\mathbf{u}_h - \mathbf{f}) \big\|^2_R$$

for $R = (t_{n-1}, t_n) \times K$, $n = 1, \dots, N$, with

$$\eta_{\mathrm{res},1,K}^2 = \frac{1}{2}\big\| M_h^{1/2}\mathbf{u}_h(0) - \mathbf{u}_0 \big\|^2_K + \frac{1}{2}\big\| M_h^{1/2}[\mathbf{u}_h]_1 \big\|^2_K\,, \quad R = (0, t_1) \times K\,,$$
$$\eta_{\mathrm{res},n,K}^2 = \frac{1}{2}\big\| M_h^{1/2}[\mathbf{u}_h]_{n-1} \big\|^2_K + \frac{1}{2}\big\| M_h^{1/2}[\mathbf{u}_h]_n \big\|^2_K\,, \quad R = (t_{n-1}, t_n) \times K\,,\ 1 < n < N\,,$$
$$\eta_{\mathrm{res},N,K}^2 = \frac{1}{2}\big\| M_h^{1/2}[\mathbf{u}_h]_{N-1} \big\|^2_K\,, \quad R = (t_{N-1}, T) \times K\,.$$

Error control

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Theorem

The error in the DG norm is bounded by

$$\|\mathbf{u}-\mathbf{u}_h\|_{h,\mathrm{DG}} \leq \left(\eta_{\mathrm{res},h}^2 + \left\|M_h^{1/2}(\mathbf{u}_h(T)-\mathbf{u}(T))\right\|_\Omega^2 + 2h\left\|M_h^{-1/2}(M-M_h)\partial_t\mathbf{u}\right\|_Q^2\right)^{1/2}.$$

Error control

Theorem

The error in the DG norm is bounded by

$$\|\mathbf{u}-\mathbf{u}_h\|_{h,\mathrm{DG}} \le \Big(\eta^2_{\mathrm{res},h} + \|M_h^{1/2}(\mathbf{u}_h(T)-\mathbf{u}(T))\|^2_\Omega + 2h\,\|M_h^{-1/2}(M-M_h)\partial_t\mathbf{u}\|^2_Q\Big)^{1/2}.$$

Let $\mathbf{u}_h^{\mathrm{cf}}$ be a conforming reconstruction.

Lemma

$$\begin{aligned}\|M_h^{1/2}\mathbf{u}_h(T)-\mathbf{u}(T)\|_\Omega &\le \|M_h^{1/2}\mathbf{u}_h(T)-\mathbf{u}_h^{\mathrm{cf}}(T)\|_\Omega\\ &\quad+\Big(\big(\|M_h^{-1/2}(L_h\mathbf{u}_h^{\mathrm{cf}}-\mathbf{f})\|_Q+\|M_h^{-1/2}(M-M_h)\partial_t\mathbf{u}\|_Q\big)\|M_h^{1/2}(\mathbf{u}-\mathbf{u}_h^{\mathrm{cf}})\|_Q\Big)^{1/2}\end{aligned}$$

$$\begin{aligned}\|M_h^{1/2}(\mathbf{u}-\mathbf{u}_h)\|_Q &\le \|M_h^{1/2}(\mathbf{u}_h-\mathbf{u}_h^{\mathrm{cf}})\|_Q\\ &\quad+2T\,\|M_h^{-1/2}(L_h\mathbf{u}_h^{\mathrm{cf}}-\mathbf{f})\|_Q+2T\,\|M_h^{-1/2}(\mathbf{u}_h^{\mathrm{cf}}(0)-\mathbf{u}_0)\|_\Omega\\ &\quad+\sup_{\mathbf{v}\in\mathcal{V}^*\setminus\{\mathbf{0}\}}\frac{\big(\underline{A}_{\mathbf{n}}\mathbf{u}_h^{\mathrm{cf}}-\mathbf{g},\mathbf{v}\big)_{I_h\times\partial\Omega}+\big(\mathbf{u},(M_h-M)\partial_t\mathbf{v}\big)_Q}{\|M_h^{-1/2}L_h^*\mathbf{v}\|_Q}\end{aligned}$$

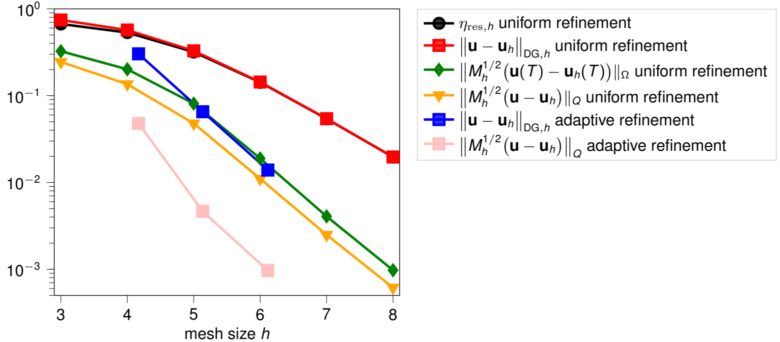
Numerical experiment 1: Smooth $\mathbf{u}_0$ and $M = M_h$

In $Q = (0,1) \times (0,1)^2$ we set $a_0(x) = \sin(3\pi x)^2$ for $x \in [0, 1/3]$ and $a_0(x) = 0$ else,

$$\varrho(\mathbf{x}) = \begin{cases} 1 & \mathbf{x}\cdot\mathbf{m} \le 1/2\,, \\ 2 & \mathbf{x}\cdot\mathbf{m} > 1/2\,, \end{cases} \qquad \mathbf{u}_0(\mathbf{x}) = a_0(\mathbf{x}\cdot\mathbf{m})\begin{pmatrix} 1 \\ \mathbf{m} \end{pmatrix}, \qquad \mathbf{m} = \begin{pmatrix} 0.0 \\ 1.0 \end{pmatrix}$$

so that with $\kappa(\mathbf{x}) = 1/\varrho(\mathbf{x})$ the impedance is constant across the interface.

The solution is given by $\mathbf{u}(t,\mathbf{x}) = \begin{cases} \mathbf{u}_0(\mathbf{x} - t\,\mathbf{m}) & \mathbf{x}\cdot\mathbf{m} \le 1/2\,, \\ \mathbf{u}_0(2\mathbf{x} - (t+1/2)\mathbf{m}) & \mathbf{x}\cdot\mathbf{m} > 1/2\,. \end{cases}$



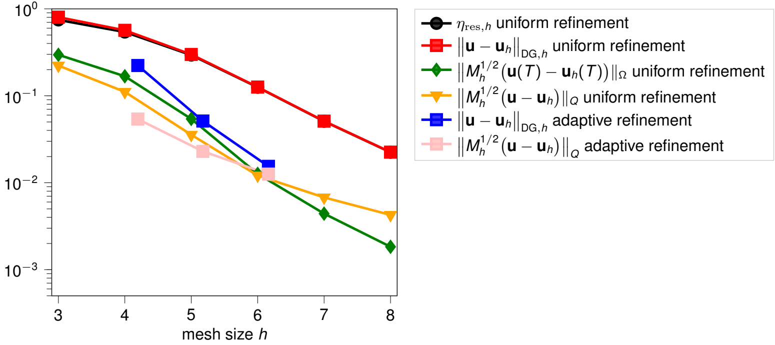
Numerical experiment 2: Smooth $\mathbf{u}_0$ and $M \neq M_h$

In $Q = (0,1) \times (0,1)^2$ we set $a_0(x) = \sin(3\pi x)^2$ for $x \in [0, 1/3]$ and $a_0(x) = 0$ else,

$$\varrho(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \cdot \mathbf{m} \leq 4/7\,, \\ 2 & \mathbf{x} \cdot \mathbf{m} > 4/7\,, \end{cases} \qquad \mathbf{u}_0(\mathbf{x}) = a_0(\mathbf{x} \cdot \mathbf{m}) \begin{pmatrix} 1 \\ \mathbf{m} \end{pmatrix}, \qquad \mathbf{m} = \begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix}$$

so that with $\kappa(\mathbf{x}) = 1/\varrho(\mathbf{x})$ the impedance is constant across the interface.

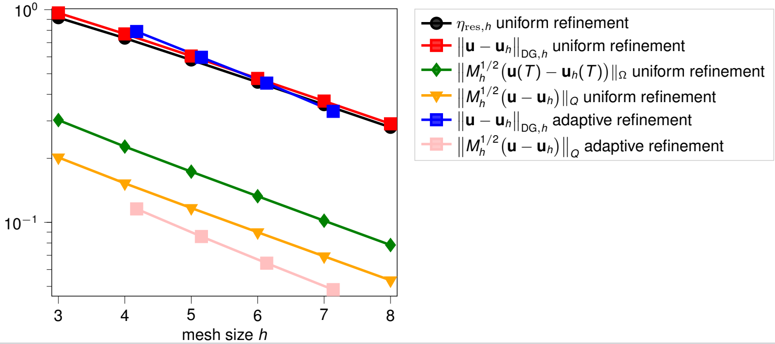
The solution is given by $\mathbf{u}(t, \mathbf{x}) = \begin{cases} \mathbf{u}_0(\mathbf{x} - t\,\mathbf{m}) & \mathbf{x} \cdot \mathbf{m} \leq 4/7\,, \\ \mathbf{u}_0(2\mathbf{x} - (t + 4/7)\mathbf{m}) & \mathbf{x} \cdot \mathbf{m} > 4/7\,. \end{cases}$



Numerical experiment 3: Riemann problem

In $Q = (0, 1/2) \times (-1, 1) \times (0, 1)$ we consider the Riemann solution

$$\mathbf{u}(t, \mathbf{x}) = \begin{cases} \begin{pmatrix} 0 \\ \mathbf{0} \end{pmatrix} & \mathbf{x} \cdot \mathbf{m} < -t\,, \\ \begin{pmatrix} 1 \\ \mathbf{m} \end{pmatrix} & -t < \mathbf{x} \cdot \mathbf{m} < t\,, \\ \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} & t < \mathbf{x} \cdot \mathbf{m}\,, \end{cases} \qquad \mathbf{m} = \begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix}, \qquad \kappa = 1\,, \quad \varrho = 1\,.$$

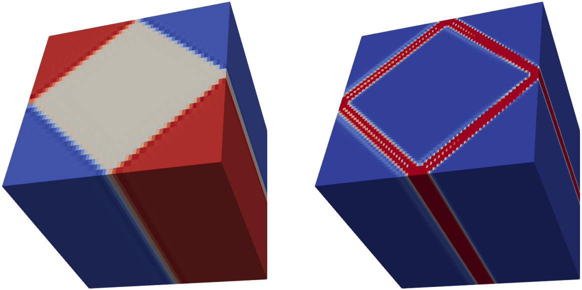


Outlook

Karlsruhe Institute of Technology

For the Riemann problem, we observe better convergence in L_1.
Can we prove this?

solution



refinement

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