On the superlinear convergence in computational elasto-plasticity
Application to non-associated models in soil mechanics

Christian Wieners (joint work with M. Sauter)
Overview

1. **Representative Models in Soil Mechanics**
   
   Drucker-Prager — Cam-clay

2. **Generalized Newton Methods for Incremental Plasticity**
   
   Semismooth Newton methods — active set strategies

3. **Nonsmooth Convergence Analysis**
   
   Semismooth functions — generalized implicit function theorem

4. **Applications and Convergence Results**
   
   Drucker-Prager — Cam-clay
Quasi-static Infinitesimal Plasticity

Let $\Omega \subset \mathbb{R}^3$ and $\partial \Omega = \Gamma_D \cup \Gamma_N$. We consider for $t \in [0, T]$

$$-\text{div} \sigma(x, t) = b(x, t) \quad \text{in} \; \Omega, \quad \sigma(x, t)n(x) = t_N(x, t) \quad \text{on} \; \Gamma_N.$$

Balance of angular momentum implies $\sigma(x, t) \in S := \{\tau \in \mathbb{R}^d : \tau = \tau^T\}$.

We only consider the small strain setting, i.e., the stress-strain relation is given by

$$\sigma = C[\varepsilon(u) - \varepsilon_p].$$

The plastic evolution is described by the plastic strain $\varepsilon_p$ and internal variables $\eta$. The generalized stress $\Sigma = (\sigma, \eta) \in \mathcal{I} = S \times \mathbb{R}^m$ is said to be admissible if

$$\Sigma \in K := \{\hat{\Sigma} \in \mathcal{I} : f_i(\hat{\Sigma}) \leq 0 \text{ for } i = 1, \ldots, p\}$$

for given yield functions $f_i : \mathcal{I} \to \mathbb{R}$. The plastic strain rate $\dot{\varepsilon}_p$ is given by

$$\dot{\varepsilon}_p = \sum_{i=1}^{p} \lambda_i r_i(\Sigma),$$

with prescribed plastic flow directions $r_i : \mathcal{I} \to S$ (possibly multi-valued) and the consistency parameters $\lambda_i \geq 0$ determined by the complementarity conditions

$$0 = \lambda_i f_i(\Sigma), \quad \lambda_i \geq 0, \quad f_i(\Sigma) \leq 0.$$

The equations of the internal variables $\eta$ depend on the model.
The Drucker-Prager Model

The classical Drucker-Prager model is defined by the single yield function

\[ f(\sigma) = |\text{dev}(\sigma)| + k_0(\tan \phi \frac{1}{3} \text{tr}(\sigma) - c) \]

defines the admissible set \( K \), which is a cone with apex \( \sigma_{\text{apex}} = \frac{c}{\tan \phi} \cdot 1 \).

\( k_0 > 0 \) is a shape factor of the cone, \( c \geq 0 \) is related to the cohesion, and \( \phi > 0 \) is the angle of friction.

The non-associated flow rule is based on the plastic potential

\[ g(\sigma) = |\text{dev}(\sigma)| + k_0(\tan \psi \frac{1}{3} \text{tr}(\sigma) - c), \]

where \( \psi \in [0, \phi] \) is the dilatancy. The incremental flow rule is then given as

\[ \varepsilon_p = \varepsilon_p^{\text{old}} + \Delta \lambda \cdot s, \quad s \in \partial g(\sigma) = \{ s \in S : g(\tau) \geq g(\sigma) + s : (\tau - s) \}. \]

Note that in this case \( \partial g(\sigma) = \{ Dg(\sigma) \} \) for \( \text{dev}(\sigma) \neq 0 \).

Since \( f \) is not differentiable, we also consider a smoothed variant with \( \theta > 0 \)

\[ f_\theta(\sigma) = \sqrt{|\text{dev}(\sigma)|^2 + \theta^2 + k_0(\tan \phi \frac{1}{3} \text{tr}(\sigma) - c)}, \]

\[ g_\theta(\sigma) = \sqrt{|\text{dev}(\sigma)|^2 + \theta^2 + k_0(\tan \psi \frac{1}{3} \text{tr}(\sigma) - c)} \]

The Drucker-Prager Model

\[ \frac{1}{k_0} | \text{dev}(\sigma) | \]

\( \dot{\varepsilon}_p \)

\( K \)

\( 0 \)

\( \frac{1}{3} \text{tr}(\sigma) \)
The smoothed Drucker-Prager Model

\[ \frac{1}{k_0} | \text{dev}(\sigma) | \]

\[ \sqrt{c^2 - \frac{\theta^2}{k_0^2}} \]

\[ \frac{c}{\tan \phi} \]

\[ \frac{1}{3} \text{tr}(\sigma) \]

\[ \frac{c - \frac{\theta}{k_0}}{\tan \phi} \]
The Modified Cam-clay Model

Cam-clay plasticity and its variants (see, e.g., Roscoe et al. 1968) are fundamental models in critical state soil mechanics. Here, we consider the modification as in Borja-Lee 1990 or Zouain-Filho-Borges-Costa 2007.

The generalized stress \( \Sigma = (\sigma, \eta) \in S \times \mathbb{R} \) includes a material strength parameter \( \eta > 0 \) (related to the pre-consolidation pressure). The yield function is given as

\[
f(\Sigma) = \frac{3}{2} |\text{dev}(\sigma)|^2 + \frac{M^2}{3} \text{tr}(\sigma) \left( \frac{1}{3} \text{tr}(\sigma) + 2\eta \right).\]

For the plastic strain rate we assume normality, and for the evolution of the strength parameter, a non-associated evolution law is proposed:

\[
\dot{\epsilon}_p = \lambda D_\sigma f(\sigma, \eta),
\]

\[
\dot{\eta} = -k \eta \text{tr}(\dot{\epsilon}_p).
\]

The parameter \( k \) related to the virgin compression and the swell-recompression index.

In the incremental approach we solve the differential equation exactly, i.e.,

\[
\eta = \eta^{\text{old}} \exp \left( -k \text{tr}(\epsilon_p - \epsilon_p^{\text{old}}) \right).
\]

The exact integration guarantees \( \eta > 0 \).
Incremental Plasticity

Our objective is to construct and to analyze generalized stress response functions

$$
R^n = (R^n, E^n) : S \rightarrow \mathcal{S}, \quad \Sigma^n = R^n(\sigma_{tr}).
$$

By means of the stress response function, the incremental elasto-plasticity problem reads as follows: depending on \( \epsilon^{n-1} \) and \( \eta^{n-1} \) find \( \sigma^n \) and \( u^n \) such that

\[
-\text{div} \sigma^n(x) = b(x, t_n), \quad x \in \Omega,
\]

\[
\sigma^n(x) = R^n(\mathbb{C}[\epsilon(u^n(x)) - \epsilon^{n-1}_p(x)]), \quad x \in \Omega,
\]

\[
u^n(x) = u_D(x, t_n), \quad x \in \Gamma_D,
\]

\[
\sigma^n(x)n(x) = t_N(x, t_n), \quad x \in \Gamma_N.
\]

The incremental solution then defines the plastic strain and history variables

\[
\epsilon^{n}_p(x) = \epsilon(u^n(x)) - \mathbb{C}^{-1}[\sigma^n(x)], \quad x \in \Omega,
\]

\[
\eta^n(x) = E^n(\mathbb{C}[\epsilon(u^n(x)) - \epsilon^{n-1}_p(x)]), \quad x \in \Omega.
\]

The FE discretization determines \( u^n \) with \( u^n|_{\Gamma_D} = u_D(t_n) \) and

\[
\int_{\Omega} R^n(\mathbb{C}[\epsilon(u^n) - \epsilon^{n-1}_p]) : \epsilon(w) \, dx = \int_{\Omega} b(t_n) \cdot w \, dx + \int_{\Gamma_N} t_N(t_n) \cdot w \, da
\]

for all test functions with \( w|_{\Gamma_D} = 0 \). In the following we write \( u = u^n, u^{old} = u^{n-1} \).
Implicit Characterization of Response Functions

We determine simultaneously the generalized stress \( \Sigma = (\sigma, \eta) \in \mathcal{S} \) and the consistency parameter \( \triangle \lambda \in \mathbb{R}^p \geq 0 \). Thus, we define the space \( \mathcal{T} = \mathcal{S} \times \mathbb{R}^p \), and we assume that a function \( G : \mathcal{T} \times \mathcal{S} \to \mathcal{S} \) exists, such that
\[
G((\Sigma, \triangle \lambda), \sigma_{tr}) = 0.
\]

Drucker-Prager
\[
G((\sigma, \triangle \lambda), \sigma_{tr}) = \mathbb{C}^{-1}[\sigma - \sigma_{tr}] + \triangle \lambda \mathbb{D}g_\theta(\Sigma)
\]

Cam-Clay
\[
G(((\sigma, \eta), \triangle \lambda), \sigma_{tr}) = \begin{bmatrix}
\mathbb{C}^{-1}[\sigma - \sigma_{tr}] + \triangle \lambda \mathbb{D}_\sigma f(\sigma, \eta) \\
\eta - \eta_{old} \exp \left( k_{tr} \left( \mathbb{C}^{-1}[\sigma - \sigma_{tr}] \right) \right)
\end{bmatrix}
\]

For \( \alpha > 0 \) we define \( \Phi_i(f, \lambda) = \max\{0, \lambda_i + \alpha f_i\} - \lambda_i \) and
\[
T : \mathcal{T} \times \mathcal{S} \to \mathcal{T}, \quad T((\Sigma, \triangle \lambda), \sigma_{tr}) = \begin{bmatrix}
G((\Sigma, \triangle \lambda), \sigma_{tr}) \\
\Phi(f(\Sigma), \triangle \lambda)
\end{bmatrix}.
\]

For the ncp-function holds \( \Phi_i(f, \lambda) = 0 \) if and only if \( 0 = \lambda_i f_i(\Sigma), \lambda_i \geq 0, f_i(\Sigma) \leq 0 \).

For a given trial stress \( \sigma_{tr} \), the solution \( (\Sigma^*, \triangle \lambda^*) \in K \times \mathbb{R}^p \geq 0 \subset \mathcal{T} \) of
\[
T((\Sigma^*, \triangle \lambda^*), \sigma_{tr}) = 0
\]
defines the response \( \Sigma^* = (\sigma^*, \eta^*) = (R(\sigma_{tr}), E(\sigma_{tr})) \).
An Active Set Method

The active index set $\mathcal{A}(\Sigma, \lambda) = \{ i \in \{1, \ldots, p\} : \lambda_i + \alpha f_i(\Sigma) > 0 \}$ and its complement $\mathcal{I}(\Sigma, \lambda)$ defines

$$T((\Sigma, \lambda), \sigma_{\text{tr}}) = \begin{bmatrix} D_\Sigma G((\Sigma, \lambda), \sigma_{\text{tr}}) & D_\lambda G((\Sigma, \lambda), \sigma_{\text{tr}}) \\ \alpha D f(\Sigma)_{\mathcal{A}(\Sigma, \lambda)} & -\text{id}_{\mathcal{I}(\Sigma, \lambda)} \end{bmatrix},$$

where $A_{\mathcal{I}}$ is defined row-wise via $(A_{\mathcal{I}})_i = A_i$ if $i \in \mathcal{I}$ and $(A_{\mathcal{I}})_i = 0$ otherwise.

(AS0) Choose $(\Sigma^0, \lambda^0) \in \mathcal{T}$, $\varepsilon \geq 0$, $\alpha > 0$ and set $k := 1$.
(AS1) If $\|T((\Sigma^{k-1}, \lambda^{k-1}), \sigma_{\text{tr}})\| \leq \varepsilon$,
set $(\Sigma^*, \lambda^*) = (\Sigma^{k-1}, \lambda^{k-1})$ and $T^* = T^{k-1}$, STOP.
(AS2) Set $T^k = T((\Sigma^{k-1}, \lambda^{k-1}), \sigma_{\text{tr}})$.
(AS3) Solve $T^k [\delta \Sigma^k, \delta \lambda^k] = -T((\Sigma^{k-1}, \lambda^{k-1}), \sigma_{\text{tr}})$.
(AS4) Set $(\Sigma^k, \lambda^k) = (\Sigma^{k-1}, \lambda^{k-1}) + (\delta \Sigma^k, \delta \lambda^k)$.
Set $k := k + 1$ and go to (AS1).

This computes the response $\Sigma^* = (\sigma^*, \eta^*) = (R(\sigma_{\text{tr}}), E(\sigma_{\text{tr}}))$ and the Jacobian

$$\begin{bmatrix} \mathcal{S}^* \\ \mathcal{E}^* \\ \Lambda^* \end{bmatrix} = -(T^*)^{-1} C^{-1}.$$
A Generalized Newton Method

If the response function $R$ is Lipschitz continuous, it is differentiable in a dense the set $\Theta_R$, and the set-valued B(ouligand)-subdifferential of $R$ is given as

$$
\partial^B R(\sigma_{\text{tr}}) = \{ S \in \text{Lin}(S, S) : S = \lim_{\theta \to \sigma_{\text{tr}}, \theta \in \Theta_R} DR(\theta) \}.
$$

Clarke’s subdifferential is its convex hull $\partial R(\sigma_{\text{tr}}) = \text{conv} \{ \partial^B R(\sigma_{\text{tr}}) \}$.

$R$ is semismooth, if for any $S \in \partial R(\sigma_{\text{tr}} + \theta)$

$$
|R(\sigma_{\text{tr}} + \theta) - R(\sigma_{\text{tr}}) - S[\theta]| = o(|\theta|) \quad \text{as} \quad \theta \to 0.
$$

(GN0) Choose $u_0^h \in X_h(u_D)$, $\varepsilon \geq 0$, and set $k := 1$.

(GN1) Compute $\sigma_{\text{tr}}^{k-1} = C[\varepsilon(u_{k-1}^h) - \varepsilon_p^{\text{old}}]$, the stress response $\sigma^{k-1} = R(\sigma_{\text{tr}}^{k-1})$, and the residual

$$
r_{k-1}(w_h) = \int_{\Omega} \sigma^{k-1} : \varepsilon(w_h) \, dx - \ell(w_h), \quad w_h \in X_h.
$$

If $\|r_{k-1}\| \leq \varepsilon$, set $u^*_h = u_{k-1}^h$, STOP.

(GN2) Choose $S \in \partial R(\sigma_{\text{tr}}^{k-1})$ and compute $\delta u_h^k \in X_h$ solving the linearization

$$
\int_{\Omega} S[C[\varepsilon(\delta u_h^k)]] : \varepsilon(w_h) \, dx = -r_{k-1}(w_h), \quad w_h \in X_h.
$$

(GN4) Set $u_h^k = u_{h}^{k-1} + \delta u_h^k$ and $k := k + 1$. Go to (GN1).
Results from Non-smooth Analysis

For $T : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ we define

$$
\partial^B_x T(x, y) = \{ A_x \in \mathbb{R}^{N,N} : A_y \text{ exists s.t. } [A_x A_y] \in \partial^B T(x, y) \}.
$$

Theorem (Gowda 04, Kanzow-Heusinger 08)

Let $T$ be semismooth in a neighborhood of a point $(x^*, y^*)$ satisfying $T(x^*, y^*) = 0$, and let all matrices $\partial^B_x T(x^*, y^*)$ be non-singular. Then, there exists an open neighborhood $U(y^*)$ of $y^*$ and a function $Y : U(y^*) \to \mathbb{R}^N$ which is locally Lipschitz and semismooth such that $Y(y^*) = x^*$ and $T(Y(y), y) = 0$ for all $y \in U(y^*)$. Moreover, if $[A_x A_y] \in \partial^B T(Y(y), y)$, we have

$$
-A_x^{-1} A_y \in \partial^B Y(y) \subset \partial Y(y).
$$

A generalized Newton iteration to compute a root $x^*$ of a Lipschitz continuous function $F : \mathbb{R}^N \to \mathbb{R}^N$ is defined by $x^k = x^{k-1} - J_k^{-1} F(x^{k-1})$ with $J_k \in \partial F(x^{k-1})$.

Theorem (Mifflin 77, Clarke 83, Qi-Sun 93 ...)

Let $x^*$ be a solution of $F(x^*) = 0$. Assume that $F$ is semismooth at $x^*$, and that all matrices in $\partial F(x^*)$ are regular. Then, provided that $|x^0 - x^*|$ is small enough, the Newton-iteration is well-defined and converges superlinearly to the solution $x^*$.  

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Semismooth Response for the Drucker-Prager Model

The non-associated deviation between flow rule and plastic potential

\[
\begin{align*}
  f(\sigma) &= |\text{dev}(\sigma)| + k_0 (\tan \phi \frac{1}{3} \text{tr}(\sigma) - c) \\
g(\sigma) &= |\text{dev}(\sigma)| + k_0 (\tan \psi \frac{1}{3} \text{tr}(\sigma) - c)
\end{align*}
\]

is measured by \( G[\varepsilon] = \text{dev}(\varepsilon) + \frac{\tan \psi}{\tan \phi} \frac{1}{3} \text{tr}(\varepsilon) \mathbf{1} \) and \( F = C \circ G \).

The plastic response \( R \) is the orthogonal projection onto the admissible set \( K = \{ \sigma \in S : f(\sigma) \leq 0 \} \) w.r.t. the inner product induced by \( F^{-1} \).

With \( \kappa^\circ(\sigma) := \kappa \tan \phi \tan \psi |\text{dev}(\sigma)| - 2\mu \left( \frac{\tan \phi}{3} \text{tr}(\sigma) - c \right) \), we obtain

\[
R(\sigma_{\text{tr}}) = \begin{cases} 
\sigma_{\text{tr}} & f(\sigma_{\text{tr}}) \leq 0, \\
\sigma_{\text{apex}} & \kappa^\circ(\sigma_{\text{tr}}) \leq 0, \\
\sigma_{\text{tr}} - \frac{f(\sigma_{\text{tr}})}{2\mu + \tan \phi \tan \psi \kappa} F[Df(\sigma_{\text{tr}})] & \text{else}
\end{cases}
\]

for the plastic response, and the consistent tangent is given by

\[
S(\sigma_{\text{tr}}) = \begin{cases} 
\mathbb{I} & f(\sigma_{\text{tr}}) \leq 0, \\
0 & \kappa^\circ(\sigma_{\text{tr}}) \leq 0, \\
\mathbb{I} - \frac{1}{2\mu + \kappa \tan \phi \tan \psi} \left( F[Df(\sigma_{\text{tr}})] \otimes Df(\sigma_{\text{tr}}) + f(\sigma_{\text{tr}})F \circ D^2 f(\sigma_{\text{tr}}) \right) & \text{else}
\end{cases}
\]
Algorithmic Response for the Cam-clay Model

We identify the triple \((\sigma, \eta, \lambda)\) with a vector \(z \in \mathbb{R}^8\), and for given \(\sigma_{tr}\) we solve

\[
T_1(z, \sigma_{tr}) = 0, \quad \ldots \quad T_8(z, \sigma_{tr}) = 0.
\]

This is done with a black box iteration, where the Jacobian \(T\) in (AS) and thus the consistent tangent \(S\) in (GN) is approximated by symmetric finite differences, i.e.,

\[
T = \frac{1}{2\delta} \left(T_i(z + \delta e^j, \sigma_{tr}) - T_i(z - \delta e^j, \sigma_{tr})\right)_{i,j=1,...,8} \in \mathbb{R}^{8 \times 8}.
\]

Here, we choose \(\delta = 5 \cdot 10^{-7}\).

```c
void Constraint (const Tensor& T, double eta_old, const Tensor& S, double eta, double Lambda, SmallVector& c) {
    Tensor DS = S;
    DS -= T;
    DS = StressStrain(DS);
    c[6] = eta - eta_old * exp(CamClay_k * trace(DS));
    DS += Lambda * CamClayFlowDirection_S(S, eta);
    c[0] = DS[0][0]; c[1] = DS[1][1];
    c[2] = DS[2][2]; c[3] = DS[0][1];
    c[4] = DS[0][2]; c[5] = DS[1][2];
    c[7] = CamClayNCP(S, eta, Lambda);
}
```
A Strip Footing with the Drucker-Prager Model

Shear modulus $\mu = 5.5$ [MPa]
Bulk modulus $\kappa = 12.07$ [MPa]
Cohesion $c = 0.01$ [MPa]
Friction angle $\phi = 30^\circ$
Dilatancy angle $\psi = 15^\circ$
Scaling factor $k_0 = 0.7$
Smoothing parameter $\theta = 0.0001$

### Convergence history of the (outer) generalized Newton iteration for Drucker-Prager and smoothed Drucker-Prager elastoplasticity.

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Configuration for a Slope Failure Problem

Symmetry w.r.t. $x_1$

Symmetry w.r.t. $x_2$

Fixed

Load functions $L_g(t)$ and $L_t(t)$

Time and load functions graph

$26.57^\circ$
A Slope Failure Problem with the Drucker-Prager Model

Newton convergence at time step 29 and accumulated plastic strain at time $t = 2.9$.

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A Slope Failure Problem with the Cam-clay Model

Evolution of the material strength parameter $\eta$

Number of Newton steps $k$ in time step $n$

Distribution of the mean stress $\sigma_m$ for the Cam-clay model on refinement level 4 at time $t = 2.9$ (and surface mesh on refinement level 2).